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Calculation of h^1 of some Anderson t-motives

S. Ehbauer

Departamento de Matemática
Universidade Federal do Amazonas, Manaus, Brasil

A. Grishkov

Departamento de Matemática e estatistica
Universidade de São Paulo
Rua de Matão 1010, CEP 05508-090, São Paulo,
Brasil and Omsk State University, N. A. F. M. Dostoevskii
Pr. Mira 55-A, Omsk 644077, Russia

D. Logachev*

Departamento de Matemática
Universidade Federal do Amazonas, Manaus, Brasil
logachev94@gmail.com

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We consider Anderson t-motives M of dimension 2 and rank 4 defined by some simple explicit equations parameterized by 2×2 matrices. We use methods of explicit calculation of $h^1(M)$ — the dimension of their cohomology group $H^1(M)$ (= the dimension of the lattice of their dual t-motive M') developed in our earlier paper. We calculate $h^1(M)$ for M defined by all matrices of the form $\begin{pmatrix} 0 & a_{12} \\ a_{21} & 0 \end{pmatrix}$, and by some matrices of the form $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & 0 \end{pmatrix}$. These methods permit to make analogous calculations for most (probably all) t-motives. h^1 of all Anderson t-motives M under consideration satisfy the inequality $h^1(M) \le 4$, while in all known examples we have $h^1(M) = 0, 1, 4$. Do there exist M of this type having $h^1 = 2, 3$? We do not know, this is a subject of further research.

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0. General Introduction

Anderson t-motives [4, 5.4.2, 5.4.18, 5.4.16] are the function field analogs of abelian varieties (more exactly, of abelian varieties with multiplication by an imaginary quadratic field (of MIQF-type), see, for example, [9]). Nevertheless, this analogy

*Corresponding author.

is not complete. For example, let M be an Anderson t-motive, respectively, A an abelian variety. We can associate them a lattice $L(M)$, respectively, $L(A)$. For abelian varieties, the functor $A \mapsto L(A)$ has a good description. For Anderson t-motives, the situation is much worse, for example, $h_1(M)$ — the dimension of $L(M)$ — can be less than it is expected to be. We started a study of the lattice map of Anderson t-motives in [6, 7]. We proved in [6] that (roughly speaking) in a system of neighborhoods of a fixed Anderson t-motive the lattice map is an isomorphism. We developed in [7] a method of calculation of $h_1(M)$, as well as of $h^1(M)$ — the dimension of the cohomology group of M, and we gave an example that not always $h_1(M) = h^1(M)$ (unlike the case of abelian varieties).

This paper is a continuation of [7]. We apply the method of calculation of $h^1(M)$ of [7] to a larger class of Anderson t-motives. We show that this method can be applied for most (probably all) t-motives.

The cases considered in this paper form a tiny part of the whole problem of finding of $h^1(M)$ for all M. This whole problem is really enormous. Clearly it cannot be solved without use of computers. Unfortunately writing of a corresponding program is not an easy task, because there exists a large diversity of cases. It is hardly likely that one scientist will be able to solve the problem, it requires a work of a large team of scientists. The authors hope that the results of this paper will stimulate further research.

The structure of the paper is the following. Sections 1–4 are introductory. Section 2 contains definitions on Anderson t-motives, Sec. 3 gives necessary results of [7], and Sec. 4 explains general methods of calculation. For the conjectures, problems of further research and justification of the subject see Problem 1.3; Conjecture 1.5; Problem 1.8 and the end of Sec. 1; Sec. 3.6; Conjecture 3.2; Sec. 4.5. In Secs. 5 and 6, we consider the case of A of the form (3.7) (all cases), and in Secs. 7 and 8, we consider the case of A of the form (3.8) (some cases).

There exists an enlarged arxiv version [3] of this paper where in Sec. 9 we give some calculations for the cases of A of the form (3.8) which are not considered in Secs. 7 and 8. They can be useful for future researchers.

1. More Detailed Introduction

Here, we give more details. First, we recall the number field case. For an abelian variety A of dimension g there exist its homology and cohomology groups $H_1(A, \mathbb{Z})$ and $H^1(A, \mathbb{Z})$ (both these groups are isomorphic to $\mathbb{Z}^{2g}$), and a $\mathbb{Z}$-perfect pairing between them:

$$H_1(A, \mathbb{Z}) \otimes_{\mathbb{Z}} H^1(A, \mathbb{Z}) \to \mathbb{Z}. \tag{1.1}$$

There is an inclusion $\gamma : H_1(A, \mathbb{Z}) \to \mathbb{C}^g$ such that $H_1(A, \mathbb{Z})$ forms a lattice in $\mathbb{C}^g$. We have $A = \mathbb{C}^g / H_1(A, \mathbb{Z})$. We have the following theorem.

Theorem 1.1. *Abelian varieties of dimension g over $\mathbb{C}$ are in 1-1 correspondence with $\mathbb{Z}$-lattices of dimension $2g$ in $\mathbb{C}^g$, satisfying the Riemann condition.*

Finally, for an abelian variety A we can define its dual variety A'. There exist canonical isomorphisms

$$H_1(A,\mathbb{Z}) \to H^1(A',\mathbb{Z}), \quad H^1(A,\mathbb{Z}) \to H_1(A',\mathbb{Z}). \tag{1.2}$$

Let us give necessary definitions for the case of Anderson t-motives. Let q be a power of a prime p, $\mathbb{F}_q$ the finite field of order q. The function field analog of $\mathbb{Z}$ is the ring of polynomials $\mathbb{F}_q[\theta]$, where θ is an abstract variable. The analog of the archimedean valuation on $\mathbb{Q}$ is the valuation at infinity on the fraction field $\mathbb{F}_q(\theta)$ of $\mathbb{F}_q[\theta]$; it is denoted by ord, it is uniquely determined by the property ord $(\theta) = -1$. The completion of an algebraic closure of the completion of $\mathbb{F}_q(\theta)$ with respect the valuation "ord" is the function field analog of $\mathbb{C}$. It is denoted by $\mathbb{C}_\infty$.

Abelian varieties have one discrete invariant — their dimension g. Unlike them, Anderson t-motives have two invariants: dimension and rank (see 2.1); Anderson t-motives of dimension n and rank r are analogs of abelian varieties of dimension r of MIQF-type, of signature $(n, r-n)$.

An Anderson t-motive M has the homology and cohomology groups (see [4, 5.9.11(2), (3)[a]] and Definition 2.3 of this paper) $H_1(M, \mathbb{F}_q[T]) = H_1(M)$ and $H^1(M, \mathbb{F}_q[T]) = H^1(M)$ which are free $\mathbb{F}_q[T]$-modules (here T is an abstract variable, it is one of the generators of the Anderson ring, see Definition 2.1). The ranks of $H_1(M)$, $H^1(M)$ are denoted by $h_1(M), h^1(M)$, respectively. By analogy with the number field case we can expect that always $h_1(M) = h^1(M) = r$. But unlike the case of abelian varieties, they can be less than r.

Like for the case of abelian varieties, for an Anderson t-motive M there exists the dual t-motive M' (see [5]). Analogs of (1.2) hold for Anderson t-motives (see, for example [7, Proposition 1.9]): there exist canonical isomorphisms

$$H_1(M) \to H^1(M'), \quad H^1(M) \to H_1(M').$$

In particular, a method of calculation of h^1 permits us to calculate the h_1 as well: we apply it to the dual t-motive.

Analog of (1.1) is a pairing

$$\pi : H_1(M) \otimes_{\mathbb{F}_q[T]} H^1(M) \to \mathbb{F}_q[T]. \tag{1.3}$$

Counterexample of [7] shows that not always $h_1(M) = h^1(M)$, hence (1.3) is not always perfect. There is the following theorem:

Theorem 1.2 (Anderson [1]; [4, 5.9.14]). *$h^1(M) = r \iff h_1(M) = r$. In this case π is perfect over $\mathbb{F}_q[T]$.*

Anderson t-motives M satisfying these conditions are called uniformizable.

[a] Goss uses a notation $H_1(E)$ instead of $H_1(M)$. This is practically the same: there is a 1-1 correspondence between t-modules E and t-motives M.

There exists a lattice map

$$H_1(M) \xrightarrow{\alpha} (\mathbb{F}_q[\theta])^{h_1(M)} \xhookrightarrow{\beta} \mathbb{C}_\infty^n,$$

where α is an (abstract) isomorphism defined by the condition $\alpha(T) = \theta$ (it serves only in order to identify T and θ) and β is an inclusion of $\mathbb{F}_q[\theta]$-modules, see [4, Sec. 5.9]. The composition inclusion $\beta \circ \alpha$ is an analog of the above γ for abelian varieties. The image $\beta \circ \alpha(H_1(M))$ is denoted by $L(M)$, it is a $\mathbb{F}_q[\theta]$-lattice of rank $h_1(M)$ in $\mathbb{C}_\infty^n$.

There is a general problem:

Problem 1.3. What is the relation between the set of $\mathbb{F}_q[\theta]$-lattices of rank r in $\mathbb{C}_\infty^n$, up to $\mathbb{C}_\infty$-isomorphisms of $\mathbb{C}_\infty^n$, and the set of uniformizable Anderson t-motives of rank r and dimension n? Have we some analog of Theorem 1.1?

There exists a notion of purity of Anderson t-motives M (see [4, 5.5.2] for the definition). We can expect that in Problem 1.3 we must consider only pure t-motives: conjecturally, the lattice map $M \mapsto L(M)$ has a fiber of dimension ≥ 1 if we consider the set of all Anderson t-motives. Taking into consideration the following theorem:

Theorem 1.4 ([8, Theorem 3.2]). *The dimension of the moduli space of pure t-motives of rank r and dimension n is equal to $n(r-n)$*

and the obvious fact that the moduli space of lattices of rank r in $\mathbb{C}_\infty^n$ has the same dimension $n(r-n)$ we can state the following conjecture.

Conjecture 1.5. The image of the lattice map $M \mapsto L(M)$ from the set of pure uniformizable t-motives to the set of lattices is open, and its fiber at a generic point is discrete.

The following is known:

Theorem 1.6 (Drinfeld [2]). *All t-motives of dimension 1 (= Drinfeld modules) are pure and uniformizable. There is a 1-1 correspondence between Drinfeld modules of rank r over $\mathbb{C}_\infty$ and lattices of rank r in $\mathbb{C}_\infty$.*

For $n = r-1$ the duality theory gives us an immediate corollary of Theorem 1.6.

Corollary 1.7 ([5, Corollary 8.4]). *All pure t-motives of rank r and dimension $r-1$ over $\mathbb{C}_\infty$ are uniformizable. There is a 1-1 correspondence between their set, and the set of lattices of rank r in $\mathbb{C}_\infty^{r-1}$ having dual.*[b]

Not all such lattices have dual, but almost all, i.e. even in this simple case the correspondence is not strictly 1-1, but only an "almost 1-1".

We see that for pure M the minimal values of r, n for which we can expect $h^1, h_1 < r$ are $r = 4$, $n = 2$. We shall consider exactly this case. More generally,

[b]There is a notion of duality of lattices, see [5, Definition 2.3 and Sec. 3]. We do not need details here.

we shall consider a class of t-motives of dimension n and rank $r = 2n$ defined by Eq. (2.2). We see that they are defined by a matrix $A \in M_{n\times n}(\mathbb{C}_\infty)$; the corresponding t-motive is denoted by $M(A)$. All these t-motives are pure, but not all uniformizable.

Now, we can formulate the results of this paper. We calculate h^1 for all t-motives $M(A)$ where A is of the form (3.7), and for some A of the form (3.8).

In particular, we describe all uniformizable $M(A)$ where A is of the form (3.7). This is a step to a solution of the Problem 1.3: clearly, we need first to describe explicitly the set of all uniformizable Anderson t-motives. Finding (description) of their lattices is a subject of further research.

Let us indicate some earlier results to a solution of Problem 1.3, and some related problems. An explicit description of lattices in $\mathbb{C}_\infty^n$ is given in terms of their Siegel matrices $\mathcal{S}$, see for example [6], Definition 1.5 (this definition is completely analogous to the definition of Siegel matrices of lattices in $\mathbb{C}^g$).

The main result of [6] is, roughly speaking, the 1-1 correspondence between the set of t-motives $M(A)$ defined by (2.2) whose A is in a neighborhood of 0, and the set of lattices whose $\mathcal{S}$ is in a neighborhood of a fixed Siegel matrix $\mathcal{S}_0$ (the main difficulty is to show that we have the same action of some groups on the set of $\mathcal{S}$ and A). In particular, it is shown that if all entries a_{ij} of A satisfy

$$\operatorname{ord} a_{ij} > \frac{q}{q^2-1} \tag{1.4}$$

then $M(A)$ is uniformizable [6, end of p. 383 and Proposition 2]. There is a natural problem to improve this estimate, i.e. to answer the following open problem:

Open Problem 1.8. Let n, q be arbitrary. What is the minimal value of $C(n,q)$ satisfying the property: If A from (2.2) satisfies $\forall\, i,j$ $\operatorname{ord} a_{ij} > C(n,q)$ (version strict inequality) or $\operatorname{ord} a_{ij} \geq C(n,q)$ (version non-strict inequality) then $M(A)$ is uniformizable.

The above result means $C(n,q) \leq \frac{q}{q^2-1}$. Theorem 1.6 implies $C(1,q) = -\infty$. [4, Example 5.9.9] means $C(n,q) \geq -\frac{q^2}{q-1}$ for $n \geq 2$, see Remark 6.4. If we restrict ourselves by A of the form (3.7) then Proposition 6.3 gives $C_{(3.7)}(2,q) = -\frac{q^2}{q-1}$, version strict inequality (here $C_{(3.7)}(2,q)$ means the minimal value of the above $C(n,q)$ for matrices of the form (3.7)).

