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Summary

In this paper, Dolby's (1976) ultrastructural model with no replications is investigated within the class of the elliptical distributions. General asymptotic results are proved for the sample covariance matrix **S**, which are used to study the asymptotic behavior of some estimators of the slope parameter. In particular, under some regularity conditions they are shown to be consistent and asymptotically normal. Some asymptotic relative efficiencies are also reported which shows that the generalized least squares and method of moment estimators can be highly inefficient under nonnormality.

Key Words: asymptotic distribution; asymptotic relative efficiency; elliptical distribution; kurthosis parameter; t-distribution

1. Introduction

As introduced by Dolby (1976), the ultrastructural model is defined by the linear relations

$$(1.1) \qquad \begin{cases} Y_i = y_i + e_i, \\ X_i = x_i + u_i, \\ y_i = \alpha + \beta x_i, \end{cases}$$

so that only (Y_i, X_i) can be observed, with $u_i \sim (0, \sigma_{uu})$, $e_i \sim (0, \sigma_{uu})$, the measurement erros, and $x_i \sim (\mu_{xi}, \sigma_{xx})$, being all independent. The notation $(0, \sigma_{uu})$, for example, is used to denote a distribution (nonspecified) with mean 0 and variance σ_{uu}. Moreover, the asymptotic distribution of estimators of the slope parameter β derived in the literature (Fuller, 1987; Gleser, 1985; Cheng and Van Ness, 1991), all depend on the asymptotic distribution of the sample 2x2 covariance matrix, **S**, of the random variables (Y_i, X_i), with entries wich we denote by $S_{YY} = \sum_{i=1}^{n}(Y_i - \bar{Y})^2/n$, $S_{XY} = \sum_{i=1}^{n}(X_i - \bar{X})(Y_i - \bar{Y})/n$ and $S_{XX} = \sum_{i=1}^{n}(X_i - \bar{X})^2/n$. This distribution is well known in the case where the random vectors $\mathbf{Z}_i = (Y_i, X_i)'$, $i = 1, \ldots, n$, are independent and identically distributed having finite fourth moment. This situation emcompasses the structural measurement error in variables model but not the functional version of the model, which considers that the random vectors $\mathbf{Z}_i$ have mean μ_i (depending on i), $i = 1, \ldots, n$. However, this situation is considered in Gleser (1985) and Cheng and Van Ness (1991) under the normality assumption and more recently by Linder and Babu (1994), assuming that the errors are uncorrelated with null thirdth moments and with finite fourth moments. Moreover, additional assumptions are considered for the mean $\bar{\mu}$ and dispersion matrix $\mathbf{S}^*$ of $\mu_1, \ldots, \mu_n$ in order to guaranty the asymptotic normality of **S**.

In this paper, we first derive the joint asymptotic distribution of the of the mean vector $\bar{\mathbf{Z}}$ and the sample covariance matrix **S** of n independent p-dimensional random vectors $\mathbf{Z}_1, \ldots, \mathbf{Z}_n$ with mean vectors $\mu_1, \ldots, \mu_n$ and common covariance matrix Σ, under the assumptions that (a) the $\epsilon_i = \mathbf{Z}_i - \mu_i$ are indepedent and identically distributed (iid) with distribution having finite fourth moments and (b) there is a $px1$ vector μ and a pxp dispersion matrix Σ^* such that $\sqrt{n}(\bar{\mu} - \mu) \to 0$ and $\sqrt{n}(\mathbf{S}^* - \Sigma^*) \to 0$, where $\bar{\mu}$ and $\mathbf{S}^*$ are the mean vector and dispersion matrix of $\mu_1, \ldots, \mu_n$, respectively. It is shown that the $\bar{\mathbf{Z}}$ and $\mathbf{S}$ are assymptotically independent when the distribution of $\epsilon_i = \mathbf{Z}_i - \mu_i$ have null thirdh moments and it is also derived the asymptotic covariance matrix of $\sqrt{n}\mathbf{S}$. These results are used to obtain the asymptotic distribution of some estimators of β in the ultrastructural model, generalizing the results of Gleser (1985), Van Ness (1991) and Linder and Babu (1994). Finally, the results are illustrated in the context of the t-distribution. We start with some notation and preliminary facts and obtain the asymptotic distribution of $\mathbf{S}$ in its full generality in Section 2. Asymptotic distributions of some estimators of β are investigated in Section 3 under elliptical structural models, with special attention to the functional and structural models. Two ways of identifying the model are investigated: known variances ratio $(\sigma_{ee}/\sigma_{uu})$ and one of the variances (σ_{uu}) known. The special case of the ultrastructural dependent t-model is investigated in some detail. In particular, it is shown that the asymptotic relative efficiency of the generalized least squares estimator (Sprent, 1966; Gleser, 1985) and a method of moments estimator (Fuller, 1987) with respect to the maximum likelihood estimator can be close to zero.

2. Asymptotic properties of S

The following notation will be used in the sequel. Let $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$, be rxs and pxq dimensional matrices, repectively, such that

(i) $Vec(\mathbf{A})$ denotes the vector given by

$$Vec(\mathbf{A}) = (\mathbf{A}_{1.}, \ldots, \mathbf{A}_{r.})',$$

where $\mathbf{A}_{i.}$ denotes the i-th row of $\mathbf{A}$;

(ii) $\mathbf{A} \otimes \mathbf{B}$ denotes the Kroenecker product of $\mathbf{A}$ and $\mathbf{B}$, that is,

$$(\mathbf{A} \otimes \mathbf{B}) = [a_{ij}\mathbf{B}];$$

(iii) the notation $\mathbf{A} * \mathbf{B}$ is used to denote the $rpxsq$ matrix given by

$$\mathbf{A} * \mathbf{B} = [\mathbf{A} \otimes \mathbf{B}_{.1}, \ldots, \mathbf{A} \otimes \mathbf{B}_{.q}],$$

where $\mathbf{B}_{.j}$ denotes the j-th column of $\mathbf{B}$. The two lemmas presented in the sequel are used in Section 3.

Lemma 2.1. (Muirhead, 1982) *Let $\epsilon = (\epsilon_1, \ldots, \epsilon_p)' \sim (E[\epsilon], Var[\epsilon]) = (0, \Sigma)$, where $\Sigma = [\sigma_{ij}]$. Suppose that $E[\epsilon_i\epsilon_j\epsilon_k\epsilon_l] < \infty$, for all $i, j, k, l = 1, \ldots, p$. Let $\mathbf{E} = [E_{ij}] = \epsilon\epsilon'$ and $\Lambda = Var[Vec(\mathbf{E})]$. Then, the matrix Λ is obtained by computing*

$$Cov[E_{ij}, E_{kl}] = \kappa_{1111}^{ijkl} + \kappa_{11}^{ik}\kappa_{11}^{jl} + \kappa_{11}^{il}\kappa_{11}^{jk}, \tag{2.1}$$

where the coefficients κ denote the cumulants of the distribution of the vector ϵ. Particularly, $\kappa_{11}^{ij} = Cov[\epsilon_i, \epsilon_j] = \sigma_{ij}$, $\kappa_{1111}^{iijk} = \kappa_{211}^{ijk}$, $\kappa_{11}^{ii} = \kappa_2^i = Var[\epsilon_i] = \sigma_{ii}$, *and so on.*

Lemma 2.2. (Muirhead, 1982; Fang et al., 1990) *Suppose that $\epsilon \sim El_p(0, \Psi; \phi)$, that is, the p-dimensional elliptical distribution with characteristic function $E[e^{it'\epsilon}] = \phi(t'\Psi t)$. Then, under the hypothesis given in Lemma 2.1, it follows that*

$$E[\epsilon] = 0, \quad Var[\epsilon] = \Sigma = (-2\phi'(0))\Psi$$

and

$$\kappa_{1111}^{ijkl} = \kappa(\sigma_{ij}\sigma_{kl} + \sigma_{ik}\sigma_{jl} + \sigma_{il}\sigma_{jk}), \tag{2.2}$$

where

$$\kappa = \frac{\phi''(0)}{(\phi'(0))^2} - 1, \tag{2.3}$$

well known as the kurthosis parameter.

