NONLINEAR VOLTERRA-STIELTJES-INTEGRAL EQUATIONS

by Chaim Samuel Hönig

Abstract - In this communication we enounce a theorem that proves the existence and unicity of the solution of the equation

$$y(t) = g(t) - \int_a^t d_sK(t,s).f(s,y(s)) \ , \quad a \leqslant t \leqslant b$$

in a Banach space context, where $y,g: [a,b] \longrightarrow X$ are regulated functions (i.e., functions that have only discontinuities of the first kind), f is regulated as a function of the first variable and lipschitzian as a function of the second variable; $K(t,s) \in L(X)$, $a \leqslant s \leqslant t \leqslant b$, is such that the integral above is a regulated function if y is regulated. We prove that the solution y depends continuously on K, f and g. This theorem extends an analogous result proved by Groh, [1], in the numerical or euclidean case ($X = \mathbb{R}^n$) for an autonomous equation and under more restrictive conditions on f, g and y.

1. Introduction.

In [1] Groh proves the existence and unicity of the solution of the Stieltjes integral equation

$$y(t) = x_0 - \int_a^t dA(s).f(s,y(s)) \ , \quad a \leqslant t \leqslant b$$

where $y,A: [a,b] \longrightarrow \mathbb{R}$ are functions of bounded variation, f is a continuous function that is lipschitzian as a function of the second variable with Lipschitz constant L. The integral is the Young integral. Groh uses the results proved by Hildebrandt, [2], in the linear case $(f(s,y(s)) = y(s))$. If

$$\sup_{a < t \leqslant b} |A(t) - A(t-)| < \frac{1}{L}$$

Groh proves the existence and unicity of the solution of the equation above in applying the Banach fixed point theorem with respect to an appropriately weighted norm. Groh also extends his results to the euclidean case and uses a generalized

Gronwall inequality to prove that the solution depends continuously on x_0.

Our extension, given in the abstract, uses the interior integral $\int\bullet$; this integral has better properties then the Young integral but is "associated" to it, see [6], Théorème 1.16. For $f(s,y(s)) = y(s)$ our theorem reduces to Théorème 3.8 of [6], the theorem of existence, unicity and continuous dependence of the linear Volterra-Stieltjes integral equation

$$y(t) - \int_a^t d_s K(t,s)\cdot y(s) = g(t) \ , \quad a \leq t \leq b \ .$$

Let us recall that this equation encompasses as particular cases the linear Volterra integral equations , linear Stieltjes-integral equations, systems of linear differential equations, linear delay differential equations, linear neutral differential equations etc.; see [3], p. 88 and [7].

2. Notations.

We follow the notations of [6], [4], [5] and [3], but for some small changes.

For the notion of regulated function and the corresponding notations see [4], p. 175, 176; or [5], p. 305; or [6], p. V.5, V.6 .

For the definitions of function of bounded semivariation, of the interior integral $\int\bullet$ and their properties see [4], p. 176, 177, Theorems 1.9, 1.10 and 1.12; or [5], p. 306,307, Theorems 2.1 and 2.2; or [6], p. V.6, V.7, Théorèmes 1.10, 1.11 and 1.13.

For operators defined by kernels K and their main properties see [4], p. 182, Theorems 2.6 and 2.10 [we take $t_0 = a$ and we may always suppose that $K(t,t) = 0$ and then we write $K \in G_0^{\sigma}\cdot SV^u(\Gamma_a^b, L(X))$]; or [5], p. 309, 310, 311, Theorem 2.6; or [6], p. V.11, Théorèmes 2.1, 2.2, 2.8 and 2.10 .

3. Auxiliary results.

We say that a function $f: [a,b] \times X \longrightarrow Y$ is lipschitzian as a function of the second variable if there exists $L \geq 0$ such that

$$\|f(t,x_2) - f(t,x_1)\| \leq L \|x_2 - x_1\| \quad \text{for all} \quad t \in [a,b] \quad \text{and} \quad x_1, x_2 \in X;$$

we denote by $[f]$ the smallest constant L that satisfies this inequality.

We denote by $G.Lip([a,b] \times X,Y)$ the set of all functions $f: [a,b]\times X \longrightarrow Y$ that are regulated as functions of the first variable and lipschitzian as functions of the second variable.

3.1 - If $f: [a,b] \times X \longrightarrow Y$ is lipschitzian as a function of the second variable, the following properties are equivalent:

1.- $f \in G.Lip([a,b] \times X,Y)$

2 - For every $\varphi \in G([a,b],X)$ the function $t \in [a,b] \longmapsto f(t,\varphi(t)) \in Y$ is regulated.

3.2 - $G.Lip([a,b] \times X,Y)$ is a Banach space when endowed with the norm $\|f\|_{Lip} = \sup\{\|f_0\|, [f]\}$ where $\|f_0\| = \sup_{a \leq t \leq b} \|f(t,0)\|$.

