

Symmetric centers on planar cubic differential systems

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Abstract

The main focus of this paper is the investigation of simultaneous existence of centers in symmetric planar polynomial differential systems. There are presented necessary and sufficient conditions for the existence of a bi-center of four families of real cubic symmetric systems. Furthermore, the conditions for the isochronicity of such bi-centers are obtained.

Keywords: planar polynomial differential systems, bi-centers, global phase portrait, time-reversible systems, equivariant systems

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1. Introduction

Symmetries create patterns that help us organize conceptually our world. Symmetric patterns occur in nature and they could be invented by artists, craftspeople, musicians, choreographers and mathematicians.

When points are moved around the plane not changing position relative to each other, more precisely, preserving distances, angles, sizes and shapes, symmetry is involved. Symmetry is presented as well in differential systems, mostly when thermodynamic, classical and quantum mechanics are involved (see, e.g. [18]). Different symmetries appear in dynamical systems, for instance, the reversibility (see [8]). The symmetry theory can be traced back to Birkhoff [2] whose work realized the existence of an involutive map that is symmetric with respect to a set of fixed points. Studies on dynamics of reversible differential systems can be found in [5, 8, 16, 18, 22].

We say that a differential system

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \quad (1)$$

where $P(x, y), Q(x, y)$ are smooth functions, has a *center* at the singular point $(x_0, y_0) \in \mathbb{R}^2$ if all solutions (orbits) in a neighbourhood of (x_0, y_0) are closed. Moreover the center is said to be an *isochronous center* if all the periodic solutions have the same period. In order to determine the overall behaviour of solutions in the neighbourhood of singular point, whose linearized systems has center,

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it is required to solve the so-called *Center Problem*. Over the years a number of criteria have been developed for deciding the issue. The development of computer algebra has derived enormous benefits for such studies as it is shown in many papers (see, for instance, Romanovsky [23]; Rousseau et al. [26]; Françoise and Pons [13]; Wang [27], Lloyd [21], among others).

Centers can be classified in terms of underlying geometry. This happens in case of time-reversible systems, where the origin of system (1) is a center if their equations are invariant under reflections through a line passing at the origin.

In general we can say that system (1) is symmetric if there exist a transformation $\Gamma : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ and a time transformation (for reversing symmetries) such that the system is invariant under these transformations, i.e., Γ leaves orbits into orbits of system (1) with the direction of time being preserved or reserved.

When system (1) is invariant under the transformation $(x, y, t) \rightarrow (-x, y, -t)$ we say that system (1) is *time-reversible* with respect to the y -axis. When (1) is invariant under the transformation $(x, y, t) \rightarrow (-x, y, t)$ we say that system (1) is *equivariant* with respect to the y -axis. In both cases the straight line $x = 0$ is called the line of symmetry.

When system (1) is invariant under the transformation $(x, y, t) \rightarrow (-x, -y, -t)$ we say that system (1) is *time-reversible* with respect to the origin. When (1) is invariant under the transformation $(x, y, t) \rightarrow (-x, -y, t)$ we say that system (1) is *equivariant* with respect to the origin. In both cases the origin is called the point of symmetry.

The concept of symmetric can also be used to study the existence of two or more centers in differential systems as (1). Assume that (1) is a symmetric planar differential system with respect to a straight line l or a point and let $P_0 = (x_0, y_0)$ be a point that does not belongs to this line of symmetry neither coincide with the point of symmetry. If P_0 is a center and $P_1 = (x_1, y_1)$ is its image by the symmetric transformation then P_1 is a center as well. The points P_0 and P_1 are called *bi-center*.

In fact, each symmetric planar differential system (1) having a center at a point P_0 (and consequently at its symmetric point P_1) can be written in a specific normal form according to the type of symmetry of such system, see Section 3 for details. Namely, there exists a change of coordinates taking the line of symmetry (respec. the point of symmetry) to the y -axis (respec. the origin) and the point P_0 to $(1, 0)$ (and consequently P_1 to $(-1, 0)$). Moreover, the symmetry is called *equivariant symmetry* if it preserves the orientation of the orbits or *time-reversible symmetry* if it reverses the orientation of the orbits.

In the literature, studies of the existence of simultaneous centers in differential systems appear with Kirnitskaya and Sibirskii in [17] and Li in [19]. They investigate conditions for planar quadratic differential system to have two centers simultaneously. Conti [3] studied a cubic system possessing two centers. Chen, Lu and Wang [4] investigated a particular family of cubic systems, the Kukles systems. They characterized when such systems possess two centers simultaneously. The $\mathbb{Z}_2$ -equivariant cubic systems possessing two simultaneous centers were characterized by Liu and Li [20]. A class of quintic systems with $\mathbb{Z}_2$ -equivariant symmetry was studied by Fernandes, Romanovski and Oliveira in [24], where they present a characterization for the existence of a bi-center for such family of systems. Giné, Llibre and Valls, [15], provided conditions for the existence of multiple centers for $\mathbb{Z}_2$ -equivariant cubic and quintic systems. They obtained conditions for the simultaneous existence of a center at the origin and two more centers at the points (a, b) and $(-a, -b)$ with $ab \neq 0$ arbitrary. In [9] Du investigated the existence of two centers and their isochronicity for a particular family of $\mathbb{Z}_2$ -equivariant system of degree seven. In [11], the authors studied the isochronicity of centers for a family of $\mathbb{Z}_2$ -equivariant quintic systems and presented their global phase portrait in the Poincaré disk.

The results obtained in [11, 24] have given rise to some questions, for instance: Is the existence of an isochronous bi-center for a quintic system possessing equivariant symmetry related with the

reversibility of the system? Should the number of isochronous centers in a symmetric system possessing symmetries with respect to a line or a point be equal to the degree of the system? Is the existence of an isochronous bi-center connected with the existence of a third or more isochronous centers in $\mathbb{Z}_2$ -equivariant systems?