The main result of [7] is finding of an explicit method (solution of an affine equation, see (3.2)) for calculation of $h^1(M)$ where M belongs to the same set of $M(A)$, case $n = 2$. There exist matrices A such that the application of this method to these A permits to show that not always $h^1(M) = h_1(M)$.

Methods of [6], [7] are essentially different. In [6], we find explicitly $L(M)$ which is the kernel of the exponential map of M (see [4, Sec. 5.9]) using a method of successive approximations. This method can be applied to M defined by (2.2), case any n, but only for A sufficiently close to 0. For all these A we have $h_1(M(A)) = h^1(M(A)) = 2n$. In [7], we calculate $h^1(M)$ solving explicitly an affine equation (it is a system of polynomial equations, see (3.2)). This method can be applied for any A, but only for $n = 2$ (for $n > 2$ the calculations seem to be too difficult).

2. Definitions on Anderson t-Motives

Definition 2.1. The Anderson ring $\mathbb{C}_\infty[T, \tau]$ is the ring of non-commutative polynomials in two variables T, τ over $\mathbb{C}_\infty$ satisfying the following relations (here $a \in \mathbb{C}_\infty$):

$$Ta = aT, \quad T\tau = \tau T, \quad \tau a = a^q \tau.$$

Subrings of $\mathbb{C}_\infty[T, \tau]$ generated by τ, respectively T, are denoted by $\mathbb{C}_\infty\{\tau\}$ (a ring of non-commutative polynomials in one variable), respectively $\mathbb{C}_\infty[T]$ (the ordinary ring of (commutative) polynomials in one variable).

Definition 2.2 ([4, 5.4.2, 5.4.18, 5.4.16]). A t-motive[c] M is a left $\mathbb{C}_\infty[T, \tau]$-module which is free and finitely generated as both $\mathbb{C}_\infty[T]$-, $\mathbb{C}_\infty\{\tau\}$-module and such that

$$\exists \mathfrak{m} = \mathfrak{m}(M) \text{ such that } (T - \theta)^{\mathfrak{m}} M/\tau M = 0. \tag{2.1}$$

We shall consider only t-motives for which $\mathfrak{m} = 1$.

The dimension of M over $\mathbb{C}_\infty\{\tau\}$ (respectively, $\mathbb{C}_\infty[T]$) is denoted by n (respectively, r), these numbers are called the dimension and rank of M.

Let $e_* = (e_1, \ldots, e_n)^t$ (here and below t means transposition) be the vector column of elements of a basis of M over $\mathbb{C}_\infty\{\tau\}$. To define M, it is sufficient to define the multiplication by T of $e_1, \ldots, e_n$. We shall consider M where the multiplication by T is given by the formula:

$$Te_* = \theta e_* + A\tau e_* + \tau^2 e_*, \tag{2.2}$$

where $A \in M_{n\times n}(\mathbb{C}_\infty)$. The Anderson t-motive defined by (2.2) is denoted by $M(A)$. It is pure of dimension n and rank $2n$. We shall consider only the case $n = 2$. For this case (2.2) becomes

$$T\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \theta \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \tau \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} + \tau^2 \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}, \tag{2.3}$$

where $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in M_{2\times 2}(\mathbb{C}_\infty)$ is a matrix.

We have $M(A)' = M(A^t)$ (see [7, Lemma 1.10.2]).

Since our purpose is to find $h^1(M)$, we repeat a definition of $H^1(M)$ here. First, we denote by $\mathbb{C}_\infty\{T\}$ a subring of $\mathbb{C}_\infty[[T]]$ formed by series $\sum_{i=0}^\infty a_i T^i$ such that $\lim a_i = 0$ ($\iff$ ord $a_i \to +\infty$). τ acts on $\mathbb{C}_\infty\{T\}$ by the formula $\tau(\sum_{i=0}^\infty a_i T^i) = \sum_{i=0}^\infty a_i^q T^i$.

[c]Terminology of Anderson; Goss calls these objects abelian t-motives.

Now, we define

$$M\{T\} := M \otimes_{\mathbb{C}_\infty[T]} \mathbb{C}_\infty\{T\}$$

τ acts on $M\{T\}$ by the standard formula of the action of an operator on tensor product: $\tau(\alpha \otimes \beta) = \tau(\alpha) \otimes \tau(\beta)$ (see [4, 5.9.11.1]).

Definition 2.3. $H^1(M) = M\{T\}^\tau$ (the set of τ-stable elements).

This definition should be understood as follows. We denote $M[[T]] := M \otimes_{\mathbb{C}_\infty[T]} \mathbb{C}_\infty[[T]]$ with the τ-action on the tensor product. We have $M\{T\}^\tau$ is $M[[T]]^\tau \cap M\{T\}$, i.e. $H^1(M)$ is the set of τ-invariant series whose coefficients tend to 0.

3. Affine Equations: Definitions and Results of [7]

For general definitions concerning the affine equations see [7, Sec. 2]. Here, we repeat them for a particular case that we need. Let $A = \left(\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right)$ be from (2.3) above. We associate it 9 numbers $a_4, \ldots, a_0, b_{14}, b_{13}, b_{12}, b_{24}$ as follows (see [7, (3.9)], case $\varepsilon = 0, a_{21} \neq 0$; if $a_{21} = 0$ then there exist analogs of these formulas based on a symmetry between a_{21} and a_{12}):

$$a_4 = \frac{\theta^{q^3+q^2}}{a_{21}^{q^2}}; \quad a_3 = \frac{a_{11}^{q^2}\theta^{q^2}}{a_{21}^{q^2}} + \frac{a_{22}^{q}\theta^{q^2}}{a_{21}^{q}}; \quad a_2 = \frac{\theta^q}{a_{21}} + \frac{\theta^{q^2}}{a_{21}^{q^2}} + \frac{a_{11}^q a_{22}^q}{a_{21}^q} - a_{12}^q;$$

$$a_1 = \frac{a_{11}}{a_{21}} + \frac{a_{22}^q}{a_{21}^q}; \quad a_0 = \frac{1}{a_{21}}; \quad b_{14} = -\frac{\theta^{q^3} + \theta^{q^2}}{a_{21}^{q^2}}; \tag{3.1}$$

$$b_{13} = -\frac{a_{11}^{q^2}}{a_{21}^{q^2}} - \frac{a_{22}^q}{a_{21}^q}; \quad b_{12} = -\frac{1}{a_{21}} - \frac{1}{a_{21}^{q^2}}; \quad b_{24} = \frac{1}{a_{21}^{q^2}}.$$

The affine equation corresponding to $M(A)$ is a series of equations with unknowns $x_0, x_1, x_2, \ldots$. The i-th equation ($i = 0, 1, 2, \ldots$) of the series has the form (here $x_\alpha = 0$ for $\alpha < 0$)

$$a_4 x_i^{q^4} + a_3 x_i^{q^3} + a_2 x_i^{q^2} + a_1 x_i^q + a_0 x_i + b_{14} x_{i-1}^{q^4} + b_{13} x_{i-1}^{q^3} + b_{12} x_{i-1}^{q^2} + b_{24} x_{i-2}^{q^4} = 0. \tag{3.2}$$

The terms $a_k x_i^{q^k}$ ($k = 0, \ldots, 4$) are called the head terms of the equations, other terms are called the tail terms. Explicitly, for $i = 0$ the equation is

$$a_4 x_0^{q^4} + a_3 x_0^{q^3} + a_2 x_0^{q^2} + a_1 x_0^q + a_0 x_0 = 0 \tag{3.3}$$

(only the head terms); for $i = 1$

$$a_4 x_1^{q^4} + a_3 x_1^{q^3} + a_2 x_1^{q^2} + a_1 x_1^q + a_0 x_1 + b_{14} x_0^{q^4} + b_{13} x_0^{q^3} + b_{12} x_0^{q^2} = 0, \tag{3.4}$$

for $i = 2$

$$a_4 x_2^{q^4} + a_3 x_2^{q^3} + a_2 x_2^{q^2} + a_1 x_2^q + a_0 x_2 + b_{14} x_1^{q^4} + b_{13} x_1^{q^3} + b_{12} x_1^{q^2} + b_{24} x_0^{q^4} = 0, \tag{3.5}$$

for $i = 3$

$$a_4 x_3^{q^4} + a_3 x_3^{q^3} + a_2 x_3^{q^2} + a_1 x_3^{q} + a_0 x_3 + b_{14} x_2^{q^4} + b_{13} x_2^{q^3} \\ + b_{12} x_2^{q^2} + b_{24} x_1^{q^4} = 0, \tag{3.6}$$

etc., the i-th equation is analogous to (3.6).

The set of solutions to (3.3) is a 4-dimensional $\mathbb{F}_q$-vector space in $\mathbb{C}_\infty$ denoted by S_0. For any such fixed solution x_0 the set of x_1 satisfying (3.4) with the given x_0 is an affine space over S_0; the same holds for all subsequent equations. This explains the terminology (affine equations).

Let $x_0, x_1, x_2, \dots$ be a solution to (3.2). We associate it an element $x_0 + x_1 T + x_2 T^2 + \cdots \in \mathbb{C}_\infty[[T]]$. The set of these elements is a free $\mathbb{F}_q[[T]]$-module of rank 4. A solution $x_0, x_1, x_2, \dots$ (and the associated element $x_0 + x_1 T + x_2 T^2 + \cdots$) is called a small solution if $\lim x_i = 0$ ($\iff$ $\lim \operatorname{ord} x_i = +\infty$). The set of small solutions is a free $\mathbb{F}_q[T]$-module.

Theorem 3.1. *$h^1(M(A))$ is the rank of the $\mathbb{F}_q[T]$-module of small solutions to* (3.2).

This follows from the calculations of [7, Sec. 3]. Non-formally, the meaning of this theorem is the following. Any (non-necessarily small) solution to the affine equation (3.2) corresponds to an element of $(M \otimes_{\mathbb{C}_\infty[T]} \mathbb{C}_\infty[[T]])^\tau$. Condition that a solution is small is equivalent to the condition that it belongs to $H^1(M) = (M \otimes_{\mathbb{C}_\infty[T]} \mathbb{C}_\infty\{T\})^\tau$.

So, our general problem is to solve (3.2) and to find $h^1(A)$ for all $A \in M_{2\times 2}(\mathbb{C}_\infty)$. This is too complicated, see below, so we consider only some particular cases. Namely, in Sec. 5, we solve this problem completely for A having the form

$$A = \begin{pmatrix} 0 & a_{12} \\ a_{21} & 0 \end{pmatrix} \tag{3.7}$$

(because the matrix A from [4, Example 5.9.9] having $h^1 = 0$ is of this form (after some $\mathbb{C}_\infty$-linear change of basis, see Remark 6.4)), and in Sec. 7 and 8, we start to solve this problem for the matrices A having the form

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & 0 \end{pmatrix} \quad \text{where } \operatorname{ord} a_{11} = -1 \tag{3.8}$$

(because the matrix A from [7, Sec. 4] — giving a counterexample to $h^1 = h_1$ — is of this form).

Our purpose is to answer some questions of [7, Sec. 0.3]. As the first step, we want to find all possible values of $h^1(A)$. There are examples $h^1 = 0, 1, 4$, but until now there is no examples $h^1 = 2$ or 3. Particularly, it turns out that h^1 of all $M(A)$ of Sec. 3 is equal to 0 or 4.

Hence, at the moment we are far from a solution of questions of [7, Sec. 0.3], for example: What are possible triples of numbers $\{h^1(M), h_1(M)$, rank of pairing $\pi\}$? Surprisingly, we have no other nontrivial pairings except the ones that are covered by Theorem 1.2. Moreover, it seems that for "almost all" matrices A we have $h^1(M(A)) = 4$. The words "almost all" should be understood as follows. Let us consider the ord_4 map from $M_{2\times 2}(\mathbb{C}_\infty)$ to $(\mathbb{Q} \cup +\infty)^4$ defined as ord's of entries of A:

$$\text{ord}_4(A) = (\text{ord } a_{11},\ \text{ord } a_{12},\ \text{ord } a_{21},\ \text{ord } a_{22}) \in (\mathbb{Q} \cup +\infty)^4.$$

Conjecture 3.2. There exists a subset $\mathfrak{U}$ of $(\mathbb{Q} \cup +\infty)^4$ which is the complement to a union of countably many (maybe even finitely many — we do not know) linear subspaces of dimension ≤ 3 such that if ord $_4(A) \in \mathfrak{U}$ then $h^1(M(A)) = 4$.

For A of the form (3.7) this conjecture is confirmed by Proposition 6.2. Moreover, results of some computer calculations (finding of minimal chains, see Sec. 4.2) made by the authors also support this conjecture.

4. Method of Calculation

We shall solve consecutively for $i = 0, 1, 2, \ldots$ eq. (3.2) for a given A. A solution $(x_0, x_1, x_2, \ldots)$ will be denoted by $\{x\}$; if $\{x\}$ carries a subscript $\{x_k\}$ then the corresponding $(x_0, x_1, x_2, \ldots)$ will be denoted as $(x_{k0}, x_{k1}, x_{k2}, \ldots)$. There is a trivial lemma:

Lemma 4.1 (=[7, Proposition 2.3]). *Let $\{x_1\}, \ldots, \{x_4\}$ be solutions to* (3.2). *This set is a basis of the set of solutions to* (3.2) *over $\mathbb{F}_q[[T]]$ iff $x_{10}, \ldots, x_{40}$ is a basis of S_0 over $\mathbb{F}_q$.* □

To calculate ord x_i we use the notion of Newton polygon. Let us give the corresponding definitions. Let $P = \sum_{i=0}^{n} c_i x^i$ be a polynomial, $c_i \in \mathbb{C}_\infty$. We associate it a set of $n+1$ points on a plane whose coordinates are $(i, \text{ord } c_i)$, $i = 0, \ldots, n$. These points are called the Newton points of P. The Newton polygon of P is the lower convex hull of its Newton points. The ord's of the roots of P are the minus slopes of the segments of its Newton polygon.