From (2.1) and (2.2) it follows that if $\epsilon \sim El_p(0, \Psi; \phi)$ has finite fourth order moments and $Var[\epsilon] = \Sigma = [\sigma_{ij}]$ then $\Lambda = Var[Vec(\mathbf{E})]$, where $\mathbf{E} = [E_{ij}] = \epsilon\epsilon'$, the entries of which are obtained by computing

$$Cov[E_{ij}, E_{kl}] = (\kappa + 1)(\sigma_{ik}\sigma_{jl} + \sigma_{il}\sigma_{jk}) + \kappa\sigma_{ij}\sigma_{kl}. \tag{2.4}$$

Note that if $\epsilon \sim El_p(0, \Psi; \phi)$ has a density, then it must be of the form $|\Psi|^{-1/2} f(\epsilon'\Psi\epsilon)$, for some function $f(u)$, $u \geq 0$. For example, if $\epsilon \sim t_p(0, \Psi; \nu)$, the p-variate t-distribution with ν degrees of freedom, then

$$f(u) = \frac{\Gamma[\frac{1}{2}(\nu + p)]}{\Gamma[\frac{1}{2}\nu]\pi^{\frac{p}{2}}} \nu^{\nu/2} \{\nu + u\}^{-\frac{(\nu+p)}{2}}, \quad u \geq 0.$$

For the above t-model (Muirhead, 1982),

$$\kappa = \frac{\nu - 2}{\nu - 4} - 1, \quad \nu > 4,$$

and for $\nu = \infty$ (normal model), $\kappa = 0$.

In the sequel, some general asymptotic results are presented for the sample mean vector and for the sample dispersion matrix of a sample of size n of independent p-dimensional random vectors. Details of the proof of some of these results can be found in Gleser (1981).

Lemma 2.3. *Let $\mathbf{V}_1, \mathbf{V}_2, \ldots$, a sequence of mutually independent p-dimensional random vectors such that $E[\mathbf{V}_i] = 0$ and $Var[\mathbf{V}_i] = \Lambda_i < \infty$, that is, with finite entries. Suppose that*

$$\frac{1}{n}\sum_{i=1}^{n} \Delta_i \to \Lambda < \infty.$$

Then,

$$\sqrt{n}\bar{\mathbf{V}} \xrightarrow{d} \mathbf{V} \sim N_p(\mathbf{0}, \Lambda),$$

where $\bar{\mathbf{V}} = \bar{\mathbf{V}}(n) = n^{-1}\sum_{i=1}^{n} \mathbf{V}_i$.

The proof follows by considering that (see Gleser, 1981)

$$\sqrt{n}\mathbf{a}'\bar{\mathbf{V}} \xrightarrow{d} \mathbf{a}'\mathbf{V} \sim N(0, \mathbf{a}'\Lambda\mathbf{a}),$$

where "$\xrightarrow{d}$" means convergence in distribution.

Now, let $\mathbf{Z}_1, \mathbf{Z}_2, \ldots$, be a sequence of px1 random vectors such that

(A.1) $\epsilon_i = \mathbf{Z}_i - \mu_i \overset{iid}{\sim} (\mathbf{0}, \Sigma)$, that is, any distribution with mean vector $\mathbf{0}$ and covariance matrix Σ;

(A.2) there are finite (finite entries) px1 vector μ and pxp matrix Σ^* such that

$$\sqrt{n}(\bar{\mu} - \mu) \longrightarrow 0, \quad \text{and} \quad \sqrt{n}(\mathbf{S}^* - \Sigma^*) \longrightarrow 0,$$

where $\bar{\mu} = \sum_{i=1}^{n} \mu_i/n$ and $\mathbf{S}^* = \sum_{i=1}^{n}(\mu_i - \bar{\mu})(\mu_i - \bar{\mu})'/n$;

(A.3) $Var[Vec(\epsilon_1\epsilon_1')] = \Lambda < \infty$,

that is, the random vectors ϵ_i have finite fourth moments;

(A.4) $E[\epsilon_1(Vec(\epsilon_1\epsilon_1'))'] = 0$,

which corresponds, for example, to symmetry of the distribution of ϵ_i. It is easy to verify that this condition is satisfied for elliptical distributions. Moreover, let

$$\bar{\mathbf{Z}} = \frac{1}{n}\sum_{i=1}^{n}\mathbf{Z}_i, \quad \text{and} \quad \mathbf{S} = \frac{1}{n}\sum_{i=1}^{n}(\mathbf{Z}_i - \bar{\mathbf{Z}})(\mathbf{Z}_i - \bar{\mathbf{Z}})'.$$

Lemma 2.4. *Under (A.1) and (A.2) it follows that*

(i) $\bar{\mathbf{Z}} \xrightarrow{a.s.} \mu$ *and (ii)* $\mathbf{S} \xrightarrow{a.s.} \Sigma + \Sigma^*$.

Proof. (i) follows basically from the identity

$$\bar{\mathbf{Z}} - \mu = (\bar{\mathbf{Z}} - \bar{\mu}) + (\bar{\mu} - \mu).$$

To prove (ii), we write

$$\mathbf{S} = \bar{\mathbf{W}} + \bar{\mathbf{W}}^* - (\bar{\mathbf{Z}} - \mu)(\bar{\mathbf{Z}} - \mu)',$$

where

$$\bar{\mathbf{W}} = \frac{1}{n}\sum_{i=1}^{n}\mathbf{W}_i, \quad \bar{\mathbf{W}}_i^* = \frac{1}{n}\sum_{i=1}^{n}\mathbf{W}_i^*,$$

with

$$\mathbf{W}_i^* = (\mu_i - \mu)(\mu_i - \mu)'$$

and

$$\mathbf{W}_i = (\mathbf{Z}_i - \mu)(\mathbf{Z}_i - \mu)' - \mathbf{W}_i^* = \epsilon_i\epsilon_i' + \epsilon_i(\mu_i - \mu)' + (\mu_i - \mu)\epsilon_i', \tag{2.5}$$

with $\epsilon_i = \mathbf{Z}_i - \mu_i$, $i = 1, \ldots, n$. The result folllows by noticing that $(\bar{\mathbf{Z}} - \bar{\mu})(\bar{\mathbf{Z}} - \bar{\mu})' \to 0$, $\bar{W} \to \Sigma$, both almost surely, and $\bar{\mathbf{W}}^* \longrightarrow \Sigma^*$. See also Gleser (1981).

The main result of this section is presented next.