The main objectives of this paper are: the characterization of the existence of simultaneous centers for some planar symmetric polynomial cubic differential systems; the study of isochronicity and; presentation of some geometric aspects and global behaviour of these systems. More precisely, the main results present here are: normal forms of a planar cubic systems possessing a straight line or a point of symmetry and, a singular point of the center or focus type - Theorem 3. The characterization of bi-centers under the conditions of Theorem 3 - Theorem 4. Theorem 5 contains results on the investigation of the isochronicity of bi-centers obtained in the previous result (Theorem 4).

Finally, in Section 6 we provide some answers on questions appearing in sections above and we give some considerations on the results obtained in this paper.

2. Preliminaries

An approach for determine conditions for the existence of a center in a planar differential system is based in Poincaré-Lyapunov Theorem, which states that the singular point at the origin of a system of the form

$$\dot{x} = -y + \sum_{p+q=2}^n a_{p,q} x^p y^q = P(x, y), \quad \dot{y} = x + \sum_{p+q=2}^n b_{p,q} x^p y^q = Q(x, y), \quad (2)$$

where $a_{p,q}, b_{p,q} \in \mathbb{R}$, is a center if and only if such system has an analytic first integral in a neighbourhood of the origin of the form

$$\psi(x, y) = x^2 + y^2 + \dots, \quad (3)$$

equivalently, if and only if

$$\mathcal{X}\psi = \frac{\partial \psi}{\partial x} P(x, y) + \frac{\partial \psi}{\partial y} Q(x, y) \equiv 0.$$

By expanding the series $\mathcal{X}\psi$ to the form

$$\mathcal{X}\psi(x, y) := \sum_{k_1+k_2 \geq 2} v_{k_1, k_2} x^{k_1} y^{k_2} \quad (4)$$

and by step-by-step construction and computation of the coefficients v_{k_1, k_2} of series (4) we see that for $k_1 \neq k_2$ the coefficients of the series $\psi(x, y)$ can be chosen so that $v_{k_1, k_2} = 0$. But in the case $k_1 = k_2 = k$ the coefficient $v_{k, k} = v_k$ depends only on the previous coefficients. When v_k is different from zero it represents an obstacle for the existence of a first integral of the form (3). We called them *focus quantities*. Thus, the simultaneous vanishing of all focus quantities provide conditions which characterize when a system of form (2) has a center at the origin. The ideal defined by the focus quantities, $\mathcal{B} = \langle v_1, v_2, \dots \rangle \subset \mathbb{C}[a, b]$, where a and b represents all the parameters $a_{p,q}$ and $b_{p,q}$ of system (2), is called the *Bautin ideal*. The affine variety $V_{\mathcal{C}} = \mathbf{V}(\mathcal{B})$, is called the *center variety* of system (2). By the Hilbert Basis Theorem there exists a positive integer k such that $\mathcal{B} = \mathcal{B}_k = \langle v_1, \dots, v_k \rangle$. Note that the inclusion $V_{\mathcal{C}} = \mathbf{V}(\mathcal{B}) \subset \mathbf{V}(\mathcal{B}_k)$ holds for any $k \geq 1$. The opposite inclusion is verified finding the irreducible decomposition of $\mathbf{V}(\mathcal{B}_k)$ and then checking that any point of each component of the decomposition corresponds to a system having a center at the origin.

To find the irreducible decomposition of $\mathbf{V}(\mathcal{B}_k)$ we perform computations with the routine `minAssGTZ`

[7] (which is based on the algorithm of [14]) of the computer algebra system SINGULAR [6]. This process provides necessary conditions for the existence of a center the origin of system (2).

The sufficiency of the obtained conditions can be proved using the method based on Darboux integrals. A polynomial $f(x, y) \in \mathbb{C}[x, y]$ is called a *Darboux factor* or *partial first integral* of system (2) if it satisfy

$$\frac{\partial f}{\partial x}(x, y)P(x, y) + \frac{\partial f}{\partial y}(x, y)Q(x, y) = K(x, y)f(x, y),$$

where $K(x, y) \in \mathbb{C}[x, y]$ is a polynomial of degree at most $\max\{P, Q\} - 1$, called the *cofactor* of $f(x, y)$.

A *Darboux first integral* of system (2) is a first integral of the form $H = f_1^{\alpha_1} \cdots f_k^{\alpha_k}$, where $f_1, f_2, \dots, f_k$ are Darboux factors of system (2) and $\alpha_i \in \mathbb{R}$, $1 \leq i \leq k$.

We remark that if $f \in \mathbb{C}[x, y]$, i. e., $f(x, y) = R(x, y) + i I(x, y)$ where $R = \operatorname{Re}(f) \in \mathbb{R}[x, y]$ is the real part of f and $I = \operatorname{Im}(f) \in \mathbb{R}[x, y]$ is the imaginary part of f then its complex conjugate is $\bar{f}(x, y) = R(x, y) - i I(x, y)$.

Proposition 1 ([10]). *For a real polynomial system (2), f is a complex Darboux factor with cofactor K if and only if $\bar{f}$ is also a Darboux factor with cofactor $\bar{K}$.*

Remark 2. *If among Darboux factors of a real system (2) a complex conjugate pair f and $\bar{f}$ occurs, the Darboux first integral has a real factor of the form $f^\alpha \bar{f}^\alpha$, which is the multivalued real function*

$$[(\operatorname{Re} f)^2 + (\operatorname{Im} f)^2]^{Re(\alpha)} \exp^{-2 \operatorname{Im}(\alpha) \arctan((\operatorname{Im} f)/(\operatorname{Re} f))},$$

where $(\operatorname{Im}(\alpha))(\operatorname{Im} f) \neq 0$. So the Darboux first integral is real if system (2) is real.

If a first integral of system (2) cannot be found using the obtained Darboux factors, sometimes it is possible to look for an integrating factor. A *Darboux integrating factor* of system (2) is an integrating factor of the form $\mu = f_1^{\beta_1} \cdots f_k^{\beta_k}$, where $f_1, f_2, \dots, f_k$ are Darboux factors of system (2) and $\beta_i \in \mathbb{R}$, $1 \leq i \leq k$.

