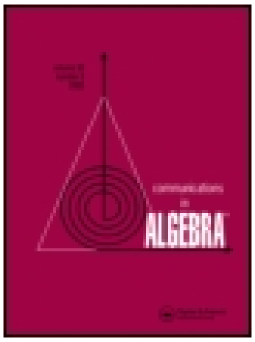

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Free generic Poisson fields and algebras

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Abstract

The free generic Poisson algebras (GP-algebras) over a field k of characteristic 0 are studied. We prove that certain properties of free Poisson algebras are true for free GP-algebras as well. In particular, the universal multiplicative enveloping algebra $U = U(GP(x_1, \ldots, x_n))$ of a free GP-field $GP(x_1, \ldots, x_n)$ is a free ideal ring. Besides, the Poisson and polynomial dependence of two elements are equivalent in $GP(x_1, \ldots, x_n)$. As a corollary, all automorphisms of the free GP-algebra $GP\{x, y\}$ are tame and we have the isomorphisms of groups of automorphisms $Aut\, GP\{x, y\} \cong Aut\, P\{x, y\} \cong Aut\, k[x, y]$.

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1. INTRODUCTION

Many interesting and important results are known about the structure of polynomial algebras, free associative algebras, and free Lie algebras. Although free Poisson algebras are closely related to these algebras, they were not studied till recently. The exact construction of a free Poisson algebra first appeared in [22], where it was used for study of Poisson Jordan superalgebras. Then they were used in [24, 25], in the proof of the Nagata conjecture on the existence of wild automorphisms in the polynomial algebra of rank three. A systematic study of free Poisson algebras (over a field k of characteristic zero) was started in [13], where the following analogue of the famous Bergman Centralizer Theorem [2] was proved: the centralizer of a nonconstant element of a free Poisson algebra is a polynomial algebra in single variable. In [12], the automorphisms and derivations of the free Poisson algebra $P\langle x, y\rangle$ of rank two were studied, and analogues of classical Jung's [8] and Rentschler's [21] theorems were proved: all automorphisms of $P\{x, y\}$ are tame and all locally nilpotent derivations are triangulable; moreover, $Aut\, P\{x, y\} \cong Aut\, k\langle x, y\rangle \cong Aut\, k[x, y]$, where $k\langle x, y\rangle$ and $k[x, y]$ denote the free associative and polynomial algebras, respectively. Later on, the automorphisms of the free Poisson field $P(x, y)$ of rank two were studied, and it was proved the isomorphism $Aut\, P(x, y) \cong Aut\, k(x, y)$, where $k(x, y)$ is the field of rational functions [14]. Observe that the last group is in turn isomorphic to the Cremona group $Cr_2(k)$ of birational automorphisms of the projective plane $\mathbf{P}_k^2$.

Furthermore, in [11] the Poisson and polynomial dependences were studied in free Poisson algebras and free Poisson fields. In [15], the Freiheitssatz was proved for free Poisson algebras. Finally, in [16, 17, 30], the universal enveloping algebras of Poisson algebras were studied.

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In this paper, we prove some analogues of the cited above results for free *generic Poisson algebras*. A generic Poisson algebra (GP-algebra) is a linear space with two operations:

- associative and commutative product $x \cdot y = xy$,
- anti-commutative bracket $\{x, y\}$,

satisfying the Leibniz identity

$$\{x, yz\} = \{x, y\}z + \{x, z\}y. \quad (1)$$

Contrary to usual Poisson algebras, it is not required here that the bracket $\{,\}$ were a Lie one. These algebras were introduced in [23] in the study of speciality and deformations of Malcev-Poisson algebras.

The variety of GP algebras contains Poisson algebras and some interesting examples of non-Poisson algebras (Poisson-Malcev algebras [26, 31], Binary-Lie Poisson algebras [1], associated graded algebras of universal enveloping algebras of Malcev algebras [20], Bol algebras [18], and Sabinin algebras [19]).

An important example of generic Poisson algebra comes from the following setting. Let A be a commutative associative algebra with $2n$ pairwise commuting derivations $d_i, \delta_j,\ i, j = 1\ldots, n$.

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Define on A a new binary operation

$$\{a,b\} = \sum_{n=1}^{n}(d_i(a)\delta_i(b) - \delta_i(a)d_i(b)),$$

then A with this new bracket is known to be a Poisson algebra. If we allow the derivations d_i, δ_i to be non-commuting then this bracket defines on a commutative algebra A a structure of a GP-algebra.

The free GP-algebras were first considered in [9], where the Freiheitssatz was proved for them. Here we continue the study of these algebras.

The paper is organized as follow:

- In section 2 we study general properties of generic Poisson modules and of universal multiplicative enveloping algebras.
- In section 3 we determine structure of free GP-algebra and of free GP-field.
- In section 4 we prove an analogue of Makar-Limanov - Shestakov's Theorem [11] that two Poisson dependent elements in a free GP-field are polynomial dependent.
- In section 5 we apply the obtained results to the study of automorphisms the free GP-algebra $GP\{x,y\}$ of rank 2 and prove that

  $$Aut\, GP\{x,y\} \cong Aut\, P\{x,y\} \cong Aut\, k\langle x,y\rangle \cong Aut\, k[x,y].$$

Throughout the paper, k denotes a field of characteristic zero.

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2. GENERIC POISSON MODULES AND UNIVERSAL MULTIPLICATIVE ENVELOPING ALGEBRAS

Following to the general conception due to Eilenberg [6], define a notion of a generic Poisson module. Let P be a GP-algebra over k. A vector space V over k is called a *generic Poisson module* or just a *GP-module* over P if there are two bilinear maps

$$P \times V \to V,\ (p, v) \to p \cdot v,$$

and

$$P \times V \to V,\ (p, v) \to \{p, v\},$$

such that *the split null extension* $E = P \oplus V$ with the operations

$$\begin{aligned} (p + v) \cdot (q + u) &= p \cdot q + p \cdot u + q \cdot v, \\ \{p + v, q + u\} &= \{p, q\} + \{p, u\} - \{q, v\}, \end{aligned}$$

is a GP-algebra. It is easy to see that this is equivalent to require that the following relations hold:

$$\begin{aligned} (x \cdot y) \cdot v &= x \cdot (y \cdot v), \\ \{x \cdot y, v\} &= x \cdot \{y, v\} + y \cdot \{x, v\}, \\ \{x, y\} \cdot v &= \{x, y \cdot v\} - y \cdot \{x, v\} \end{aligned}$$

for all $x, y \in P$ and $v \in V$.

Let V be a GP-module over a GP-algebra P. For every $x \in P$ we denote by M_x and H_x the following operators acting on V: $M_x(v) = x \cdot v,\ H_x(v) = \{x, v\}$ for any $v \in V$. Then the defining relations for

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a GP-module can be written as

$$M_{xy} = M_x M_y, \tag{2}$$

$$H_{xy} = M_y H_x + M_x H_y, \tag{3}$$

$$M_{\{x,y\}} = [H_x, M_y], \tag{4}$$

respectively.

A pair of mappings $(M, H) : P \to End\,V;\ x \mapsto M_x,\ x \mapsto H_x$ which satisfies identities (2) - (4) we will call a *representation* of a GP-algebra P associated to a GP-module V. Clearly, the notions of a module and of an associated representation define each other. Sometimes we will use a term "representation" in more general situation, just for a pair of mappings $(M, H) : P \to A$ of a GP-algebra P into an associative algebra A which satisfies identities (2) - (4).

If P is a unitary GP-algebra with the unit element 1, then we will assume that $M_1 = Id_V$, that is, V is unitary. It is easy to see that in this case also $H_1 = 0$.

Following to [7] (see also [30]), define the (unitary) universal multiplicative enveloping algebra $U(P)$ of a GP-algebra P. Let $m_P = \{m_a | a \in P\}$ and $h_P = \{h_a | a \in P\}$ be two copies of the vector space P endowed with two linear isomorphisms $m : P \to m_P$ $(a \mapsto m_a)$ and $h : P \to h_P$ $(a \mapsto h_a)$. Then $U(P)$ (or P^e in notation of [30]) is an associative algebra over k, with identity 1, generated

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by the direct sum of two linear spaces m_P and h_P, subject to the relations

$$m_{xy} = m_x m_y, \tag{5}$$

$$h_{xy} = m_y h_x + m_x h_y, \tag{6}$$

$$m_{\{x,y\}} = h_x m_y - m_y h_x = [h_x, m_y], \tag{7}$$

$$m_1 = 1 \tag{8}$$

for all $x, y \in P$.