Theorem 2.1. *Under (A.1)-(A.3) we have that*

$$\sqrt{n}(\bar{\mathbf{Z}} - \mu) \xrightarrow{d} N_z \sim N_p(0, \Sigma)$$

and

$$\sqrt{n}(\mathbf{S} - \Sigma - \Sigma^*) \xrightarrow{d} N_{zz} \sim N_p(0, \Lambda + \Lambda^*),$$

where $\Lambda = Var[Vec(\epsilon\epsilon')]$ *is given in (2.1) and*

$$\Lambda^* = \Sigma \otimes \Sigma^* + \Sigma^* \otimes \Sigma + \Sigma * \Sigma^* + \Sigma^* * \Sigma. \tag{2.6}$$

Moreover, considering (A.4) we have that N_z *and* N_{zz} *are independent.*

Proof. Let

$$\mathbf{V}_i = \begin{pmatrix} \epsilon_i \\ Vec(\mathbf{W}_i - \Sigma) \end{pmatrix},$$

where $\mathbf{W}_i$ is as given in (2.5). Thus, (A.1) and (A.3) imply that $\mathbf{V}_1, \mathbf{V}_2, \ldots,$ are independent $(p + p^2)$-dimensional random vectors with $E[\mathbf{V}_i] = 0$ and $Var[\mathbf{V}_i] = \boldsymbol{\Delta}_i$, which is finite, $i = 1, 2, \ldots$. Notice that

$$\boldsymbol{\Delta}_i = \begin{pmatrix} \Sigma & \Upsilon_i \\ \Upsilon_i' & \Lambda_i \end{pmatrix},$$

where $\Upsilon_i = Cov[\epsilon_i, Vec(\mathbf{W}_i)]$ and $\Lambda_i = Var[Vec(\mathbf{W}_i)]$. With $\mathbf{W}_i$ as given in (2.5), (A.1) implies that we can write

$$\begin{aligned}\Upsilon_i &= E[\epsilon_1\{Vec(\epsilon_1\epsilon_1')\}'] + E[\epsilon_1\{Vec(\epsilon_1(\mu_i - \mu)'\}'] + E[\epsilon_1\{Vec((\mu_i - \mu)\epsilon_1')\}'] \\ &= \Upsilon + E[\epsilon_1\{Vec(\epsilon_1(\mu_i - \mu)')\}'] + E[\epsilon_1\{Vec((\mu_i - \mu)\epsilon_1')\}'],\end{aligned}$$

where $\Upsilon = E[\epsilon_1\{Vec(\epsilon_1\epsilon_1')\}']$. Thus, it follows that

$$\bar{\Upsilon} = \frac{1}{n}\sum_{i=1}^{n}\Upsilon_i = \Upsilon + E[\epsilon_1\{Vec(\epsilon_1(\bar{\mu} - \mu)'\}'] + E[\epsilon_1\{Vec((\bar{\mu} - \mu)\epsilon_1'\}'] \longrightarrow \Upsilon.$$

Similarly, we have that

$$\Lambda_i = Var[Vec(\mathbf{W}_i)] = \Lambda + \Lambda_{1i} + \Lambda_{2i},$$

where

$$\Lambda = Var[Vec(\epsilon_1\epsilon_1')],$$

$$\Lambda_{1i} = Var[Vec(\epsilon_1(\mu_i - \mu)' + (\mu_i - \mu)\epsilon_1')]$$

and

$$\Lambda_{2i} = Cov[Vec(\epsilon_1\epsilon_1'), Vec(\epsilon_1(\mu_i-\mu)'+(\mu_i-\mu)\epsilon_1')]+Cov[Vec(\epsilon_1(\mu_i-\mu)'+(\mu_i-\mu)\epsilon_1'), Vec(\epsilon_1\epsilon_1')].$$

It is easy to see under the stated assumptions that

$$\frac{1}{n}\sum_{i=1}^{n}\Lambda_{1i} \longrightarrow \Lambda^* = \Sigma\otimes\Sigma^* + \Sigma^*\otimes\Sigma + \Sigma * \Sigma^* + \Sigma^* * \Sigma,$$

and

$$\frac{1}{n}\sum_{i=1}^{n}\Lambda_{2i} \longrightarrow 0,$$

from where it follows that

$$\bar{\Lambda} = \frac{1}{n}\sum_{i=1}^{n}\Lambda_i \longrightarrow \Lambda + \Lambda^* = \Lambda + \Sigma\otimes\Sigma^* + \Sigma^*\otimes\Sigma + \Sigma * \Sigma^* + \Sigma^* * \Sigma,$$

so that

$$\bar{\Delta} = \frac{1}{n}\sum_{i=1}^{n}\Delta_i = \begin{pmatrix} \Sigma & \bar{\Upsilon} \\ \bar{\Upsilon}' & \bar{\Lambda} \end{pmatrix} \longrightarrow \Delta = \begin{pmatrix} \Sigma & \Upsilon \\ \Upsilon' & \Lambda + \Lambda^* \end{pmatrix}.$$

Using Lemma 2.3 we have that

$$\sqrt{n}\bar{\mathbf{V}} = \sqrt{n}\begin{pmatrix} \bar{\epsilon} \\ Vec(\bar{\mathbf{W}} - \Sigma) \end{pmatrix} \xrightarrow{d} \begin{pmatrix} N_z \\ N_{zz} \end{pmatrix} \sim N_{p+p^2}(0, \Delta),$$

from where it follows that

$$\sqrt{n}\bar{\epsilon} = \sqrt{n}(\bar{\mathbf{Z}} - \bar{\mu}) \xrightarrow{d} N_p(0, \Sigma)$$

and

$$\sqrt{n}(\mathbf{S} - \Sigma - \Sigma^*) \xrightarrow{d} N_{zz} \sim N_{p^2}(0, \Lambda + \Lambda^*),$$

with N_z and N_{zz} being independent, since, under (A.4), $\Upsilon = 0$. Thus, the result follows after observing that

$$\sqrt{n}\begin{pmatrix} \bar{\mathbf{Z}} - \mu \\ Vec(\mathbf{S} - \Sigma - \Sigma^*) \end{pmatrix} = \sqrt{n}\begin{pmatrix} \bar{\epsilon} \\ Vec(\bar{\mathbf{W}} - \Sigma) \end{pmatrix} + \sqrt{n}\begin{pmatrix} \bar{\mu} - \mu \\ Vec(\bar{\mathbf{W}}^* - \Sigma^*) \end{pmatrix}$$

$$-\sqrt{n}\begin{pmatrix} 0 \\ Vec((\bar{\mathbf{Z}} - \mu)(\bar{\mathbf{Z}} - \mu)') \end{pmatrix} \xrightarrow{d} \begin{pmatrix} N_z \\ N_{zz} \end{pmatrix},$$

and using Lemma 2.4 and Slutsky theorem.

Remark 2.1. Since S is symmetric, the distribution of N_{zz} is singular because $\mathbf{A}+\mathbf{A}^*$ is singular. However, considering only the $p(p+1)/2$ different entries of S (as in $Vec(\mathbf{S})$), a nonsingular N_{zz} distribution can be obtained by deleting the corresponding rows and columns in $\mathbf{A}+\mathbf{A}^*$.

Remark 2.2. In the *iid* case with $\mu_i = \mu$, it follows that $\mathbf{A}^* = 0$, so that the asymptotic covariance matrix of $\sqrt{n}\mathbf{S}$ becomes $\mathbf{A}$.

3. Ultrastructrural elliptical models

In this section, we consider the relation

$$\mathbf{Z}_i = \mathbf{a} + \mathbf{B}\mathbf{r}_i, \tag{3.1}$$

where

$$\mathbf{Z}_i = \begin{pmatrix} Y_i \\ X_i \end{pmatrix}, \quad \mathbf{r}_i = \begin{pmatrix} x_i \\ e_i \\ u_i \end{pmatrix}, \quad \mathbf{a} = \begin{pmatrix} \alpha \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} \beta & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix},$$

$i = 1, \ldots, n$. Suppose that:

(S.1) $\mathbf{r}_i \overset{iid}{\sim} El_3(\eta_i, \Psi; \phi)$, with $E[\mathbf{r}_i] = \eta_i = (\mu_{xi}, 0, 0)'$ and

$$Var[\mathbf{r}_i] = \Omega = (-2\phi'(0))\Psi = \text{diag}(\sigma_{xx}, \sigma_{ee}, \sigma_{uu})',$$

both finite, where $\text{diag}(a_1, \ldots, a_p)$ denotes a diagonal matrix with $a_1, \ldots, a_p$, as diagonal elements;

(S.2) There are μ_x and $\sigma^*_{xx} \geq 0$ such that as $n \to \infty$

$$\sqrt{n}(\bar{\mu}_n - \mu_x) \longrightarrow 0 \quad \text{and} \quad \sqrt{n}(S^*_{xx} - \sigma^*_{xx}) \longrightarrow 0,$$

where $\bar{\mu}_x = \sum_{i=1}^n \mu_{xi}/n$ and $S^*_{xx} = \sum_{i=1}^n (\mu_{xi} - \bar{\mu}_x)^2/n$;

(S.3) $\mathbf{r}_1 - \eta_1 = (x_1 - \mu_{x1}, e_1, u_1)' \sim El_3(0, \Psi; \phi)$ has finite fourth order moments.