Finding $f_1, f_2, \dots, f_k$ Darboux factors of system (2) with corresponding cofactors $K_1, K_2, \dots, K_k$, satisfying

$$\sum_{i=1}^k \alpha_i K_i = 0, \quad (5)$$

where $\alpha_i \in \mathbb{R}$, $1 \leq i \leq k$, then $H = f_1^{\alpha_1} \cdots f_k^{\alpha_k}$ is a Darboux first integral of (2). If

$$\sum_{i=1}^k \beta_i K_i + \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = 0, \quad (6)$$

where $\beta_i \in \mathbb{R}$, $1 \leq i \leq k$, then $\mu = f_1^{\beta_1} \cdots f_k^{\beta_k}$ is a Darboux integrating factor of system (2).

Another way to prove the integrability of a system is to find a Hamiltonian function. We remind that a system of the form (2) is *Hamiltonian* if there exists a function $H : \mathbb{R}^2 \rightarrow \mathbb{R}$, called the *Hamiltonian* of the system, such that $\dot{x} = -\frac{\partial H}{\partial y}$ and $\dot{y} = \frac{\partial H}{\partial x}$. It follows immediately that

$$\frac{\partial H}{\partial x}P + \frac{\partial H}{\partial y}Q \equiv 0,$$

hence H is a first integral of system (2).

System (2) is said to be *linearizable* if there exists an analytic change of coordinates

$$x_1 = x + \sum_{m+n \geq 2} c_{m,n} x^m y^n, \quad y_1 = y + \sum_{m+n \geq 2} d_{m,n} x^m y^n, \quad (7)$$

that reduces (2) to the canonical linear center $\dot{x}_1 = -y_1$, $\dot{y}_1 = x_1$.

Poincaré and Lyapunov have shown that the isochronicity problem for system (2) is equivalent to the linearizability problem (see e.g. [25] for a proof). Obstacles for existence of a change of coordinates (7) are polynomials on the parameters of (2) called *linearizability quantities* and denoted by i_k, j_k ($k = 1, 2, \dots$).

The process to compute the linearizability quantities for system (2) can be done by a step-by-step process. By differentiating both sides of each equation of (7) with respect to t we obtain

$$\begin{aligned} \dot{x}_1 &= \dot{x} + \left(\sum_{m+n \geq 2} m c_{m,n} x^{m-1} y^n \right) \dot{x} + \left(\sum_{m+n \geq 2} n c_{m,n} x^m y^{n-1} \right) \dot{y}, \\ \dot{y}_1 &= \dot{y} + \left(\sum_{m+n \geq 2} m d_{m,n} x^{m-1} y^n \right) \dot{x} + \left(\sum_{m+n \geq 2} n d_{m,n} x^m y^{n-1} \right) \dot{y}, \end{aligned}$$

and then by substituting the expressions of system (2) and $\dot{x}_1 = -y_1$, $\dot{y}_1 = x_1$ in the above equations (see [12] for more details).

System (2) admits a linearizing change of coordinates (7) if and only if $i_k = j_k = 0$, for all $k \geq 1$, that is, the simultaneous vanishing of all linearizability quantities provide conditions which characterize when a system of the form (2) is linearizable. The ideal defined by the linearizability quantities, $\mathcal{L} = \langle i_1, j_1, i_2, j_2, \dots \rangle \subset \mathbb{C}[a, b]$, is called the *linearizability ideal* and its affine variety, $V_{\mathcal{L}} = \mathbf{V}(\mathcal{L})$ is called the *linearizability variety*.

Therefore the linearizability problem is completely solved by finding the variety $V_{\mathcal{L}}$. By the Hilbert Basis Theorem there exists a positive integer k such that $\mathcal{L} = \mathcal{L}_k = \langle i_1, j_1, \dots, i_k, j_k \rangle$. Thus, computing the irreducible decomposition of the variety $\mathbf{V}(\mathcal{L}_k)$ we find necessary conditions for the existence of a linearizable change of coordinates of the form (7). The sufficiency of the obtained conditions is proven checking that any point of each component of $\mathbf{V}(\mathcal{L}_k)$ corresponds to a linearizable system. The irreducible decomposition can be found using the routine `minAssGTZ` of the computer algebra system SINGULAR.

The method of Darboux linearization is used to find a linearizing change of coordinates. To construct Darboux linearization it is convenient to complexify the real system (2) so we introduce the complex variable $z = x + iy$, $w = x - iy$, which together with a time rescaling by i transforms (2) into a system of the form

$$\dot{z} = z - X(z, w), \quad \dot{w} = -w + Y(z, w). \quad (8)$$

System (8) is called the *complexification* of real system (2).

The linearizability problem for complex system (8) is to find, if it exists, an analytic change of coordinates of the form $z_1 = z + Z_1(z, w)$, $w_1 = w + W_1(z, w)$ that transforms system to the linear system $\dot{z}_1 = z_1$, $\dot{w}_1 = -w_1$.

3. Normal forms for symmetric cubic systems

Each planar symmetric differential system can be written in a specific normal form according to the type of symmetry. In this paper we are interested in planar systems which are invariant under the

following transformations (i.e. systems with the following type of symmetry):

- (i) $(x, y, t) \rightarrow (-x, y, t)$ - equivariant symmetry with respect to the y-axis;
- (ii) $(x, y, t) \rightarrow (-x, -y, t)$ - equivariant symmetry with respect to the origin;
- (iii) $(x, y, t) \rightarrow (-x, y, -t)$ - time-reversible symmetry with respect to the y-axis;
- (iv) $(x, y, t) \rightarrow (-x, -y, -t)$ - time-reversible symmetry with respect to the origin.

The following result provides the normal forms for the systems investigated in this paper.