In a standard way, the universal multiplicative enveloping algebra is uniquely defined up to isomorphism by its universal property:

Proposition 2.1. *Let P be a GP-algebra. For any representation $(M, H) : P \to A$ of P into an associative algebra A there exists a unique algebra homomophism $\varphi : U(P) \to A$ such that $\varphi(m_x) = M_x,\ \varphi(h_x) = H_x$ for any $x \in P$.*

By the definition of the universal multiplicative enveloping algebra, the notion of a GP-module over a GP-algebra is equivalent to the notion of a left module over its universal enveloping algebra. Let V be an arbitrary GP-module over P. Then V becomes a left $U(P)$-module under the action

$m_x v = x \cdot v,\ h_x v = \{x, v\},$

for all $x \in P$ and $v \in V$. Conversely, if V is a left $U(P)$-module then the same formulas turn V to a GP-module over P. Therefore, we have the following

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Proposition 2.2. *The category of unitary GP-modules over a unitary GP-algebra P and the category of (left) unitary modules over the universal enveloping algebra $U(P)$ are equivalent.*

The first example of a GP-module over P is the regular module $Reg\, P = P$, under the actions $x \cdot v$ and $\{x, v\}$. Since $1 \in P$ and $m_x 1 = x$, it follows that the mapping

$$m : P \to U(P),\ x \to m_x,$$

is an injection. Therefore we may identify m_x with x. After this identification, the essential part of the defining relations of the $U(P)$ are

$$h_{xy} = yh_x + xh_y, \tag{9}$$

$$\{x, y\} = h_x y - yh_x = [h_x, y], \tag{10}$$

for all $x, y \in P$. In particular, it follows from (9) that the mapping h is a derivation of P into the left P-module $U(P)$. Moreover, if $x \in P$ is an invertible element then

$$h_{x^{-1}} = -x^{-2}h_x,\ h_{x^{-1}y} = -yx^{-2}h_x + x^{-1}h_y. \tag{11}$$

Recall the notion of Hopf smash product. Let H be a Hopf algebra. A (left) *H-module algebra A* is an algebra which is a (left) module over the algebra H such that

$$h \cdot 1_A = \epsilon(h)1_A \qquad \text{and} \qquad h \cdot (ab) = (h_{(1)} \cdot a)(h_{(2)} \cdot b)$$

whenever $a, b \in A, h \in H, \epsilon$ is the counit of H, and $\Delta(h) = h_{(1)} \otimes h_{(2)}$ is sumless Sweedler notation for the coproduct in H.

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Now, let H be a Hopf algebra and A be a left Hopf H-module algebra. The *smash product* algebra $A \sharp H$ is the vector space $A \otimes H$ with the product

$$(a \otimes h)(b \otimes k) := a(h_{(1)} \cdot b) \otimes h_{(2)}k,$$

and we write $a\sharp h$ for $a \otimes h$ in this context.

It is easy to see that $A \cong A \sharp 1$ and $H \cong 1 \sharp H$; moreover, $A \sharp H \cong A \otimes H$ as a left A-module. Besides, A is a left $A \sharp H$-module under $(a \sharp h) \otimes b \mapsto a(h \cdot b)$.

Observe that the smash product $A \sharp H$ is characterized by the following universal property:

For any associative algebra C and homomorphisms of algebras $\varphi : A \to C$, $\psi : H \to C$ such that

$$\psi(h)\varphi(a) = \varphi(h_{(1)} \cdot a)\psi(h_{(2)}) \qquad \textit{for any } a \in A, h \in H, \tag{12}$$

there exists a unique homomorphism $\varphi\sharp\psi : A \sharp H \to C$ such that $(\varphi\sharp\psi)(a\sharp 1_H) = \varphi(a)$, $(\varphi\sharp\psi)(1_A\sharp h) = \psi(h)$.

An important example of smash-product is obtained when a Lie algebra L acts by derivations on an algebra A. In this case A becomes a Lie module over L and also a left module over the enveloping algebra $U(L)$. It is well known that this $U(L)$-module structure is in fact a left Hopf module structure, and we may construct the smash Hopf product $A \sharp U(L)$.

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In particular, let $i : V \to Der\,A$ be a linear mapping of a vector space V to the algebra of derivations of an associative algebra A. We can extend i to a homomorphism of the free Lie algebra $Lie(V)$ over V to $Der\,A$. Then, as above, the algebra A has a left Hopf module structure over $U(Lie(V)) \cong k\langle V\rangle$, the free associative algebra (tensor algebra) over the space V, and we may consider the smash Hopf product $A \sharp k\langle V\rangle$.

We will often use the following simple fact:

Lemma 2.3. *Let $A \sharp k\langle V\rangle$ be a smash Hopf product constructed as above via a linear mapping $i : V \to Der\,A$, and let B be an associative algebra. For any algebra homomorphism $\varphi : A \to B$ and a linear mapping $\psi : V \to B$ such that*

$$[\psi(v), \varphi(a)] = \varphi(i(v)(a)) = \varphi(v \cdot a) \qquad \text{for any } v \in V,\ a \in A, \tag{13}$$

there exists a unique homomorphism $\varphi \sharp \psi : A \sharp k\langle V\rangle \to B$ extending the mappings φ and ψ.

Proof. The mapping ψ is extended to an algebra homomorphism of $k\langle V\rangle$ to B, which we will denote by ψ as well. Let us check that the pair (φ, ψ) satisfies condition (12). Let $a \in A,\ u \in k\langle V\rangle$. It suffices to consider the case when u is a monomial: $u = v_1 \cdots v_n,\ v_i \in V$. For $n = 1$ the condition holds by (13). If $n > 1$, we write $u = wv,\ v \in V$, then

$$\Delta(u) = \Delta(w)\Delta(v) = (w_{(1)} \otimes w_{(2)})(1 \otimes v + v \otimes 1) = w_{(1)} \otimes w_{(2)}v + w_{(1)}v \otimes w_{(2)},$$

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and we have by (13) and induction

$$\begin{aligned}\psi(u)\varphi(a) &= \psi(w)\psi(v)\varphi(a) = \psi(w)(\varphi(a)\psi(v) + \varphi(i(v)(a)) \\ &= \varphi(w_{(1)} \cdot a)\psi(w_{(2)})\psi(v) + \varphi(w_{(1)} \cdot (v \cdot a))\psi(w_{(2)}) \\ &= \varphi(w_{(1)} \cdot a)\psi(w_{(2)}v) + \varphi(w_{(1)} \cdot (v \cdot a))\psi(w_{(2)}) = \varphi(u_{(1)} \cdot a)\psi(u_{(2)}).\end{aligned}$$

By the universal property of smash product, there exists a homomorphism $\varphi\sharp\psi : A \sharp k\langle V\rangle \to B$ extending φ and ψ. □

Theorem 2.4. *Let P be a GP-algebra. Then $U(P) \cong (P \sharp k\langle P\rangle)/I$, where $k\langle P\rangle$ is the free associative algebra over the space P with the canonical Hopf algebra structure and I is the ideal of $P \sharp k\langle P\rangle$ generated by the set $\{1\sharp(ab) - a\sharp b - b\sharp a \mid a, b \in P,\ ab \text{ is the product in } P\}$.*

Proof. Consider the linear mapping $\tau : P \to Der\, P,\ \tau(a)(x) = \{a, x\}$. As above, we may construct the smash Hopf product $P \sharp k\langle P\rangle$. For any $a, v \in P$ we have by (7)

$[h_v, m_a] = m_{\{v,a\}} = m_{v\cdot a}$,

which proves that the applications $m, h : P \to U(P)$, $m : p \mapsto m_p$, $h : p \mapsto h_p$ satisfy identity (13). By lemma 2.3, there exists a homomorphism $m\sharp h : P \sharp k\langle P\rangle \to U(P)$ extending homomorphisms m and h. Clearly, $m\sharp h$ is surjective. By (9), the ideal I is contained in $\ker(m\sharp h)$, hence $U(P)$ is a homomorphic image of $(P \sharp k\langle P\rangle)/I$. Finally, consider the mappings $\tilde{m}, \tilde{h} : P \to (P \sharp k\langle P\rangle)/I$, $\tilde{m} : p \mapsto p\sharp 1 + I$, $\tilde{h} : p \mapsto 1\sharp p + I$. One can easily check that these mappings satisfy relations (5)–(8), hence there exists a homomorphism $\phi : U(P) \to (P \sharp k\langle P\rangle)/I$ such that $\phi(m(p)) = \tilde{m}(p)$, $\phi(h(p)) = \tilde{h}(p)$ for any $p \in P$. This implies clearly that $U(P) \cong (P \sharp k\langle P\rangle)/I$. □

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Let L be an anticommutative algebra. The anticommutative product $[,]$ on L can be extended via Leibniz identity (1) toan anticommutative bracket $\{,\}$ on the (associative and commutative) symmetric algebra $k[L]$. In this way, the algebra $k[L]$ is endowed with the structure of a GP-algebra. If $e_1, e_2, \ldots, e_k, \ldots$ is a linear basis of L, then $k[L]$ is the polynomial algebra $k[e_1, e_2, \ldots, e_k, \ldots]$ on the variables $e_1, e_2, \ldots, e_k, \ldots$.