Under assumptions (S.1)-(S.3) we have that the sequence of random vectors $\mathbf{Z}_1, \mathbf{Z}_2, \ldots$, defined by (3.1) satisfy conditions (A.1)-(A.4). In fact, considering well known properties of elliptical distributions (Fang et al., 1990), it follows that

(S.1)* $\mathbf{Z}_i = \mathbf{a} + \mathbf{B}\mathbf{r}_i \sim El_2(\mathbf{a} + \mathbf{B}\eta_i, \mathbf{B}\Psi\mathbf{B}'; \phi)$ with $E[\mathbf{Z}_i] = \mu_i$ and $Var[\mathbf{Z}_i] = \Sigma$, given by

$$\mu_i = \mathbf{a} + \mathbf{B}\eta_i = \begin{pmatrix} \alpha + \beta\mu_{xi} \\ \mu_{xi} \end{pmatrix}, \quad \Sigma = \mathbf{B}\Omega\mathbf{B}' = \begin{pmatrix} \beta^2\sigma_{xx} + \sigma_{ee} & \beta\sigma_{xx} \\ & \sigma_{xx} + \sigma_{uu} \end{pmatrix}.$$

Here as elsewhere, symmetric matrices are written in upper triangular forms. In particular,

$$\epsilon_i = \mathbf{Z}_i - \mu_i = \mathbf{B}(\mathbf{r}_i - \eta_i) \overset{iid}{\sim} El_2(0, \mathbf{B}\Psi\mathbf{B}'; \phi),$$

with $E[\epsilon_i] = 0$ and $Var[\epsilon_i] = \Sigma$;

(S.2)* Let $\bar{\mu} = \sum_{i=1}^n \mu_i = (\alpha + \beta\bar{\mu}_x, \bar{\mu}_x)'$, $\mu = (\alpha + \beta\mu_x, \mu_x)'$, $S^* = \sum_{i=1}^n (\mu_i - \bar{\mu})(\mu_i - \bar{\mu})' = S_{xx}\mathbf{b}$ and $\Sigma^* = \sigma_{xx}^* \mathbf{b}\mathbf{b}'$, where $\mathbf{b} = (\beta, 1)'$. Thus, (S.2) implies that

$$\sqrt{n}(\bar{\mu} - \mu) = \sqrt{n}(\bar{\mu}_x - \mu_x)\mathbf{b} \longrightarrow 0 \quad \text{and} \quad \sqrt{n}(S^* - \Sigma^*) = \sqrt{n}(S_{xx}^* - \sigma_{xx}^*)\mathbf{b}\mathbf{b}' \longrightarrow 0;$$

(S.3)* $\Lambda = Var[Vec(\epsilon_1 \epsilon_1')] = Var[Vec(\mathbf{B}(\mathbf{r}_1 - \eta_1)(\mathbf{r}_1 - \eta_1)'\mathbf{B}')] < \infty$.

Note that (A.4) is a consequence of the fact that $\mathbf{r}_i$ is distributed according to an elliptical distribution. Thus, being $\bar{\mathbf{Z}}$ and $\mathbf{S}$ the sample mean and sample covariance of $\mathbf{Z}_1, \ldots, \mathbf{Z}_n$, as defined in (3.1), it follows that under (S.1)-(S.3), the results in Lemma 2.4 and Theorem 2.1 hold for $\bar{\mathbf{Z}}$ and $\mathbf{S}$. The matrix Λ which is involved in the asymptotic covariance matrix of $\mathbf{S}$ can be obtained by using (2.4). To see this, note that

$$\epsilon_1'\epsilon_1 = [E_{ij}] = \begin{pmatrix} (Y_1 - \alpha - \beta\mu_{x1})^2 & (Y_1 - \alpha - \beta\mu_{x1})(X_1 - \mu_{x1}) \\ & (X_1 - \mu_{x1})^2 \end{pmatrix}$$

$$= \begin{pmatrix} (\beta(x_1 - \mu_{x1}) + e_1)^2 & \beta(x_1 - \mu_{x1})^2 + (x_1 - \mu_{x1})(\beta u_1 + e_1) + e_1 u_1 \\ & (x_1 - \mu_{x1} + u_1)^2 \end{pmatrix},$$

where the last equality follows from the fact that $\epsilon_1 = \mathbf{B}(\mathbf{r}_1 - \eta_1)$. Using (2.4) and since $Var[\epsilon] = \Sigma = [\sigma_{ij}]$ we have that

$$\Delta = \begin{pmatrix} Var[E_{11}] & Cov[E_{11}, E_{12}] & Cov[E_{11}, E_{21}] & Cov[E_{11}, E_{22}] \\ & Var[E_{12}] & Cov[E_{12}, E_{21}] & Cov[E_{12}, E_{22}] \\ & & Var[E_{21}] & Cov[E_{21}, E_{22}] \\ & & & Var[E_{22}] \end{pmatrix} \tag{3.2}$$

$$= \begin{pmatrix} (3\kappa+2)\sigma_{YY}^2 & (3\kappa+2)\sigma_{YY}\sigma_{YX} & (3\kappa+2)\sigma_{YY}\sigma_{YX} & \kappa\sigma_{YY}\sigma_{XX} + c_1 \\ & (\kappa+1)\sigma_{YY}\sigma_{XX} + c_2 & (\kappa+1)\sigma_{YY}\sigma_{XX} + c_2 & (3\kappa+2)\sigma_{YX}\sigma_{XX} \\ & & (\kappa+1)\sigma_{YY}\sigma_{XX} + c_2 & (3k+2)\sigma_{YX}\sigma_{XX} \\ & & & (3\kappa+2)\sigma_{XX}^2 \end{pmatrix},$$

where $c_1 = 2(\kappa+1)\sigma_{YX}^2$, $c_2 = (2\kappa+1)\sigma_{YX}^2$, $\sigma_{11} = \sigma_{YY} = Var[Y_i] = \beta^2\sigma_{xx} + \sigma_{ee}$, $\sigma_{12} = \sigma_{YX} = Cov[Y_i, X_i] = \beta\sigma_{xx}$, $\sigma_{22} = \sigma_{XX} = Var[X_i] = \sigma_{xx} + \sigma_{uu}$, and κ is the kurthosis coefficient defined in (2.3). Moreover, since $\Sigma^* = \sigma_{xx}^*\mathbf{b}\mathbf{b}'$, $\mathbf{b} = (\beta, 1)'$, we have from (2.6) that

$$\Lambda^* = \sigma_{xx}^* \begin{pmatrix} \sigma_{YY}\beta^2 & \sigma_{YY}\beta & \sigma_{YX}\beta^2 & \sigma_{YX}\beta \\ & \sigma_{YY} & \sigma_{YX}\beta & \sigma_{YX} \\ & & \sigma_{xx}\beta^2 & \sigma_{xx}\beta \\ & & & \sigma_{xx} \end{pmatrix} + \sigma_{xx}^* \begin{pmatrix} \beta^2\sigma_{YY} & \beta^2\sigma_{YX} & \beta\sigma_{YY} & \beta\sigma_{YX} \\ & \beta^2\sigma_{XX} & \beta\sigma_{YX} & \beta\sigma_{XX} \\ & & \sigma_{YY} & \sigma_{YX} \\ & & & \sigma_{XX} \end{pmatrix}$$

