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Exposé n°2
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STRICTLY p-INTEGRAL AND p-NUCLEAR OPERATORS

ABSTRACT. In this note we present conditions under which every strictly p-integral operator is compact or p-nuclear, first independently of the range space and afterwards of the domain.

I. DEFINITIONS AND EXAMPLES

In all that follows X, Y and Z are Banach spaces and $L(X,Y)$, $K(X,Y)$ and $W(X,Y)$ denote respectively the spaces of all bounded linear operators, of the compact operators, and of the weakly compact operators from X into Y. Also, RNP and wRNP are shortenings for the Radon-Nikodym and the weak Radon-Nikodym properties, respectively.

DEFINITION I.1. A linear mapping $T : X \to Y$ is <u>p-nuclear</u>, $1 \leqslant p \leqslant \infty$, if there are sequences $\{x_n^\star\}$ in $X^\star$ and $\{y_n\}$ in Y such that

(i) $Tx = \sum_n \langle x_n^\star , x \rangle y_n$, for every $x \in X$;

(ii) $\sum_n \|x_n^\star\|^p < \infty$, if $1 \leqslant p < \infty$,

and $\lim_n \|x_n^\star\| = 0$, if $p = \infty$;

(iii) $\sup \{\sup_n |\langle y^\star, y_n \rangle| : y^\star \in B_{Y^\star}\} = \sup_n \|y_n\| < \infty$, if $p = 1$,

and $\sup\{(\sum_n |\langle y^\star, y_n\rangle|^q)^{1/q} : y^\star \in B_{Y^\star}\} < \infty$, if $1 < p \leqslant \infty$, where q is the conjugate exponent of p.

In this case, the <u>p-nuclear norm</u> of T is given by

$n_1(T) = \inf\{(\sum_n \|x_n^\star\|) \sup_n \|y_n\|\}$, if $p = 1$,

$n_p(T) = \inf\{(\sum_n \|x_n^\star\|^p)^{1/p} \sup(\sum_n |\langle y^\star, y_n\rangle|^q)^{1/q} : y^\star \in B_{Y^\star}\}$, if $1 < p < \infty$, and

$n_\infty(T) = \inf\{(\sup_n \|x_n^\star\|) \sup \{\sum_n |\langle y^\star, y_n\rangle| : y^\star \in B_{Y^\star}\}$, if $p = \infty$, where the infima are taken over all possible representations of T as in (i), satisfying (ii) and (iii).

The set of p-nuclear linear mappings from X into Y is denoted by $N_p(X,Y)$.

DEFINITION I.2. A linear mapping $T : X \to Y$ is strictly p-integral, $1 \leq p \leq \infty$, if there is a σ-additive vector measure $G : \mathcal{B}(B_{X^\star}) \to Y$ such that

(i) $Tx = \int_{B_{X^\star}} \langle x^\star, x \rangle \, dG(x^\star)$, for every $x \in X$;

(ii) If $p < \infty$, there exists $\mu \in rca^+(\mathcal{B}(B_{X^\star}))$ such that

$$\left\| \int_{B_{X^\star}} f \, dG \right\| \leq \left[\int_{B_{X^\star}} |f|^p \, d\mu \right]^{1/p} ,$$

for every $f \in C(B_{X^\star})$.

In this case, the strictly p-integral norm of T is given by

$si_p(T) = \inf\{\mu(B_{X^\star})^{1/p} : G$ and μ satisfy (i) (ii)$\}$, if $1 \leq p < \infty$, and

$si_\infty(T) = \inf\{\|G\|(B_{X^\star}) : G$ satisfies (i)$\}$, if $p = \infty$.

The set of strictly p-integral linear mappings from X into Y is denoted by $SI_p(X,Y)$.

DEFINITION I.3. A linear mapping $T : X \to Y$ is p-integral, $1 \leq p \leq \infty$, if $J_Y \circ T : X \to Y^{\star\star}$ is strictly p-integral, where J_Y is the canonical injection of Y into $Y^{\star\star}$.

In this case, the p-integral norm of T is given by $i_p(T) = si_p(J_Y \circ T)$.

The set of p-integral linear mappings from X into Y is denoted by $I_p(X,Y)$.

REMARK. It can be shown that $N_p(X,Y) \subset SI_p(X,Y) \subset I_p(X,Y) \subset L(X,Y)$, with $\|.\| \leq i_p(.) \leq si_p(.) \leq n_p(.)$, and that all these sets of operators are Banach spaces when endowed with their respective norms.

Also, $N_p(X,Y)$, $SI_p(X,Y)$ and $I_p(X,Y)$ have an ideal structure, ie, the composition either on the left or on the right side of an arbitrary operator with one in any of these classes produces a new element of the same class. For this reason they are usually refered as operator ideals.

It is worthy to note that $N_p(X,Y) \subset K(X,Y)$. Also, $SI_p(X,Y) \subset W(X,Y)$ and $SI_p(X,Y) \subset C(X,Y)$, where $C(X,Y)$ denotes the set of completely continuous operators from X into Y.

EXAMPLE I.4. Canonical p-nuclear operators, $1 \leqslant p \leqslant \infty$.

Let $\delta = \{\delta_n\} \varepsilon \ell_p$, if $1 \leqslant p < \infty$, and $\delta \varepsilon c_0$, if $p = \infty$, and define diagonal operators $\Delta_p : \ell_\infty \to \ell_p$, $1 \leqslant p < \infty$, and $\Delta_\infty : \ell_\infty \to c_0$ by

$$\Delta_p\{a_n\} = \{\delta_n a_n\}, \text{ for all } \{a_n\} \varepsilon \ell_\infty.$$

Then $\Delta_p \varepsilon N_p(\ell_\infty, \ell_p)$, $1 \leqslant p < \infty$, and $\Delta_\infty \varepsilon N_\infty(\ell_\infty, c_0)$, with $\|\Delta_p\| = \|\delta\|_p = n_p(\Delta_p)$.

They are called canonical because every p-nuclear operator admits Δ_p as a factor (Theorem I.7).

EXAMPLE I.5. Canonical strictly p-integral operators, $1 \leqslant p \leqslant \infty$.

Let K be a compact Hausdorff space, $\nu \varepsilon rca^+(\mathscr{B}(K))$, (Ω,Σ,μ) be a finite measure space and J_p be either the inclusion of $C(K)$ into $L_p(K,\nu)$ or the inclusion of $L_\infty(\Omega,\mu)$ into $L_p(\Omega,\mu)$, for $1 \leqslant p < \infty$. Then J_p is a strictly p-integral operator, with $\|J_p\| = si_p(J_p)$. In the first case, $si_p(J_p) = \nu(K)^{1/p}$ and in the second one, $si_p(J_p) = \mu(\Omega)^{1/p}$. They are canonical because any strictly p-integral operator, $1 \leqslant p < \infty$, admits J_p as a factor, for some compact Hausdorff space K and some finite measure space (Ω,Σ,μ) (Theorem I.8).