Let us apply this notion to an affine equation. The 0-th polynomial of an affine equation gives us a set of Newton points $(1, \text{ord } a_0), (q, \text{ord } a_1), \ldots,\ (q^4, \text{ord } a_4)$. If the Newton polygon of the 0-th polynomial consists of 4 segments then their minus slopes are the ord's of the four elements of a basis of S_0 over $\mathbb{F}_q$. According the above notations, we denote these basis elements by $x_{10}, \ldots, x_{40}$. If the Newton polygon of the 0-th polynomial consists of less than 4 segments then some of ord's of $x_{10}, \ldots, x_{40}$ are equal.

Now, let us consider the i-th affine equation. Its unknown is x_i, we consider $x_0, \ldots, x_{i-1}$ as already known (fixed). We denote the ord of the sum of the tail terms of the i-th equation by g_i, hence the Newton points of the i-th equation are $(0, g_i)$, $(1, \text{ord } a_0)$, $(q, \text{ord } a_1), \ldots,$ $(q^4, \text{ord } a_4)$.

Let us recall a notion of a minimal solution from [7, Definition 2.5]: A solution to (3.2) $\{x\} = (x_0, x_1, x_2, \ldots)$ is called a minimal solution (generated by x_0) if $\forall\, i > 0$ it satisfies the following condition: ord x_i corresponds to the leftmost segment of the Newton polygon of the i-th equation (3.2) (the i-th formula of the series (3.3)–(3.6)), i.e. ord x_i has the maximal possible value amongst ord's of solutions to the i-th affine equation for fixed $x_0, x_1, x_2, \ldots, x_{i-1}$.

Clearly a minimal solution generated by x_0 is not unique. Moreover, it can happen that even ord x_i of a minimal solution vary (i is fixed, minimal solutions vary): this can occur if there exists a jump of valuation[d] of the tail terms.

Conversely, let us formalize the situation when ord's of tail terms are different. We repeat [7, Definition 2.4]: A solution $\{x\} = (x_0, x_1, x_2, \ldots)$ is called simple if for all i we have: among all tail members of the i-th equation (3.2) for this $\{x\}$ (i.e. obtained while we substitute $x_0, x_1, \ldots, x_{i-1}$) there exists only one term whose ord is the minimal one.

Clearly if one minimal solution generated by x_0 is simple then all minimal solutions generated by x_0 are also simple, and for any i the ord x_i are the same for all minimal solutions generated by x_0. They depend only on i and on ord's of $a_0, \ldots, a_4, b_{12}, b_{13}, b_{14}, b_{24}$.

4.1

Further, we can define a sequence of ord's (called a simple minimal sequence) as follows. We start from ord x_{i0} for some fixed i, $i = 1, \ldots, 4$. We define $\tilde{g}_1$ as the minimum of the ord's of the tail terms of (3.4). We define v_1 (abbreviation of "valuation of x_{i1}") as the minus slope of the leftmost segment of the convex hull of $(0, \tilde{g}_1)$, $(1, \text{ord } a_0)$, $(q, \text{ord } a_1), \ldots,$ $(q^4, \text{ord } a_4)$. Now, we consider the ord's of the tail terms of (3.5) where instead of ord x_1 we substitute v_1. Exactly, we define $\tilde{g}_2 := \min$ (ord $b_{1k} + q^k v_1$, $k = 2, 3, 4$, and ord $b_{24} + q^4 \text{ord } x_{i0}$). We define v_2 as the minus slope of the leftmost segment of the convex hull of $(0, \tilde{g}_2)$, $(1, \text{ord } a_0)$, $(q, \text{ord } a_1), \ldots,$ $(q^4, \text{ord } a_4)$. Continuing this process, we define $\tilde{g}_3 := \min$ (ord $b_{1k} + q^k v_2$, $k = 2, 3, 4$, and ord $b_{24} + q^4 v_1$), we define v_3 etc.

For any minimal solution $x_{i0}, x_{i1}, x_{i2}, \ldots$ generated by x_{i0} the following inequality holds:

$$\forall\, j \quad \text{ord } x_{ij} \geq v_j$$

[d] A jump of valuation is a situation when ord $a =$ ord b and ord $a + b >$ ord a.

(it is proved immediately by induction by j). In particular, if $v_j \to +\infty$ then a minimal solution generated by x_{i0} is a small solution. Moreover, we have the following proposition.

Proposition 4.2. *If ord $x_{10} \geq \cdots \geq$ ord x_{40} and the sequence v_j for x_{40} tends to $+\infty$ then $h^1 = 4$.* □

4.2

This simple criterion was used in computer calculations. We chose random values of ord a_{ij}, $i, j = 1, 2$ (i.e. 4 random rational numbers). We assumed that there were no jumps of valuation in terms of the formulas (3.1), i.e. we put ord of $a_0, \ldots, a_4, b_{12}, b_{13}, b_{14}, b_{24}$ as the minimum of the ord's of their terms. Further, we calculated (according the algorithm described in Sec. 4.1) the values of v_j starting from ord x_{40} (the minimal value, see Proposition 4.2). In all considered cases we had $v_j \to +\infty$. This supports Conjecture 3.2.

4.3

It is easy to see that if the structure of the convex hull of Newton points does not depend on i then the behavior of v_i is either linear or exponential:

$v_i = \alpha + \beta i$ (linear, α, β are constants), or

$v_i = \alpha + \beta \cdot \gamma^i$ (exponential, $\alpha, \beta, \gamma > 0$ are constants). It can happen $\beta > 0$ or $\beta < 0$, $\gamma > 1$ or $\gamma < 1$.

4.4

Proofs of all propositions of this paper are similar. We use induction to find the above α, β, γ and according the cases $\gamma > 1$ or $\gamma < 1$ we get the result. Difficulties occur if there are different types of complex hulls, or jumps of valuation.

4.5

Further research. The reader will see that in order to solve the problem of calculation of h^1 completely (for all A) we must use computers. There are following questions.

4.5.1

Is it true that for any fixed A and for all minimal solutions there exists i_0 such that for $i > i_0$ the types of convex hulls of the Newton polygons of the i-th equation (3.2), i.e. the i-th formula of the series (3.3)–(3.6), are the same (and hence v_i behave like in Sec. 4.3)?

4.5.2

Does there exist an algorithm of calculation of $h^1(M(A))$ for any fixed A?

Even if the answer to (4.5.1) is YES, we cannot guarantee existence of such algorithm, because a linear combination of non-small solutions can give a small solution, i.e. $h^1(M(A))$ can be bigger than the quantity of independent small minimal solutions.

4.5.3

Is the quantity of types of the matrices A (see below for the types) finite or not?

In order to prove or to disprove Conjecture 1.5, we should study the lattices corresponding to uniformizable t-motives. The first question in this direction is

Problem 4.3. Find (describe) Siegel matrices corresponding to $M(A)$ where A is of the form (3.7) such that $M(A)$ is uniformizable.

5. Case of A of the Form (3.7)

We use notations v, t where $v = \operatorname{ord} a_{21}$ and $t := \operatorname{ord} a_2 = \operatorname{ord}\left(\frac{\theta^q}{a_{21}} + \frac{\theta^{q^2}}{a_{21}^{q^2}} - a_{12}^q\right)$. Since a_{12} appears in the formulas (3.1) only once (in a_2), we get that a_2 defines a_{12} uniquely.

Theorem 5.1. *For $t \geq -v$, $v \leq -\frac{q^2}{q-1}$ and for $t \geq -q^2 v$, $v \geq \frac{1}{q-1}$, we have $h^1(M(A)) = 0$, for other values of t, v, we have $h^1(M(A)) = 4$.*

Remark. The set of points on (v,t)-plane having $h^1 = 0$ is the union of interiors and borders of two angles (denoted by $\mathcal{A}_-$, respectively $\mathcal{A}_+$ according the sign of v) whose intersection is empty.

Proof. It will consist of Propositions 5.3, 5.4, 5.6. Formulas (3.1) for the present case give us

$\operatorname{ord} a_4 = -q^2 v - q^3 - q^2$; $a_3 = 0$; $\operatorname{ord} a_2 = t$; $a_1 = 0$; $\operatorname{ord} a_0 = -v$;

$\operatorname{ord} b_{14} = -q^2 v - q^3$; $b_{13} = 0$; $\operatorname{ord} b_{12} = -\delta v$ where $\delta = 1$ for $v < 0$, $\delta = q^2$ for $v > 0$, and $\operatorname{ord} b_{12} \geq 0$ for $v = 0$; $\operatorname{ord} b_{24} = -q^2 v$.

We denote vertices of the angles $\mathcal{A}_-$, respectively $\mathcal{A}_+$ by V_-, respectively V_+. We have

$$t < -\frac{2q^2}{q^2+1} v - \frac{q^3+q^2}{q^2+1}, \tag{5.1}$$

is the condition that the Newton polygon of (3.3) consists of two segments, their vertices are $(1,-v)$, (q^2,t), $(q^4, -q^2 v - q^3 - q^2)$.

5.1a.

We have: the straight line $t = -\frac{2q^2}{q^2+1} v - \frac{q^3+q^2}{q^2+1}$ passes through V_-, V_+.

If (5.1) holds then the ord's of x_{j0} are

$$\text{ord } x_{10} = \text{ord } x_{20} = -\frac{1}{q^2-1}t - \frac{1}{q^2-1}v, \tag{5.2}$$

$$\text{ord } x_{30} = \text{ord } x_{40} = \frac{1}{q^4-q^2}t + \frac{1}{q^2-1}v + \frac{1}{q-1}. \tag{5.3}$$

Case 5.2. $t \le -\frac{2q^2}{q^2+1}v - \frac{q^3+q^2}{q^2+1}$ and $t \ge -\delta v$. 5.1a implies that the set of (v, t) satisfying these conditions is the union of two angles (denoted by $\mathcal{A}_{1-}, \mathcal{A}_{1+}$) such that $\mathcal{A}_{1-} \subset \mathcal{A}_-$, $\mathcal{A}_{1+} \subset \mathcal{A}_+$, their vertices are V_-, respectively V_+, and they have common rays with $\mathcal{A}_-$, respectively $\mathcal{A}_+$. Let us prove that in this case $h^1 = 0$. We use notations of Sec. 4: $x_0, x_1, x_2, \dots$ is any solution to the affine equation (3.2), and for any fixed $i > 0$ let $g_i := \text{ord } (b_{14}x_{i-1}^{q^4} + b_{13}x_{i-1}^{q^3} + b_{12}x_{i-1}^{q^2} + b_{24}x_{i-2}^{q^4})$.

Proposition 5.3. *For Case* 5.2 *for any solution* $x_0, x_1, x_2, \dots,$ *we have*

$$\forall i \quad \frac{1}{q^2}v + \frac{1}{q-1} \le \mathit{ord}\ x_i \le -\frac{1}{q^2-1}t - \frac{1}{q^2-1}v \quad \mathit{if}\ v < 0, \tag{5.4}$$

$$\frac{1}{q-1} \le \mathit{ord}\ x_i \le -\frac{1}{q^2-1}t - \frac{1}{q^2-1}v \quad \mathit{if}\ v > 0; \tag{5.5}$$

$$\forall i \quad \mathit{ord}\ x_i \le \mathit{ord}\ x_{i-1}. \tag{5.6}$$

Proof. First, we check that conditions of Case 5.2 imply

$$\frac{1}{q^2}v + \frac{1}{q-1} \le -\frac{1}{q^2-1}t - \frac{1}{q^2-1}v \quad \text{if } v < 0, \tag{5.7}$$

$$\frac{1}{q-1} \le -\frac{1}{q^2-1}t - \frac{1}{q^2-1}v \quad \text{if } v > 0. \tag{5.8}$$

Case $v < 0$: (5.7) is $\frac{-2q^2+1}{q^2}v - q - 1 \ge t$. Because of (5.1), it is sufficient to prove that $\frac{-2q^2+1}{q^2}v - q - 1 \ge -\frac{2q^2}{q^2+1}v - \frac{q^3+q^2}{q^2+1}$. This is equivalent $v \le -\frac{q^2}{q-1}$, i.e. (5.7) holds. Case $v > 0$: (5.8) is $t \le -v - q - 1$, this holds for Case 5.2, $v > 0$.

Now, we use induction by i. Let $i = 0$. Condition $t \le -\frac{2q^2}{q^2+1}v - \frac{q^3+q^2}{q^2+1}$ implies that ord $x_0 = -\frac{1}{q^2-1}t - \frac{1}{q^2-1}v$ or ord $x_0 = \frac{1}{q^4-q^2}t + \frac{1}{q^2-1}v + \frac{1}{q-1}$, see (5.2), (5.3). Both these values satisfy (5.4) and (5.5), if the conditions of Case 5.2 hold.

Now, we assume that (5.4)–(5.6) hold for a fixed value of i (where for $i = 0$ condition (5.6) is omitted), and prove that they hold for $i + 1$. First, let us prove that (5.4)–(5.6) imply that

$$\forall i \quad \text{ord } b_{14}x_i^{q^4} > \text{ord } b_{12}x_i^{q^2} < \text{ord } b_{24}x_{i-1}^{q^4}.$$

This is immediate. For $v < 0$, we have ord $b_{14}x_i^{q^4} >$ ord $b_{12}x_i^{q^2}$ is equivalent to ord $x_i > \frac{1}{q^2}v + \frac{q}{q^2-1}$, which holds because of (5.4). For $v > 0$ we have ord $b_{14}x_i^{q^4} >$ ord $b_{12}x_i^{q^2}$ is equivalent to ord $x_i > \frac{q}{q^2-1}$, which holds because of (5.5).

Further, we have

$$\begin{aligned}\text{ord } b_{24}x_{i-1}^{q^4} &= -q^2v + q^4\text{ord } x_{i-1} \geq -q^2v + q^4\text{ord } x_i \\ &> -\delta v + q^2\text{ord } x_i = \text{ord } b_{12}x_i^{q^2}\end{aligned}$$

(the inner inequality because of ord $x_i > \frac{1}{q^2}v$ for $v < 0$, ord $x_i > 0$ for $v > 0$).