$$+\sigma_{xx}^* \begin{pmatrix} \beta^2\sigma_{YY} & \beta^2\sigma_{YX} & \beta\sigma_{YY} & \beta\sigma_{YX} \\ \beta\sigma_{YY} & \beta\sigma_{YX} & \sigma_{YY} & \sigma_{YX} \\ \beta^2\sigma_{YX} & \beta^2\sigma_{XX} & \beta\sigma_{YX} & \beta\sigma_{XX} \\ \beta\sigma_{YX} & \beta\sigma_{XX} & \sigma_{YX} & \sigma_{XX} \end{pmatrix} + \sigma_{xx}^* \begin{pmatrix} \beta^2\sigma_{YY} & \beta\sigma_{YY} & \beta^2\sigma_{YX} & \beta\sigma_{YX} \\ \beta^2\sigma_{YX} & \beta\sigma_{YX} & \beta^2\sigma_{XX} & \beta\sigma_{XX} \\ \beta\sigma_{YY} & \sigma_{YX} & \beta\sigma_{XX} & \sigma_{XX} \end{pmatrix}$$

$$= \sigma^*_{zz} \begin{pmatrix} 4\beta^2\sigma_{YY} & 2\beta\sigma_{YY} + 2\beta^2\sigma_{YX} & 2\beta\sigma_{YY} + 2\beta^2\sigma_{YX} & 4\beta\sigma_{YX} \\ & \sigma_{YY} + 2\beta\sigma_{YX} + \beta^2\sigma_{XX} & \sigma_{YY} + 2\beta\sigma_{YX} + \beta^2\sigma_{XX} & 2\sigma_{YX} + 2\beta\sigma_{XX} \\ & & \sigma_{YY} + 2\beta\sigma_{YX} + \beta^2\sigma_{XX} & 2\sigma_{YX} + 2\beta\sigma_{XX} \\ & & & 4\sigma_{XX} \end{pmatrix}.$$

Notice that $\Lambda + \Lambda^*$ is singular since $\mathbf{S}$ is symmetric. To obtain a nonsingular distribution corresponding to the $p(p+1)/2 = 3$ distinct elements of $\mathbf{S}$, namely, (S_{YY}, S_{YX}, S_{XX}), it sufficies to delete the second (or thirdth) row and column of $\Lambda + \Lambda^*$, so that we obtain

$$\Lambda + \Lambda^* = \begin{pmatrix} (3\kappa+2)\sigma^2_{YY} & (3\kappa+2)\sigma_{YY}\sigma_{YX} & \kappa\sigma_{YY}\sigma_{XX} + 2(\kappa+1)\sigma^2_{YX} \\ & (\kappa+1)\sigma_{YY}\sigma_{XX} + (2\kappa+1)\sigma^2_{YX} & (3\kappa+2)\sigma_{YX}\sigma_{XX} \\ & & (3\kappa+2)\sigma^2_{XX} \end{pmatrix}$$

$$(3.3) \qquad +\sigma^*_{zz} \begin{pmatrix} 4\beta^2\sigma_{YY} & 2\beta\sigma_{YY} + 2\beta^2\sigma_{YX} & 4\beta\sigma_{YX} \\ & \sigma_{YY} + 2\beta\sigma_{YX} + \beta^2\sigma_{XX} & 2\sigma_{YX} + 2\beta\sigma_{XX} \\ & & 4\sigma_{XX} \end{pmatrix}.$$

Remark 3.1 We call attention to the fact that the above results are independent of the fact that the ultrastructural model considered in this section is not identifiable. As is well known, to make the ultrastructural model identifiable and the estimation of β feasible, some additional assumptions are required. We rewrite expression (3.3) by considering the reparametrization where $\lambda_e = \sigma_{ee}/\sigma_{uu}$, $\lambda_z = \sigma_{zz}/\sigma_{uu}$ and $\lambda^*_z = \sigma^*_{zz}/\sigma_{uu}$. This implies that $\sigma_{YY} = \sigma_{uu}(\beta^2\lambda_z + \lambda_e)$, $\sigma_{YX} = \beta\sigma_{uu}\lambda_z$ and $\sigma_{XX} = \sigma_{uu}(\lambda_z + 1)$. One assumption commonly considered in the literature to make the model identifiable is to consider λ_e known. Considering σ_{uu} (or σ_{ee}) known is another typical assumption that makes the model identifiable. Both cases will be considered in the next section. With the above parametrization, the nonsingular asymptotic covariance matrix of $\sqrt{n}\mathbf{S}$ follows from (3.3) and can be written as

$$\Lambda + \Lambda^* =$$

$$= \sigma^2_{uu} \begin{pmatrix} (3\kappa+2)(\beta^2\lambda_z + \lambda_e)^2 & (3\kappa+2)(\beta^2\lambda_z + \lambda_e)\beta\lambda_z & \kappa(\beta^2\lambda_z + \lambda_e)(\lambda_z + 1) + c_1 \\ & (\kappa+1)(\beta^2\lambda_z + \lambda_e)(\lambda_z + 1) + c_2 & (3\kappa+2)\beta\lambda_z(\lambda_z + 1) \\ & & (3\kappa+2)(\lambda_z + 1)^2 \end{pmatrix}$$

$$+\lambda^*_z\sigma^2_{uu} \begin{pmatrix} 4\beta^2(\beta^2\lambda_z + \lambda_e) & 2\beta(\beta^2\lambda_z + \lambda_e) & 4\beta^2\lambda_z \\ & \beta^2(4\lambda_z + 1) + \lambda_e & 2\beta(2\lambda_z + 1) \\ & & 4(\lambda_z + 1) \end{pmatrix},$$

with c_1 and c_2 given in (3.2). Thus, we have two special cases:

The structural model: By taking $\mu_{zi} = \mu_z$, $i = 1, \ldots, n$, so that $\sigma^*_{zz} = 0$, we have that $\Lambda^* = 0$ and the structural model follows (Fuller, 1987). Thus, the (nonsingular) asymptotic covariance matrix of $\sqrt{n}\mathbf{S}$ under normality ($\kappa = 0$) becomes

$$\Lambda + \Lambda^* = \Lambda_N = \sigma^2_{uu} \begin{pmatrix} 2(\beta^2\lambda_z + \lambda_e)^2 & 2(\beta^2\lambda_z + \lambda_e)\beta\lambda_z & 2(\beta\lambda_z)^2 \\ & (\beta^2\lambda_z + \lambda_e)(\lambda_z + 1) + (\beta\lambda_z)^2 & 2\beta\lambda_z(\lambda_z + 1) \\ & & 2(\lambda_z + 1)^2 \end{pmatrix}.$$

The functional Model: The ultrastructural model reduces to the functional model when $\lambda_x = 0$, $\mu_{xi} = x_i$, $\mu_x = \lim_{n\to\infty} \bar{x}$ and $\sigma^*_{xx} = \lim_{n\to\infty} \sum_{i=1}^n (x_i - \bar{x})^2/n$. In this case, it follows that

$$\Lambda + \Lambda^* = \sigma^2_{uu} \begin{pmatrix} (3\kappa+2)\lambda_e^2 & 0 & \kappa\lambda_e \\ & (\kappa+1)\lambda_e & 0 \\ & & 3\kappa+2 \end{pmatrix} + \lambda_x^2 \sigma^2_{uu} \begin{pmatrix} 4\beta^2\lambda_e & 2\beta\lambda_e & 0 \\ & \beta^2+\lambda_e & 2\beta \\ & & 4 \end{pmatrix}.$$

Moreover, under normality,

$$\Lambda_N + \Lambda_N^* = \sigma^2_{uu} \begin{pmatrix} 2\lambda_e^2 & 0 & 0 \\ & \lambda_e & 0 \\ & & 2 \end{pmatrix} + \lambda_x^* \sigma^2_{uu} \begin{pmatrix} 4\beta^2\lambda_e & 2\beta\lambda_e & 0 \\ & \beta^2+\lambda_e & 2\beta \\ & & 4 \end{pmatrix}. \tag{3.4}$$

Notice that expression (3.4) coincides with a corresponding result given in Gleser (1985), with the notation $\sigma_{uu} = \theta$, $\lambda_e = \lambda$, $\lambda_x = \nu$, $\sigma^*_{xx} = v_*$ and $\lambda^*_x = \theta\nu$, and the observation that the matrix Λ in equation (9) of that paper should have $2\tau^2 - v_*^2$ replaced by $2(\tau^2 - v_*^2)$ (see also Cheng and Van Ness, 1991).

