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Definability in o-minimal expansions of the real numbers

Ricardo Bianconi [3,2,1]

Instituto de Matemática e Estatística
Universidade de São Paulo
São Paulo, SP, Brazil

Abstract

We present some recent results and problems concerning definable sets and functions in o-minimal expansions of the field of real numbers by analytic or smooth functions. We deal with geometry (cell decomposition), Schanuel's conjecture and complex analytic varieties.

Key words: definable sets, o-minimal expansions, analytic functions, Schanuel's conjecture, complex analytic varieties

1 Introduction

We survey here some results and problems in the model theory of o-minimal expansions of $\bar{R} = \langle \mathbb{R}, +, \cdot, 0, 1, < \rangle$, the field of the real numbers, by adding analytic or only smooth functions. This subject is of interest not only in model theory but also in the now rapidly growing field of real algebraic and analytic geometry.

We recall that a linearly ordered structure $M = \langle |M|, <, \ldots \rangle$ is o-minimal if the definable sets $X \subseteq |M|$ (in the language of M) are the union of finitely many points and open intervals with endpoints in $|M| \cup \{\pm\infty\}$. Let $X \subseteq |M|^n$ be a definable set. Then it admits a cell-decomposition [16], that is X can be written as the disjoint union of finitely many definable sets called cells $X = \bigcup_{1 \leq j \leq k} X_j$, each X_j being an open set, or the embedding the graph of a definable function $\phi_j : U_j \subseteq |M|^{n_j} \to |M|$, with U_j a definable open set, or isolated points.

[1] Email: bianconi@ime.usp.br
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We focus this paper in three interconnected topics. In section 2, we deal with geometry (or rather differential topology) of definable sets in such expansions, that is, on cell decomposition. In section 3, we recall Schanuel's conjecture on transcendental number theory and point out applications to model theory. This conjecture is still open for the field of complex numbers, but a somewhat weaker statement was proved for power series rings by J. Ax in [2]. This result has been applied to nondefinability results and this is the subject of section 4. We make some final remarks on problems not treated here in the conclusion, section 5.

2 Geometry

One of the main problems in the model theory of certain class of structures is to describe the definable sets, their topology and geometry. Tarski's result on elimination of quantifiers for $\bar{R}$ showed that we can describe easily the definable sets in any real closed field, being an important result in real algebraic geometry. Namely, the theorem says that the definable sets on a real closed field are exactly the semialgebraic sets, see [16]. If we introduce the exponential function, then we lose the elimination of quantifiers but mantain model completeness, see [18], so the definable sets are reasonably easy to describe. The expansion of $\bar{R}$ with all real analytic functions restricted to convenient compact domains regains the elimination of quantifiers, see [6]. All these examples are o-minimal structures. Until recently we only knew o-minimal expansions of $\bar{R}$ admitting analytic cell decomposition. In [15] there is the first example of o-minimal expansions of $\bar{R}$ admitting nonanalytic but smooth cell decomposition. A. J. Wilkie has shown in [17] that the concept of smooth function is not first order definable. So we may ask the following question.

Problem 2.1 *Is there an expansion of $\bar{R}$ with a nonsmooth cell decomposition?*

We recall that a function $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ is of class C^k $(k \geq 1)$ if it has all partial derivatives continuous up to order k. If f is of order C^k for all $k > 0$, then we say that f is smooth or of order C^∞. There is a restriction to the class of functions definable in an o-minimal expansion of $\bar{R}$.

Lemma 2.2 *Let M be an o-minimal expansion or $\bar{R}$ and $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ be a definable function, where U is a definable open set. Then, for all $k > 0$, f is piecewise C^k, that is, there is a definable open subset $V \subseteq U$, such that V is dense in U, $U \setminus V$ is definable and f is of class C^k in V.*

(Sketch of proof) We first consider $n = 1$. By o-minimality, f is piecewise monotonic and therefore piecewise C^1. Its derivative is also piecewise monotonic, and so on. For the general case, since the partial derivatives of order k are definable, [17], restricted to one-dimensional linear sets in U we conclude the result. □

There is another restriction on o-minimal expansions of $\bar{R}$, and points to another line of research in this topic. The following theorem shows an interesting relation between semialgebraic sets and semianalytic sets.