Theorem 2.5. *Let L be an anticommutative algebra and* $k[L]$ *be the symmetric algebra over L equipped with the GP-algebra structure. Then*

$U(k[L]) \cong k[L] \sharp k\langle L\rangle,$

where $k\langle L\rangle$ *is the free associative algebra over the space L with the canonical Hopf algebra structure.*

Proof. Consider the linear mapping $\tau : L \to Der\, k[L]$, $\tau(a)(x) = \{a, x\}$. As above, we may construct the smash Hopfproduct $k[L] \sharp k\langle L\rangle$, and a homomorphism $\psi : k[L] \sharp k\langle L\rangle \to U(k[L])$ such that $\psi : a \sharp l \mapsto m_a h_l$ for any $a \in k[L]$, $l \in L$.

On the other hand, consider the mapping $h : L \to k[L] \sharp k\langle L\rangle$, $h : l \mapsto 1 \sharp l$ for $l \in L$. By the propertiesof the symmetric algebra $k[L]$, it is extended uniquely to a derivation H of the algebra $k[L]$ into the left $k[L]$-module $k[L] \sharp k\langle L\rangle$. Let $M : a \mapsto a \sharp 1$, then the pair of mappings $(M, H) : k[L] \to k[L] \sharp k\langle L\rangle$ evidently satisfies identities (2), (3). To prove that it satisfies (4), consider, for any $a \in k[L]$, the mapping $T_a : k[L] \to k[L] \sharp k\langle L\rangle$, $T_a : b \mapsto M_{\{b,a\}} - [H_b, M_a]$. It is

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easy to see that T_a is a derivation of the algebra $k[L]$ into the left $k[L]$-module $k[L] \sharp k\langle L\rangle$. For any $l \in L,\ T_a(l) = 0$, which implies that $T_a = 0$ and the pair (M, H) satisfies (4). By Proposition 2.1, there exists a homomorphism $\varphi : U(k[L]) \to k[L] \sharp k\langle L\rangle$ such that $\varphi(m_a) = a \sharp 1,\ \varphi(h_l) = 1 \sharp l$ for any $a \in k[L],\ l \in L$. One can easily see that the homomorphisms ψ, φ are mutually inverse. □

Corollary 2.6. *Let L be an anticommutative algebra with a base $e_1, \ldots, e_n, \ldots$. The subalgebra A of $U(k[L])$ generated by the elements $h_{e_1}, \ldots, h_{e_n}, \ldots$ is a free associative algebra in variables $h_{e_1}, \ldots, h_{e_n}, \ldots$. The left $k[L]$-module $U(k[L])$ is isomorphic to the left commutative $k[L]$-module $k[L] \otimes A$.*

If L is a Lie algebra then the GP-algebra $k[L]$ is in fact a Lie-Poisson algebra (a P-algebra). Denote in this case by $U_{Lie}(k[L])$ its universal multiplicative enveloping algebra as a P-algebra considered in [16, 17, 30]. Similar to the proof of the theorem, we get the following

Remark 2.7. Let L be a Lie algebra. Then $U_{Lie}(k[L]) \cong k[L] \sharp U(L)$ where $U(L)$ is the universal enveloping associative algebra of the Lie algebra L. In particular, for a free P-algebra $P\langle X\rangle$ we have $U_{Lie}(P\langle X\rangle) \cong P\langle X\rangle \sharp k\langle X\rangle$.

Let P be a GP-algebra and $S \subset P$ be a multiplicative subset of P. The bracket $\{.,.\}$ can be extended to the algebra of fractions $S^{-1}P$ via

$$\{s^{-1}a, t^{-1}b\} = (st)^{-2}(\{a, b\}st - \{a, t\}sb - \{s, b\}at + \{s, t\}ab),\ s, t \in S,\ a, b \in P.$$

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In this way, $S^{-1}P$ also has a structure of a GP-algebra. In particular, the field $k(L)$ of rational functions on an anticommutative algebra L may be considered as a GP-algebra. As above, we may construct the smash Hopf product $k(L) \sharp k\langle L\rangle$ via the linear mapping $i : L \to Der\, k(L),\ i(l)(a) = \{a, l\}$.

Theorem 2.8. $U(k(L)) \cong k(L) \sharp k\langle L\rangle$.

Proof. By Lemma 2.3 again, there exists a homomorphism $\psi : k(L) \sharp k\langle L\rangle \to U(k(L))$ such that $\psi : a \sharp l \mapsto m_a h_l$ for any $a \in k(L)$, $l \in L$. On the other hand, the mapping $l \mapsto 1 \sharp l,\ l \in L$, is extended uniquely first to a derivation of the algebra $k[L]$ and then to a derivation H of the field $k(L)$ into the left vector $k(L)$-space $k(L) \sharp k\langle L\rangle$. Let $M : a \mapsto a \sharp 1$, then, as in the proof of Theorem 2.5, one can verify that the pair of mappings $(M, H) : k(L) \to k(L) \sharp k\langle L\rangle$ satisfies identities (2) - (4). By Proposition 2.1 again, there exists a homomorphism $\varphi : U(k(L)) \to k(L) \sharp k\langle L\rangle$ such that $\varphi(m_a) = a \sharp 1$, $\varphi(h_l) = 1 \sharp l$ for any $a \in k(L)$, $l \in L$. One can easily see that the homomorphisms ψ, φ are mutually inversible. □

Corollary 2.9. *Let L be an anticommutative algebra. Then $U(k[L])$ as a left $k[L]$-module is isomorphic to $k[L] \otimes_k k\langle L\rangle$. Similarly, $U(k(L)) \cong k(L) \otimes_k k\langle L\rangle$ as a left $k(L)$-vector space.*

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3. FREE *GP*-ALGEBRAS AND FREE *GP*-FIELDS

As usually, we call a *GP*-algebra P on a set of generators X the free *GP*-algebra with the set of free generators X, if for any *GP*-algebra A, any mapping ϕ of X into A is uniquely extended to a *GP*-algebra homomorphism $\phi^* : P \to A$. From now on, $AC\langle X\rangle$ and $GP\langle X\rangle$ will denote the free anticommutative algebra and the free *GP*-algebra on the set of free generators X over k.

Lemma 3.1. $GP\langle X\rangle \cong k[AC\langle X\rangle]$.