3.1. Known variances ratio

Let $\mathbf{Z}_i = \mathbf{a} + \mathbf{B}\mathbf{r}_i$, where $\mathbf{Z}_i = (Y_i, X_i)'$, $\mathbf{r}_i = (x_i, e_i, u_i)'$, $i = 1, \ldots, n$ $\mathbf{a} = (\alpha, 0)'$ and $\mathbf{B} = [\mathbf{b}, \mathbf{I}_2]$, with $\mathbf{b} = (\beta, 1)'$ and $\mathbf{I}_2$ the 2-dimensional identity matrix. Assuming that $\mathbf{r}_1, \ldots, \mathbf{r}_n$ are random vectors with $E[\mathbf{r}_i] = \eta_i = (\mu_{xi}, 0, 0)'$ and $Var[\mathbf{r}_i] = \sigma_{uu}\mathrm{diag}(\lambda_x, \lambda_e, 1)$, Gleser (1985) show that for λ_e known, the generalized least square error estimator of β which is obtained by minimizing the function

$$\mathbf{Q}_G(\alpha, \beta, \mu'_*, \lambda_x, \sigma_{uu}) = \sum_{i=1}^n \epsilon_i' \Sigma^{-1} \epsilon_i,$$

with $\epsilon_i = \mathbf{Z}_i - \mu_i = \mathbf{B}(\mathbf{r}_i - \eta_i)$, $\mu_i = \mathbf{a} + \mathbf{B}\eta_i$, $\Sigma = \mathbf{B}\Omega\mathbf{B}'$ and $\mu_* = (\mu_{x1}, \ldots, \mu_{xn})'$, is given by

$$\hat{\beta}_{GLS} = \frac{1}{2}\{U_n + (U_n^2 + 4\lambda_e)^{1/2}\}, \tag{3.5}$$

where

$$U_n = U(\mathbf{S}) = \frac{S_{YY} - \lambda_e S_{XX}}{S_{YX}}. \tag{3.6}$$

If the sequence $\mu_{x1}, \mu_{x2}, \ldots$, satisfy condition (2.2) then from Lemma 2.4, (i), $\mathbf{S} \longrightarrow \Sigma + \Sigma^*$, almost surely, as $n \to \infty$, where

$$\Sigma = \sigma_{uu} \begin{pmatrix} \beta^2\lambda_x^2 + \lambda_e & \beta\lambda_x \\ & \lambda_x + 1 \end{pmatrix} \quad \text{and} \quad \Sigma^* = \sigma_{uu} \begin{pmatrix} \beta^2\lambda_x^* & \beta\lambda_x^* \\ & \lambda_x^* \end{pmatrix}.$$

Thus, if $\sigma_{xx} + \sigma^*_{xx} = \sigma_{uu}\lambda_x + \sigma_{uu}\lambda^*_x > 0$, then

$$U(\mathbf{S}) \xrightarrow{a.s.} U(\boldsymbol{\Sigma} + \boldsymbol{\Sigma}^*) = \frac{\beta^2 - \lambda_e}{\beta},$$

since $U(\mathbf{S})$ is (almost surely) a continuous function of $\mathbf{S}$. This implies that

$$\hat{\beta}_{GLS} \xrightarrow{a.s.} \beta.$$

In the sequel we show that if the sequence $\mathbf{Z}_i = \mathbf{a} + \mathbf{B}\mathbf{r}_i$, $i = 1, \ldots, n$, satisfy conditions (A.1)-(A.3) then $\hat{\beta}_{GLS}$ is asymptotically normally distributed. As seen in Section 2, (A.2) holds as long as the sequence $\mu_{x1}, \mu_{x2}, \ldots$, satisfy condition (S.2). Moreover, (A.1) and (A.3) hold as long as assumptions (S.1) and (S.3) are satisfied by the random vectors $\mathbf{r}_1, \mathbf{r}_2, \ldots$, which is the case for the elliptical family of distributions.

Theorem 3.1. *Consider the ultrastructural model defined in (3.1) and suppose that λ_e is known and that assumptions (S.1)-(S.3) are satisfied. If $\sigma_{xx} + \sigma^*_{xx} = \sigma_{uu}\lambda_x + \sigma_{uu}\lambda^*_x > 0$ then we have that*

$$\sqrt{n}(\hat{\beta}_{GLS} - \beta) \xrightarrow{d} N(0, \sigma_{\beta\beta}), \tag{3.7}$$

where

$$\sigma_{\beta\beta} = (\kappa + 1)\beta^2 \frac{\sigma_{YY}\sigma_{XX} - \sigma^2_{YX}}{(\sigma_{YX} + \sigma^*_{YX})^2} + \beta^2 \frac{\sigma^*_{xx}(\sigma_{YY} + \beta^2\sigma_{XX} - 2\beta\sigma_{YX})}{(\sigma_{YX} + \sigma^*_{YX})^2}$$
$$= \frac{(\kappa + 1)(\beta^2\lambda_x + \lambda_e\lambda_x + \lambda_e) + (\beta^2 + \lambda_e)\lambda^*_x}{(\lambda_x + \lambda^*_x)^2}.$$

Proof. Since $\mathbf{Z}_i = \mathbf{a} + \mathbf{B}\mathbf{r}_i$, $i = 1, 2, \ldots$, satisfy conditions (A.1)-(A.3), then Theorem 2.1 and the delta method applies for the sequence $U_n = U(\mathbf{S})$ defined in (3.6), leading to

$$\sqrt{n}(U(\mathbf{S}) - U(\boldsymbol{\Sigma} + \boldsymbol{\Sigma}^*)) \xrightarrow{d} N(0, \mathbf{d}'\mathbf{A}\mathbf{d} + \mathbf{d}'\mathbf{A}^*\mathbf{d}),$$

where $\mathbf{d} = \{1/(\sigma_{YX} + \sigma^*_{YX})\}(1, -u, -\lambda_e)'$, $\sigma_{YX} = \beta\sigma_{uu}\lambda_x$, $\sigma^*_{YX} = \beta\sigma_{uu}\lambda^*_x$ and $u = U(\boldsymbol{\Sigma} + \boldsymbol{\Sigma}^*) = (\beta^2 - \lambda_e)/\beta$. Note that

$$\mathbf{d}'\mathbf{A}\mathbf{d} = (\kappa + 1)((u^2 + 4\lambda_e)\frac{\sigma_{YY}\sigma_{XX} - \sigma^2_{YX}}{(\sigma_{YX} + \sigma^*_{YX})^2},$$

and

$$\mathbf{d}'\mathbf{A}^*\mathbf{d} = (u^2 + 4\lambda_e)\frac{\sigma^*_{xx}(\sigma_{YY} + \beta^2\sigma_{XX} - 2\beta\sigma_{YX})}{(\sigma_{YX} + \sigma^*_{YX})^2},$$

where $u^2 + 4\lambda_e = (\beta^2 + \lambda_e)^2/\beta^2$. Using again the delta method to obtain the asymptotic distribution of $\hat{\beta}_{GLS} = g(U_n) = g(U(\mathbf{S}))$, where $g(t) = \{t + (t^2 + 4\lambda_e)^{1/2}\}/2$, it follows that

$$\sqrt{n}(\hat{\beta}_{GLS} - \beta) \xrightarrow{d} N(0, (g'(u))^2\mathbf{d}'(\mathbf{A} + \mathbf{A}^*)\mathbf{d}).$$

The proof follows by noticing that $g'(t) = \{1+t(t^2+4\lambda_e)^{-1/2}\}/2$, so that $g'(u) = \beta^2/(\beta^2+\lambda_e)$.