For $p = \infty$, we have $W(C(K),Y) = SI_\infty(C(K),Y)$ and $W(L_\infty(\mu),Y) = SI_\infty(L_\infty(\mu),Y)$, for any Banach space Y, and $\|.\| = si_\infty(.)$. Any strictly ∞-integral operator $T : X \to Y$ is such that $T = B \circ A$, where $A \varepsilon L(X,C(K))$ and $B \varepsilon W(C(K),Y)$ for some compact Hausdorff space K, or $A \varepsilon L(X,L_\infty(\mu))$ and $B \varepsilon W(L_\infty(\mu),Y)$, for some finite measure space (Ω,Σ,μ) (Theorem I.8).

EXAMPLE I.6.

Let $i_r : \ell_1 \to \ell_r$, $2 \leqslant r < \infty$ and $i_0 : \ell_1 \to c_0$ be the canonical inclusions.

Then $i_r \in SI_p(\ell_1,\ell_r)$, for any p, $1 < p \leqslant \infty$, and $i_0 \in SI_p(\ell_1,c_0)$, for any p, $1 < p \leqslant \infty$.

Also, $i_2 \notin SI_1(\ell_1,\ell_2)$, but $K_{\ell_2} \circ i_2 \in SI_1(\ell_1,\ell_\infty(B_{\ell_2}))$, where $\ell_\infty(B_{\ell_2})$ is the canonical injective space associated to ℓ_2 and K_{ℓ_2} is the canonical injection of ℓ_2 into $\ell_\infty(B_{\ell_2})$. Recall that for a Banach space Z , $\ell_\infty(B_{Z^\star})$ is the space of bounded families of scalars $\{a_{z^\star}\}_{z^\star \in Z^\star}$, with norm $\|\{a_{z^\star}\}\| = \sup\{|a_{z^\star}| : z^\star \in Z^\star\}$ and that Z can be identified with a subspace of $\ell_\infty(B_{Z^\star})$ through the mapping $K_Z : Z \to \ell_\infty(B_{Z^\star})$, given by $K_Z(z) = \{z^\star(z)\}_{z^\star \in Z^\star}$.

THEOREM I.7

I. Let $1 \leqslant p < \infty$. An operator T belongs to $N_p(X,Y)$ if and only if it admits a factorization as

$$\begin{array}{ccc} X & \xrightarrow{\ T\ } & Y \\ p\downarrow & & \uparrow Q \\ \ell_\infty & \xrightarrow[\Delta_p]{} & \ell_p \end{array}$$

where $P \in L(X,\ell_\infty)$, $Q \in L(\ell_p,Y)$ and Δ_p is a diagonal operator, ie, there is $\delta = \{\delta_n\} \in \ell_p$ such that $\Delta_p\{a_n\} = \{a_n\delta_n\}$, for $\{a_n\} \in \ell_\infty$.

II. An operator T belongs to $N_\infty(X,Y)$ if and only if it admits a factorization as

$$\begin{array}{ccc} X & \xrightarrow{\ T\ } & Y \\ p\downarrow & & \uparrow Q \\ \ell_\infty & \xrightarrow[\Delta_\infty]{} & c_0 \end{array}$$

where $P \in L(X,\ell_\infty)$, $Q \in L(c_0,Y)$ and Δ_∞ is a diagonal operator as in Example I.4.

Moreover, given $\varepsilon > 0$, a factorization can be chosen in I or II such that $\|P\| \leqslant 1$, $\|Q\| \leqslant 1$ and $n_p(\Delta_p) \leqslant n_p(T) + \varepsilon$.

THEOREM I.8.

I. Let $1 \leq p < \infty$. The following conditions are equivalent :

(1) $T \varepsilon SI_p(X,Y)$;

(2) T admits a factorization as

$$\begin{array}{ccc} X & \xrightarrow{\ T\ } & Y \\ A \big\downarrow & & \big\uparrow B \\ C(K) & \xrightarrow[J_p]{} & L_p(\mu) \end{array}$$

where K is a compact Hausdorff space, $\mu \varepsilon rca^+(\mathscr{B}(K))$, J_p is the canonical injection of $C(K)$ into $L_p(K,\mu)$, $A \varepsilon L(X,L_\infty(K,\mu))$ and $B \varepsilon L(L_p(\Omega,\mu),Y)$;

(3) T admits a factorization as

$$\begin{array}{ccc} X & \xrightarrow{\ T\ } & Y \\ A \big\downarrow & & \big\uparrow B \\ L_\infty(\mu) & \xrightarrow[J_p]{} & L_p(\mu) \end{array}$$

where (Σ,Ω,μ) is a finite measure space, J_p is the canonical injection of $L_\infty(\Omega,\mu)$ into $L_p(\Omega,\mu)$, $A \varepsilon L(X,L_\infty(\Omega,\mu))$ and $B \varepsilon L(L_p(\Omega,\mu),Y)$.

In this case, given $\varepsilon > 0$, it is possible to choose a factorization as in (2) or (3) such that $\|A\| \leq 1$,$\|B\| \leq 1$ and $si_p(J_p) \leq si_p(T) + \varepsilon$.

II. The following conditions are equivalent :

(1) $T \varepsilon SI_\infty(X,Y)$;

(2) T admits a factorization as

$$\begin{array}{ccc} X & \xrightarrow{\ T\ } & Y \\ & A \searrow \quad \nearrow B & \\ & C(K) & \end{array}$$

where K is a compact Hausdorff space, $A \varepsilon L(X,C(K))$ and $B \varepsilon W(C(K),Y)$;

(3) T admits a factorization as

$$\begin{array}{ccc} X & \xrightarrow{\ T\ } & Y \\ & A \searrow \quad \nearrow B & \\ & L_\infty(\mu) & \end{array}$$

where (Ω,Σ,μ) is a finite measure space, $A \in L(X,L_\infty(\Omega,\mu))$ and $B \in W(L_\infty(\Omega,\mu),Y)$.

In this case, given $\varepsilon > 0$, it is possible to choose a factorization as in (2) or (3) such that $\|A\| \leq 1$ and $\|B\| = si_\infty(B) \leq si_\infty(T) + \varepsilon$.

REMARK. More information about these operator ideals can be found in [PP] and [CA].

In this work, we present conditions under which $SI_p(X,.) = N_p(X,.)$ and $SI_p(.,Y) = N_p(.,Y)$, for $1 < p \leq \infty$. These problems were suggested by the following well-known results.

THEOREM A. $X^\star$ has RNP $\Leftrightarrow SI_1(X,.) = N_1(X,.)$, with $si_1 = n_1$.

THEOREM B. Y has RNP $\Leftrightarrow SI_1(.,Y) = N_1(.,Y)$, with $si_1 = n_1$.