Hence, we have $g_{i+1} = \text{ord } b_{12}x_i^{q^2} = -\delta v + q^2\text{ord } x_i$. (5.4), (5.5) imply

$$\frac{q^2}{q-1} \leq g_{i+1} \leq -\frac{q^2}{q^2-1}t - \frac{2q^2-1}{q^2-1}v \quad (\text{case } v < 0),$$

$$q^2 - q^2v \leq g_{i+1} \leq -\frac{q^2}{q^2-1}t - \frac{q^4}{q^2-1}v \quad (\text{case } v > 0).$$

The line defined by the points (q^2, t), $(q^4, -q^2v - q^3 - q^2)$ (the second segment of the Newton polygon) crosses the t-axis at the point

$$\left(0, \frac{q^2}{q^2-1}t + \frac{q^2}{q^2-1}v + \frac{q^2}{q-1}\right) \tag{5.9}$$

and the line defined by points $(1, -v)$, (q^2, t) (the first segment of the Newton polygon) crosses the t-axis at the point

$$\left(0, -\frac{1}{q^2-1}t - \frac{q^2}{q^2-1}v\right).$$

The fact that for (t, v) defined by the conditions of Case 5.2, we have the inequalities

$$-\frac{1}{q^2-1}t - \frac{q^2}{q^2-1}v \geq -\frac{q^2}{q^2-1}t - \frac{2q^2-1}{q^2-1}v \quad (\text{case } v < 0),$$

$$-\frac{1}{q^2-1}t - \frac{q^2}{q^2-1}v \geq -\frac{q^2}{q^2-1}t - \frac{q^4}{q^2-1}v \quad (\text{case } v > 0),$$

implies that the convex hull of the point $(0, g_{i+1})$ and the Newton polygon of the head of the Eq. (3.2) consists of the points

$$(0, g_{i+1}), (q^2, t), (q^4, -q^2v - q^3 - q^2) \quad \text{if } g_{i+1} > \frac{q^2}{q^2-1}t + \frac{q^2}{q^2-1}v + \frac{q^2}{q-1},$$

$$(0, g_{i+1}), (q^4, -q^2v - q^3 - q^2) \quad \text{if } g_{i+1} \leq \frac{q^2}{q^2-1}t + \frac{q^2}{q^2-1}v + \frac{q^2}{q-1}.$$

Hence, if $g_{i+1} > \frac{q^2}{q^2-1}t + \frac{q^2}{q^2-1}v + \frac{q^2}{q-1}$ then there are two values of ord x_{i+1}:

$$\text{ord } x_{i+1,1} = \text{ord } x_i - \frac{1}{q^2}t - \frac{\delta}{q^2}v \quad \text{and ord } x_{i+1,2} = \frac{1}{q^4-q^2}t + \frac{1}{q^2-1}v + \frac{1}{q-1},$$

where ord $x_{i+1,1} >$ ord $x_{i+1,2}$.

If $g_{i+1} \le \frac{q^2}{q^2-1}t + \frac{q^2}{q^2-1}v + \frac{q^2}{q-1}$ then there is one value of ord x_{i+1}:

$$\text{ord } x_{i+1} = \frac{1}{q^2}\text{ord } x_i + \frac{q^2-\delta}{q^4}v + \frac{q+1}{q^2}.$$

In all cases, we have (5.4)–(5.6) for i imply (5.4)–(5.6) for $i+1$: This is checked immediately. □

Proposition 5.3 implies that for the domain $t \le -\frac{2q^2}{q^2+1}v - \frac{q^3+q^2}{q^2+1}$ and $t \ge -\delta v$, we have $h^1 = 0$.

Proposition 5.4. *If $t < -\frac{2q^2}{q^2+1}v - \frac{q^3+q^2}{q^2+1}$, $t < -\delta v$ then $h^1 = 4$.*

Proof. We consider a minimal chain generated by x_{40}, see Proposition 4.2. We abbreviate $\mathfrak{A} := \frac{q^2}{q^2-1}t + \frac{q^2}{q^2-1}v + \frac{q^2}{q-1}$ ($\mathfrak{A}$ from (5.9)) and $\alpha :=$ min (ord $b_{14}x_{40}^{q^4}$, ord $b_{12}x_{40}^{q^2}$) $- \mathfrak{A}$ where ord x_{40} is from (5.3). Condition $t < -\delta v$ implies $\alpha > 0$. We shall prove by induction that for all i

$$g_i \ge \mathfrak{A} + \alpha i, \tag{5.10}$$

$$\text{ord } x_{4i} \ge \text{ord } x_{40} + \frac{\alpha}{q^2}i. \tag{5.11}$$

Inequality (5.11) holds for $i = 0$ and (5.10) holds for $i = 1$. Further, truth of (5.10) for i implies truth of (5.11) for i (if the convex hull of $(0, g_i); (1, -v); (q^2, t)$ is the segment $(0, g_i); (q^2, t)$ and $g_i = \mathfrak{A} + \alpha i$ then in (5.11) for i we have equality; if not then ord x_{4i} is higher).

So, we should prove that truth of (5.11) for i implies truth of (5.10) for $i+1$. We have: truth of (5.10) for $i+1$ is equivalent to

$$\text{ord } b_{14}x_{4i}^{q^4} \ge \mathfrak{A} + \alpha(i+1),$$

$$\text{ord } b_{12}x_{4i}^{q^2} \ge \mathfrak{A} + \alpha(i+1),$$

$$\text{ord } b_{24}x_{4,i-1}^{q^4} \ge \mathfrak{A} + \alpha(i+1).$$

All these inequalities follow immediately from (5.11) for i and the conditions of the proposition.

Finally, (5.11) shows that x_{j0} generates a small solution for $j = 3, 4$. For $j = 1, 2$ the same inequalities (5.10), (5.11) also hold, hence the proposition (also, we can use Proposition 4.2). □

Case 5.5. $t \ge -\frac{2q^2}{q^2+1}v - \frac{q^3+q^2}{q^2+1}$. This is a condition that the Newton polygon of the head of the equation consists of one segment, its vertices are $(1, -v)$, $(q^4, -q^2v - q^3 - q^2)$, and

$$\forall j \quad \text{ord } x_{j0} = \frac{1}{q^2+1}v + \frac{q^2}{(q-1)(q^2+1)}. \tag{5.12}$$

Proposition 5.6. *For Case* 5.5, *we have* $h^1 = 0$ *if* $v \geq \frac{1}{q-1}$, $v \leq -\frac{q^2}{q-1}$, *and* $h^1 = 4$ *otherwise.*

Proof. Induction. We consider five cases; the proofs for all these cases are similar. The following equalities (cases (a)–(d)) and inequalities (case (e)) hold for all solutions of (3.2), not necessarily for minimal solutions.

(a) Case $v \geq \frac{1}{q-1}$: We have

$$\forall\, j,\ i \quad \operatorname{ord} x_{ji} = \frac{1}{(q^2+1)q^{2i}}v + \frac{1}{q-1} - \frac{1}{(q^2+1)(q-1)q^{2i}}. \tag{5.13}$$

Really, for $i = 0$ this is (5.12); this argument will be also valid for the below cases (b)–(e). The induction step from i to $i+1$: we have

$$\operatorname{ord} b_{14}x_{ji}^{q^4} = \left(\frac{q^4}{(q^2+1)q^{2i}} - q^2\right)v + \frac{q^3}{q-1} - \frac{q^4}{(q^2+1)(q-1)q^{2i}}, \tag{5.14}$$

$$\operatorname{ord} b_{12}x_{ji}^{q^2} = \left(\frac{q^2}{(q^2+1)q^{2i}} - q^2\right)v + \frac{q^2}{q-1} - \frac{q^2}{(q^2+1)(q-1)q^{2i}}, \tag{5.15}$$

$$\operatorname{ord} b_{24}x_{j,i-1}^{q^4} = \left(\frac{q^6}{(q^2+1)q^{2i}} - q^2\right)v + \frac{q^4}{q-1} - \frac{q^6}{(q^2+1)(q-1)q^{2i}}. \tag{5.16}$$

5.2a.

The below arguments hold for all cases (a)–(d), and they are slightly modified for the case (e).

For v of the present case, we have

$$\operatorname{ord} b_{12}x_{ji}^{q^2} < \operatorname{ord} b_{14}x_{ji}^{q^4}, \quad \operatorname{ord} b_{12}x_{ji}^{q^2} < \operatorname{ord} b_{24}x_{j,i-1}^{q^4} \quad (\text{here } i \geq 1). \tag{5.1}$$

We get that for v of the present case the Newton polygon of the equation for $x_{j,i+1}$ is a segment $(0, \operatorname{ord} b_{12}x_{ji}^{q^2}), (q^4, -q^2v - q^3 - q^2)$, hence, we get the induction proposition.

(b) Case $v \leq -\frac{q^2}{q-1}$. Analogs of (5.13)–(5.16) are

$$\forall\, j,\ i \quad \operatorname{ord} x_{ji} = \left(\frac{1}{q^2} - \frac{1}{(q^4+q^2)q^{2i}}\right)v + \frac{1}{q-1} - \frac{1}{(q^2+1)(q-1)q^{2i}}, \tag{5.2}$$

$$\operatorname{ord} b_{14}x_{ji}^{q^4} = -\frac{q^2}{(q^2+1)q^{2i}}v + \frac{q^3}{q-1} - \frac{q^4}{(q^2+1)(q-1)q^{2i}}, \tag{5.3}$$

$$\operatorname{ord} b_{12}x_{ji}^{q^2} = -\frac{1}{(q^2+1)q^{2i}}v + \frac{q^2}{q-1} - \frac{q^2}{(q^2+1)(q-1)q^{2i}}, \tag{5.4}$$

$$\operatorname{ord} b_{24}x_{j,i-1}^{q^4} = -\frac{q^4}{(q^2+1)q^{2i}}v + \frac{q^4}{q-1} - \frac{q^6}{(q^2+1)(q-1)q^{2i}}. \tag{5.5}$$

Arguments 5.2a are the same.

(c) Case $0 < v < \frac{1}{q-1}$. Analogs of (5.13)–(5.16) are

$$\forall j,\ i \quad \operatorname{ord} x_{ji} = \left(1 - \frac{q^2}{q^2+1}q^{2i}\right)v + \frac{q^2}{(q^2+1)(q-1)}q^{2i}, \tag{5.6}$$

$$\operatorname{ord} b_{14}x_{ji}^{q^4} = \left(q^4 - q^2 - \frac{q^6}{q^2+1}q^{2i}\right)v + \frac{q^6}{(q^2+1)(q-1)}q^{2i} - q^3, \tag{5.7}$$

$$\operatorname{ord} b_{12}x_{ji}^{q^2} = -\frac{q^4}{q^2+1}q^{2i}v + \frac{q^4}{(q^2+1)(q-1)}q^{2i}, \tag{5.8}$$

$$\operatorname{ord} b_{24}x_{j,i-1}^{q^4} = \left(q^4 - q^2 - \frac{q^4}{q^2+1}q^{2i}\right)v + \frac{q^4}{(q^2+1)(q-1)}q^{2i}. \tag{5.9}$$

Arguments 5.2a are the same.

(d) Case $-\frac{q^2}{q-1} < v < 0$. Analogs of (5.13)–(5.16) are

$$\forall j,\ i \quad \operatorname{ord} x_{ji} = \frac{1}{q^2+1}q^{2i}v + \frac{q^2}{(q^2+1)(q-1)}q^{2i}, \tag{5.10}$$

$$\operatorname{ord} b_{14}x_{ji}^{q^4} = \left(\frac{q^4}{q^2+1}q^{2i} - q^2\right)v + \frac{q^6}{(q^2+1)(q-1)}q^{2i} - q^3, \tag{5.11}$$

$$\operatorname{ord} b_{12}x_{ji}^{q^2} = \left(\frac{q^2}{q^2+1}q^{2i} - 1\right)v + \frac{q^4}{(q^2+1)(q-1)}q^{2i}, \tag{5.12}$$

$$\operatorname{ord} b_{24}x_{j,i-1}^{q^4} = \left(\frac{q^2}{q^2+1}q^{2i} - q^2\right)v + \frac{q^4}{(q^2+1)(q-1)}q^{2i}. \tag{5.13}$$

Arguments 5.2a are the same; in order to prove (5.1) for this case it is sufficient to check these inequalities for the values $v = 0$ and $v = -\frac{q^2}{q-1}$; truth of (5.1) for intermediate values of v holds by linearity.

(e) Case $v = 0$. It is slightly different from the above cases because we have $\operatorname{ord} b_{12} \geq 0$ and equalities (5.13)–(5.16), (5.2)–(5.13) are replaced by inequalities:

$$\forall j,\ i \quad \operatorname{ord} x_{ji} \geq \frac{q^2}{(q^2+1)(q-1)}q^{2i}, \tag{5.14}$$

$$\operatorname{ord} b_{14}x_{ji}^{q^4} \geq \frac{q^6}{(q^2+1)(q-1)}q^{2i} - q^3, \tag{5.15}$$

$$\operatorname{ord} b_{12}x_{ji}^{q^2} \geq \frac{q^4}{(q^2+1)(q-1)}q^{2i}, \tag{5.16}$$

$$\operatorname{ord} b_{24}x_{j,i-1}^{q^4} \geq \frac{q^4}{(q^2+1)(q-1)}q^{2i}. \tag{5.17}$$

Formulas (5.1) are modified as follows:

$$\operatorname{ord} b_{14}x_{ji}^{q^4} \geq \frac{q^4}{(q^2+1)(q-1)}q^{2i}; \quad \operatorname{ord} b_{24}x_{j,i-1}^{q^4} \geq \frac{q^4}{(q^2+1)(q-1)}q^{2i}.$$

All other arguments are the same.