Proof. Let A be a *GP*-algebra, and let $\phi : X \to A$ be a mapping. By the universal property of free algebras, ϕ extends uniquely to an anticommutative algebra homomorphism $\phi_1 : AC\langle X\rangle \to \langle A, +, \{,\}\rangle$. Similarly, by the universal property of symmetric algebras, a linear mapping ϕ_1 extends uniquely to a homomorphism of associative algebras $\phi^* : k[AC\langle X\rangle] \to A$. Finally, the Leibniz identity readily implies that ϕ^* preserve brackets. The lemma is proved. □

The corresponding field of fractions $k(AC\langle X\rangle)$ we will call *the free GP-field* on the set of free generators X and will denote as $QGP\langle X\rangle$. Theorems 2.5, 2.8 imply

Corollary 3.2. $U(GP\langle X\rangle) \cong GP\langle X\rangle \sharp\, k\langle AC\langle X\rangle\rangle,\ U(QGP\langle X\rangle) \cong QGP\langle X\rangle \sharp\, k\langle AC\langle X\rangle\rangle$

Let $U = A \sharp\, k\langle V\rangle$ be, as above, a smash Hopf product induced by a linear mapping $V \to Der\, A$. We know that $k\langle V\rangle = \oplus_{i\geq 0} V^{\otimes i}$, hence $U = \oplus_{i\geq 0} A \sharp\, V^{\otimes i}$. Denote $U_n = \oplus_{i=0}^{n} A \sharp\, V^{\otimes i}$, then one can

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easily check that

$$A \sharp 1 = U_0 \subset U_1 \subset \cdots \subset U_n \subset \cdots \tag{14}$$

is a filtration on U, that is, $U_i U_j \subseteq U_{i+j}$ for all $i, j \geq 0$ and $\cup_{i \geq 0} U_i = U$. Put

$$\mathrm{gr}\, U = \mathrm{gr}\, U_0 \oplus \mathrm{gr}\, U_1 \oplus \cdots \oplus \mathrm{gr}\, U_n \oplus \cdots,$$

where $\mathrm{gr}\, U_0 = A \sharp 1$ and $\mathrm{gr}\, U_i = U_i / U_{i-1}$ for $i \geq 1$. The multiplication on U induces a multiplication on $\mathrm{gr}\, U$ and the graded vector space $\mathrm{gr}\, U$ becomes a graded algebra.

Proposition 3.3. *The graded algebra* $\mathrm{gr}\, U$ *is isomorphic to* $A \otimes k\langle V \rangle$.

Proof. Every element $a \in \mathrm{gr}\, U_k$ may be written in the form $a = \sum_i a_i \sharp v_i + U_{k-1}$, for some $a_i \in A$, $v_i \in V^{\otimes k}$. Define $\varphi_k(a) = \sum_i a_i \otimes v_i$, then it is easy to see that the mapping $\varphi_k : \mathrm{gr}\, U_k \to A \otimes V^{\otimes k}$ is correctly defined and is an isomorphism of left A-modules. Let $\varphi = \{\varphi_k\}_{k \geq 0} : \mathrm{gr}\, U \to A \otimes k\langle V \rangle$, $\varphi(u) = \varphi_k(u)$ if $u \in \mathrm{gr}\, U_k$; then evidently φ is an isomorphism of A-modules. Furthermore, for any $a, b \in A$, $v \in V^{\otimes i}$, $u \in V^{\otimes j}$, consider the product

$$(a \sharp v)(b \sharp u) = \sum_{(v)} a v_{(1)}(b) \sharp v_{(2)} u = ab \sharp vu + \sum_{v_{(2)} \in \sum_{k<i} V^{\otimes k}} a v_{(1)}(b) \sharp v_{(2)} u \in ab \sharp vu + U_{i+j-1}.$$

This proves that $\varphi((a \sharp v + U_{i-1})(b \sharp u + U_{j-1})) = ab \otimes vu = (a \otimes v)(b \otimes u) = \varphi(a \sharp v + U_{i-1}) \varphi(b \sharp u + U_{j-1})$. Therefore, φ is an algebra isomorphism. □

Associate with filtration (14) the corresponding value function $v : (U \backslash \{0\}) \to \mathbf{N}$; $v(u) = n$ if $u \in U_n \backslash U_{n-1}$ (see [4]).

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Theorem 3.4. *Let A be a field, then the above smash product $U = A \sharp k\langle V\rangle$ satisfies the weak algorithm for the value function v and it is a free ideal ring.*

Proof. We consider only the left dependence in U since the right dependence can be treated similarly. We have to prove that for any finite set of left U-dependent elements $s_1, s_2, \ldots, s_k$ of U with $v(s_1) \leq v(s_2) \leq \cdots \leq v(s_k)$ there exist $i \in \{1, \cdots, k\}$ and $t_1, \ldots, t_{i-1} \in U$ such that $v(s_i - t_1 s_1 - \cdots - t_{i-1} s_{i-1}) < v(s_i)$ and $v(t_j) + v(s_j) \leq v(s_i)$ for all $j \in \{1, \ldots, i-1\}$ (see [4]).

For an element $u \in U$ with $v(u) = n$ we denote by $\bar{u} = u + U_{n-1}$ its projection in $\operatorname{gr} U_n = U_n/U_{n-1}$.

Assume that $\sum_{r=1}^{k} u_r s_r = 0$. Put $m = \max\{v(u_r) + v(s_r) \mid 1 \leq r \leq k\}$. Let $r_1, \ldots, r_l$ be the set of indices r with $v(u_r) + v(s_r) = m$. Consider

$$\sum_{j=1}^{l} \bar{u}_{r_j} \bar{s}_{r_j} = \left(\sum_{j=1}^{l} u_{r_j} s_{r_j}\right) + U_{m-1} = \left(\sum_{r=1}^{k} u_r s_r\right) + U_{m-1} = \bar{0}.$$

This equality holds in the algebra $\operatorname{gr} U \cong A \otimes k\langle V\rangle \cong A\langle V\rangle$ which is a free associative A-algebra (a tensor algebra) over the space of V. It gives a nontrivial left dependence of the homogeneous elements $\bar{s}_{r_1}, \bar{s}_{r_2}, \ldots, \bar{s}_{r_l}$. The free algebra $A\langle V\rangle$ has a natural degree function deg with respect to the generating space V and satisfies the weak algorithm for it [4]. Therefore, there exist i and elements $t_{r_1}, \cdots, t_{r_{i-1}} \in U$ such that

$\bar{s}_{r_i} = \bar{t}_{r_1} \bar{s}_{r_1} + \cdots + \bar{t}_{r_{i-1}} \bar{s}_{r_{i-1}}$

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and $\deg(\bar{t}_{r_j}) + \deg(\bar{s}_{r_j}) \leq \deg(\bar{s}_{r_i})$ for all $j \in \{1, \ldots, i-1\}$. Observe that $\deg(\bar{u}) = v(u)$ for any $u \in U$. Therefore, we have

$v(s_{r_i} - t_{r_1} s_{r_1} - \cdots - t_{r_{i-1}} s_{r_{i-1}}) < v(s_{r_i})$

and $v(t_{r_j}) + v(s_{r_j}) \leq v(s_{r_i})$ for all $j < i$. □

The proof of the theorem is constructive. Hence the standard algorithms (see, for example [29]) give the next result.

Corollary 3.5. *Let A be a field and $U = A \sharp k\langle V \rangle$ the above smash product. Then*

(i) *The left ideal membership problem for U is algorithmically decidable;*

(ii) *The left dependence of a finite system of elements of U is algorithmically recognizable.*

Taking in account Corollary 3.2, we have

Corollary 3.6.

(i) *The two problems from the previous corollary have positive solution in the universal multiplicative enveloping algebra $U(QGP\langle X \rangle)$ of the free GP-field $QGP\langle X \rangle$;*

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(ii) *The left dependence of a finite system of elements is also algorithmically recognizable in the universal multiplicative enveloping algebra $U(GP\langle X\rangle)$ of the free GP-algebra $U(GP\langle X\rangle)$.*

Proof. It suffices only to prove (ii). Since $U(GP\langle X\rangle) \subset U(QGP\langle X\rangle)$, for any finite system $s_1, \ldots, s_k \in U(GP\langle X\rangle)$ we may recognize whether it is left dependent in $U(QGP\langle X\rangle)$. If it is left independent in $U(QGP\langle X\rangle)$ then clearly it is so in $U(GP\langle X\rangle)$. Assume now that the system $s_1, \ldots, s_k$ is left dependent in $U(QGP\langle X\rangle)$; then there exist $u_1, \ldots, u_k \in U(QGP\langle X\rangle)$ such that $\sum_{i=1}^{k} u_i s_i = 0$. The algebras $U(QGP\langle X\rangle)$ and $U(GP\langle X\rangle)$ are isomorphic, as left $GP\langle X\rangle$-modules, to $QGP\langle X\rangle \otimes k\langle AC\langle X\rangle\rangle$ and $GP\langle X\rangle \otimes k\langle AC\langle X\rangle\rangle$, respectively. Therefore, there exist $0 \neq g \in GP\langle X\rangle$ such that $gu_i \in U(GP\langle X\rangle)$ for all $i = 1, \ldots, k$, and we have $0 = g(\sum_{i=1}^{k} u_i s_i) = \sum_{i=1}^{k} (gu_i) s_i$, which gives left dependence of $s_1, \ldots, s_k$ over $U(GP\langle X\rangle)$. □

4. POISSON DEPENDENCE OF TWO ELEMENTS

Let S be an arbitrary GP-algebra over k. Elements $a_1, a_2, \ldots, a_m$ of S are called *Poisson dependent* if there exists a non-zero element $p(x_1, x_2, \ldots, x_m)$ in a free GP-algebra $GP\langle x_1, x_2, \ldots, x_m\rangle$ such that $p(a_1, a_2, \ldots, a_m) = 0$ in S; otherwise $a_1, a_2, \ldots, a_m$ are called *Poisson free* or *Poisson independent*. If $a_1, a_2, \ldots, a_m$ are Poisson free then the Poisson subalgebra of S generated by these elements is a free GP-algebra in these variables.