II. SPACES X SUCHT THAT $SI_p(X,.) = N_p(X,.)$

Our first result is a characterization of the spaces X for which $SI_p(X,.) \subset K(X,.)$, for one or all p, $1 \leq p \leq \infty$. It is interesting to note that the characterization does not depend on p, and that it is equivalent to $I_p(X,.) \subset K(X,.)$.

THEOREME II.1. The following conditions on X are equivalent :

(1) $X^\star$ has the weak Radon-Nikodym property ;

(2) $X \not\supset \ell_1$;

(3) for every p, $1 \leq p \leq \infty$, $SI_p(X,.) \subset K(X,.)$;

(4) there exists p, $1 \leq p \leq \infty$, such that $SI_p(X,.) \subset K(X,.)$;

(5) for every p, $1 \leq p \leq \infty$, $I_p(X,.) \subset K(X,.)$;

(6) there exists p, $1 \leq p \leq \infty$, such that $I_p(X,.) \subset K(X,.)$.

PROOF. (1) $\Leftrightarrow$ (2) It is well-known [DUP-page 151].

(2) $\Rightarrow$ (3) Let p, $1 \leq p \leq \infty$, Y and $T \in SI_p(X,Y)$ be given.

Then $T = B \circ A$ where $A \in L(X, C(K))$ and $B \in W(C(K), Y)$, for some compact Hausdorff space K.

Let $\{x_n\}$ be a bounded sequence in X. Since $X \not\hookleftarrow \ell_1$, from Rosenthal's Theorem it follows that $\{x_n\}$ has a weakly Cauchy subsequence $\{x_{n_k}\}$, so that $\{Ax_{n_k}\}$ is a weakly Cauchy sequence in $C(K)$. But $W(C(K),Y) = C(C(K),Y)$ [DU-VI.2.17] and then $\{Tx_{n_k}\} = \{B(Ax_{n_k})\}$ is norm convergent in Y because B is completely continuous. Hence T is a compact operator.

(3) ⇒ (4) It is obvious.

(4) ⇒ (2) From Example I.6, for any p, $1 \leqslant p \leqslant \infty$, there are a space Y and an operator $T : \ell_1 \to Y$ such that $T \in SI_p(\ell_1, Y)$ but $T \notin K(\ell_1, Y)$. Namely, $Y = \ell_2$ and $T = i_2$, if $1 < p \leqslant \infty$, and $Y = \ell_\infty(B_{\ell_2})$ and $T = K_{\ell_2} \circ i_2$, if $p = 1$.

Then T admits a factorization $T = B \circ A$, with $A \in L(\ell_1, L_\infty(\mu))$ and $B \in L(L_\infty(\mu), Y)$, for some finite measure space (Ω, Σ, μ).

Suppose $X \hookleftarrow \ell_1$. By the extension property of $L_\infty(\mu)$, there is an operator $\hat{A} \in L(X, L_\infty(\mu))$, such that $\hat{A}|_{\ell_1} = A$.

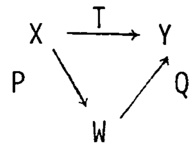
The operator $\hat{T} = B \circ \hat{A} : X \to Y$ is strictly p-integral because B is, and is not compact, otherwise $\hat{T}|_{\ell_1} = T$ would be compact, too.

This contradicts (4). Hence $X \not\hookleftarrow \ell_1$.

(3) ⇔ (5) and (4) ⇔ (6) Since $T \in I_p(X,Y)$ if and only if $J_Y \circ T \in SI_p(X, Y^{**})$, for any p, $1 \leqslant p \leqslant \infty$, and $T \in K(X,Y)$ if and only if $J_Y \circ T \in K(X, Y^{**})$ for J_Y is an isometry, it follows that $I_p(X,.) \subset K(X,.)$ if and only if $SI_p(X,.) \subset K(X,.)$. □

While a necessary condition for $SI_p(X,.) = N_p(X,.)$ is given by Theorem II.1, we have a sufficient condition in Theorem II.4, as a consequence of a multiplication theorem.

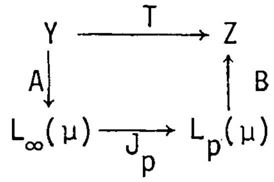
DEFINITION II.2. An operator $T : X \to Y$ is an Asplund operator if it admits a factorization as



where $P \in L(X,W)$, $Q \in L(W,Y)$ and $W^\star$ has RNP (ie, W is an Asplund space).

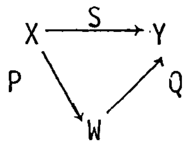
THEOREM II.3. Let $1 \leqslant p \leqslant \infty$. If $S \in L(X,Y)$ is an Asplund operator, and $T \in SI_p(Y,Z)$, then $T_\circ S \in N_p(X,Z)$ and $n_p(T_\circ S) \leqslant \|S\|\ si_p(T)$.

PROOF. Let $p < \infty$. If $T \in SI_p(Y,Z)$ and $\varepsilon > 0$ are given, choose a factorization



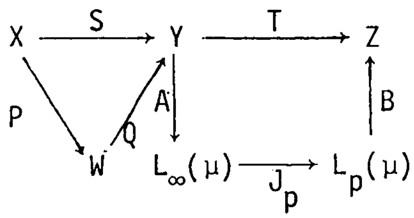
where (Ω,Σ,μ) is a finite measure space, $A \in L(Y,L_\infty(\mu))$; $\|A\| \leqslant 1$, $B \in L(L_p(\mu),Z)$, $\|B\| \leqslant 1$ and $\mu(\Omega)^{1/p} \leqslant si_p(T) + \varepsilon$.

Since S is an Asplund operator, it admits a factorization



where $W^\star$ has RNP , $P \in L(X,W)$, $Q \in L(W,Y)$, $\|Q\| \leqslant 1$ and $\|P\| \leqslant \|S\| + \varepsilon$.

Then



Consider $A_\circ Q : W \to L_\infty(\mu)$ and $(A_\circ Q)^\star : L_\infty(\mu)^\star \to W^\star$. Let $R = (A_\circ Q)^\star|_{L_1(\mu)}$. Since $W^\star$ has RNP, R is representable by a bounded μ-mesurable function $g : \Omega \to W^\star$,

$$Rf = \int_\Omega f(t)g(t)d\mu(t), \text{ for } f \in L_1(\mu) ,$$

with $\|g\|_\infty = \|R\| \leqslant \|(A_\circ Q)^\star\| = \|A_\circ Q\| \leqslant 1.$

Defining $F : \Omega \to W^\star$ by

$$\langle F(t),w\rangle = [(A_\circ Q)w](t) \ , \text{ for } t \in \Omega \text{ and } w \in W \ ,$$

one has $\langle F(.),w\rangle \in L_\infty(\mu)$ for every $w \in W$, so that the integral $\int_\Omega f(t)\langle F(t),w\rangle d\mu(t)$ exists, for every $f \in L_1(\mu)$ and $w \in W$.