Theorem 3. *Consider a planar polynomial differential system of degree three that has:*

- 1) *a singular point with pure imaginary eigenvalues on the Jacobian matrix;*
- 2) *a straight line (or point) of symmetry such that the singular point in 1) does not belong to it.*

Then there exists an affine change of coordinates such that the straight line (respec. point) of symmetry is taking to y-axis (respec. the origin) and the cubic system can be written in one of the following forms according to the type of symmetry:

(i) *Equivariant symmetry with respect to a straight line:*

$$\begin{aligned}\dot{x} &= -xy + a_8xy^2, \\ \dot{y} &= -\frac{1}{2} + b_2y + \frac{x^2}{2} + b_5y^2 - b_2x^2y + b_9y^3.\end{aligned}\tag{9}$$

(ii) *Equivariant symmetry with respect to a point:*

$$\begin{aligned}\dot{x} &= a_2y - (1 + a_2)x^2y + a_8xy^2 + a_9y^3, \\ \dot{y} &= -\frac{x}{2} + b_2y + \frac{x^3}{2} - b_2x^2y + b_8xy^2 + b_9y^3.\end{aligned}\tag{10}$$

(iii) *Time-reversible symmetry with respect to a straight line:*

$$\begin{aligned}\dot{x} &= a_2y - (1 + a_2)x^2y + a_5y^2 + a_9y^3, \\ \dot{y} &= -\frac{x}{2} + \frac{x^3}{2} + b_8xy^2.\end{aligned}\tag{11}$$

(iv) *Time-reversible symmetry with respect to a point:*

$$\begin{aligned}\dot{x} &= -xy + a_5y^2, \\ \dot{y} &= -\frac{1}{2} + \frac{x^2}{2} + b_5y^2,\end{aligned}\tag{12}$$

where $a_2, a_5, a_8, a_9, b_2, b_5, b_8, b_9 \in \mathbb{R}$ are parameters of the systems.

Proof. Let

$$\begin{aligned}\dot{x} &= A_0 + A_1x + A_2y + A_3x^2 + A_4xy + A_5y^2 + A_6x^3 + A_7x^2y + A_8xy^2 + A_9y^3, \\ \dot{y} &= B_0 + B_1x + B_2y + B_3x^2 + B_4xy + B_5y^2 + B_6x^3 + B_7x^2y + B_8xy^2 + B_9y^3,\end{aligned}\tag{13}$$

where $A_i, B_i \in \mathbb{R}$ for $i = 0, \dots, 9$, be a planar polynomial differential system of degree 3 possessing a singular point (x_0, y_0) with pure imaginary eigenvalues on the Jacobian matrix. Let us split our study in two cases: a) systems with symmetry with respect to a straight line and; b) systems with symmetry with respect to a point.

a) Assume that system (13) possess $r : ax + by = c$, $a, b, c \in \mathbb{R}$ as a straight line of symmetry and $(x_0, y_0) \notin r$. Generically we can suppose $a \neq 0$ (otherwise we apply the change of coordinates $(x, y) \rightarrow (y, x)$). Under such conditions, the following affine change of coordinates is

$$\begin{aligned} x_1 &= \frac{1}{\sqrt{\left(\left(x_0 - \frac{c}{a}\right) \cos \theta - y_0 \sin \theta\right)^2}} \left(\left(x - \frac{c}{a}\right) \cos \theta - y \sin \theta \right), \\ y_1 &= \frac{1}{\sqrt{\left(\left(x_0 - \frac{c}{a}\right) \cos \theta - y_0 \sin \theta\right)^2}} \left(\left(x - \frac{c}{a}\right) \sin \theta + y \cos \theta - \left(x_0 - \frac{c}{a}\right) \sin \theta - y_0 \cos \theta \right), \end{aligned} \quad (14)$$

where $\theta = \arctan\left(-\frac{b}{a}\right)$, transforms system (13) into a cubic system of the form

$$\begin{aligned} \dot{x} &= a_0 + a_1x + a_2y + a_3x^2 + a_4xy + a_5y^2 + a_6x^3 + a_7x^2y + a_8xy^2 + a_9y^3, \\ \dot{y} &= b_0 + b_1x + b_2y + b_3x^2 + b_4xy + b_5y^2 + b_6x^3 + b_7x^2y + b_8xy^2 + b_9y^3, \end{aligned} \quad (15)$$

where parameters a_i, b_i , $i = 0, \dots, 9$ are written in function of parameters A_i, B_i , $i = 0, \dots, 9$ of system (13) and x and y are written as x_1 and y_1 , respectively. Observe that affine transformation (14) move the straight r into the straight line $x = 0$ and the singular point (x_0, y_0) into the point $(1, 0)$.

Now, since system (15) is symmetric by the y -axis we have two different types of reflections, equivariant and time-reversible.

(i) Considering that system (15) has equivariant symmetry with respect to the y -axis, that is, it is invariant under the transformation $(x, y, t) \rightarrow (-x, y, t)$, we obtain $a_0 = a_2 = a_3 = a_5 = a_7 = a_9 = b_1 = b_4 = b_6 = b_8 = 0$. Assuming that $(1, 0)$ is a singular point such that the linear part of system (15) at this point is a linear center, that is the Jacobian Matrix is given by

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

we conclude that $a_1 = a_6 = 0, a_4 = -1, b_0 = -\frac{1}{2}, b_3 = \frac{1}{2}, b_7 = -b_2$. This process leads us to the normal form (9).