Similarly, elements $a_1, a_2, \ldots, a_m$ are called *polynomially dependent* if there exist a non-zero polynomial $f(x_1, x_2, \ldots, x_m) \in k[x_1, x_2, \ldots, x_m]$ such that $f(a_1, a_2, \ldots, a_m) = 0$.

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It was proved in [11] that in a free Poisson field any two Poisson dependent elements are polynomially dependent. In this section we will prove that the same fact is true in a free *GP*-field.

A family of polynomial weight degree functions can be defined on $k[y_1, y_2, \ldots]$ by giving arbitrary real weight $w_i = w(y_i)$ to the generators and extending it on monomials $M = y_1^{j_1} y_2^{j_2} \ldots$ by $w(M) = \sum_i j_i w(y_i)$. Then for $f \in k[y_1, y_2, \ldots]$ degree can be defined as $D(f) = \max\{w(M) | M \in f\}$, i.e. maximum by all monomials contained in f with non-zero coefficients. Of course not all of these functions make sense for *GP* as a generic Poisson algebra. We say that a weight degree function D on *GP* is *compatible with the generic Poisson structures* if it satisfies the following natural condition:

for any two monomials $M_1, M_2 \in GP$ (as a polynomial algebra) the bracket $\{M_1, M_2\}$ is D-homogeneous.

For example, the weight which is defined on an anticommutative monomial y as the number of appearances of a free generator x_k in y defines a compatible degree function d_{x_k}. It is easy to check (see [11]) that in order to define a compatible degree function the weight should be given on an anticommutative monomial by

$$w(y) = \sum (w(x_i) - c) d_{x_i}(y) + c,$$

where $w(x_i)$ and c are arbitrary real numbers.

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Examples of compatible degree function are d_{x_k} defined above and the Poisson degree which corresponds to $w(x_1) = \ldots = w(x_n) = 1$, $c = 0$. Total polynomial degree deg is also compatible and corresponds to $w(x_1) = \ldots = w(x_n) = c = 1$.

Observe that $\deg(\{f, g\}) = \deg(f) + \deg(g) - 1$ for homogeneous f and g if $\{f, g\} \neq 0$. Similar relation is true for any compatible weight degree function:

$D(\{f, g\}) = D(f) + D(g) - c$

if $\{f, g\} \neq 0$ and f and g are D-homogeneous.

We will consider only weight functions for which all parameters are integers.

Below $\mathcal{P}$ and $\mathcal{Q}$ denote the free GP-algebra $GP\langle X\rangle$ and the free GP-field $QGP\langle X\rangle$ on a set of free generators $X = \{x_1, x_2 \ldots\}$.

Lemma 4.1. *If $f, g \in \mathcal{Q}$ are Poisson dependent and $r_1(x_1, x_2)$, $r_2(x_1, x_2) \in k(x_1, x_2)$ are rational functions then $r_1(f, g)$, $r_2(f, g) \in \mathcal{Q}$ are also Poisson dependent.*

Proof. We modify the proof for free Poisson algebras in [11, Lemma 1]. Elements f, g are Poisson dependent if the basic anticommutative monomials of f, g are algebraically dependent. Denote by $y_1, \ldots, y_{N(a)}$ the set of all basic anticommutative monomials in two variables with $d(y_j) \leq a$. Consider the smallest A for which $y_1(f, g), y_2(f, g), \ldots, y_{N(A)}(f, g)$

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are algebraically dependent. It is easy to check using induction on $a_i = d(y_i(x_1, x_2))$ that $y_i(r_1(f,g), r_2(f,g)) \in k(f,g)[y_3(f,g), \dots, y_{N(a_i)}(f,g)]$. Hence there is an algebraic dependence between $y_1(r_1(f,g), r_2(f,g)), \dots, y_{N(A)}(r_1(f,g), r_2(f,g))$. □

For $f \in \mathcal{Q}$ denote by supp (f) the minimal set of polynomial variables on which f depends.

Lemma 4.2 ([11, Lemma 2]). *Let $f, g \in \mathcal{Q}$ be elements which are algebraically independent. Then for a given polynomial weight degree function D there exists an element $h \in k[f, g]$ such that the leading form f_D, h_D are algebraically independent.*

Lemma 4.3 ([11, Lemma 3]). *Let $f, g \in \mathcal{Q}$ be elements which are Poisson dependent but not algebraically dependent. Then there exists a pair of elements which are homogeneous relative to any compatible weight degree function D with the same property.*

We call elements which are homogeneous relative to all compatible degree functions *completely homogeneous*.

Lemma 4.4. *Let $f, g \in \mathcal{Q}$ be a Poisson dependent pair and x be the smallest element in supp (f). Write $f = x^n f_x + \cdots$, $g = x^m g_x + \cdots$, where f_x, g_x do not contain x and dots stand for terms with smaller (polynomial) degrees in x. Then the pair f_x, g_x is Poisson dependent as well.*

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Proof. Consider the Poisson polynomial $P(x_1, x_2)$ for which $P(f, g) = 0$; it is a sum of monomials of the type

$$u = y_1^{k_1} y_2^{k_2} \dots y_s^{k_s},$$

where y_i are anticommutative monomials in x_1, x_2. We have

$$\{f, g\} = x^{n+m}\{f_x, g_x\} + \cdots ,$$

$$y_i(f, g) = x^{N_i} y_i(f_x, g_x) + \cdots , N_i = nd_{x_1}(y_i) + md_{x_2}(y_i),$$

$$u(f, g) = x^{N(u)} u(f_x, g_x) + \cdots , N(u) = \sum k_i N_i,$$

where again dots means terms of smaller degree in x. Observe that x cannot appear in $\{f_x, g_x\}$ or in $y_i(f_x, g_x)$ when $d(y_i) > 1$ since for any $y \in \text{supp}\,(f), z \in \text{supp}\,(g)$ we have $\{y, z\} > y \geq x$. Therefore

$$0 = P(f, g) = Q(f_x, g_x)x^N + \cdots ,$$

where $N = \max\{N(u)|\ u$ monomial in $P(x_1, x_2)\}$, $Q(x_1, x_2) = \sum_{N(u)=N} u(x_1, x_2)$. Since all monomials u in $P(x_1, x_2)$ are linearly independent, we have $Q(x_1, x_2) \neq 0$ and hence f_x, g_x are Poisson dependent. □

Lemma 4.5. *In the conditions of Lemma 4.4, assume that* $f_x = 1, f = x^n + \alpha x^{n-1} + \cdots$, $\deg(g_x) \neq 0$. *Then the pair* $nx + \alpha, g_x$ *is Poisson dependent.*

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Proof. We modify the proof for free Poisson algebras in [11, Lemma 5]. Let us check by induction on the Poisson degree that

$$y_i(f,g) = x^{N_i} y_i(nx+\alpha, g_x) + \cdots$$

for any anticommutative monomial $y_i(x_1,x_2) \neq x_1$, where $N_i = (n-1)d_{x_1}(y_i) + md_{x_2}(y_i)$ and dots stand for the terms of smaller degree in x (recall that $deg_x(g) = m$). The base of induction for $y_2(f,g) = g$ is clear. An anticommutative monomial $y_k(f,g)$ with $k > 2$ can be presented as $\{y_l(f,g), y_j(f,g)\}$, where y_l, y_j are monomials with a smaller Poisson degree and $l < j$. If $k = 3$, then $l = 1, j = 2$ and

$$\begin{aligned} y_3(f,g) &= \{f,g\} = nx^{n-1+m}\{x,g_x\} + x^{n-1+m}\{\alpha,g_x\} + \cdots = x^{n-1+m}\{nx+\alpha,g_x\} + \cdots \\ &= x^{n-1+m}y_3(nx+\alpha,g_x) + \cdots . \end{aligned}$$

If $k > 3$ then by induction

$$y_j(f,g) = x^{N_j} y_j(nx+\alpha, g_x) + \cdots$$

and either $l = 1$ and $y_l(f,g) = f$, or $l > 1$ and

$$y_l(f,g) = x^{N_l} y_l(nx+\alpha, g_x) + \cdots .$$

In both cases, the similar computations verify the claim. It is essential that x is the smallest element in supp (f) because supp $(\{y,z\})$ does not contain x if $y \geq x$ and no additional powers of x may appear as results of Poisson brackets.