But

$$\int_\Omega f(t)\langle F(t),w\rangle d\mu(t) = \int_\Omega f(t)\langle (A_\circ Q)w\rangle(t)d\mu(t)$$

$$= \langle f,(A_\circ Q)w\rangle = \langle Rf,w\rangle \ ,$$

for every $f \in L_1(\mu)$ and $w \in W$.

Then $\langle F(t),w\rangle = \langle g(t),w\rangle$, μ-a.e. in Ω, for every $w \in W$. Hence $F = g$ μ-a.e. and $F \in L_\infty(\mu,W^\star)$.

Case 1. F has countable range.

Let $\{u_n^\star : n \in \mathbb{N}\}$ be the set of distinct nonzero essential values of F. Defining, for each $n \in \mathbb{N}$, $E_n = F^{-1}\{u_n^\star\}$, one has that E_n is μ-measurable and $\mu(E_n) > 0$. Moreover, $\{E_n\}$ is a pairwise disjoint sequence in Σ.

Let the sequences $\{w_n^\star\}$ in $W^\star$ and $\{f_n\}$ in $L_p(\mu)$ be given by

$$w_n^\star = \mu(E_n)^{1/p}\, u_n^\star \quad \text{and} \quad f_n = \mu(E_n)^{-1/p}\chi E_n \ ,$$

for $n \in \mathbb{N}$. Then

$$\sum_n \|w_n^\star\|^p = \sum_n \|u_n^\star\|^p\, \mu(E_n) = \|F\|_p^p \leqslant \|F\|_\infty^p\, \mu(\Omega) \ ,$$

and, for $h \in L_q(\mu) = L_p(\mu)^\star$,

$$\sum_n |\langle h,f_n\rangle|^q = \sum_n \mu(E_n)^{-q/p}\left|\int_{E_n} h\,d\mu\right|^q$$

$$\leqslant \sum_n \mu(E_n)^{-q/p}\left[\left(\int_{E_n} 1^p d\mu\right)^{1/p}\left(\int_{E_n} |h|^q\, d\mu\right)^{1/q}\right]^q$$

$$= \sum_n \int_{E_n} |h|^q\, d\mu \leqslant \|h\|_q^q \quad , \text{ if } 1 < q < \infty,$$

and

$$\sup_n |\langle h,f_n\rangle| = \sup_n \mu(E_n)^{-1}\left|\int_{E_n} h\,d\mu\right| \leqslant \|h\|_\infty \ , \text{ if } q = \infty.$$

Hence

$$\sup\{(\sum_n |\langle h,f_n\rangle|^q)^{1/q} : h \in B_{L_q}\} \leq 1 \text{ , if } 1<q<\infty ,$$

and

$$\sup\{\sup_n |\langle h,f_n\rangle| : h \in B_{L_\infty}\} \leq 1 \text{ , if } q=\infty ,$$

so that $\{w_n^\star\}$ and $\{f_n\}$ satisfy (ii) and (iii) of Definition I.1.

Since, for $w \in W$,

$$(J_p \circ A \circ Q)w = \langle F(.),w\rangle = \sum_n \langle u_n^\star,w\rangle \chi_{E_n}(.) = \sum_n \langle w_n^\star,w\rangle f_n ,$$

it follows that $J_p \circ A \circ Q \in N_p(W,L_p(\mu))$, with $n_p(J_p \circ A \circ Q) \leq \|F\|_\infty \mu(\Omega)^{1/p} \leq si_p(T) + \varepsilon$, for $\|F\|_\infty = \|g\|_\infty \leq 1$.

Then $T \circ S = B \circ J_p \circ A \circ Q \circ P \in N_p(X,Z)$ and $n_p(T \circ S) \leq (\|S\| + \varepsilon)(si_p(T) + \varepsilon)$. Since ε is arbitrary, $n_p(T \circ S) \leq \|S\| si_p(T)$.

<u>Case 2</u>. F does not have necessarily countable range.

Since F is μ-measurable, there is a sequence $\{F_n\}$ in $L_\infty(\mu,W^\star)$, each F_n with countable range, such that $\|F_n - F\|_\infty \to 0$ [DU-II.1.3].

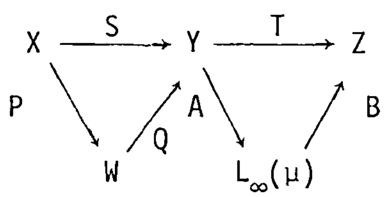
Defining $R_n : W \to L_\infty(\mu)$ for each $n \in \mathbb{N}$ by

$$R_n w = \langle F_n(.),w\rangle , \quad \text{for } w \in W ,$$

one has $\|R_n\| = \|F_n\|_\infty$ and $J_p \circ R_n \in N_p(W,L_p(\mu))$, with $n_p(J_p \circ R_n) \leq \|F_n\|_\infty \mu(\Omega)^{1/p}$, from Case 1. Also, $J_p \circ (R_n - R_m) \in N_p(W,L_p(\mu))$, with $n_p(J_p \circ (R_n - R_m)) \leq \|F_n - F_m\|_\infty \mu(\Omega)^{1/p}$, for any $n, m \in \mathbb{N}$, for the same reason, so that $\{J_p \circ R_n\}$ is a Cauchy sequence in $N_p(W,L_p(\mu))$, thus n_p-norm convergent to some element in this space of operators. But $\{J_p \circ R_n\}$ is norm convergent to $J_p \circ A \circ Q$ and since $\|.\| \leq n_p(.)$, $\{J_p \circ R_n\}$ converges in norm to $J_p \circ A \circ Q$.

Hence, $J_p \circ A \circ Q \in N_p(W,L_p(\mu))$ and $n_p(J_p \circ A \circ Q) \leq \|F\|_\infty \mu(\Omega)^{1/p} \leq si_p(T) + \varepsilon$. As before, it follows that $T \circ S \in N_p(X,Z)$ and $n_p(T \circ S) \leq \|S\| si_p(T)$.

Now let $p = \infty$. Given $\varepsilon > 0$, choose factorizations for S and T



where $W^\star$ has RNP, $P \in L(X,W)$, $Q \in L(X,Y)$, $\|Q\| \leq 1$, $\|P\| \leq \|S\| + \varepsilon$, (Ω,Σ,μ) is a finite measure space, $A \in L(Y,L_\infty(\mu))$, $\|A\| \leq 1$, and $B \in W(L_\infty(\mu),Z)$ is represented by a σ-additive vector measure $G : \Sigma \to Z$, with $\|G\|(\Omega) \leq si_\infty(T) + \varepsilon$.