Remark 3.2. Note that according to Theorem 2.1, $\hat{\beta}_{GLS}$ is asymptotically independent of $\bar{\mathbf{Z}}$ since condition (A.4) is satisfied under elliptical models.

Remark 3.3. In the case of the structural model, the asymptotic variance of $\sqrt{n}\hat{\beta}_{GLS}$ can be written as

$$\sigma_{\beta\beta} = (\kappa+1)\beta^2\frac{(1-\rho_{YX}^2)}{\rho_{YX}^2},$$

where $\rho_{YX} = Corr[Y_i, X_i] = \beta\lambda_x\{(\beta^2\lambda_x+\lambda_e)(\lambda_x+1)\}^{-1/2}$.

Remark 3.4. Gleser (1985) shows that in the ultrastructural normal model with λ_e known, $\hat{\beta}_{GLS}$ also is the maximum likelihood estimator of β. In the special case of the structural model with $\eta_{xi} = \eta_x$, $i = 1, \ldots, n$, ($\mu_{xi} = \mu_x$, $i = 1, \ldots, n$ and $\lambda_x^* = 0$), Arellano-Valle and Bolfarine (1994) (see also Anderson et al., 1986) show that $\hat{\beta}_{GLS}$ is the maximum likelihood estimator of β under the structural dependent elliptical model which assumes that $\mathbf{Z} = (\mathbf{Z}_1', \ldots, \mathbf{Z}_n') \sim EL_{2n}(\mathbf{1}_n \otimes \mu, \mathbf{I}_n \otimes \mathbf{V}; \phi)$, with density given by

$$|\mathbf{V}|^{-n/2} f(\sum_{i=1}^{n}(\mathbf{Z}_i-\mu)'\mathbf{V}^{-1}(\mathbf{Z}_i-\mu)),$$

where $\mu = \mathbf{a}+\mathbf{B}\eta$ and $\mathbf{V} = \mathbf{B}\Psi\mathbf{B}'$. Since, in such case $\hat{\beta}_{GLS}(a\mathbf{Z}) = \hat{\beta}_{GLS}(\mathbf{Z})$, for any real a, it follows (Fang et al., 1990) that $\hat{\beta}_{GLS}(\mathbf{Z}) \stackrel{d}{=} \hat{\beta}_{GLS}(\mathbf{Z}_N)$ where $\mathbf{Z}_N \sim N_{2n}(\mathbf{1}_n \otimes \mu, \mathbf{I}_n \otimes \mathbf{V})$ and $Y \stackrel{d}{=} X$ means that X and Y are identically distributed. Thus, if $\hat{\beta}_\phi$ and $\hat{\beta}_N$ are the maximum likelihood estimators under the elliptical dependent and normal models, respectively, then $\hat{\beta}_\phi = \hat{\beta}_N = \hat{\beta}_{GLS}$ and from (3.6) with $\kappa = 0$,

$$\sqrt{n}(\hat{\beta}_\phi - \beta) \stackrel{d}{=} \sqrt{n}(\hat{\beta}_N - \beta) \xrightarrow{d} N(0, \beta^2(\frac{1-\rho_{YX}^2}{\rho_{YX}^2})).$$

Remark 3.5. It is worth remarking that in the case of elliptical independent structural models (the case considered in this paper) the maximum likelihood estimator of β does not necessarily coincides with the generalized least squares estimator $\hat{\beta}_{GLS}$ given in (3.5). This is the case when $\mathbf{Z}_1, \ldots, \mathbf{Z}_n$, are independent with $\mathbf{Z}_i \sim t_2(\mu, \mathbf{V}; \nu)$, $0 < \nu < \infty$, where $\mu = E[\mathbf{Z}_i]$ and $\Sigma = Var[\mathbf{Z}_i] = (\nu/(\nu-2))\mathbf{V}$, for $\nu > 2$. In such case the maximum likelihood estimator has no closed form and has to be obtained iteratively. However, as illustrated in Lange et al. (1989), for finite values of ν the t-model assures robust (outliers resistant) estimation of β. In this case, Arellano-Valle and Bolfarine (1994) shows that the asymptotic distribution of the maximum likelihood estimator of β, which we denote by $\hat{\beta}_t$ is such that

$$\sqrt{n}(\hat{\beta}_t - \beta) \xrightarrow{d} N(0, (\frac{\nu+4}{\nu+2})\beta^2(\frac{1-\rho_{YX}^2}{\rho_{YX}^2})). \tag{3.8}$$

Thus, comparing results (3.6) and (3.7) where $\kappa+1=(\nu-2)/(\nu-4)$, $\nu>4$, we have that the asymptotic relative efficiency of $\hat{\beta}_{GLS}$ with respect to the estimator $\hat{\beta}_t$ is given by

$$e_{\hat{\beta}_{GLS},\hat{\beta}_t}=(\frac{\nu-4}{\nu-2})(\frac{\nu+4}{\nu+2}),$$

$\nu>4$. For $\nu=4.5$, $e_{\hat{\beta}_{GLS},\hat{\beta}_t}=0.26$ and as $\nu\to\infty(4)$, $e_{\hat{\beta}_{GLS},\hat{\beta}_t}\to 1(0)$. Finding the asymptotic distribution of the maximum likelihood estimator under the elliptical functional model still is an open problem.

Remark 3.6. Linder and Babu (1994) obtain the asymptotic distribution of $\hat{\beta}_{GLS}$ for the special case of the functional model assuming that the errors e_i and u_i are uncorrelated and have null thirdth moments (a kind of symmetry property). However, in the context of elliptical distributions, these assumptions imply that the errors are independent and normally distributed (Fang et al., 1990).

3.2. One of the error variances known

In this section, we consider that the error variance σ_{uu} is known. As pointed out by Fuller (1987), a reasonable value for σ_{uu} can be determined by making a large number of independent repeated measurements of the independent variable. Let $\mathbf{Z}_i=(Y_i,X_i)'$, $\mathbf{B}$, $\mathbf{a}$ and $\mathbf{r}_i$, $i=1,\ldots,n$, as in Section 3.1. In the sequel, we assume that the sequence $\mu_{x1},\mu_{x2},\ldots$, satisify condition (A.2) and that the sequence $\mathbf{r}_1,\mathbf{r}_2,\ldots$, satisfy conditions (S.1)-(S.3), so that $\mathbf{Z}_1,\mathbf{Z}_2,\ldots$, satisfy (A.1) and (A.3) in the context of elliptical distributions. The main result of this section is stated next. It presents asymptotic propreties of the estimator

$$\hat{\beta}_M=\frac{S_{YX}}{S_{YX}-\sigma_{uu}},$$

which is a consistent estimator under the assumption that $\mathbf{r}_i$ is normally distributed, $i=1,\ldots,n$. Asymptotic properties of $\hat{\beta}_M$ under normality are investigated by Chang and Van Ness (1991).