We get that for the cases (c)–(e), we have ord $x_{ji} \to +\infty$ as $i \to \infty$, while for the cases (a), (b) ord x_{ji} is bounded. This proves the proposition. □

Propositions 5.3, 5.4, 5.6 cover the whole plane (v, t), hence we get Theorem 5.1. □

6. Description of the sets $h^1 = 4$, $h^1 = 0$ in terms of a_{12}, a_{21}

Let $u = \text{ord}\, a_{12}$ and $v = \text{ord}\, a_{21}$ as above. Here, we find the sets of $h^1 = 4$, $h^1 = 0$ on (u, v)-plane. This will give us an improved estimate (1.4) for the matrices $A = \begin{pmatrix} 0 & a_{12} \\ a_{21} & 0 \end{pmatrix}$, see Proposition 6.3.

Further, we verify that the symmetry with respect to $a_{12} \longleftrightarrow a_{21}$ really takes place. Although the t-motives $M(A)$, $M(A)'$ are isomorphic (a matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ gives a change of basis between $M(A)$ and $M(A^t)$), the equality $h^1(M(A)) = h^1(M(A'))$ is not seen beforehand, because formulas (3.1) are not symmetric with respect to the transposition of A.

6.1

We have $t = \text{ord}\, a_2 = \text{ord}\,(\frac{\theta^q}{a_{21}} + \frac{\theta^{q^2}}{a_{21}^{q^2}} - a_{12}^q)$. There are 3 domains on the (u, v)-coordinate plane, according the minimality of ord's of the 3 summands of a_2. Let us describe them. There are 3 rays (half-lines) R_1, R_2, R_3 having the same initial point $(u, v) = (-\frac{q}{q+1}, -\frac{q}{q+1})$.

The ray R_1 has the equation $v = -\frac{q}{q+1}$, $u \geq -\frac{q}{q+1}$;

The ray R_2 has the equation $v = -qu - q$, $u \geq -\frac{q}{q+1}$;

The ray R_3 has the equation $v = -\frac{1}{q}u - 1$, $u \leq -\frac{q}{q+1}$;

The open domain between the rays R_i and R_j is denoted by D_{ij}.

In Domain D_{12}, we have ord $\frac{\theta^q}{a_{21}} <$ ord $\frac{\theta^{q^2}}{a_{21}^{q^2}}$, ord $\frac{\theta^q}{a_{21}} <$ ord a_{12}^q, i.e. $t = -v - q$. The ray $t = -v - q$, $v < -\frac{q}{q+1}$ on the (v, t)-coordinate plane is entirely in the domain $h^1 = 4$ (see Theorem 5.1).

In Domain D_{13}, we have ord $\frac{\theta^{q^2}}{a_{21}^{q^2}} <$ ord $\frac{\theta^q}{a_{21}}$, ord $\frac{\theta^{q^2}}{a_{21}^{q^2}} <$ ord a_{12}^q, i.e. $t = -q^2v - q^2$. The ray $t = -q^2v - q^2$, $v > -\frac{q}{q+1}$ on the (v, t)-coordinate plane is entirely in the domain $h^1 = 4$ (see Theorem 5.1).

In Domain D_{23}, we have ord $a_{12}^q <$ ord $\frac{\theta^{q^2}}{a_{21}^{q^2}}$, ord $a_{12}^q <$ ord $\frac{\theta^q}{a_{21}}$, i.e. $t = qu$. The image of D_{23} under the map $(u, v) \to (t, v)$ defined by $(u, v) \mapsto (qu, v)$, is the open interior of the angle formed by the rays $t = -v - q$, $v < -\frac{q}{q+1}$ and $t = -q^2v - q^2$, $v > -\frac{q}{q+1}$. It is entirely in the domain $h^1 = 4$ (see Theorem 5.1).

In particular, we get that Conjecture 3.2 holds for the case under consideration.

Proposition 6.2. *The set of points on (u, v)-plane such that $M(A)$ can be non-uniformizable is of dimension less than* 2 (*i.e.* "*almost all*" $M(A)$, A *of the form* (3.7), *are uniformizable*). □

For the points of the ray R_1, we have $v = -\frac{q}{q+1}$, t can vary. According Theorem 5.1, all these points have $h^1 = 4$. Moreover, Theorem 5.1 shows that if $-\frac{q^2}{q-1} < v < \frac{1}{q-1}$ then $h^1 = 4$. This means that the only points on (u,v)-coordinate plane where it can happen $h^1 = 0$ are subrays $\bar{R}_2$, respectively $\bar{R}_3$ of R_2, respectively R_3 having initial points

$$u = -\frac{q^2}{q-1}, \quad v = \frac{1}{q-1} \quad \text{for } \bar{R}_2,$$

$$u = \frac{1}{q-1}, \quad v = -\frac{q^2}{q-1} \quad \text{for } \bar{R}_3.$$

Remark. (1) We see that $\bar{R}_2$, $\bar{R}_3$ are symmetric with respect to the symmetry $u \longleftrightarrow v$ as it must be.

(2) Clearly not for all a_{12}, a_{21} such that their ord's belong to $\bar{R}_2$, $\bar{R}_3$ we have $h^1 = 0$.

Further, we get the following proposition.

Proposition 6.3. *Let $A = \begin{pmatrix} 0 & a_{12} \\ a_{21} & 0 \end{pmatrix}$. If ord a_{12}, ord $a_{21} > -\frac{q^2}{q-1}$ then $M(A)$ is uniformizable.* □

Remark 6.4. Example [4, 5.9.9] shows that this estimate is exact. Really, we have (c is of [4]; r of [4] is q of this paper[e]) ord $c = 1$. After a change of the basis with the matrix $\begin{pmatrix} c^{\frac{1+q+q^2}{1-q^2}} & 0 \\ 0 & c^{\frac{q}{1-q^2}} \end{pmatrix}$ the equation of [4, 5.9.9] becomes (2.3) of this paper, with ord $a_{12} = \frac{1}{q-1}$, ord $a_{21} = -\frac{q^2}{q-1}$. We see that the example [4, 5.9.9] belongs to the initial point of the ray $\bar{R}_3$.

Second proof of the equality $h^1(M(A)) = h^1(M(A'))$ for A of the form (3.7).

We consider a subset U (described below) of $\mathcal{A}_+$ on the (v,t)-coordinate plane, and we give a direct proof (without using Theorem 1.2) of the following proposition.

Proposition 6.5. *Let a_{12}, a_{21} be such that their (v,t) belong to U, and let A be of the form* (3.7). *In this case, we have $h^1(M(A)) = h^1(M(A'))$.*

Remark. The restriction $(v,t) \in U$ is not essential, the same methods will give us a proof for any values of (v,t).

Proof. Conditions ord $a_{21} = v$, ord $a_2 = t$ mean that $\exists\ x, y \in \mathbb{C}_\infty$ such that ord $x =$ ord $y = 0$ and such that

$$a_{21} = \theta^{-v} x, \quad a_2 = \theta^{-t} y,$$

[e] The equation of [4, 5.9.9] contains a misprint: The (2,1)-term of the first matrix is $1 - c^{r+1}$ and not $1 - c^r$ as it is printed.

we have

$$a_{12} = \left(\frac{\theta^q}{a_{21}} + \frac{\theta^{q^2}}{a_{21}^{q^2}} - a_2\right)^{\frac{1}{q}} = \theta^{1+\frac{1}{q}v}x^{-\frac{1}{q}} + \theta^{qv+q}x^{-q} - \theta^{-\frac{1}{q}t}y^{\frac{1}{q}}.$$

For $(v,t) \in \mathcal{A}_+$, we have $-(1+\frac{1}{q}v) > -(qv+q) < \frac{1}{q}t$, hence $u = \text{ord } a_{12} = -(qv+q)$. We have $(v,t) \in \mathcal{A}_+ \implies (u,v) \in \bar{R}_3$.

We denote objects for the dual t-motive by prime, i.e. $a'_{12} = a_{21}$, $a'_{21} = a_{12}$, etc. We have[f]

$$\begin{aligned} a'_2 &= \frac{\theta^q}{a'_{21}} + \frac{\theta^{q^2}}{{a'_{21}}^{q^2}} - {a'_{12}}^q = \frac{\theta^q - {a'_{12}}^q a'_{21}}{a'_{21}} + \frac{\theta^{q^2}}{{a'_{21}}^{q^2}} \\ &= \frac{\theta^{(-q+\frac{1}{q})v+1}x^{q-\frac{1}{q}} + \theta^{-qv-\frac{1}{q}t}x^q y^{\frac{1}{q}}}{a'_{21}} + \frac{\theta^{q^2}}{{a'_{21}}^{q^2}}. \end{aligned}$$

We consider only the case $t < -v - q$. We denote $\mathcal{A}_+ \cap \{t < -v - q\}$ by U. For $(v,t) \in U$, we have

$$\text{ord } \theta^{-qv-\frac{1}{q}t}x^q y^{\frac{1}{q}} < \text{ord } \theta^{(-q+\frac{1}{q})v+1}x^{q-\frac{1}{q}}$$

hence

$$t' = \text{ord } a'_2 = \text{ord } \frac{\theta^{-qv-\frac{1}{q}t}x^q y^{\frac{1}{q}}}{a'_{21}} = 2qv + \frac{1}{q}t + q$$

(because $\text{ord } \frac{\theta^{q^2}}{{a'_{21}}^{q^2}} > \text{ord } \frac{\theta^{-qv-\frac{1}{q}t}x^q y^{\frac{1}{q}}}{a'_{21}}$).

We get that for $(v,t) \in U$ the numbers v', t' are given by the formulas

$$\begin{aligned} v' &= -qv - q, \\ t' &= 2qv + \frac{1}{q}t + q. \end{aligned}$$

The easiest way to find the image of U under this linear transformation is to find images of some points P_* on borders of U. We have

P_1: $v = q$, $t = -q^3$ (side of both U, $\mathcal{A}_+$). Its image: $v' = -q^2 - q$, $t' = q^2 + q$ (side of $\mathcal{A}_-$).

P_2: $v = \frac{1}{q-1}$, $t = -\frac{q^2}{q-1}$ (vertex of both U, $\mathcal{A}_+$). Its image: $v' = -\frac{q^2}{q-1}$, $t' = \frac{q^2}{q-1}$ (vertex of $\mathcal{A}_-$).

P_3: $v = \frac{1}{q-1}$, $t = \frac{-q^2+q-1}{q-1}$ (vertex of U, side of $\mathcal{A}_+$). Its image: $v' = -\frac{q^2}{q-1}$, $t' = \frac{q^3+q-1}{q^2-q}$ (side of $\mathcal{A}_-$).

P_4: $v = q$, $t = -2q$ (side of U, interior of $\mathcal{A}_+$). Its image: $v' = -q^2-q$, $t' = 2q^2+q-2$ (interior of $\mathcal{A}_-$).

[f] The reader can see that in this elementary calculation some terms cancel, just as teeth of a gear fit into one another. This is quite typical for mathematics; this cancelation exposes a symmetry of the present construction, not seen in (3.9), (3.10) of [7].

We see that the image of U under this linear transformation is in $\mathcal{A}_-$, as it must be. $\square$

7. Case of A of the Form (3.8)

For simplicity, we consider the case $q = 2$. We use notations $u = \operatorname{ord} a_{12}$, $v = \operatorname{ord} a_{21}$ as in Sec. 6. Since $a_{22} = 0$ the value of a_2 for the present case is the same as in Sec. 6, hence we have the same R_i, D_{ij} $(i, j = 1, 2, 3)$ as in (6.1). We have: $\operatorname{ord} a_4 = -4v - 12$; $\operatorname{ord} a_3 = -4v - 8$; $\operatorname{ord} a_2 \geq \min\,(-4v - 4; -v - 2; 2u)$; $\operatorname{ord} a_1 = -v-1$; $\operatorname{ord} a_0 = -v$. The tail coefficients are: $\operatorname{ord} b_{14} = -4v-8$, $\operatorname{ord} b_{13} = -4v - 4$, $\operatorname{ord} b_{12} = -4v$ for $v > 0$, $\operatorname{ord} b_{12} = -v$ for $v < 0$, $\operatorname{ord} b_{12} \geq 0$ for $v = 0$, $\operatorname{ord} b_{24} = -4v$.

Case of Domain D_{13}.

For $(u, v) \in D_{13}$, we have $\operatorname{ord} a_2 = -4v - 4$. If $v > -\frac{1}{3}$ then the Newton polygon of (3.3) has vertices $(1, -v)$, $(4, -4v - 4)$, $(8, -4v - 8)$, $(16, -4v - 12)$ and $\operatorname{ord} x_{j0}$ are respectively $v + \frac{4}{3}, v + \frac{4}{3}, 1, \frac{1}{2}$.

If $v \leq -\frac{1}{3}$ then the Newton polygon of (3.3) has vertices $(1, -v)$, $(2, -v - 1)$, $(8, -4v - 8)$, $(16, -4v - 12)$ and $\operatorname{ord} x_{j0}$ are respectively $1, \frac{1}{2}v + \frac{7}{6}, \frac{1}{2}v + \frac{7}{6}, \frac{1}{2}$.