As before, the operator $R = (A_\circ Q)^\star|_{L_1(\mu)}$ is representable by a function $F \in L_\infty(\mu,W^\star)$ given by

$$\langle F(t),w \rangle = [(A_\circ Q)w](t), \text{ for } t \in \Omega \text{ and } w \in W,$$

through

$$\int_\Omega f(t)\langle F(t),w \rangle d\mu(t) = \langle f,(A_\circ Q)w \rangle = \langle Rf,w \rangle ,$$

for every $f \in L_1(\mu)$ and $w \in W$.

Again, there are two cases.

<u>Case 1'</u>. F has countable range.

Let $\{u_n^\star : n \in \mathbb{N}\}$ be the set of distinct nonzero essential values of F. Defining, for each $n \in \mathbb{N}$, $E_n = F^{-1}\{u_n^\star\}$, one has that $\{E_n\}$ is a pairwise disjoint sequence in Σ, with $\mu(E_n) > 0$, for $n \in \mathbb{N}$.

For $w \in W$,

$$(B_\circ A_\circ Q)w = \int_\Omega [(A_\circ Q)w](t)dG(t) = \int_\Omega \langle F(t),w \rangle dG(t)$$

$$= \sum_n \langle u_n^\star,w \rangle G(E_n) ,$$

and $\sup_n \|u_n^\star\| = \|F\|_\infty \leq \|A_\circ Q\| \leq 1$.

Since G is σ-additive, and $\{E_n\}$ is a pairwise disjoint sequence,

$$\sup \{\sum_n |\langle z^\star,G(E_n)\rangle| : z^\star \in B_{Z^\star}\} \leq \|G\|(\Omega) < \infty$$

and

$$\lim_n \sum_{m>n} |\langle z^\star,G(E_m)\rangle| = 0 \text{ , uniformly in } z^\star \in B_{Z^\star}$$

[DU-I.1.18] . Let $\delta > 0$ be given and choose $1 \leq n_1 < n_2 < \ldots$ such that

$$\sum_{n>n_k} |\langle z^\star,G(E_n)\rangle| \leq \delta/_{k2^k} \text{ , for } k \in \mathbb{N} \text{ and } z^\star \in B_{Z^\star} .$$

Defining sequence $\{w_n^\star\}$ in $W^\star$ and $\{z_n\}$ in Z by

$$w_n^\star = \begin{cases} u_n^\star , & \text{if } 1 \leq n \leq n_1 , \\ k^{-1} u_n^\star , & \text{if } n_k < n \leq n_{k+1} , \text{ for } k \geq 1, \end{cases}$$

and

$$z_n = \begin{cases} G(E_n) , & \text{if } 1 \leq n \leq n_1 , \\ kG(E_n) , & \text{if } n_k < n \leq n_{k+1} , \text{ for } k \geq 1 , \end{cases}$$

one has $\lim_n \|w_n^\star\| = 0$, $\sup_n \|w_n^\star\| \leq 1$, because

$$\|w_n^\star\| \leq \begin{cases} \|F\|_\infty , & \text{if } 1 \leq n \leq n_1 , \\ k^{-1}\|F\|_\infty , & \text{if } n_k < n \leq n_{k+1} , \text{ for } k \geq 1 , \end{cases}$$

and $\|F\|_\infty \leq 1$; also, for each $z^\star \in B_{Z^\star}$,

$$\sum_n |\langle z^\star, z_n \rangle| = \sum_{n=1}^{n_1} |\langle z^\star, G(E_n)\rangle| + \sum_k \sum_{n=n_k+1}^{n_{k+1}} |\langle z^\star, G(E_n)\rangle|$$

$$\leq \sum_n |\langle z^\star, G(E_n)\rangle| + \sum_k k(\sum_{n>n_k} |\langle z^\star, G(E_n)\rangle|)$$

$$\leq \|G\|(\Omega) + \sum_k k\delta/_{k2^k} = \|G\|(\Omega) + \delta .$$

Then $\{w_n^\star\}$ and $\{z_n\}$ satisfy (ii) and (iii) of Definition I.1 and since $(B\circ A\circ Q)w = \sum_n \langle u_n^\star, w\rangle G(E_n) = \sum_n \langle w_n^\star, w\rangle z_n$, for $w \in W$, it follows that $B\circ A\circ Q \in N_\infty(W,Z)$ with $n_\infty(B\circ A\circ Q) \leq \|F\|_\infty \|G\|(\Omega)$, since δ is arbitrary.

It follows that $T\circ S = B\circ A\circ Q \circ P \in N_\infty(X,Z)$ and $n_\infty(T\circ S) \leq \|P\| n_\infty(B\circ A\circ Q) \leq (\|S\| + \varepsilon)(si_\infty(T) + \varepsilon)$. Since ε is arbitrary, $n_\infty(T\circ S) \leq \|S\| si_\infty(T)$.

<u>Case 2'</u>. F does not have necessarily countable range.

Again, from [DU.II.1.3] there is a sequence $\{F_n\}$ in $L_\infty(\mu, W^\star)$, each F_n with countable range, such that $\|F_n - F\|_\infty \to 0$. For each $n \in \mathbb{N}$, define $R_n : W \to Z$ by

$$R_n w = \int_\Omega \langle F_n(t), w\rangle dG(t) , \text{ for } w \in W ,$$

so that

$$\begin{array}{ccc} W & \xrightarrow{R_n} & Z \\ & {\scriptstyle A_n}\searrow \quad \nearrow{\scriptstyle B} & \\ & L_\infty(\mu) & \end{array}$$

where $A_n w = \langle F_n(.), w\rangle$, for $w \in W$.

From Case 1', for any $n, m \in \mathbb{N}$, $R_n \in N_\infty(W,Z)$ and $R_n - R_m \in N_\infty(W,Z)$, with $n_\infty(R_n) \leqslant \|F_n\|_\infty \|G\|(\Omega)$ and $n_\infty(R_n - R_m) \leqslant \|F_n - F_m\|_\infty \|G\|(\Omega)$. Then $\{R_n\}$ is a Cauchy sequence in $N_\infty(W,Z)$ that converges in n_∞-norm to some element $U \in N_\infty(W,Z)$, with $n_\infty(U) \leqslant \|F\|_\infty \|G\|(\Omega)$. But $\{R_n\}$ converges in norm to $B \circ A \circ Q$ and $\|.\| \leqslant n_\infty(.)$, so that $U = B \circ A \circ Q$. Then $T \circ S = B \circ A \circ Q \circ P \in N_\infty(X,Z)$, with $n_\infty(T \circ S) \leqslant \|P\| \|G\|(\Omega) \leqslant (si_\infty(T) + \varepsilon)(\|S\| + \varepsilon)$. Since ε is arbitrary, $n_\infty(T \circ S) \leqslant \|S\| si_\infty(T)$. □

THEOREM II.4. If $X^\star$ has RNP, then $SI_p(X,.) = N_p(X,.)$, for every p, $1 \leqslant p \leqslant \infty$, with $si_p(.) = n_p(.)$.