(ii) Considering that system (15) has time-reversible symmetry with respect to the y -axis, that is, it is invariant under the transformation $(x, y, t) \rightarrow (-x, y, -t)$, we get $a_1 = a_4 = a_6 = a_8 = b_0 = b_2 = b_3 = b_5 = b_7 = b_9 = 0$. As in the previous case, assuming that $(1, 0)$ is a linear center, then we have $a_0 = a_3 = 0, a_7 = -1 - a_2, b_1 = -\frac{1}{2}, b_4 = 0, b_6 = \frac{1}{2}$. Thus, the normal form (11) is obtained.

b) Assume that system (13) possess a point $(\bar{x}_0, \bar{y}_0)$ of symmetry such that $(x_0, y_0) \neq (\bar{x}_0, \bar{y}_0)$. In

this case, the affine change of coordinates

$$\begin{aligned} x_1 &= \frac{1}{\sqrt{((x_0 - \bar{x}_0) \cos \theta + (y_0 - \bar{y}_0) \sin \theta)^2}} ((x - \bar{x}_0) \cos \theta + (y - \bar{y}_0) \sin \theta), \\ y_1 &= \frac{1}{\sqrt{((x_0 - \bar{x}_0) \cos \theta + (y_0 - \bar{y}_0) \sin \theta)^2}} (-(x - \bar{x}_0) \sin \theta + (y - \bar{y}_0) \cos \theta), \end{aligned} \quad (16)$$

where $\theta = \arctan\left(\frac{y_0 - \bar{y}_0}{x_0 - \bar{x}_0}\right)$, transforms system (13) into a system of the form (15), where the parameters a_i, b_i , $i = 0, \dots, 9$, are written in function of the parameters A_i, B_i , $i = 0, \dots, 9$ of system (13) and we write x and y instead of x_1 and y_1 , respectively. Observe that such affine transformation moves $(\bar{x}_0, \bar{y}_0)$ into the origin and the singular point (x_0, y_0) into $(1, 0)$. Now, as before we study the types of reflections: equivariant and time-reversible.

(iii) Considering that system (15) has equivariant symmetry with respect to the origin, that is, it is invariant under the transformation $(x, y, t) \rightarrow (-x, -y, t)$, we obtain $a_0 = a_3 = a_4 = a_5 = b_0 = b_3 = b_4 = b_5 = 0$. Assuming that $(1, 0)$ is a linear center, we conclude that $a_1 = a_6 = 0, a_7 = -1 - a_2, b_1 = -1/2, b_6 = 1/2, b_7 = -b_2$. This process leads us to the normal form (10).

(iv) Considering that system (15) has time-reversible symmetry with respect to the origin, that is, it is invariant under the transformation $(x, y, t) \rightarrow (-x, -y, -t)$, we get $a_1 = a_2 = a_6 = a_7 = a_8 = a_9 = b_1 = b_2 = b_6 = b_5 = b_7 = b_8 = b_9 = 0$. Assuming that $(1, 0)$ is a linear center we obtain $a_0 = a_3 = 0, a_4 = -1, b_0 = -1/2, b_4 = 0, b_3 = 1/2$. Thus, normal form (12) is obtained. $\square$

4. The center conditions

For each normal form described in Theorem 3 we investigate necessary and sufficient conditions for the existence of a bi-center.

Theorem 4. *For each one of four normal forms described in Theorem 3 we obtain the following necessary and sufficient conditions for the existence of a bi-center.*

(i) *System (9) has a bi-center if and only if one the conditions holds:*

1. $b_2 = 2b_5a_8 + 3b_9 = 0$,
2. $b_5 = b_9 = 0$,
3. $a_8 - 2b_2 = a_8b_5 + b_9 = 0$.

(ii) *System (10) has a bi-center if and only if one of the conditions of Theorem 11 [20] holds.*

(iii) *System (11) has a bi-center if and only if one the conditions holds:*

1. $a_2 - b_8 + 1 = 0$,
2. $a_5 = 0$.

(iv) *System (12) has a bi-center if and only if one the conditions holds:*

1. $2b_5 - 1 = 0$,
2. $a_5 = 0$.

Proof. Following the approach described in Section 2 we compute focus quantities for each one of four normal forms presented in Theorem 3. We start by applying the transformation $\tilde{x} = x - 1$, $\tilde{y} = y$ to move the singular point at $(1, 0)$ to the origin.

Due to the complicated and long polynomials we present here only first two focus quantities of system (9):

$$\begin{aligned} v_1 &= 2a_8b_5 + 2b_2b_5 - 3b_9, \\ v_2 &= 22a_8b_5 + 22b_2b_5 - 80a_8b_2^2b_5 - 32b_2^3b_5 + 56a_8b_5^2 + 56b_2b_5^2 + 280a_8b_5^3 + 280b_2b_5^3 - 33b_9 + 40a_8b_2b_9 + \\ &\quad 16b_2^2b_9 - 84b_5b_9 - 420b_5^2b_9. \end{aligned}$$

Using the radical ideal membership test [25] we check that the ideal generated by first k focus quantities, $\mathbb{B}_k$, stabilizes for $k = 3$, hence $\mathbb{B}_3 \subseteq \mathbb{B}$. Consequently $V(\mathbb{B}) \subseteq V(\mathbb{B}_3)$. By computing the variety of ideal $\mathbb{B}_3 = \langle v_1, v_2, v_3 \rangle$ we obtain the necessary conditions for the existence of a bi-center for the systems (9).

The approach for obtaining the necessary conditions for the existence of a bi-center for systems (11) and (12) are the same. Using radical ideal membership test we have checked that for system (11) and system (12) only one focus quantity is enough to obtain necessary conditions, that is $\mathbb{B}_1 \subseteq \mathbb{B}$.

The next step is to prove the sufficiency of each of the conditions obtained and stated in Theorem 4.

(i) 1. System (9) under conditions in Theorem 4 (i) 1. becomes

$$\dot{x} = -xy + a_8xy^2, \quad \dot{y} = -\frac{1}{2} + \frac{x^2}{2} + b_5y^2 - \frac{2}{3}a_8b_5y^3. \quad (17)$$

System (17) posses the Darboux factor $f(x, y) = x$, with corresponding cofactor $K(x, y) = y(-1 + a_8y)$. It is easy to verify that equation (6) is satisfied by $\beta = -1 + 2b_5$. So system (17) posses the integrating factor $\mu = x^{-1+2b_5}$, which allows to construct the first integral

$$H(x, y) = \frac{x^{2b_5}}{4b_5} - \frac{x^{2+2b_5}}{2(2+2b_5)} - \frac{1}{2}x^{2b_5}y^2 + \frac{1}{3}ax^{2b_5}y^3.$$

(i) 2. System (9) under conditions presented in Theorem 4 (i) 2. becomes system

$$\dot{x} = -xy + a_8xy^2, \quad \dot{y} = -\frac{1}{2} + \frac{x^2}{2} + b_2y - b_2x^2y. \quad (18)$$

System (18) has the Darboux factors $f_1(x, y) = x$, $f_2(x, y) = 1 - 2b_2y$, with corresponding cofactors $K_1(x, y) = -y + a_8y^2$, $K_2(x, y) = -b_2(x - 1)(x + 1)$, which allows to construct the integrating factor $\mu = x^{-1}(1 - 2b_2y)^{-1}$. Then,

$$H(x, y) = \left(\frac{1}{2b_2} - \frac{a_8}{4b_2^2} \right) y - \frac{x^2}{4} - \frac{a_8y^2}{4b_2} + \frac{\ln(x)}{2} + \left(\frac{1}{4b_2^2} - \frac{a_8}{8b_2^3} \right) \ln(1 - 2b_2y),$$

is a first integral of system (18).