Proposition 7.1. *For the subdomain $(u, v) \in \{\text{Domain } D_{13},\ v \geq 0\}$, we have $h^1 = 4$.*

Proof. Let x_0 be a solution to (3.3). We consider its minimal solution $x_0, x_1, x_2, \ldots$. We shall prove by induction that $\operatorname{ord} x_i \geq \frac{1}{2}i + \frac{1}{2}$; this implies the proposition. For $i = 0$ this is true: $\operatorname{ord} x_0 \geq \frac{1}{2}$. Let us prove that if this is true for some i then this is true for $i + 1$. First, we consider the case $i = 0$. Ord's of the tail terms are:

$\operatorname{ord} b_{14}x_0^{16} \geq -4v$; $\operatorname{ord} b_{13}x_0^{8} \geq -4v$; $\operatorname{ord} b_{12}x_0^{4} \geq -4v + 2$,

hence $g_1 \geq -4v$. The Newton polygon has a point $(4, -4v - 4)$, hence we get $\operatorname{ord} x_1 \geq 1$, the induction supposition is true for $i = 1$. So, now we consider the case $i > 0$. For this case ord's of the tail terms are

$\operatorname{ord} b_{14}x_i^{16} \geq -4v + 8i$;

$\operatorname{ord} b_{13}x_i^{8} \geq -4v + 4i$;

$\operatorname{ord} b_{12}x_i^{4} \geq -4v + 2 + 2i$,

$\operatorname{ord} b_{24}x_{i-1}^{16} \geq -4v + 8i$

hence $g_{i+1} \geq -4v + 2 + 2i$. Again, joining the points $(0, -4v + 2 + 2i)$ and $(4, -4v - 4)$ we get $\operatorname{ord} x_{i+1} \geq \frac{1}{2}(i + 1) + \frac{1}{2}$ — the induction supposition for $i + 1$. $\square$

Proposition 7.2. *For the subdomain* $(u, v) \in \{$*Domain* D_{13}, $-\frac{2}{3} < v \leq 0\}$, *we have* $h^1 = 4$.

Proof. Similar to the proof of Proposition 7.1, we use the same notations. As above we have ord $x_0 \geq \frac{1}{2}$. Let us evaluate ord x_1. We have

ord $b_{14}x_0^{16} \geq -4v$;

ord $b_{13}x_0^{8} \geq -4v$;

ord $b_{12}x_0^{4} \geq -v + 2$.

If $-\frac{1}{3} \leq v \leq 0$ then $-4v < -v + 2$, hence $g_1 \geq -4v$. As in Proposition 7.1, we get ord $x_1 \geq 1$ (if $v < -\frac{1}{3}$ then the point $(4, -4v - 4)$ is above the Newton polygon. In this case ord $x_1 \geq 1$ also holds).

Now, the induction supposition is the following: ord $x_i \geq 4^{i-1}$. This is true for $i = 1$. For a fixed i, we have

ord $b_{14}x_i^{16} \geq -4v - 8 + 4^{i+1}$;

ord $b_{13}x_i^{8} \geq -4v - 4 + 2 \cdot 4^i$;

ord $b_{12}x_i^{4} \geq -v + 4^i$,

ord $b_{24}x_{i-1}^{16} \geq -4v + 4^i$.

For $i \geq 1$ all these numbers are $\geq -v + 4^i$. Joining the points $(0, -v + 4^i)$ and $(1, -v)$ we get ord $x_{i+1} \geq 4^i$. □

Propositions 7.1, 7.2 show that for all $(u, v) \in D_{13}$ we have $h^1 = 4$.

Now, we consider the case of Domain D_{12}. For $(u, v) \in D_{12}$ we have ord $a_2 = -v - 2$. First, we consider the case $v \geq -\frac{4}{3}$.

Proposition 7.3. *If* $(u, v) \in D_{12}$ *and* $v \geq -\frac{4}{3}$ *then* $h^1 = 4$.

Proof. If $(u, v) \in D_{12}$ and $v \geq -\frac{4}{3}$ then the Newton polygon of (3.3) has vertices $(1, -v)$, $(2, -v - 1)$, $(8, -4v - 8)$, $(16, -4v - 12)$ and ord x_{j0} are respectively $1, \frac{1}{2}v + \frac{7}{6}, \frac{1}{2}v + \frac{7}{6}, \frac{1}{2}$.

We consider consecutively minimal solutions generated by x_{j0} for all $j = 1, \ldots, 4$.

Lemma 7.3.1. *All minimal solutions generated by* x_{10} *are small.*

Proof. Since ord $x_{10} = 1$, we have ord $b_{14}x_{10}^{16} = -4v + 8$, ord $b_{13}x_{10}^{8} = -4v + 4$, ord $b_{12}x_{10}^{4} = -v + 4$ and

ord $(b_{14}x_{10}^{16} + b_{13}x_{10}^{8} + b_{12}x_{10}^{4}) = \min\{-4v + 8, -4v + 4, -v + 4\} = -v + 4$. Hence, $g_1 = -v + 4$ and ord $x_{11} = v + (-v + 4) = 4$. By induction, continuing this calculation we get a small solution x_{1i}, $i \in \mathbb{N}$, with ord $x_{1,i+1} = 4 \cdot$ ord x_{1i}. Really, for $i = 0$ we just showed that ord $x_{10} = 1$ and ord $x_{11} = 4$.

Induction step $i \implies i+1$: The free term in the Newton polygon for ord $x_{1,i+1}$ has the form ord $b_{14}x_{1i}^{16} + b_{13}x_{1i}^{8} + b_{12}x_{1i}^{4} + b_{24}x_{1,i-1}^{16}$. Therefore its order is

min $\{-4v-8+16\cdot\text{ord } x_{1i}, -4v-4+8\cdot\text{ord } x_{1i}, -v+4\cdot\text{ord } x_{1i}, -4v+16\cdot\text{ord } x_{1,i-1}\}$ (because there exists only one minimal value, see below).

By induction, the assumption $v \le -\frac{2}{3}$ and the fact ord $x_{1i} \ge 0$, this is equal min $\{-4v-8+16\cdot\text{ord } x_{1i}, -4v-4+8\cdot\text{ord } x_{1i}, -v+4\cdot\text{ord } x_{1i}, -4v+4\cdot\text{ord } x_{1i}\} = -v+4\cdot\text{ord } x_{1i}$ (the only minimal value).

The leftmost segment of the Newton polygon gives us a solution $x_{1,i+1}$ with ord $x_{1,i+1} = v + (-v + 4\cdot\text{ord } x_{1i}) = 4\cdot\text{ord } x_{1i}$. □

Lemma 7.3.2. *All minimal solutions generated by x_{j0}, $j = 2, 3$, are small.*

Proof. Since ord $x_{j0} = \frac{1}{2}v + \frac{7}{6}$ (here and below $j = 2, 3$) we have ord $b_{14}x_{j0}^{16} = 4v+\frac{32}{3}$, ord $b_{13}x_{j0}^{8} = \frac{16}{3}$, ord $b_{12}x_{j0}^{4} = v+\frac{14}{3}$ hence $g_1 = v+\frac{14}{3}$. The leftmost segment of the Newton polygon gives us a solution x_{j1} with ord $x_{j1} = v+(v+\frac{14}{3}) = 2v+\frac{14}{3}$ which is again $4\cdot$ ord x_{j0}.

As earlier, by induction, continuing this calculation we get small solutions x_{ji}, $i \in \mathbb{N}$, with ord $x_{j,i+1} = 4\cdot\text{ord } x_{ji}$. Really, for $i = 0$ we just showed that ord $x_{j1} = 4\cdot\text{ord } x_{j0}$.

Induction step $i \implies i+1$: The free term in the Newton polygon for ord $x_{j,i+1}$ has the form ord $b_{14}x_{ji}^{16} + b_{13}x_{ji}^{8} + b_{12}x_{ji}^{4} + b_{24}x_{j,i-1}^{16}$. Therefore its order is

min $\{-4v-8+16\cdot\text{ord } x_{ji}, -4v-4+8\cdot\text{ord } x_{ji}, -v+4\cdot\text{ord } x_{ji}, -4v+16\cdot\text{ord } x_{j,i-1}\}$ (because there exists only one minimal value, see below).

By induction, the assumption $-\frac{4}{3} \le v \le -\frac{2}{3}$ and the fact ord $x_{ji} \ge 0$, this is equal min $\{-4v-8+16\cdot\text{ord } x_{ji}, -4v-4+8\cdot\text{ord } x_{ji}, -v+4\cdot\text{ord } x_{ji}, -4v+4\cdot\text{ord } x_{ji}\} = -v+4\cdot\text{ord } x_{ji}$ (the only minimal value).

The leftmost segment of the Newton polygon gives us a solution $x_{j,i+1}$ with ord $x_{j,i+1} = v + (-v + 4\cdot\text{ord } x_{ji}) = 4\cdot\text{ord } x_{ji}$. □

Lemma 7.3.3. *All minimal solutions generated by x_{40} are small.*

Proof. It is completely analogous to the proofs of the above lemmas. Since ord $x_{40} = \frac{1}{2}$ we have ord $b_{14}x_{40}^{16} = -4v$, ord $b_{13}x_{40}^{8} = -4v$, ord $b_{12}x_{40}^{4} = -v+2$.

Hence $g_1 = -v+2$, because $-v+2 \le -4v$ if and only if $v \le -\frac{2}{3}$. The leftmost segment of the Newton polygon gives us a solution x_{41} with ord $x_{41} = v+(-v+2) = 2$ which is as before $4\cdot\text{ord } x_{40}$.

As earlier, by induction, continuing this calculation we get small solutions x_{4i}, $i \in \mathbb{N}$, with ord $x_{4,i+1} = 4\cdot\text{ord } x_{4i}$. Really, for $i = 0$ we just showed that ord $x_{41} = 4\cdot\text{ord } x_{40}$.

Induction step $i \implies i+1$: The free term in the Newton polygon for ord $x_{4,i+1}$ has the form ord $b_{14}x_{4i}^{16} + b_{13}x_{4i}^{8} + b_{12}x_{4i}^{4} + b_{24}x_{4,i-1}^{16}$. Therefore its order is

min $\{-4v-8+16\cdot\text{ord } x_{4i}, -4v-4+8\cdot\text{ord } x_{4i}, -v+4\cdot\text{ord } x_{4i}, -4v+16\cdot\text{ord } x_{4,i-1}\}$ (because there exists only one minimal value, see below).

By induction, the assumption $-\frac{4}{3} \le v \le -\frac{2}{3}$ and the fact ord $x_{4i} \ge \frac{1}{2}$, this is equal min $\{-4v - 8 + 16 \cdot \text{ord } x_{4i}, -4v - 4 + 8 \cdot \text{ord } x_{4i}, -v + 4 \cdot \text{ord } x_{4i}, -4v + 4 \cdot \text{ord } x_{4i}\} = -v + 4 \cdot \text{ord } x_{4i}$ (the only minimal value).

The leftmost segment of the Newton polygon gives us a solution $x_{4,i+1}$ with ord $x_{4,i+1} = v + (-v + 4 \cdot \text{ord } x_{4i}) = 4 \cdot \text{ord } x_{4i}$. □

These three lemmas show that there are 4 linearly independent small solutions, i.e. $h^1 = 4$. □

Proposition 7.4. *If* $(u, v) \in D_{12}$ *and* $v \le -\frac{4}{3}$ *then* $h^1 = 4$.

Proof. If $(u, v) \in D_{12}$ and $v \le -\frac{4}{3}$ then the Newton polygon of (3.3) has vertices $(1, -v)$, $(2, -v-1)$, $(4, -v-2)$, $(16, -4v-12)$ and ord x_{j0} are respectively $1, \frac{1}{2}, \frac{1}{4}v + \frac{5}{6}, \frac{1}{4}v + \frac{5}{6}$.

For $j = 1$, respectively 2, we have the same values of ord x_{ji} (minimal solutions) as in Lemma 7.3.1, respectively 7.3.2, i.e. these minimal solutions are small.

Let us consider the case $j = 3, 4$, i.e. ord $x_{j0} = \frac{1}{4}v + \frac{5}{6}$. We have ord $b_{14}x_{j0}^{16} = \frac{16}{3}$, ord $b_{13}x_{j0}^{8} = -2v + \frac{8}{3}$, ord $b_{12}x_{j0}^{4} = \frac{10}{3}$. Since $v \le -\frac{2}{3}$ the minimum of these three values is $\frac{10}{3}$ and we have $g_1 = \frac{10}{3}$.

Therefore, there are three cases of the Newton polygon:

(a) If $v + \frac{10}{3} > 1$ equivalently $v > -\frac{7}{3}$ or ord $x_{j0} > \frac{1}{4}$, then the Newton polygon for x_{j1} has the first two vertices $(0, \frac{10}{3})$ and $(1, -v)$ and the order of the small solution x_{j1} is $v + \frac{10}{3} = 4 \cdot \text{ord } x_{j0}$.

(b) If $1 \ge v + \frac{10}{3}$ and $\frac{v+1+\frac{10}{3}}{2} > \frac{1}{2}$, equivalently $-\frac{7}{3} \ge v > -\frac{10}{3}$ or $\frac{1}{4} \ge \text{ord } x_{j0} > 0$, then the Newton polygon for x_{j1} has the first two vertices $(0, \frac{10}{3})$ and $(2, -v-1)$ and the order of the small solution x_{j1} is $\frac{1}{2}(v + \frac{10}{3}) = \frac{1}{2}(4 \cdot \text{ord } x_{j0}) + \frac{1}{2}$.

(c) If $\frac{v+1+\frac{10}{3}}{2} \le \frac{1}{2}$, equivalently $v \le -\frac{10}{3}$ or ord $x_{j0} \le 0$, then the Newton polygon for x_{j1} has the first two vertices $(0, \frac{10}{3})$ and $(4, -v-2)$ and the order of the small solution x_{j1} is $\frac{1}{4}(v + \frac{10}{3}) + \frac{1}{2} = \frac{1}{4}(4 \cdot \text{ord } x_{j0}) + \frac{1}{2} = \text{ord } x_{j0} + \frac{1}{2}$.

Lemma 7.4.1. *If there exists* $i \in \mathbb{N}$ *such that ord* $x_{ji} = 4 \cdot$ *ord* $x_{j,i-1} > 1$ *then ord* $x_{j,i+1} = 4 \cdot$ *ord* x_{ji}.