PROOF. Since $X^\star$ has RNP, Id_X is an Asplund operator and the result follows from Theorem II.3. □

REMARK. When $X^\star$ is complemented in a Banach lattice, RNP and wRNP coincide in $X^\star$ and in this case Theorems II.1 and II.4 give a complete description of $SI_p(X,.) = N_p(X,.)$. Important classes of spaces are included in this situation, such as $C(K)$-spaces, L_p-spaces, Lorentz spaces, Orlicz spaces, Marcinkiewicz spaces, and Banach spaces with unconditional basis.

III. SPACES Y SUCH THAT $SI_p(.,Y) = N_p(.,Y)$.

The problem $SI_p(.,Y) = N_p(.,Y)$ defines completely different situations for $p = 1$ and $p > 1$. As example I.6 shows, when $p > 1$, wRNP, RNP, reflexive or even Hilbert spaces can fail to satisfy $SI_p(.,Y) = N_p(.,Y)$. Moreover, RNP is not a necessary condition, either, since the space $JH^\star$, where JH is Hagler's space, is not a RNP space, but it has Schur property, so that $SI_p(.,JH^\star) = N_p(.,JH^\star)$ (Theorem III.3).

Let us present first a sufficient condition for $1<p\leq\infty$. For $p=\infty$ we will need the following Lemma.

LEMMA III.1.

(a) $N_\infty(C(K),.) = K(C(K),.)$, with $n_\infty(.) = \|.\|$, for every compact Hausdorff space K ;

(b) $N_\infty(L_\infty(\ ),.) = K(L_\infty(\ ),.)$, with $n_\infty(.) = \|.\|$, for every finite measure space (Ω,Σ,μ).

PROOF.

(a) It is known that $N_\infty(C(K),.)\subset K(C(K),.)$, with $\|.\|\leq n_\infty(.)$.

For the reverse inclusion and inequality, let $T\in K(C(K),Y)$ be given, where Y is any Banach space . Since $C(K)^\star$ has the approximation property, there is a sequence $\{T_n\}$ of finite rank operators from $C(K)$ into Y such that $\|T_n-T\|\to 0$. Also, $C(K)^\star$ has the metric approximation property, so that for any $n,\ m\in\mathbb{N}$, $si_\infty(T_n) = n_\infty(T_n)$ and $si_\infty(T_n-T_m) = n_\infty(T_n-T_m)$ [PP-Lemma 11].

From Example I.5 one has $si_\infty(T_n) = \|T_n\|$ and $si_\infty(T_n-T_m) = \|T_n-T_m\|$, for any $n,\ m\in\mathbb{N}$. Then $\{T_n\}$ is a Cauchy sequence in $N_\infty(C(K),Y)$ that converges in norm to T. Therefore, T is also the limit in n_∞-norm of $\{T_n\}$, so that $T\in N_\infty(C(K),Y)$ and $n_\infty(T) = \|T\|$.

(b) It follows from (a) and the identification of any $L_\infty(\mu)$, for a finite measure space (Ω,Σ,μ), with $C(K)$, for a convenient compact Hausdorff space K. □

THEOREME III.2. Let $1<p\leq\infty$. If $T\ SI_p(X,Y)$ and $S\in C(Y,Z)$, then $S\circ T\in N_p(X,Z)$, with $n_p(S\circ T)\leq\|S\|si_p(T)$.

PROOF. Let $p<\infty$. Given $T\in SI_p(X,Y)$ and $\varepsilon>0$, choose a factorization of T as

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ A \downarrow & & \uparrow B \\ C(K) & \xrightarrow[J_p]{} & L_p(\mu) \end{array}$$

where K is a compact Hausdorff space, $\mu \varepsilon rca^+(\mathcal{B}(C(K)))$, J_p is the inclusion, $A \varepsilon L(X,C(K))$, $B \varepsilon L(L_p(\mu),Y)$, $\|A\| \leq 1$, $\|B\| \leq 1$ and $si_p(J_p) \leq si_p(T) + \varepsilon$.

Hence $S \circ T$ has a factorization

$$\begin{array}{ccc} X & \xrightarrow{S \circ T} & Z \\ A \downarrow & & \uparrow S \circ B \\ C(K) & \xrightarrow[J_p]{} & L_p(\mu) \end{array}$$

Since S is completely continuous and B is weakly compact, $S \circ B$ is a compact operator. By the approximation property of $L_q(\mu) = L_p(\mu)^{\star}$ there is a sequence $\{R_n\}$ of finite rank operators in $L(L_p(\mu),Z)$, with $\|R_n - S \circ B\| \to 0$.

Then $\{R_n \circ J_p\}$ is a sequence of strictly p-integral finite rank operators from $C(K)$ into Z. The metric approximation property of $C(K)^{\star}$ and [PP-Lemma 7] give, for any $n, m \varepsilon N$,

$$n_p(R_n \circ J_p) \leq si_p(R_n \circ J_p) \leq \|R_n\| si_p(J_p) ,$$

and

$$n_p[(R_n - R_m) \circ J_p] \leq si_p[(R_n - R_m) \circ J_p] \leq \|R_n - R_m\| si_p(J_p)$$

so that $\{R_n \circ J_p\}$ is a Cauchy sequence in n_p-norm, thus convergent to some element in $N_p(C(K),Z)$.

But $\{R_n \circ J_p\}$ converges in norm to $S \circ B \circ J_p$, and $\|.\| \leq n_p(.)$, so that $S \circ B \circ J_p \varepsilon N_p(C(K),Z)$, with $n_p(S \circ B \circ J_p) \leq \|S \circ B\| si_p(J_p) \leq \|S\|(si_p(T) + \varepsilon)$. Hence $S \circ T = S \circ B \circ J_p \circ A \varepsilon N_p(X,Z)$ and $n_p(S \circ T) \leq \|S\| si_p(T)$, since $\|A\| \leq 1$ and ε is arbitrary.

Now let $p = \infty$. For given $T \varepsilon SI_{\infty}(X,Y)$ and $\varepsilon > 0$, choose a factorization

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ & A \searrow \quad \nearrow B & \\ & C(K) & \end{array}$$

where K is a compact Hausdorff space, $A \in L(X,C(K))$, $\|A\| \leq 1$, and $B \in W(C(K),Y)$ is given by $Bf = \int_K fdG$, with $\|B\| = \|G\|(K) \leq si_\infty(T) + \varepsilon$.

Since B is weakly compact and S is completely continuous, $S \circ B \in K(C(K),Z)$. From Lemma III.1, $S \circ B \in N_\infty(C(K),Z)$ and $n_\infty(S \circ B) = \|S \circ B\| \leq \|S\|(si_\infty(T) + \varepsilon)$.