(i) 3. System (9) under conditions presented in Theorem 4 (i) 3. is of the form

$$\dot{x} = -xy + a_8xy^2, \quad \dot{y} = \left(-\frac{1}{2} + \frac{a_8}{2}y \right) (1 - x^2) + b_5y^2 - a_2b_5y^3. \quad (19)$$

System (19) possesses the Darboux factors $f_1(x, y) = x$, $f_2(x, y) = 1 - \frac{b_5 x^2}{1 + b_5} - 2b_5 y^2$, with corresponding cofactors $K_1(x, y) = -y + a_8 y^2$, $K_2(x, y) = 2b_8 y(1 - a_8 y)$. It is simple to verify that equation (5) is satisfied by the constants $\alpha_1 = 2b_5$ and $\alpha_2 = 1$. Then,

$$H(x, y) = x^{2b_5} \left(1 - \frac{b_5 x^2}{1 + b_5} - 2b_5 y^2 \right),$$

is a first integral of system (19).

(ii) The first integrals for this case are presented in [20].

(iii) 1. System (11) under conditions presented in Theorem 4 (iii) 1. becomes system

$$\dot{x} = a_2 y - (1 + a_2)x^2 y + a_5 y^2 + a_9 y^3, \quad \dot{y} = -\frac{x}{2}(1 - x^2) + (1 + a_2)xy^2. \quad (20)$$

System (20) is Hamiltonian system and,

$$H(x, y) = -\frac{x^2}{4} - \frac{a_2 y^2}{2} - \frac{a_5 y^3}{3} + \frac{x^4}{8} + \left(\frac{1}{2} + \frac{a_2}{2} \right) x^2 y^2 - \frac{a_9 y^4}{4},$$

is a first integral of system (20).

(iii) 2. System (11) under conditions presented in Theorem 4 (iii) 2. is of the form

$$\dot{x} = a_2 y - (1 + a_2)x^2 y + a_9 y^3, \quad \dot{y} = -\frac{x}{2}(1 - x^2) + b_8 xy^2. \quad (21)$$

System (21) possess the Darboux factors

$$f_1(x, y) = 1 - \frac{A + \sqrt{B}}{2a_2 + a_9 + 2a_2 b_8} x^2 - \frac{(1 + a_2 + b_8 - \sqrt{B})(A + \sqrt{B})}{2a_2 + a_9 + 2a_2 b_8} y^2,$$

$$f_2(x, y) = 1 + \frac{-A + \sqrt{B}}{2a_2 + a_9 + 2a_2 b_8} x^2 - \frac{(1 + a_2 + b_8 + \sqrt{B})(A - \sqrt{B})}{2a_2 + a_9 + 2a_2 b_8} y^2,$$

where $A = 1 + a_2 + a_9 + b_8 + 2a_2 b_8$ and $B = 1 + 2a_2 + a_2^2 + 2a_9 + 2b_8 + 2a_2 b_8 + b_8^2$, with corresponding cofactors $K_1(x, y) = (-1 - a_2 + b_8 - \sqrt{B})xy$ and $K_2(x, y) = (-1 - a_2 + b_8 + \sqrt{B})xy$. Then, it easy to verify that

$$H(x, y) = f_1(x, y)^{\alpha_1} f_2(x, y)^{\alpha_2},$$

where $\alpha_1 = -1 - a_2 + b_8 + \sqrt{B}$ and $\alpha_2 = 1 + a_2 - b_8 + \sqrt{B}$, is a first integral of system (21).

(iv) 1. System (12) under conditions presented in Theorem 4 (iv) 1. is of the form

$$\dot{x} = -xy + a_5 y^2, \quad \dot{y} = -\frac{1}{2} + \frac{x^2}{2} + \frac{y^2}{2}. \quad (22)$$

System (22) is Hamiltonian and $H(x, y) = -\frac{x}{2} + \frac{x^3}{6} + \frac{xy^2}{2} - \frac{a_5 y^3}{3}$ is a first integral of system (22).

(iv) 2. System (12) under conditions presented in Theorem 4 (iv) 2. is

$$\dot{x} = -xy, \quad \dot{y} = -\frac{1}{2} + \frac{x^2}{2} + b_5 y^2. \quad (23)$$

System (23) has the Darboux factors $f_1(x, y) = x$, $f_2(x, y) = 1 - \frac{b_5 x^2}{1 + b_5} - 2b_5 y^2$, with corresponding cofactors $K_1(x, y) = -y$, $K_2(x, y) = 2b_5 y$, which allows to construct the first integral

$$H(x, y) = x^{2b_5} \left(1 - \frac{b_5 x^2}{1 + b_5} - 2b_5 y^2 \right).$$

□

5. The isochronous center conditions

For each normal form described in Theorem 3 we investigate necessary and sufficient conditions for the existence of an isochronous bi-center.

Theorem 5. *For each one of the four normal forms described in Theorem 3 we obtain the following necessary and sufficient conditions for the existence of an isochronous bi-center.*

(i) *System (9) has an isochronous bi-center if and only if one of the following conditions holds:*

1. $a_8 = b_2 = b_9 = 0$, $b_5 = -2$.
2. $a_8 = b_2 = b_9 = 0$, $b_5 = -\frac{1}{2}$.