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Therefore, for $u = y_1^{k_1} y_2^{k_2} \dots y_s^{k_s}$ the leading form of $u(f, g)$ relative to x is

$$x^{nk_1+N_u} y_2(nx+\alpha, g_x)^{k_2} \dots y_s(nx+\alpha, g_x)^{k_s},$$

where

$$N_u = (n-1)d_{x_1}(y_2^{k_2} \dots y_s^{k_s}) + md_{x_2}(y_2^{k_2} \dots y_s^{k_s}).$$

Hence different monomials of $P(f, g)$ cannot cancel in the x-leading form of $P(f, g)$ and the elements $nx+\alpha, g_x$ are Poisson dependent. □

Consider now a pair of algebraically independent elements $f, g \in \mathcal{Q}$. By Shirshov Theorem a subalgebra of a free anticommutative algebra is a free anticommutative algebra (see [27, 28]) so the elements of supp $(f) \cup$ supp (g) generate a free anticommutative algebra $\mathcal{A}$ with the free basis which contains two smallest elements x, y of supp $(f) \cup$ supp (g). Elements x and y are different since otherwise supp $(f) \cup$supp $(g) = x$ and f, g are algebraically dependent. If $\mathcal{P}$ is the free generic Poisson algebra which correspond to $\mathcal{A}$ and $\mathcal{Q}$ is the field of fraction of $\mathcal{P}$ then $f, g \in \mathcal{Q}$. Though f, g are possibly written through different generators, the size of supp $(f) \cup$supp (g) did not change.

Assume that there exist a pair of algebraically independent Poisson dependent elements in a free generic Poisson field $\mathcal{Q}$. Then we can find a pair which is minimal in the following sense: the size $|f, g|$ of supp $(f) \cup$ supp (g) is minimal possible, Q is generated by supp $(f) \cup$ supp (g), elements f and g are completely homogeneous.

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As we observed $|f, g|$ does not change when we replace the original generic Poisson field with the minimal one. The elements may stop being completely homogeneous but by Lemma 4.3 we can produce a completely homogeneous pair which belongs to $k[f, g]$, hence the union of supports of these two elements belongs to the union of supports of the original elements. Since $|f, g|$ is minimal it implies that the size cannot become smaller, so the union of supports of a completely homogeneous pair is the same as for the original pair.

Recall (see [2]) that if two homogeneous polynomials $f, g \in k[X]$ are algebraically dependent then there exist a homogeneous polynomial $h \in k[X]$ such that $f = \alpha h^k, g = \beta h^l$ for some $\alpha, \beta \in k$ and natural numbers k, l. Similar statement is true for two algebraically dependent homogeneous rational functions $f, g \in k(X)$ if one of them, say f, has a non-zero degree (see [11]).

Lemma 4.6. *Let $f, g \in \mathcal{Q}$ be a minimal pair. If x is the smallest element in supp $(f) \cup$ supp (g), then there exists a minimal pair $\widetilde{f} = x + f_1, \widetilde{g}$ where $x \notin$ supp $(f_1) \cup$ supp $(\widetilde{g})$.*

Proof. Write $f = x^n f_x + \cdots, g = x^m g_x + \cdots$, where f_x, g_x do not contain x and dots stand for terms with smaller (polynomial) degrees in x. Then the pair f_x, g_x is Poisson dependent by Lemma 4.4 and is algebraically dependent since $|f_x, g_x| < |f, g|$. If $D(f_x) = 0$ for any compatible degree function consider the second smallest element $y \in$ supp $(f) \cup$ supp (g) and present $f = y^{n_1} f_y + \cdots$ where f_y does not contain y and dots stand for terms with smaller degrees in y. If $D(f_y) = 0$ for any compatible degree function then $D(x^n) = D(y^{n_1})$ for any compatible degree. But x, y are elements of a free basis, so the Poisson degrees d_x and d_y are compatible degree functions and either $x = y$

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which is impossible or $n = n_1 = 0$. Similar considerations for g show that either $D(g_x)$ or $D(g_y)$ is not identically zero or $m = m_1 = 0$. If $n = m = 0$ consider polynomial dependence q between f_x, g_x and a minimal pair $f, g_1 = q(f, g)$. Then $g_1 = x^k g_{1x} + \cdots$ where $k < 0$. So either $D(g_{1x})$ or $D(g_{1y})$ is not identically zero. Since x, y are elements of a free basis we can reorder them as well as f, g_1 and assume that $D(f_x) \neq 0$ for some compatible degree function. Then by above there exist a completely homogeneous element $h \in \mathcal{Q}$ such that $f_x = c_1 h^a, g_x = c_2 h^b$ where $c_1, c_2 \in k\backslash\{0\}, D(h) \neq 0$ for some compatible degree function, and $a \neq 0$. Without loss of generality we may assume that $c_1 = c_2 = 1$. Hence $f = x^n h^a + \cdots$, $g = x^m h^b + \cdots$. The pair $f^b g^{-a}, f \in \mathcal{Q}$ is Poisson dependent by Lemma 4.1. We can write $f^b g^{-a} = x^{bn-am} + \alpha x^{bn-am-1} + \cdots$. Hence by Lemma 4.5 the pair $(bn - am)x + \alpha, h^a$ is Poisson dependent. Recall that $x \notin \operatorname{supp}(\alpha) \cup \operatorname{supp}(h)$. Therefore $(bn - am)x + \alpha$ and h are algebraically independent if $bn - am \neq 0$. So if $bn - am \neq 0$ we proved the lemma.

If $bn - am = 0$ then algebraically independent rational functions f, g have algebraically dependent leading forms relative to polynomial $\deg_x$. According to lemma 4.2 ring $k[f, g]$ contains an element g' such that $\deg_x$-leading forms of f and g' are algebraically independent. Since $\operatorname{supp}(g') \subset \operatorname{supp}(f) \cup \operatorname{supp}(g)$ the pair f, g' is minimal and we can use it to prove the lemma. $\square$

Theorem 4.7. *Every two Poisson dependent elements in free generic Poisson field $\mathcal{Q}$ are algebraically dependent.*

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Proof. Assume that the theorem is not true. Then by the previous lemmas there exist a completely homogeneous Poisson dependent algebraically independent pair $f = x + f_1$, $g \in \mathcal{Q}$, where the size $|f, g|$ is minimal possible, x is the minimal element in $\mathrm{supp}\,(f) \cup \mathrm{supp}\,(g)$ and $x \notin \mathrm{supp}\,(f_1) \cup \mathrm{supp}\,(g)$, and x is an element of the basis of $\mathcal{P}$.

Consider the smallest element $y \in \mathrm{supp}\,(g)$ and write $f = y^n f_y + \cdots$, $g = y^m g_y + \cdots$ where $y \notin \mathrm{supp}\,(f_y) \cup \mathrm{supp}\,(g_y)$. Elements f_y and g_y should be Poisson dependent by Lemma 4.4 and algebraically dependent since $|f_y, g_y| < |f, g|$.