Proof. By hypothesis, we have ord $x_{ji} = 4 \cdot \text{ord } x_{j,i-1}$. Under this condition the ord's of the tail terms:

$-4v - 8 + 16 \cdot \text{ord } x_{ji}$, $-4v - 4 + 8 \cdot \text{ord } x_{ji}$, $-v + 4 \cdot \text{ord } x_{ji}$, $-4v + 4 \cdot \text{ord } x_{ji}$ are different, and conditions $v \le -\frac{4}{3}$ and ord $x_{ji} > 1$ imply that their minimum is $-v + 4 \cdot \text{ord } x_{ji}$. Since $-v - (-v + 4 \cdot \text{ord } x_{ji}) < -1$ then the Newton polygon for $x_{j,i+1}$ has the first two vertices $(0, -v + 4 \cdot \text{ord } x_{ji})$ and $(1, -v)$ and the order of the minimal solution $x_{j,j+1}$ is $4 \cdot \text{ord } x_{ji}$. □

Lemma 7.4.2. *If there exists* $i \in \mathbb{N}$ *such that ord* $x_{ji} = 2 \cdot$ *ord* $x_{j,i-1} + \frac{1}{2} > \frac{1}{2}$, *then ord* $x_{j,i+1} = 4 \cdot$ *ord* $x_{ji} > 2$.

Proof. The ord's of the tail terms are the following:

$-4v-8+16\cdot\text{ord }x_{ji}$, $-4v-4+8\cdot\text{ord }x_{ji}$, $-v+4\cdot\text{ord }x_{ji}$, $-4v+16\cdot\text{ord }x_{j,i-1}$. By hypothesis, we have $-4v-4+8\cdot\text{ord }x_{ji}=-4v+16\cdot\text{ord }x_{j,i-1}$ and with $\text{ord }x_{ji}>\frac{1}{2}$ we get $-4v-4+8\cdot\text{ord }x_{ji}<-4v-8+16\cdot\text{ord }x_{ji}$. Because of $v\le-\frac{4}{3}$ and $\text{ord }x_{ji}>0$ we have $-v+4\cdot\text{ord }x_{ji}<-4v-4+8\cdot\text{ord }x_{ji}$. We get

$\min\{-4v-8+16\cdot\text{ord }x_{ji},\ -4v-4+8\cdot\text{ord }x_{ji},\ -v+4\cdot\text{ord }x_{ji},\ -4v+16\cdot\text{ord }x_{j,i-1}\}=-v+4\cdot\text{ord }x_{ji}$ (the only minimal value). Since $-v-(-v+4\cdot\text{ord }x_{ji})<-2$ then the Newton polygon for $x_{j,i+1}$ has the first two vertices $(0,-v+4\cdot\text{ord }x_{ji})$ and $(1,-v)$ and the order of the minimal solution $x_{j,j+1}$ is $4\cdot\text{ord }x_{ji}$ which is ≥ 2. $\square$

Lemma 7.4.3. *If ord $x_{ji}=$ ord $x_{j,i-1}+\frac{1}{2}\le\frac{1}{2}$ and if ord $x_{ji}=\frac{1}{4}v+c$ where $c>\frac{2}{3}$ then*

$$\mathit{ord}\ x_{j,i+1}=\begin{cases}4\cdot \mathit{ord}\ x_{ji} & \text{if}\quad \mathit{ord}\ x_{ji}>\dfrac{1}{4},\\[2ex] 2\cdot \mathit{ord}\ x_{ji}+\dfrac{1}{2} & \text{if}\quad 0<\mathit{ord}\ x_{ji}\le\dfrac{1}{4},\\[2ex] \mathit{ord}\ x_{ji}+\dfrac{1}{2} & \text{if}\quad \mathit{ord}\ x_{ji}\le 0.\end{cases}$$

Proof. The ord's of the tail terms are the following:

$-4v-8+16\cdot\text{ord }x_{ji}$, $-4v-4+8\cdot\text{ord }x_{ji}$, $-v+4\cdot\text{ord }x_{ji}$, $-4v+16\cdot\text{ord }x_{j,i-1}$. By hypothesis, we have $-4v-8+16\cdot\text{ord }x_{ji}=-4v+16\cdot\text{ord }x_{j,i-1}$ and with $\text{ord }x_{ji}\le\frac{1}{2}$ we get $-4v-4+8\cdot\text{ord }x_{ji}\ge-4v-8+16\cdot\text{ord }x_{ji}$. Because of $\text{ord }x_{ji}=\frac{1}{4}v+c$ we have $-4v-8+16\cdot\text{ord }x_{ji}=-8+16c$ and $-v+4\cdot\text{ord }x_{ji}=4c$.

Condition $c>\frac{2}{3}$ implies $-4v-8+16\cdot\text{ord }x_{ji}>-v+4\cdot\text{ord }x_{ji}$. Therefore $\min\{-4v-8+16\cdot\text{ord }x_{ji},\ -4v-4+8\cdot\text{ord }x_{ji},\ -v+4\cdot\text{ord }x_{ji},\ -4v+16\cdot\text{ord }x_{j,i-1}\}=-v+4\cdot\text{ord }x_{ji}$ (the only minimal value). If $-v-(-v+4\cdot\text{ord }x_{ji})<-1$, equivalently $\text{ord }x_{ji}>\frac{1}{4}$, then the Newton polygon for $x_{j,i+1}$ has the first two vertices $(0,-v+4\cdot\text{ord }x_{ji})$ and $(1,-v)$ and the order of the minimal solution $x_{j,i+1}$ is $4\cdot\text{ord }x_{ji}$ which is ≥ 2.

If $-v-(-v+4\cdot\text{ord }x_{ji})\ge-1$ and $\frac{-v-1-(-v+4\cdot\text{ord }x_{ji})}{2}<-\frac{1}{2}$, equivalently $0<\text{ord }x_{ji}\le\frac{1}{4}$, then the Newton polygon for $x_{j,i+1}$ has the first two vertices $(0,-v+4\cdot\text{ord }x_{ji})$ and $(2,-v-1)$ and the order of the minimal solution $x_{j,i+1}$ is $\frac{4\cdot\text{ord }x_{ji}+1}{2}=2\cdot\text{ord }x_{ji}+\frac{1}{2}$.

If $(-v-1-(-v+4\cdot\text{ord }x_{ji}))/2\ge-\frac{1}{2}$, equivalently $0\ge\text{ord }x_{ji}$, then the Newton polygon for $x_{j,i+1}$ has the first two vertices $(0,-v+4\cdot\text{ord }x_{ji})$ and $(4,-v-2)$ and the order of the minimal solution $x_{j,i+1}$ is $(4\cdot\text{ord }x_{ji}+2)/4=\text{ord }x_{ji}+\frac{1}{2}$. $\square$

Now, we can finish the proof. We have $\text{ord }x_{j0}=\frac{1}{4}v+\frac{5}{6}$, $j=3,4$. Condition $v\le-\frac{4}{3}$ implies $\text{ord }x_{j0}\le\frac{1}{2}$. Lemma 7.4.3 implies: If $\text{ord }x_{j0}>\frac{1}{4}$ then $\text{ord }x_{j1}=4\cdot\text{ord }x_{j0}>1$. Lemma 7.4.1 shows that the corresponding minimal solution is small, it satisfies $\text{ord }x_{ji}=4^i\cdot\text{ord }x_{j0}$.

If $0 < \text{ord } x_{j0} \leq \frac{1}{4}$ then $\text{ord } x_{j1} = 2 \cdot \text{ord } x_{j0} + \frac{1}{2} > \frac{1}{2}$. Applying Lemma 7.4.2, we get $\text{ord } x_{j2} = 4 \cdot \text{ord } x_{j1} > 2$. Lemma 7.4.1 shows that the corresponding minimal solution is small, it satisfies $\text{ord } x_{ji} = 4^{i-1} \cdot \text{ord } x_{j1}$.

If $\text{ord } x_{j0} \leq 0$ then $\text{ord } x_{j1} = \text{ord } x_{j0} + \frac{1}{2}$. There exists a number $k \in \mathbb{N}$ such that $0 < \text{ord } x_{j0} + k \cdot \frac{1}{2} \leq \frac{1}{2}$. By Lemma 7.4.3, we have $\text{ord } x_{jk} = \text{ord } x_{j,k-1} + \frac{1}{2}$. Also, by Lemma 7.4.3 if $\text{ord } x_{jk} \leq \frac{1}{4}$ then $\text{ord } x_{j,k+1} = 2 \cdot \text{ord } x_{jk} + \frac{1}{2} > \frac{1}{2}$ and therefore $\text{ord } x_{j,k+2} = 4 \cdot \text{ord } x_{j,k+1} > 1$. Hence, we get a small solution with $\text{ord } x_{j,k+i} = 4^{i-1} \cdot \text{ord } x_{j,k+1}$. Otherwise, if $\text{ord } x_{jk} > \frac{1}{4}$ then $\text{ord } x_{j,k+1} = 4 \cdot \text{ord } x_{j,k} > 1$ and therefore we get a small solution with $\text{ord } x_{j,k+i} = 4^i \cdot \text{ord } x_{j,k}$. □

These propositions show that for all $(u, v) \in D_{12}$ we have $h^1 = 4$.

Remark 7.4.4. End of the proof of Proposition 7.4 (case of Lemma 7.4.3 and the very end of the proof) shows that two types of the Newton polygon and respectively two types of the growth of $\text{ord } x_i$ can occur: linear for small i, $i < k$ and exponential for $i \geq k$.

Now, we consider the case $(u, v) \in D_{23}$. We get a result only for a subset of D_{23} denoted in [3, Sec. 9] by $D_{23}1.1.2$. It is defined by the inequalities $v \leq -\frac{4}{3}$, $-2u - 4 \geq v \geq -2u - 3$.

Proposition 7.5. *For all* $(u, v) \in D_{23}1.1.2$, *we have* $h^1 = 4$.

Proof. We have (see [3, Sec. 9]: If $v \leq -\frac{4}{3}$ then the convex hull of the points $(1, -v)$, $(2, -v-1)$, $(8, -4v-8)$, $(16, -4v-12)$ consists of the points $(1, -v)$, $(2, -v-1)$, $(16, -4v-12)$ (case $D_{23}1$). If $v \leq -\frac{4}{3}$, $2u \leq -v - 3$ the Newton polygon of (3.3) has vertices $(1, -v)$, $(4, 2u)$, $(16, -4v-12)$ (case $D_{23}1.1$ of [3, Sec. 9]).

We have $\text{ord } x_{j0}$ are respectively, $\frac{1}{6}u + \frac{1}{3}v + 1, \frac{1}{6}u + \frac{1}{3}v + 1, -\frac{2}{3}u - \frac{1}{3}v, -\frac{2}{3}u - \frac{1}{3}v$.

We prove by induction by i that for all $(u, v) \in D_{23}1.1.2$, for all solutions $\{x\}$, for all $i \geq 0$ we have $\text{ord } x_i \geq (\frac{1}{6} - \frac{1}{2}i)u + (\frac{1}{3} - \frac{1}{4}i)v + 1$. This is true for $i = 0$. Using the induction assumption for i, we get

$$\text{ord } b_{14}x_i^{16} \geq (\tfrac{8}{3} - 8i)u + (\tfrac{4}{3} - 4i)v + 8,$$

$$\text{ord } b_{13}x_i^{8} \geq (\tfrac{4}{3} - 4i)u + (-\tfrac{4}{3} - 2i)v + 4,$$

$$\text{ord } b_{12}x_i^{4} \geq (\tfrac{2}{3} - 2i)u + (\tfrac{1}{3} - i)v + 4,$$

$$\text{ord } b_{24}x_{i-1}^{16} \geq (\tfrac{32}{3} - 8i)u + (\tfrac{16}{3} - 4i)v + 16.$$

For all $(u, v) \in D_{23}1.1.2$, for all $i \geq 0$ a minimal of these four numbers is the third one (for $i = 0$ the fourth number is not considered, and the first number is equal to the third one). This means that g_{i+1} is $\geq (\frac{2}{3} - 2i)u + (\frac{1}{3} - i)v + 4$. Further, for all $(u, v) \in D_{23}1.1.2$, for all $i \geq 0$ we have $(\frac{2}{3} - 2i)u + (\frac{1}{3} - i)v + 4 > \frac{8}{3}u + \frac{4}{3}v + 4$, hence $\text{ord } x_{i+1} \geq$ the minus slope of the segment $(0, (\frac{2}{3} - 2i)u + (\frac{1}{3} - i)v + 4); (4, 2u)$ which is equal $(\frac{1}{6} - \frac{1}{2}(i+1))u + (\frac{1}{3} - \frac{1}{4}(i+1))v + 1$. This implies the induction supposition.

Finally, we have $\forall\ (u,v) \in D_{23}1.1.2$ $(\frac{1}{6}-\frac{1}{2}i)u+(\frac{1}{3}-\frac{1}{4}i)v+1$ tends to $+\infty$ as i tends to $+\infty$, hence any x_0 generates a small solution, hence all Anderson t-motives in $D_{23}1.1.2$ are uniformizable. $\square$

8. Case of Jump of Valuation of the Terms of a_2

Here, we continue to study the case of A of the form (3.8), but now we consider the case when there exists a jump of valuation of the terms of a_2, i.e. ord's of some of the terms of a_2 are equal. This can occur if $(u,v) \in R_1 \cup R_2 \cup R_3$, where R_i are from Sec. 6. As in Sec. 5, we denote $t := \text{ord } a_2$.