(ii) *System (10) has an isochronous bi-center if and only if one of the conditions of Theorem 3.1 [24] holds.*

(iii) *System (11) has an isochronous bi-center if and only if one of the following conditions holds:*

1. $a_2 = \frac{1}{2}$, $a_5 = 0$, $a_9 = \frac{1}{2}$, $b_8 = -\frac{3}{2}$.
2. $a_2 = 2$, $a_5 = a_9 = 0$, $b_8 = -9$.

(iv) *System (12) has an isochronous bi-center if and only if one of the following conditions holds:*

1. $a_5 = 0$, $b_5 = -\frac{1}{2}$.
2. $a_5 = 0$, $b_5 = -2$.

Proof. To prove this result we first compute the linearizability quantities for each one of four normal forms described in Theorem 3. So we split the proof according to each normal form.

(i) We compute the first eight pairs of linearizability quantities for system (9). Due to the long expression of the polynomials we present here only the first pair

$$\begin{aligned} i_1 &= \frac{2}{9}(2 + 5a_8^2 + a_8 b_2 + 2b^2 + 5b_5 + 2b_5^2), \\ j_1 &= \frac{1}{3}(2a_8 b_5 + 2b_2 b_5 + 3b_9). \end{aligned}$$

The next computational step is to compute the irreducible decomposition of the variety of the ideal $\mathcal{L}_8$. Using the routine `minAssGTZ` of SINGULAR we obtain that the irreducible decomposition of

the variety $\mathbf{V}(\mathcal{L}_8)$ is composed of the following ideals

$$\begin{aligned} I_1 &= \left\langle \frac{2}{3}a_8b_5 + \frac{2}{3}b_2b_5 + b_9, 1 + 3b_5, b_2, 1 + 9a_2^2 \right\rangle, & I_2 &= \left\langle \frac{2}{3}a_8b_5 + \frac{2}{3}b_2b_5 + b_9, 2 + b_5, b_2, a_2 \right\rangle, \\ I_3 &= \left\langle \frac{2}{3}a_8b_5 + \frac{2}{3}b_2b_5 + b_9, 3 + b_5, b_2, 1 + a_2^2 \right\rangle, & I_4 &= \left\langle \frac{2}{3}a_8b_5 + \frac{2}{3}b_2b_5 + b_9, 1 + 2b_5, b_2, a_2 \right\rangle, \\ I_5 &= \left\langle \frac{2}{3}a_8b_5 + \frac{2}{3}b_2b_5 + b_9, b_5, 1 + b_2^2, a_2 \right\rangle, & I_6 &= \left\langle \frac{2}{3}a_8b_5 + \frac{2}{3}b_2b_5 + b_9, b_5, 1 + 4b_2^2, a_2 - b_2 \right\rangle. \end{aligned}$$

It is easy to see that the varieties of the ideals I_1 , I_3 , I_5 and I_6 are the empty set in $\mathbb{R}^4$. The varieties of ideals I_2 and I_4 provide conditions (i) 1. and (i) 2. of Theorem 5.

System (9) under conditions in Theorem 5 (i) 1. is

$$\dot{x} = -xy, \quad \dot{y} = -\frac{1}{2} + \frac{x^2}{2} - 2y^2 \quad (24)$$

and under conditions in Theorem 5 (i) 2. is

$$\dot{x} = -xy, \quad \dot{y} = -\frac{1}{2} + \frac{x^2}{2} - \frac{y^2}{2}. \quad (25)$$

Translating the point (1,0) to the origin using the substitution $u = x - 1$, $v = y$, and then performing the complexification $z = u + iv$, $w = u - iv$, we change systems (24) and (25) into systems

$$\dot{z} = z + \frac{7}{8}z^2 - \frac{3}{4}zw + \frac{3}{8}w^2, \quad \dot{w} = -w - \frac{3}{8}z^2 + \frac{3}{4}zw - \frac{7}{8}w^2, \quad (26)$$

and

$$\dot{z} = z + \frac{z^2}{2}, \quad \dot{w} = -w - \frac{w^2}{2}, \quad (27)$$

respectively.

Systems (26) and (27) are particular cases of the system considered in [25] (namely, they satisfy the conditions 5. and 8. of Theorem 4.5.1 in [25], respectively). Therefore, systems (24) and (25) posses an isochronous bi-center.

(ii) The investigation of existence of an isochronous bi-centers for such system is done in [24], where the authors found two systems possessing an isochronous bi-center,

$$\dot{x} = \frac{y}{2} - \frac{3}{2}x^2y + \frac{1}{2}y^3, \quad \dot{y} = -\frac{x}{2} + \frac{1}{2}x^3 - \frac{3}{2}xy^2, \quad (28)$$

and

$$\dot{x} = 2y - 3x^2y, \quad \dot{y} = -\frac{x}{2} + \frac{1}{2}x^3 - 9xy^2. \quad (29)$$

(iii) Using approach explained in case (i) we compute the first eight pairs of linearizability quantities for system (11). The irreducible decomposition of the variety $\mathbf{V}(\mathcal{L}_8)$ is composed by varieties of the two ideals

$$\begin{aligned} I_1 &= \left\langle 2b_8 + 3, \frac{4}{9}a_2^2 + \frac{10}{9}a_5^2 + \frac{10}{9}a_2d + \frac{4}{9}b_8^2 + \frac{8}{9}a_2 + a_9 + \frac{22}{9}b_8 + \frac{22}{9}, a_5b, 2a_2 - 1 \right\rangle, \\ I_2 &= \left\langle b_8 + 9, \frac{4}{9}a_2^2 + \frac{10}{9}a_5^2 + \frac{10}{9}a_2b_8 + \frac{4}{9}b_8^2 + \frac{8}{9}a_2 + ca_9 + \frac{22}{9}b_8 + \frac{22}{9}, a_5, a_2 - 2 \right\rangle. \end{aligned}$$

The varieties of ideals I_1 and I_2 provide conditions (iii) 1. and (iii) 2. of Theorem 5. Moreover, system (11) under conditions (iii) 1. and (iii) 2. become systems (28) and (29), respectively. Therefore, system (11) under conditions in Theorem 5 (iii) possess an isochronous bi-center.