Theorem 3.2. *Consider the ultrastructural model model defined by (3.1) and supposed that σ_{uu} is known and that assumptions (S.1)-(S.3) are satisfied. If $\sigma_{xx}+\sigma^*_{xx}>0$ then*

$$\sqrt{n}(\hat{\beta}_M-\beta)\xrightarrow{d}N(0,\Delta_M), \tag{3.9}$$

where

$$\Delta_M=\frac{(\kappa+1)(\sigma_{XX}\sigma_{YY}-\sigma^2_{XY})+(3\kappa+2)(\beta\sigma_{XX}-\sigma_{XY})^2}{(\sigma_{xx}+\sigma^*_{xx})^2}$$
$$+\frac{\sigma^*_{xx}(\beta^2\sigma_{XX}-2\beta\sigma_{XY}+\sigma_{YY})}{(\sigma_{xx}+\sigma^*_{xx})^2}$$
$$=\frac{(\kappa+1)(\beta^2\lambda_x+\lambda_e\lambda_x+\lambda_e)+(3\kappa+2)\beta^2+(\beta^2+\lambda_e)\lambda^*_x}{(\lambda_x+\lambda^*_x)^2}.$$

Proof. Considering $\hat{\beta}_M = S_{XY}/(S_{XX} - \sigma_{uu}) = g(\mathbf{S})$, where $g(x,y,z) = y/(z-\sigma_{uu})$, it follows easily that

$$\hat{\beta}_M \xrightarrow{a.s.} \beta.$$

Now, using the delta method to obtain the asymptotic distribution of $\hat{\beta}_M = g(\mathbf{S})$, it follows that

$$\sqrt{n}(\hat{\beta}_M - \beta) \xrightarrow{d} N(0, \mathbf{d}'(\mathbf{\Lambda} + \mathbf{\Lambda}^*)\mathbf{d}),$$

where

$$\mathbf{d} = 1/(\sigma_{XX} + \sigma^*_{XX} - \sigma_{uu})(0,1,-\beta)'.$$

Thus, since $\sigma^*_{XX} = \sigma^*_{xx}$ and $\sigma_{XX} - \sigma_{uu} = \sigma_{xx}$, it follows that

$$\Delta_M = \mathbf{d}'(\mathbf{\Lambda} + \mathbf{\Lambda}^*)\mathbf{d} = \frac{(\kappa+1)(\sigma_{XX}\sigma_{YY} - \sigma^2_{XY}) + (3\kappa+2)(\beta\sigma_{XX} - \sigma_{XY})^2}{(\sigma_{xx} + \sigma^*_{xx})^2}$$

$$+ \frac{\sigma^*_{xx}(\beta^2\sigma_{XX} - 2\beta\sigma_{XY} + \sigma_{YY})}{(\sigma_{xx} + \sigma^*_{xx})^2}.$$

Remark 3.6. Under normality, that is, $\kappa = 0$, it follows that

$$\Delta_M = \frac{\sigma_{XX}\sigma_{YY} - \sigma^2_{XY} + 2(\beta\sigma_{XX} - \sigma_{XY})^2 + \sigma^*_{xx}(\beta^2\sigma_{XX} - 2\beta\sigma_{XY} + \sigma_{YY})}{(\sigma_{xx} + \sigma^*_{xx})^2},$$

which coincides with the corresponding result given in Cheng and Van Ness (1991).

Remark 3.7. In the case of the elliptical structural model, the asymptotic variance of $\sqrt{n}\hat{\beta}$ can be written as

$$\Delta_M = \frac{(\kappa+1)(\sigma_{XX}\sigma_{YY} - \sigma^2_{XY}) + (3\kappa+2)(\beta\sigma_{XX} - \sigma_{XY})^2}{\sigma^2_{xx}}$$

$$= (\kappa+1)\beta^2\left(\frac{1-\rho^2_{YX}}{\rho^2_{YX}}\right) + (3\kappa+2)\frac{\beta^2}{\lambda^2_x},$$

where $\lambda_x = \sigma_{xx}/\sigma_{uu}$. Under normality, $\kappa = 0$ and the asymptotic variance reduces to

$$\Delta_M = \frac{(\sigma_{XX}\sigma_{YY} - \sigma^2_{XY}) + 2(\beta\sigma_{XX} - \sigma_{XY})^2}{\sigma^2_{xx}}$$

$$= \beta^2\left(\frac{1-\rho^2_{YX}}{\rho^2_{YX}}\right) + 2\frac{\beta^2}{\lambda^2_x},$$

which coincides with the corresponding asymptotic variance given in Fuller (1987).

Remark 3.8. In the special case of the independent structural model with $\mathbf{Z}_1, \ldots, \mathbf{Z}_n$ independent and $\mathbf{Z}_i \sim t_2(\mu_z, \mathbf{V}; \nu)$, $0 < \nu < \infty$, as in Remark 3.5, the maximum likelihood estimator, $\hat{\beta}_t$, has no closed form and has to be obtained iteratively by using, for example, the EM algorithm. However, after lengthy algebraic manipulation, it can be shown that

$$\sqrt{n}(\hat{\beta}_t - \beta) \xrightarrow{d} N(0, \Delta_t), \tag{3.10}$$

where

$$\Delta_t = (\frac{\nu+4}{\nu+2})(\frac{\nu+1}{\nu})\frac{2(\beta\sigma_{XX} - \sigma_{XY})^2 + \frac{\nu}{\nu+1}(\sigma_{XX}\sigma_{YY} - \sigma_{XY}^2)}{\sigma_{xx}^2}$$

$$= (\frac{\nu+4}{\nu+2})(\frac{\nu+1}{\nu})\{2\frac{\beta^2}{\lambda_x^2} + \frac{\nu}{\nu+1}\beta^2(\frac{1-\rho_{YX}^2}{\rho_{YX}^2})\}.$$

Since under the t distribution with ν degrees of freedom $\kappa + 1 = (\nu - 2)/(\nu - 4)$, $\nu > 4$, it follows from (3.9) that

$$\Delta_M = \frac{(\frac{\nu-2}{\nu-4})(\sigma_{XX}\sigma_{YY} - \sigma_{XY}^2) + 2(\frac{\nu-1}{\nu-4})(\beta\sigma_{XX} - \sigma_{YY})^2}{\sigma_{xx}^2}$$

$$= (\frac{\nu-2}{\nu-4})\beta^2(\frac{1-\rho_{YX}^2}{\rho_{YX}^2}) + 2(\frac{\nu-1}{\nu-4})\frac{\beta^2}{\lambda_x^2}. \tag{3.11}$$

Thus, by comparing (3.11) and (3.10), it follows that the asymptotic relative efficiency of $\hat{\beta}_M$ with respect to $\hat{\beta}_t$ is given by

$$e_{\hat{\beta}_M,\hat{\beta}_t} = \frac{(\nu-4)(\nu+4)(\nu+1)}{\nu(\nu-1)(\nu+2)}\{\frac{1 + (\frac{\nu+2}{\nu+4})(\frac{\nu}{\nu+1})^2 a_{YX}}{1 + (\frac{\nu-2}{\nu-1})a_{YX}}\},$$

where $a_{YX} = (1 - \rho_{YX}^2)/\lambda_x^2\rho_{YX}^2$. It follows that $e_{\hat{\beta}_M,\hat{\beta}_t} \leq 1$, for all $\nu > 4$, since

$$\frac{(\nu+1)(\nu-4)(\nu+4)}{\nu(\nu-1)(\nu+2)} \leq 1 \quad \text{and} \quad (\frac{\nu-2}{\nu-1}) \geq \frac{\nu+2}{\nu+4}(\frac{\nu}{\nu+1})^2,$$

$\nu > 4$. Morever, as $\nu \to \infty(4)$, $e_{\hat{\beta}_M,\hat{\beta}_t} \to 1(0)$.

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