If $n = 0$ then $f_y = x + f_{1y}$ and g_y are algebraically dependent and $d_x(f_y) = 1$. Hence $g_y = cf_y^b$ and $b = 0$ since otherwise $x \in \mathrm{supp}\,(g_y)$. If furthemore $m = 0$ then $g = c + \cdots$ where $c \in F$ and we will replace g by $\widetilde{g} = g - c$. Then $\widetilde{g} = y^{\widetilde{m}}\widetilde{g}_y + \cdots$ where $\widetilde{m} < 0$. Furthemore, $f_y = x + f_{1y}$ and $\widetilde{g}_y$ are Poisson and algebraically dependent, which as above is possible only if $\widetilde{g}_y \in k$. Since y is an anticommutative monomial and $y \neq x$, there exist an element $z \neq x$ in the free basis for which $d_z(y) \neq 0$ and $d_z(\widetilde{g}) = \widetilde{m}d_z(y) \neq 0$. But $D(\widetilde{g}) = 0$ for any compatible weight degree function (since g is completely homogeneous and $D(\widetilde{g}) = D(g) = D(c) = 0$). Therefore $\widetilde{g}_y \notin k$, $\widetilde{g}_y = cf_y^b$ where b is a non-zero integer, and $x \in \mathrm{supp}\,(\widetilde{g}_y)$, a contradiction. We can conclude that $m \neq 0$ and that f_y, $my + \beta$ are Poisson dependent by Lemma 4.5. Since $f_y = x + f_{1y}$ where all elements of $\mathrm{supp}\,(f_{1y})$ are larger than x and all elements of $\mathrm{supp}\,(\beta)$ are larger than y, we can see that $y_i(x + f_{1y}, y + m^{-1}\beta) = y_i(x, y) + \cdots$ for any anticommutative monomials y_i where dots stand for anticommutative monomials larger than $y_i(x, y)$. Hence these elements are Poisson independent and $n = 0$ is impossible.

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The condition $n \neq 0$ implies that $x \notin \text{supp}\,(f_y)$; otherwise $1 = f_x = y^n g_1$ where $y \notin \text{supp}\,(g_1)$ which is impossible. Since x is an element of the free basis, it follows from the complete homogeneity that $0 = d_z(x) = d_z(y^n f_y)$. Therefore $d_z(f_y) = -nd_z(y) \neq 0$ and $f_y \notin k$. Elements f_y, g_y are algebraically dependent, hence $f_y = c_1 h^a, g_y = c_2 h^b$ for some element h where $a \neq 0$ and we may assume that $c_1 = c_2 = 1$.

If $b = 0$ and $m \neq 0$ then $g = y^m + \cdots$ and by Lemma 4.5 f_y and $my + \beta$ are Poisson dependent. They are algebraically independent since $y \notin \text{supp}\,(f_y),\ y \in \text{supp}\,(my + \beta)$. But $x \notin \text{supp}\,(f_y) \cup \text{supp}\,(g_y)$, and we have a contradiction with the minimality of the pair f, g. If $m = 0$, consider $\widetilde{g} = g - 1$. Then $\widetilde{g} = y^{\widetilde{m}}\widetilde{g}_y + \cdots$ where $\widetilde{m} \neq 0$ and $\widetilde{g}_y = h^{\widetilde{b}}$ because $f_y = h^a$ where $a \neq 0$. Now,

$$
\begin{aligned}
d_x(f) &= d_x(x) = 1 = d_x(y^n f_y) = d_x(y^n h^a) = nd_x(y) + ad_x(h),\\
d_z(f) &= d_z(x) = 0 = d_z(y^n f_y) = d_z(y^n h^a) = nd_z(y) + ad_z(h),\\
d_x(\widetilde{g}) &= d_x(1) = 0 = d_x(y^{\widetilde{m}} h^{\widetilde{b}}) = \widetilde{m} d_x(y) + \widetilde{b} d_x(h),\\
d_z(\widetilde{g}) &= d_z(1) = 0 = d_z(y^{\widetilde{m}} h^{\widetilde{b}}) = \widetilde{m} d_z(y) + \widetilde{b} d_z(h).
\end{aligned}
$$

Since $(\widetilde{m}, \widetilde{b}) \neq (0,0)$, we have $(n, a) = \lambda(\widetilde{m}, \widetilde{b})$ for some $\lambda \in k$ and then $nd_x(y) + ad_x(h) = 0$, a contradiction. Therefore $b \neq 0$.

Replace now g by $\widetilde{g} = g^{-a} f^b$. Then $\widetilde{g} = y^k + y^{k-1}\widetilde{g}_1 + \cdots$. The case $k = 0$ could be brought to a contradiction just as the case $b = m = 0$ above. Therefore $k \neq 0$.

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Elements $ky+\widetilde{g}_1, f_y$ are algebraically independent since $y \notin \text{supp}\,(f_y)$ and $k \neq 0$. Since supp $(ky+\widetilde{g}_1)\cup\text{supp}\,(f_y) \subseteq \text{supp}\,(f)\cup\text{supp}\,(g)$, we should have supp $(ky+\widetilde{g}_1)\cup\text{supp}\,(f_y) = \text{supp}\,(f)\cup\text{supp}\,(g)$ by the minimality condition, and thus $x \in \text{supp}\,(\widetilde{g}_1)$. Recall that $\widetilde{g} = g^{-a}f^b$ and therefore $x \in \text{supp}\,(\widetilde{g}_1)$ only if $n = 1$, i.e. if $f = yh^a + (x+\delta) + \cdots$ where dots stand for the terms with negative powers in y. Hence

$$\widetilde{g} = \frac{(yh^a + (x+\delta) + \cdots)^b}{(y^m h^b + \epsilon y^{m-1} + \cdots)^a} = y^k + [b(x+\delta)h^{-a} - a\epsilon h^{-b}]y^{k-1} + \cdots$$

and $\widetilde{g}_1 = b(x+\delta)h^{-a} - a\epsilon h^{-b}$.

The elements $(ky+\widetilde{g}_1)f_y, f_y$ are Poisson dependent by Lemma 4.1. Hence

$$[ky + b(x+\delta)h^{-a} - a\epsilon h^{-b}]h^a = b(x+\delta) - a\epsilon h^{a-b} + kyh^a$$

and h^a are Poisson dependent.

It is clear that supp (h^a) is a proper subset of supp (g). So we may apply induction on the size of supp (g) to prove the theorem. The base of induction when $|g| = 1$ corresponds to $g = y^m, m \neq 0$. As we have seen above in order to avoid a contradiction we should have $f = y^n f_y + \cdots$ where $n \neq 0, f_y \notin k$, and $g_y \notin k$, But $g_y = 1$ and we have a contradiction which proves the theorem. □

Corollary 4.8. *Let $f, g \in \mathcal{Q}$, $\{f, g\} \neq 0$. Then f, g generate a free anticommutative algebra with respect to the bracket $\{,\}$, and they generate a free generic Poisson subalgebra in $\mathcal{Q}$ in complete analogy to the case of free associative algebras [3].*

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Proof. It suffices to notice that f, g are algebraically independent. In fact, let $F(x, y) \neq 0$ be a polynomial of minimal degree such that $F(f, g) = 0$. Then $0 = \{F(f, g), g\} = F_f(f, g)\{f, g\}$ and $F_f(f, g) = 0$, a contradiction. □

Remark 4.9. Observe that the theorem is evidently not true for more than two elements: the elements $x_1, x_2, \{x_1, x_2\}$ are Poisson dependent but are algebraically independent. It is not true as well if $char(k) = p > 0$; the elements x_1, x_2^p are algebraically independent but $\{x_1, x_2^p\} = px_2^{p-1}\{x_1, x_2\} = 0$.

5. APPLICATION TO AUTOMORPHISMS

In this section we prove the following analogue of Makar-Limanov - Turusbekova - Umirbaev's Theorem [12]) for ordinary Poisson algebras:

Theorem 5.1. *Automorphisms of the free generic Poisson algebra $GP\langle x, y\rangle$ of rank two over a field k of characteristic 0 are tame.*

Proof. The proof repeats the proof given for ordinary Poisson algebras in [11]. We will give it for the sake of complicity.