(iv) Following the same procedure described in (i) and (iii), we compute the first eight pairs of the linearizability quantities for system (12). We obtain that the irreducible decomposition of the variety $V(\mathcal{L}_8)$ is composed by the varieties of three ideals

$$I_1 = \langle 2b_5 - 1, a_5^2 + 1 \rangle, \quad I_2 = \langle 2b_5 + 1, a_5 \rangle, \quad I_3 = \langle b_5 + 2, a_5 \rangle,$$

It is easy to see that the variety of the ideal I_1 is the empty set in $\mathbb{R}^2$. The varieties of the ideals I_2 and I_3 provide conditions (iv) 1. and (iv) 2. of Theorem 5. Moreover, system (12) under conditions (iv) 1. and (iv) 2. are of form (25) and (24), respectively. Therefore, system (12) under conditions in Theorem 5 (iv) possess an isochronous bi-center. □

6. Conclusions

Using the software P4 [1] we obtain the global phase portraits of planar cubic symmetric systems possessing isochronous bi-centers.

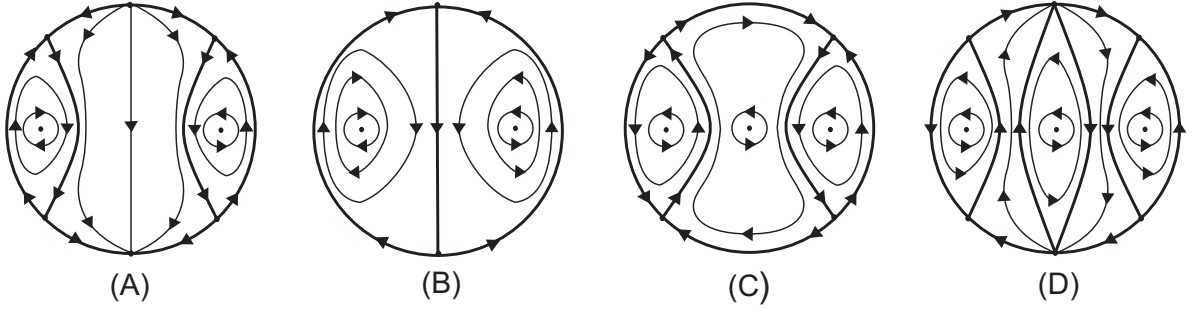


Figure 1: Global phase portraits of systems (24), (25), (28) and (29), respectively.

Figure 1 (A) and (B), represents the global phase portraits of systems (24) and (25), respectively. It is important to notice that both of them are quadratic systems and not cubic systems. That is, they are non-equivalent quadratic systems possessing two isochronous centers.

Figure 1 (C) and (D), represents the global phase portraits of systems (28) and (29), respectively. In [24] the authors proved that the third center at the origin of such systems is isochronous as well, that is, the cubic systems (28) and (29) admit three isochronous centers.

In [11] the authors conjecture that a quintic system with equivariant symmetry with respect to the origin possess at most 5 isochronous centers. Our results presented here have fortified Remark 6.4 in [11]. Specified as: "Should the number of isochronous centers in a system possessing symmetries with respect to a line or a point be equal to the degree of the system?" Moreover, also from [11], Remark 4.5, where they inquired if the existence of an isochronous bi-center for a quintic system possessing equivariant symmetry is related with the reversibility of the system. In this paper we answer such question for planar cubic systems with 4 types of symmetry.

Finally, from the proof of Theorem 3 and conditions in Theorem 4 (or Theorem 5) we provide the following algorithm:

1. insert parameters a , b and c of the straight line of symmetry $r : ax + by = c$ (or coordinates of the symmetry point $(\bar{x}_0, \bar{y}_0)$);

2. choose the type of symmetry (time-reversible or equivariant);
3. insert coordinates of the singular point (x_0, y_0) .

This way we obtain conditions for the existence of a bi-center (or an isochronous bi-center) of system (13) satisfying the assumptions above.

For example: choosing $a = -1$, $b = 1$, $c = 0$ (straight line of symmetry $x = y$) with equivariant symmetry and $x_0 = 0$, $y_0 = 2$ (singular point at $(0, 2)$). Obtaining conditions for the existence of an isochronous bi-center are:

	A_0	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9
1.	$-\frac{5}{2}$	$\frac{5}{2}$	$\frac{3}{2}$	$-\frac{5}{8}$	$-\frac{5}{4}$	$-\frac{1}{8}$	0	0	0	0
2.	-1	1	0	$-\frac{1}{4}$	$-\frac{1}{2}$	$\frac{1}{4}$	0	0	0	0

	B_0	B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9
1.	$-\frac{5}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	$-\frac{1}{8}$	$-\frac{5}{4}$	$-\frac{5}{8}$	0	0	0	0
2.	-1	0	1	$\frac{1}{4}$	$-\frac{1}{2}$	$-\frac{1}{4}$	0	0	0	0

Such conditions provide systems

$$\begin{aligned}\dot{x} &= -\frac{5}{2} + \frac{5}{2}x + \frac{3}{2}y - \frac{5}{8}x^2 - \frac{5}{4}xy - \frac{1}{8}y^2, \\ \dot{y} &= -\frac{5}{2} + \frac{3}{2}x + \frac{5}{2}y - \frac{1}{8}x^2 - \frac{5}{4}xy - \frac{5}{8}y^2,\end{aligned}\tag{30}$$

and

$$\begin{aligned}\dot{x} &= -1 + x - \frac{1}{4}x^2 - \frac{1}{2}xy + \frac{1}{4}y^2, \\ \dot{y} &= -1 + y + \frac{1}{4}x^2 - \frac{1}{2}xy - \frac{1}{4}y^2,\end{aligned}\tag{31}$$

respectively.

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