Theorem 2.3 (Piêkosz, [14]) *If the o-minimal structure M admits analytic cell decomposition and $A \subset M^n$ is a definable set, then the set $Alg(A)$ of points $x \in A$, such that the germ A_x is algebraic (see [5]), is definable.*

3 Schanuel's conjecture

Now we deal with transcendental number theory and applications to definability.

The still open Schanuel's conjecture is the following statement (see [7]).

Conjecture 3.1 (Schanuel) *If $x_1, \dots, x_n \in \mathbb{C}$ are linearly independent over $\mathbb{Q}$, then the transcendence degree of $\mathbb{Q}(x_1, \dots, x_n, \exp(x_1), \dots, \exp(x_n))$ over $\mathbb{Q}$ is at least n.*

This conjecture was inspired by the Lindemann-Weierstrass theorem which states that this conjecture is true if $x_1, \dots, x_n$ are algebraic numbers, see [7].

James Ax has proved a power series version of the conjecture (see [2, theorem 3]):

Theorem 3.2 (Ax, [2]) *Let $\mathbb{Q} \subseteq C \subseteq K$ be a tower of fields and Δ a set of derivations of K with $\bigcap_{D \in \Delta} \ker D = C$. Let $f_1, \dots, f_n, g_1, \dots, g_n \in F^\times$ be such that*

(i) *for all $D \in \Delta$, $j = 1, \dots, n$, $Df_j = Dg_j/g_j$ and*

(ii) *the f_j are $\mathbb{Q}$-linearly independent modulo C (that is, no nontrivial $\mathbb{Q}$-linear combination of the f_j is in C).*

Then the transcendence degree over C of $C(f_1, \dots, f_n,\ g_1, \dots, g_n)$ is at least $n + \operatorname{rank}(Df_i)_{i,D}$.

The conjecture 3.1 implies the decidability of $R_{\exp}$ (see [10]). In [4] we have proved that this conjecture is true for the infinitesimals of a nonstandard extension of $\mathbb{C}$. Namely,

Theorem 3.3 (Bianconi, [4]) *Let $K = M(\mathbf{i})$, $\mathbf{i} = \sqrt{-1}$, be such that M is a nonarchimedean model of a expansion of $\bar{R}_{\exp}$ with restricted sine function, and let $\mu(K)$ be the set of infinitesimal elements of K. Let $a_1, \dots, a_n \in \mu(K)$ linearly independent over $\mathbb{Q}$, $F = \mathbb{Q}(a_1, \dots, a_n,\ \exp a_1, \dots, \exp a_n)$. Then the transcendence degree of F over $\mathbb{Q}$ is at least $n+1$*

Its proof relies on theorem 3.2 and suggests the following conjecture.

Conjecture 3.4 *There is a countable subfield K of $\mathbb{C}$ such that if $x_1, \dots, x_n \in \mathbb{C}$ are such that no nontrivial Q-linear combination of $x_1, \dots, x_n$ is in K, then* *the transcendence degree of* $\mathbb{Q}(x_1, \ldots, x_n, \exp(x_1), \ldots, \exp(x_n))$ *over* K *is at least* $n+1$.

I believe that a fine analysis of these techniques may provide us with a proof of the full Schanuel's conjecture.

4 Complex analysis

The problems treated in this section were asked by Chris Miller. These are motivated by the nondefinability of the sine function in $R_{\exp}$ in [3]. The technique used is the same, the proof of Schanuel's conjecture for some fields in [2], see theorem 3.2 above.

The motivating problem for these results is the following.

Problem 4.1 *How does the o-minimal machinery apply to the (model theoretic, algebraic, analytic) study of complex analysis and complex analytic manifolds or varieties?*

The following result shows some of the limitations on dealing with this problem.