Proposition 8.1. *If (v,t) satisfy $0 \le v < 2$, $t \ge -\frac{16}{7}v - \frac{24}{7}$ (this is a subset of $D_{23}3.3$ of* [3, *Sec.* 9]*) then $h^1 = 4$.*

Proof. Condition $v \ge -\frac{1}{3}$, $t \ge -\frac{16}{7}v - \frac{24}{7}$ means that the vertices of the convex hull of the Newton polygon of (3.3) are $(1,-v)$, $(8,-4v-8)$, $(16,-4v-12)$ and ord's of x_{j0} are $\frac{1}{2}$, $\frac{3}{7}v+\frac{8}{7}$ (three times). In particular, $\forall\, i = 1,\dots,4$ we have ord $x_{i0} \ge \frac{1}{2}$. Before starting induction, we evaluate ord x_{i1}, ord x_{i2}. We have

ord $b_{14}x_{i0}^{16} \ge -4v$,

ord $b_{13}x_{i0}^{8} \ge -4v$,

ord $b_{12}x_{i0}^{4} \ge -4v+2$ (because $v \ge 0$), hence $g_1 \ge -4v$. Joining the vertices $(0,g_1)$ and $(8,-4v-8)$ we get ord $x_{i1} \ge 1$.

Further, we have

ord $b_{14}x_{i1}^{16} \ge -4v+8$,

ord $b_{13}x_{i1}^{8} \ge -4v+4$,

ord $b_{12}x_{i1}^{4} \ge -4v+4$,

ord $b_{24}x_{i0}^{16} \ge -4v+8$. By the same reason, we get ord $x_{i2} \ge \frac{3}{2}$.

The first induction supposition is: ord $x_{ij} \ge 2 - \frac{1}{2^{j-1}}$. This is true for $j = 1,2$. We assume that this is true for $j-1, j$, and prove that this is true for $j+1$. We have ord $x_{i,j+1} \ge \min\ ((\text{ord } b_{14}x_{ij}^{16} - (-4v-8))/8,\ (\text{ord } b_{13}x_{ij}^{8} - (-4v-8))/8,$

$(\text{ord } b_{12}x_{ij}^{4} - (-4v-8))/8,\ (\text{ord } b_{24}x_{i,j-1}^{16} - (-4v-8))/8)$. These four numbers satisfy inequalities

$(\text{ord } b_{14}x_{ij}^{16} - (-4v-8))/8 \ge 2(2-\frac{1}{2^{j-1}})$,

$(\text{ord } b_{13}x_{ij}^{8} - (-4v-8))/8 \ge 2-\frac{1}{2^{j-1}}+\frac{1}{2}$,

$(\text{ord } b_{12}x_{ij}^{4} - (-4v-8))/8 \ge 2-\frac{1}{2^{j}}$,

$(\text{ord } b_{24}x_{i,j-1}^{16} - (-4v-8))/8 \ge 5-\frac{1}{2^{j-3}}$. The minimal is the third number; this implies the induction supposition.

Now, we consider the segment of the Newton polygon $(0,g_i)$ — $(1,-v)$. We denote $\varepsilon = 2-v$. There exists j_0 such that ord $x_{i,j_0-1} > 2-\frac{\varepsilon}{4}$. Let us evaluate ord x_{i,j_0+1}, ord x_{i,j_0+2}: For them the point $(1,-v)$ belongs to the convex hull of the Newton polygon for these x_{ij}. Really,

min (ord $b_{14}x_{i,j_0}^{16}$, ord $b_{13}x_{i,j_0}^8$, ord $b_{12}x_{i,j_0}^4$, ord $b_{24}x_{i,j_0-1}^{16}$) = ord $b_{12}x_{i,j_0}^4$

(because ord $x_{i,j_0-1} \geq \frac{3}{2}$), and the slope of the segment of the Newton polygon $(0,g_i)$ — $(1,-v)$ is $\geq 2+2\varepsilon$, i.e. ord $x_{i,j_0+1} \geq 2+2\varepsilon$. Analogously, ord $x_{i,j_0+2} \geq 2+2\varepsilon$ (we use the above arguments for j_0+1 instead of j_0).

The second induction supposition is: ord $x_{i,j_0+j} \geq 2+2^{j-1}\varepsilon$ $(j>0)$. Really, this is true for $j=1,2$. We assume that this is true for $j-1,j$, and prove that this is true for $j+1$. We have

ord $b_{14}x_{i,j_0+j}^{16} - (-v) \geq -3v + 24 + 2^{j-1}\cdot 16\varepsilon > 2+2^j$,

ord $b_{13}x_{i,j_0+j}^{8} - (-v) \geq -3v + 12 + 2^{j-1}\cdot 8\varepsilon > 2+2^j$,

ord $b_{12}x_{i,j_0+j}^{4} - (-v) \geq -3v + 8 + 2^{j-1}\cdot 4\varepsilon > 2+2^j$,

ord $b_{24}x_{i,j_0+j-1}^{16} - (-v) \geq -3v + 32 + 2^{j-2}\cdot 16\varepsilon > 2+2^j$.

This proves the induction supposition, and hence the proposition. □

Proposition 8.2. *If (v,t) satisfy $v \geq 2$, $t \geq -\frac{16}{7}v - \frac{24}{7}$ (this is a subset of $D_{23}3.3$) then $h^1 = 0$.*

Proof. First, we consider the maximal value of ord x_0 and their minimal solution x_i. We denote $\alpha_i :=$ ord $x_i - 2$. We have $\alpha_0 = \frac{3}{7}v - \frac{6}{7} \geq 0$. The induction supposition is $\alpha_{i+1} = \alpha_i/2$. Let us prove it. We have

ord $b_{14}x_i^{16} = -4v + 16\alpha_i + 24$,

ord $b_{13}x_i^{8} = -4v + 8\alpha_i + 12$,

ord $b_{12}x_i^{4} = -4v + 4\alpha_i + 8$,

ord $b_{24}x_{i-1}^{16} = -4v + 32\alpha_i + 16$.

According the induction supposition, we have $\alpha_i > 0$, hence the above numbers are different and $g_{i+1} = -4v + 4\alpha_i + 8$. The convex hull of $(0,g_{i+1}),(1,-v),(2,-v-1),(4,t),(8,-4v-8),(16,-4v-12)$ consists of the points $(0,g_{i+1}),(8,-4v-8),(16,-4v-12)$ — this follows immediately from $\frac{3}{7}v - \frac{6}{7} \geq \alpha_i \geq 0$ and $v \geq 2$. Hence, the maximal value of ord x_{i+1} is $2+\alpha_i/2$, hence the supposition.

This means that these minimal solutions are not small.

Let us consider the case of ord $x_0 = \frac{1}{2}$ and its minimal solution. We have in this case

ord $b_{14}x_0^{16} = -4v$,

ord $b_{13}x_0^{8} = -4v$,

ord $b_{12}x_0^{4} = -4v+2$, i.e. a jump can occur. Hence, we have to consider one more term in an approximation. We denote $c_2 := a_2\theta^{-\frac{16}{7}v-\frac{24}{7}}$, i.e. ord $c_2 \geq 0$. The first approximation to x_0 is $x_0 \approx \frac{a_{11}^{1/2}}{\theta}$ (it is obtained if we consider only the first two terms of (3.3)). Hence, we let $\Delta_0 := x_0 - \frac{a_{11}^{1/2}}{\theta}$, i.e. $x_0 = \Delta_0 + \frac{a_{11}^{1/2}}{\theta}$. Substituting this value to (3.3), we get

$$a_4\Delta_0^{16} + a_3\Delta_0^8 + a_2\Delta_0^4 + a_1\Delta_0^2 + a_0\Delta_0 + \left(\theta^{\frac{16}{7}v-\frac{4}{7}}a_{11}^2c_2 + \frac{a_{11}^2}{a_{21}\theta^2} + \frac{a_{11}^{1/2}}{a_{21}\theta}\right) = 0 \tag{8.2.1}$$

we have ord $\theta^{\frac{16}{7}v-\frac{4}{7}}a_{11}^2c_2 \geq \frac{-16v-10}{7}$ which is always less than ord $\frac{a_{11}^2}{a_{21}\theta^2}$, ord $\frac{a_{11}^{1/2}}{a_{21}\theta}$, hence ord of the free term of (8.2.1) is $\geq \frac{-16v-10}{7}$. Comparing with a Newton point $(8, -4v-8)$ we get that ord $\Delta_0 \geq \frac{6v+23}{28} \geq \frac{5}{4}$, because $v \geq 2$.

Now, we substitute the obtained value of x_0 to (3.4). The sum of the tail terms is

$$\frac{\theta^8+\theta^4}{a_{21}^4}\left(\frac{a_{11}^8}{\theta^{16}} + \Delta_0^{16}\right) + \frac{a_{11}^4}{a_{21}^4}\left(\frac{a_{11}^4}{\theta^8} + \Delta_0^8\right) + \left(\frac{1}{a_{21}} + \frac{1}{a_{21}^4}\right)\left(\frac{a_{11}^2}{\theta^4} + \Delta_0^4\right). \tag{8.2.2}$$

Taking into consideration that ord $\Delta_0 \geq \frac{5}{4}$ we get that the term $\frac{1}{a_{21}^4}\frac{a_{11}^2}{\theta^4}$ of (8.2.2) has the minimal ord $= -4v+2$, all other terms of (8.2.2) have the higher ord's. This means that $g_1 = -4v+2$ and ord $x_1 = \frac{5}{4}$.

The continuation is simpler because there will be no more jumps of valuation. The induction supposition is: for $i \geq 1$ we have ord $x_i = 2 - \frac{3}{2^{i+1}}$. This is true for $i = 1$. Let us assume that this is true for some i, and prove that this is true for $i+1$. We have

ord $b_{14}x_i^{16} = -4v + 24 - \frac{3}{2^{i-3}}$,

ord $b_{13}x_i^8 = -4v + 12 - \frac{3}{2^{i-2}}$,

ord $b_{12}x_i^4 = -4v + 8 - \frac{3}{2^{i-1}}$

ord $b_{24}x_0^{16} = -4v + 8$ for $i = 1$, ord $b_{24}x_{i-1}^{16} = -4v + 24 + \frac{3}{2^{i-2}}$ for $i > 1$.

For $i \geq 1$ the minimal of the above four numbers is $-4v+8-\frac{3}{2^{i-1}}$. The leftmost segment of the convex hull of $(0, -4v+8-\frac{3}{2^{i-1}})$, $(1, -v)$, $(2, -v-1)$, $(8, -4v-8)$, $(16, -4v-12)$ is $[(0, -4v+8-\frac{3}{2^{i-1}}), (8, -4v-8)]$, hence ord $x_{i+1} = 2 - \frac{3}{2^i}$ — the induction supposition is proved. We see that a minimal solution is not small.

Let us prove that there are no small solutions at all (in principle, it can happen that a linear combination of non-small solutions is small). The proof is exactly the same as the proof of [7, Lemma 4.6], so we repeat it here. Namely, we denote by $\{x_j\}$ $(j = 1, 2, 3)$ linearly independent minimal solutions corresponding to ord $x_{j0} = \frac{3}{7}v+\frac{8}{7}$, and by $\{x_4\}$ a minimal solution corresponding to ord $x_{40} = \frac{1}{2}$. Let us assume that $\exists\, C_1, \ldots, C_4 \in \mathbb{F}_2[[T]]$ such that $\sum_{j=1}^4 C_j\{x_j\}$ is a small solution. We consider $S_{123} := \sum_{j=1}^3 C_j\{x_j\}$, we denote $S_{123} = \sum_{i=0}^\infty \bar{x}_{1,2,3;i}T^i$, We have ord $\bar{x}_{1,2,3;i} \geq 2$, because $\forall\, i$ elements $\bar{x}_{1,2,3;i}$ are linear combinations of x_{jk} for $j = 1, 2, 3$, $k \leq i$ with coefficients in $\mathbb{F}_2$.

Further, we denote $S_4 := C_4\{x_4\} = \sum_{i=0}^{\infty} \bar{x}_{4i} T^i$. The above considerations show that $\forall\ i$ ord x_{4i} are different and $\frac{1}{2} \leq$ ord $x_{4i} < 2$, hence $\forall\ i$ we have $\frac{1}{2} \leq$ ord $\bar{x}_{4i} < 2$. This means that $\sum_{j=1}^{4} C_j\{x_j\} = S_{123} + S_4$ cannot be a small solution. $\square$

Proposition 8.3. *If* (v, t) *satisfy conditions* $-\frac{4}{3} < v < -\frac{1}{3}$, $t \geq -4v - 6$ *then* $h^1 = 4$.

Remark. The conditions $-\frac{4}{3} < v < -\frac{1}{3}$, $t \geq -4v - 6$ correspond to the types D_{23}2a.3, D_{23}2b.2, D_{23}2b.3, D_{23}2.5 of [3, Sec. 9].

Proof. For all these cases, we have ord $x_{i0} \geq \frac{1}{2}$. First, we evaluate ord x_{i1}. We have

ord $b_{14} x_{i0}^{16} \geq -4v$,

ord $b_{13} x_{i0}^{8} \geq -4v$,

ord $b_{12} x_{i0}^{4} \geq -v + 2$, hence ord $x_{i1} \geq \min\,(-3v, 2) = 1$. Induction supposition for any j: ord $x_{ij} \geq 2^{j-1}$. This is true for $j = 0, 1$; let us prove that it holds for $j + 1$. We have

ord $b_{14} x_{ij}^{16} \geq \frac{4}{3} - 8 + 16 \cdot 2^{j-1}$,

ord $b_{13} x_{ij}^{8} \geq \frac{4}{3} - 4 + 8 \cdot 2^{j-1}$,

ord $b_{12} x_{ij}^{4} \geq \frac{1}{3} + 4 \cdot 2^{j-1}$

ord $b_{24} x_{i,j-1}^{16} \geq \frac{4}{3} + 16 \cdot 2^{j-2}$.

To prove the induction supposition, we must prove that for $j \geq 1$ we have $\min\,(\frac{4}{3} - 8 + 16 \cdot 2^{j-1},\ \frac{4}{3} - 4 + 8 \cdot 2^{j-1},\ \frac{1}{3} + 4 \cdot 2^{j-1},\ \frac{4}{3} + 16 \cdot 2^{j-2}) - \frac{4}{3} \geq 2^j$. This is straightforward. $\square$

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