Let α be an automorphism of $\mathcal{P}_2 = GP\langle x, y\rangle$. Since any (tame) automorphism of $k[x, y]$ can be lifted to a (tame) automorphism of $\mathcal{P}_2$, we can assume without loss of generality that the abelianization

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of α (that is, its homomorphic image under the natural epimorphism $Aut\,(\mathcal{P}_2) \rightarrow Aut\,(k[x,y])$) is the identity automorphism of $k[x,y]$. It remains to show that then α is the identity automorphism

Let $\alpha(x) = f,\ \alpha(y) = g$. Assume that either $f \neq x$ or $g \neq y$. If we take weights $w(x) = \rho, w(y) = 1$ where $\rho > 0$ then $f = x$ and $g = y$ where f and g are the lowest Poisson forms of f and g with respect to w. If we start now to decrees ρ then for some non-positive value of ρ either $f \neq x$ or $g \neq y$ for the corresponding f and g. Let us take the largest ρ with this property. Then f and g are Poisson w-homogeneous, $d_w(f) = \rho,\ d_w(g) = 1,\ f = x + f_1,\ g = y + g_1$, where at least one of $f_1,\ g_1$ is non-zero and their abelianizations in $k[x,y]$ are both zero. Clearly, f and g are Poisson independent.

Let $x = X(f,g)$ for some Poisson polynomial $X(x_1,x_2)$, then $x = (X(f,g)) = X(f,g)$ since f and g are Poisson independent. Similarly, y belongs to the Poisson subalgebra generated by f and g. Therefore, the w-homogeneous Poisson forms $f,\ g$ generate $\mathcal{P}_2$.

Consider now the Poisson leading forms $\widetilde{(f)}$ and $\widetilde{(g)}$ of f and g with respect to the Poisson degree, when $d(x) = d(y) = 1$. If they were Poisson independent, then as above they would generate $\mathcal{P}_2$. But this is impossible since otherwise their abelianizations, the images under the epimorphism $\mathcal{P}_2 \twoheadrightarrow k[x,y]$, would generate $k[x,y]$, while at least one of them is 0.

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Next we can use Theorem 4.7 and conclude that $\widetilde{(f)}$ and $\widetilde{(g)}$ are algebraically dependent. Therefore up to scalars they are h^a, h^b for a certain Poisson-homogeneous element $h \in \mathcal{P}_2$ and non-negative integers a, b. Then we have $ad_w(h) = \rho$, $bd_w(h) = 1$ where $\rho < 0$, which is impossible. □

From this theorem and the previous results [5, 10, 12], we have

Corollary 5.2. *Let k be a field of characteristic zero. Then*

$$Aut\, GP\{x, y\} \cong Aut\, P\{x, y\} \cong Aut\, k\langle x, y\rangle \cong Aut\, k[x, y],$$

where $k[x, y]$ is the polynomial algebra; $k\langle x, y\rangle$ is the free associative algebra; $P\{x, y\}$ is the free Poisson algebra, and $GP\{x, y\}$ is the free generic Poisson algebra, respectively, on variables x, y.

We will finish with the following open question:

Let $GP(x, y)$ be the free generic Poisson field on two generators. Is it true that $Aut\, GP(x, y) \cong Aut\, P(x, y)$? Observe that by [14], the last group is isomorphic to the Cremona Group $Cr_2(k) = Aut\, k(x, y)$.

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REFERENCES

1. Arenas, M., Arenas-Cartona, L. (2013). Universal Poisson envelope for binary Lie algebras. *Comm. Alg.* 41(5):1781–1789.
2. Bergman, G. M. (1969). Centralizers in free associative algebras. *Trans. Amer. Math. Soc.* 137:327–344.
3. Cohn, P. M. (1964). Subalgebras of free associative algebras. *Proc. London Math. Soc.* 56:618–632.
4. Cohn, P. M. (2006). *Free Ideal Rings and Localization in General Rings*. New Mathematical Monographs, Vol. 3. Cambridge: Cambridge University Press.
5. Czerniakiewicz, A. J. (1971/1972). Automorphisms of a free associative algebra of rank 2. I. *Trans. Amer. Math. Soc.* 160:393–401; II, *Trans. Amer. Math. Soc.* 171:309–315.
6. Eilenberg, S. (1948). Extensions of algebras. *Ann. Soc. Pol. Math.* 21:125–134.
7. Jacobson, N. (1969). *Structure and Representation of Jordan Algebras*. Providence, R.I.: Amer. Math. Soc.
8. Jung, H. W. E. (1942). Uber ganze birationale Transformationen der Ebene. *J. Reine Angew. Math.* 184:161–174.
9. Kolesnikov, P., Makar-Limanov, L., Shestakov, I. (2014). *The Freiheitssatz for Generic Poisson Algebras*. Symmetry, Integrability and Geometry: Methods and Applications, Vol. 10, p. 1–15.

Accepted Manuscript

10. Makar-Limanov, L. (1970). The automorphisms of the free algebra with two generators. *Funct. Anal. Appl.* 4:262–263.

11. Makar-Limanov, L., Shestakov, I. (2012). Polynomial and Poisson dependence in free Poisson algebras and free Poisson fields. *J. Algebra* 349:372–379.

12. Makar-Limanov, L., Turusbekova, U., Umirbaev, U. (2009). Automorphisms and derivations of free Poisson algebras in two variables. *J. Algebra* 322(9):3318–3330.

13. Makar-Limanov, L., Umirbaev, U. (2007). Centralizers in free Poisson algebras. *Proc. Amer. Math. Soc.* 135(7):1969–1975.

14. Makar-Limanov, L., Umirbaev, U. *Free Poisson fields and their automorphisms*, http://arxiv.org/abs/1202.4382

15. Makar-Limanov, L., Umirbaev, U. (2011). The Freiheitssatz for Poisson algebras. *J. Algebra* 328:495–503.

16. Oh, S.-Q. (1999). Poisson enveloping algebras. *Comm. Algebra* 27(5):2181–2186.

17. Oh, S.-Q., Park, C.-G., Shin, Y.-Y. (2002). A Poincare-Birkhoff-Witt theorem for Poisson enveloping algebras. *Comm. Alg.* 30(10):4867–4887.

18. Pérez-Izquierdo, J. M. (2005). An envelope for Bol algebras. *J. Algebra* 284(2):480–493.

19. Pérez-Izquierdo, J. M. (2007). Algebras, hyperalgebras, nonassociative bialgebras and loops. *Adv. Math.* 208(2):834–876.

20. Pérez-Izquierdo, J. M., Shestakov I. P. (2004). An envelope for Malcev algebras. *J. Algebra* 272(1):379–393.

21. Rentschler, R. (1968). Operations du groupe additif sur le plan. *C. R. Acad. Sci. Paris* 267:384–387.

Accepted Manuscript

22. Shestakov, I. P. (1993). Quantization of Poisson superalgebras and the specialty of Jordan superalgebras of Poisson type. *Algebra and Logic* 32(5):309–317.

23. Shestakov, I. (2000). Speciality problem for Malcev algebras and Poisson Malcev algebras. In: *Nonassociative Algebra and Its Applications (São Paulo, 1998)*. Lecture Notes in Pure and Appl. Math., Vol. 211. New York: Dekker, pp. 365–371.

24. Shestakov, I., Umirbaev, U. (2004). Poisson brackets and two-generated subalgebras of rings of polynomials. *J. Amer. Math. Soc.* 17(1):181–196.

25. Shestakov, I., Umirbaev, U. (2004). The tame and the wild automorphisms of polynomial rings in three variables. *J. Amer. Math. Soc.* 17(1):197–227.

26. Shestakov, I., Zhukavets, N. (2006). The Malcev Poisson superalgebra of the free Malcev superalgebra on one odd generator. *J. Algebra Appl.* 5(4):521–535.

27. Shirshov, A. I. (1954). Subalgebras of free commutative and free anticommutative algebras. *Mat. Sbornik N.S.* 34(76):81–88 (Russian).

28. Shirshov, A. I. (2009). *Selected Works of A. I. Shirshov*. In: Bokut, L., Latyshev, V., Shestakov, I., Zelmanov, E., eds. *Contemporary Mathematicians*. Basel: Birkhäuser Verlag, viii+242 pp.

29. Umirbaev, U. U. (1994). On Schreier varieties of algebras. *Algebra and Logic* 33(3):180–193.

30. Umirbaev, U. (2012). Universal enveloping algebras and universal derivations of Poisson algebras. *J. Algebra* 354:77–94.

31. Vershinin, V. (2003). On Poisson-Malcev structures. In: *Monodromy and Differential Equations (Moscow, 2001)*. Acta Appl. Math., Vol. 75, nos. 1–3, pp. 281–292.

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