For $K \in G_0^{\sigma}.SV^u = G_0^{\sigma}.SV^u(\Gamma_a^b, L(X))$, $f \in G.Lip = G.Lip([a,b] \times X,X)$ and $g,u \in G = G([a,b],X)$ we define

$$(Tu)(t) = (T^{K,f,g}\, u)(t) = g(t) - \int_a^t d_s K(t,s).f(s,u(s)) \ , \quad a \leq t \leq b$$

3.3 - The mapping

$$(K,f,g,u) \in G_0^{\sigma}.SV^u \times G.Lip \times G \times G \longmapsto T^{K,f,g}\, u \in G$$

is continuous.

4. The main theorem.

Theorem - Let $K \in G_0^{\sigma}.SV^u$ and $f \in G.Lip$ be such that there exists a division $d: t_0 = a < t_1 < t_2 < \ldots < t_n = b$ of $[a,b]$ such that $\bar{c}(K,d^*).[f] < 1$, where

$$\bar{c}(K,d^*) = \sup\left\{ \|K(t_i^+,t_i)\|,\ \sup_{t_i \leq t \leq t_{i+1}} SV_{]t_i,t]}[K^t] \ \middle|\ 0 \leq i \leq n-1 \right\} .$$

Then for every $g \in G$ there exists one and only one $y = y^{K,f,g} \in G$ that is solution of

$$y(t) = g(t) - \int_a^t d_s K(t,s).f(s,y(s)) \ , \quad a \leq t \leq b .$$

There exist neighborhoods V_K and V_f of K and f, respectively, such that the result above is still true for every $(\tilde{K},\tilde{f},\tilde{g}) \in V_K \times V_f \times G$; the mapping

$$(\tilde{K},\tilde{f},\tilde{g}) \in V_K \times V_f \times G \longmapsto \tilde{y} = y^{\tilde{K},\tilde{f},\tilde{g}} \in G$$

is continuous.

Sketch of proof: we prove that for $c > 0$ such that $\bar{c}(K,d^{\bullet}).[f] < c < 1$ there exist $n \geq 1$ and a neighborhood $V_K \times V_f$ of (K,f) such that for all $(\tilde{K},\tilde{f}) \in V_K \times V_f$ we have $\bar{c}(\tilde{K},d^{\bullet}).[\tilde{f}] < c$ and such that $\tilde{T}^n$ (cf. 3.3) is a contraction of G with the same contraction constant < 1 ; the result follows from the uniform contraction theorem; see [3], p. 5 .

Remarks: 1 - The theorem remains true if we suppose only that $\bar{c}(K^-,d^{\bullet}).[f^-] < 1$, where $K^-(t,s) = K(t-,s)$ for $a \leq s < t \leq b$ and where for $\varphi \in G$ we define $f^-(t,\varphi(t)) = \lim_{s \uparrow t} f(s,\varphi(s)) = f(t-,\varphi^-(t))$ if $a < t \leq b$ and $f^-(a,\varphi(a)) = \lim_{s \downarrow a} f(s,\varphi(s)) = f(a+,\varphi(a+))$; cf. [6], Théorème 3.6 .

2 - In the linear case, i.e., for $f(s,y(s)) = y(s)$, the theorem above reduces to Théorème 3.8 of [6].

3 - The results of this communication may be extended to integral equations with respect to vector measures; this generalization encompasses hyperbolic partial differential equations.

REFERENCES

[1] J.Groh, A nonlinear Volterra-Stieltjes integral equation and a Gronwall inequality in one dimension, Illinois J. of Math., 24(1980), 244-263

[2] T.H.Hildebrandt, On systems of linear differentio-Stieltjes-integral equations, Illinois J. of Math., 3(1959), 352-373.

[3] C.S.Hönig, Volterra-Stieltjes integral equations. Mathematics Studies 16, North-Holland Publishing Comp., Amsterdam, 1975

[4] ———— Volterra-Stieltjes integral equations; in Functional Differential Equations and Bifurcation; Proceedings of the São Carlos Conference, 1979, Springer Lecture Notes in Mathematics, 799, pp. 173-216

[5] ———— The resolvent of a linear Stieltjes-integral equation, 12º Semin. Bras. de Análise, 1980, pp. 301-344.

[6] ———— Equations intégrales généralisées et applications. Publications Mathématiques d'Orsay, 83-01 (1983), pp. V.1-V.50.

[7] L.Fichmann, Equações integrais de Volterra-Stieltjes e equações de tipo neutro. Master thesis, Instituto de Matemática e Estatística da USP, 1984

Instituto de Matemática e Estatística da Universidade de São Paulo
Caixa Postal 20.570
São Paulo - Brazil