Theorem 4.2 *Let* $u, v : D \to \mathbb{R}$ *be two definable functions in* $R_{\exp}$ *such that* $u + iv$ *is holomorphic,* $D \subset \mathbb{R}^{2n}$ *is a definable polydisc. Then* u *and* v *are already definable in* $\bar{R}$.

Proof. Let u and v be the real and imaginary parts of a holomorphic function definable in $R_{\exp}$ around the origin. Assume that they are not semialgebraic. Suppose that they are defined with $Z = (Z_1, \ldots, Z_p)$ as witnesses, and variables $X = (X_1, \ldots, X_n)$, $Y = (Y_1, \ldots, Y_n)$, U and V, that is, their graph are defined by existential formulas $\exists Z \psi_u(X, Y, U, Z)$ and $\exists Z \psi_v(X, Y, V, Z)$, respectively. Then, modulo desingularization tricks (as in [3]), we have polynomials P_j, $j = 1, \ldots, p+2$ such that $F_j(X, Y, U, V, Z) = P_j(X, Y, U, V, Z, \exp X, \exp Y, \exp U, \exp V, \exp Z) = 0$ and nonsingular Jacobian matrix $\partial(F_1, \ldots, F_{p+2})/\partial(U, V, Z)$. Imposing Cauchy Riemmann equations, we introduce $2n$ new equations, with polynomials P_j, $j = p+3, \ldots, p+2+2n$. So we have $p+2+2n$ polynomial relations on $X, Y, U, V, Z, \exp X$, $\exp Y$, $\exp U$, $\exp V$, $\exp Z$, totalling $2(p+2+2n) = 2p+4n+4$ terms, giving at most $p+2n+2$ for the transcendence degree of $\mathbb{Q}(X, Y, U, V, Z, \exp X, \exp Y, \exp U, \exp V, \exp Z)$, over Q. By the implicit function theorem for power series, see [1], the solutions to those equations are analytic functions, so represented by power series over $\mathbb{R}$. Then we can apply theorem 3.2, giving transcendence degree over R at least $p + 2n + 3$, a contradiction. So u, v, z must be definable in $\bar{R}$. □

This means that there is no complex analytic variety defined in $R_{\exp}$ which is not already defined over $\bar{R}$. But this does not rules out the following problem.

Problem 4.3 *Does the algebraic closure in $\mathbb{C}$ of the prime the model of R_{exp} support a (complex) exponential function?*

In particular, we can ask:

Problem 4.4 *Is the number π definable in R_{exp}?*

If we go into more general algebraically closed fields with exponential, we have:

Problem 4.5 *Let $M = \langle |M|, +, \cdot, exp \rangle$ a proper elementary extension of R_{exp} and let $r > \mathbb{R}$ (r is an element of M bigger than all standard real numbers). Consider the structure $\langle |M|, +, \cdot, x^r \rangle$. If $U \subseteq |M|^{2n}$ and $f : U \to |M|^2$ is definable and complex differentiable, is f definable in $\langle |M|, +, \cdot \rangle$?*

The following conjecture is suggested by the work of Arthur Piêkosz [14], see theorem 2.3 above.

Conjecture 4.6 (C. Miller) *Let M be a complex analytic (embedded) submanifold of $\mathbb{C}^m$ such that M is a definable (in R_{exp}) subset of $\mathbb{R}^{2m}$. Then M is definable is $\langle \mathbb{C}, +\cdot \rangle$ (that is, M is algebraic).*

5 Conclusion

In this paper we have touched only a small part of the research on o-minimality and it reflects the preferences of the author. We have not touched other important lines of research. The interested reader should consult for instance [9] on differential equations over (polynomially bounded) o-minimal structures, or [8,12,11] on extensions of the notion of o-minimality, or the book [16] on a more topological and geometric approaches. For a more algebraic geometric approach, the reader should consult [13] and references therein.

The reference list is far from complete. The reader should consult the references in the book [16], and in the papers cited in order to delve deeper into the literature.

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