Hence $S \circ T = S \circ B \circ A \in N_\infty(X,Z)$ with $n_\infty(S \circ T) \leq \|S\| si_\infty(T)$, since $\|A\| \leq 1$ and ε is arbitrary. □

A moment of reflection about the precedint proof shows that the essential hypothesis was the compactness of $S \circ B$. When $p = \infty$, a $L_\infty(\mu)$-space could have been used instead of a $C(K)$-espace. These remarks lead us to the following theorem.

THEOREM III.3.

(a) Let $1 < p < \infty$. If $L(L_p(\mu),Y) = K(L_p(\mu),Y)$ for every finite measure space (Ω,Σ,μ), then $SI_p(.,Y) = N_p(.,Y)$, with $si_p(.) = n_p(.)$.

(b) If $W(C(K),Y) = K(C(K),Y)$ for every compact Hausdorff space K or $W(L_\infty(\mu),Y) = K(L_\infty(\mu),Y)$ for every finite measure space (Ω,Σ,μ), then $SI_\infty(.,Y) = N_\infty(.,Y)$, with $si_\infty(.) = n_\infty(.)$.

PROOF. It follows as in Theorem III.2. □

The next theorem gives a necessary condition to $SI_p(.,Y) \subset K(.,Y)$ and, in particular, to $SI_p(.,Y) = N_p(.,Y)$, since p-nuclear operators are compact.

THEOREM III.4. If for some r, $1 < r \leq \infty$, $SI_r(.,Y) \subset K(.,Y)$, then $L(L_p(\mu),Y) = K(L_p(\mu),Y)$, for every p, $2 \leq p < \infty$, and every finite measure space (Ω,Σ,μ).

PROOF. Let (Ω,Σ,μ) be a finite measure space and p, $2 \leq p < \infty$, be a fixed index.

Suppose that there is a non-compact operator $T : L_p(\mu) \to Y$. Then there is a bounded sequence $\{f_n\}$ in $L_p(\mu)$ such that $\{Tf_n\}$ has no norm convergent subsequence. Taking subsequence, translating and/or normalizing, one can assume that

(i) $\{f_n\}$ is weakly convergent to zero, since $L_p(\mu)$ is reflexive ;

(ii) there is an $\varepsilon_0 > 0$ such that $\|f_n\|_p \geqslant \varepsilon_0$, for every $n \in \mathbb{N}$, condition that holds for some subsequence of $\{f_n\}$, otherwise $\|f_n\| \to 0$ and $Tf_n \to 0$;

(iii) $\|f_n\|_p = 1$, for $n \in \mathbb{N}$, otherwise consider $g_n : f_n/\|f_n\|_p$, so that $\|g_n\|_p = 1$, for $n \in \mathbb{N}$, and still $g_n \xrightarrow{w} 0$.

By Bessaga-Pelczynski Selection Principle [DI-page 42], there is a subsequence of $\{f_n\}$, which can still be called $\{f_n\}$, that is a basic sequence, ie, it is a basis for the subspace it generates. Let Z be this subspace of $L_p(\mu)$, $Z = [f_n]$.

If $p = 2$, Z is isomorphic to ℓ_2, because Z is a separable Hilbert space. Moreover, the isomorhism takes the canonical basis of ℓ_2 into the basis of Z.

If $2 < p < \infty$, Z contains a subspaces that is isomorphis to ℓ_2 or ℓ_p [KP-Corollary 6], and the isomorhism takes the canonical basis of ℓ_2 or ℓ_p into a sequence $\{f_{n_k}\}$, still a basic sequence. Although Kadec and Pelczynski work only with $L_p[0,1]$, they remark that their results hold for any measure space [KP-page 162] .

In any case, there is an isomorphism $R : \ell_2 \to L_p(\mu)$ or $R : \ell_p \to L_p(\mu)$ such that $Re_k = f_{n_k}$, for any $k \in \mathbb{N}$, where $\{e_k\}$ is the basis of ℓ_2 or ℓ_p.

Defining an operator $S : \ell_1 \to Y$ by $S = T \circ R \circ i_p$, one has, for $1 < r \leqslant \infty$, that $S \in SI_r(\ell_1, Y)$, since $i_p \in SI_r(\ell_1, \ell_p)$, for $p \geqslant 2$. But S is not a compact operator, because $\{Se_k\} = \{Tf_{n_k}\}$ and no subsequence of $\{Tf_n\}$ is norm convergent.

This contradicts the hypothesis that there is r, $1 < r \leqslant \infty$, such that $SI_r(\cdot, Y) \subset K(\cdot, Y)$, so that there is no non-compact operator $T : L_p(\mu) \to Y$, or $L(L_p(\mu), Y) = K(L_p(\mu), Y)$. □

REMARK. Other necessary conditions to $SI_p(.,Y) = N_p(.,Y)$, for any p, $1 < p \leqslant \infty$, are that $Y \not\supset c_0$ and $Y \not\supset \ell_r$, $r \geqslant 2$, shown by Example I.6, and that $Y \not\supset L_1[0,1]$, because J_1 is strictly p-integral for any p, $1 \leqslant p \leqslant \infty$, but it is not compact.

We have the following characterization of the spaces Y such that $SI_p(.,Y) = N_p(.,Y)$, when $p \geqslant 2$.

THEOREM III.5.

I. The following conditions on Y are equivalent :

(1) for every p, $2 \leqslant p < \infty$, $SI_p(.,Y) = N_p(.,Y)$, with $si_p(.) = n_p(.)$;

(2) there exists p, $2 \leqslant p < \infty$, such that $SI_p(.,Y) = N_p(.,Y)$, with $si_p(.) = n_p(.)$;

(3) for every p, $2 \leqslant p < \infty$, $SI_p(.,Y) = N_p(.,Y)$;

(4) there exists p, $2 \leqslant p < \infty$, such that $SI_p(.,Y) = N_p(.,Y)$;

(5) for every p, $1 < p < \infty$, $SI_p(.,Y) \subset K(.,Y)$;

(6) there exists p, $1 < p < \infty$, such that $SI_p(.,Y) \subset K(.,Y)$;

(7) $L(\ell_2,Y) = K(\ell_2,Y)$;

(8) $L(L_2(\mu),Y) = K(L_2(\mu),Y)$, for every finite measure space (Ω,Σ,μ) ;

(9) for every p, $2 \leqslant p < \infty$, $L(L_p(\mu),Y) = K(L_p(\mu),Y)$, for every finite measure space (Ω,Σ,μ) ;

(10) there exists p, $2 \leqslant p < \infty$, such that $L(L_p(\mu),Y) = K(L_p(\mu),Y)$, for every finite measure space (Ω,Σ,μ).

II. If $Y = Z^\star$ for some space Z, the conditions (1) to (10) imply

(11) for every q, $1 < q \leqslant 2$, $L(Z,L_q(\mu)) = K(Z,L_q(\mu))$, for every finite measure space (Ω,Σ,μ) ;

(12) there exists q, $1 < q \leqslant 2$, such that $L(Z,L_q(\mu)) = K(Z,L_q(\mu))$, for every finite measure space (Ω,Σ,μ) ;

(13) $L(Z,\ell_2) = K(Z,\ell_2)$.

III. If Y is a reflexive space, conditions (1) to (13) are equivalent, where, in (11) to (13), $Z = Y^{\star}$.

PROOF.

I. The chains (1) ⇒ (2) ⇒ (4) ⇒ (6) and (1) ⇒ (3) ⇒ (4) ⇒ (6) and the implications (5) ⇒ (6) and (9) ⇒ (10) are trivial.

(6) ⇒ (9) This is Theorem III.4.

(10) ⇒ (7) From Theorem III.3 it follows that, for such a p, $SI_p(.,Y) \subset K(.,Y)$. The result follows then from Theorem III.4, applied to $L_2[0,1]$, which is isomorphic to ℓ_2.

(7) ⇒ (8) If there is a finite measure space (Ω,Σ,μ) and a non-compact operator $T : L_2(\mu) \to Y$, the same argument of Theorem III.4 says that there is a normalized weakly convergent basic sequence $\{f_n\}$ in $L_2(\mu)$ sucn that $\{Tf_n\}$ has no norm convergent subsequence and, moreover, $Z = [f_n]$ is isomorphic to ℓ_2. Hence the operator $S : \ell_2 \to Y$ given by $S = T|_Z$ is not compact, which contradicts the hypothesis.

(8) ⇒ (5) From Theorem III.3 $SI_2(.,Y) \subset K(.,Y)$. If $1 < p < 2$, $SI_p(.,Y) \subset K(.,Y)$, since $SI_p \subset SI_2$. On the other hand, for $2 \leqslant p < \infty$, it follows that $L(L_p(\mu),Y) = K(L_p(\mu),Y)$, from Theorem III.4 . Hence $SI_p(.,Y) \subset K(.,Y)$, $2 \leqslant p < \infty$ [Theorem III.3].

It remains to show that (6) ⇒ (1). But (6) is equivalent to (9), that implies (1), as seen in Theorem III.4.

II. Assume $Y = Z^{\star}$.

To show (9) ⇒ (11), let q, $1 < q \leqslant 2$, a finite measure space (Ω,Σ,μ) and $T \in L(Z,L_q(\mu))$ be given. Then $T^{\star} \in K(L_p(\mu),Y)$, by hypothesis, because p , the conjugate exponent of q , is such that $2 \leqslant p < \infty$. Hence T is compact.

The same argument shows that (7) implies (13), and the implication (11) ⇒ (12) is obvious.

III. Assume that Y is reflexive.

From II, it follows that the equivalent conditions (1) to (10) imply

(11) to (13), with $Z = Y^\star$.

To show that (12) implies (10), let a finite measure space (Ω,Σ,μ) be given and p be the conjugate exponent of the index q, $1<q\leq 2$, that exists by hypothesis. If $T \in L(L_p(\mu),Y)$, then $T^\star \in L(Y^\star,L_q(\mu))$, so that $T^\star$ is compact. Hence $T \in K(L_p(\mu),Y)$.

The implication (13) ⇒ (7) follows the same pattern and it is obvious that (11) ⇒ (12). □

REMARKS. Conditions (9) and (10) in Theorem III.5 can not be extended to $1<p<2$. Indeed, for these indices p, we do have $L(\ell_2,\ell_p) = K(\ell_2,\ell_p)$, but $L(L_p[0,1],\ell_p) \neq K(L_p[0,1],\ell_p)$, since ℓ_p is complemented in $L_p[0,1]$ and the corresponding projection can not be compact.

When $1<p<2$, we do not know a necessary and sufficient condition for $SI_p(.,Y) = N_p(.,Y)$ yet.

For $p=\infty$, we have the following result.

THEOREM III.6.

I. The following conditions on Y are equivalent :

(1) $SI_\infty(.,Y) = N_\infty(.,Y)$, with $si_\infty(.) = n_\infty(.)$;

(2) $SI_\infty(.,Y) = N_\infty(.,Y)$;

(3) $SI_\infty(.,Y) \subset K(.,Y)$;

(4) $L(C(K),Y) = K(C(K),Y)$, for every compact Hausdorff space K ;

(5) $W(C(K),Y) = K(C(K),Y)$, for every compact Hausdorff space K ;

(6) $L(L_\infty(\mu),Y) = K(L_\infty(\mu),Y)$, for every finite measure space (Ω,Σ,μ) ;

(7) $W(L_\infty(\mu),Y) = K(L_\infty(\mu),Y)$, for every finite measure space (Ω,Σ,μ).

II. If $Y = Z^\star$ for some space Z, the conditions (1) to (7) imply

(8) $L(Z,L_1(\mu)) = K(Z,L_1(\mu))$, for every finite measure space (Ω,Σ,μ) ;

(9) $W(Z,L_1(\mu)) = K(Z,L_1(\mu))$, for every finite measure space (Ω,Σ,μ).

III. If Y is reflexive, conditions (1) to (9) are equivalent, where, in (8) and (9), $Z = Y^\star$.

PROOF.

I. The chain (1) ⇒ (2) ⇒ (3) and the implications (4) ⇒ (5) and (6) ⇒ (7) are trivial.

(3) ⇒ (4) Let $X = C(K)$ in the hypothesis to get the inclusion $W(C(K),Y) \subset K(C(K),Y)$, since $SI_\infty(C(K),Y) = W(C(K),Y)$, as seen in Example I.5. Moreover, $Y \not\hookrightarrow c_0$, by the remark before Theorem III.5, so that $L(C(K),Y) = W(C(K),Y)$ [DU-VI.2.1.5].

(3) ⇒ (6) Use the preceding argument with $X = L_\infty(\mu)$.

(5) ⇒ (7) It follows from the identification of $L_\infty(\mu)$ with $C(K)$, for a convenient compact Hausdorff space K.

(7) ⇒ (1) This is just Theorem III.3.

II. Assume $Y = Z^\star$.

(8) ⇒ (9) It is trivial.

(6) ⇒ (9) Let a finite measure space (Ω,Σ,μ) be given. If $T \in L(Z,L_1(\mu))$, then $T^\star \in L(L_\infty(\mu),Y) = K(L_\infty(\mu),Y)$, by hypothesis. Hence T is compact.

III. Assume Y reflexive.

From II, the quivalent conditions (1) to (7) imply (8) and (9) and (8) ⇒ (9), with $Z = Y^\star$.

To show that (9) implies (5), let a compact Hausdorff space K and $T \in W(C(K),Y)$ be given. Then $T^\star \in L(Y^\star,L_1(\mu))$, for some finite measure space (Ω,Σ,μ). By hypothesis, $T^\star$ is compact and so is T. □

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