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EXISTENCE, MULTIPLICITY AND REGULARITY FOR SUB-RIEMANNIAN GEODESICS BY VARIATIONAL METHODS

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ABSTRACT. We develop a variational theory for geodesics joining a point and a one dimensional submanifold of a sub-Riemannian manifold. Given a Riemannian manifold $(\mathcal{M}, g)$, a smooth distribution $\Delta \subset T\mathcal{M}$ of codimension one in $\mathcal{M}$, a point $p \in \mathcal{M}$ and a smooth inextendible curve $\gamma : I\!R \mapsto \mathcal{M}$ which is everywhere transversal to Δ, we look for curves in $\mathcal{M}$ that are stationary with respect to the Riemannian energy functional among all the absolutely continuous curves *horizontal* with respect to Δ and that join p and γ. If $(\mathcal{M}, g)$ is complete, such extremizers exist, and they are curves of class C^2 characterized as the solutions of an integro-differential equation or by a system of ordinary differential equations. We present some results concerning a sort of *exponential map* relative to the integro-differential equation and some applications. In particular, we obtain that if p and γ are sufficiently close in $\mathcal{M}$, then there exists a unique length minimizer. We obtain existence and multiplicity results by means of the Ljusternik–Schnirelman theory.

1. INTRODUCTION

The goal of this paper is to develop a variational theory for sub-Riemannian geodesics in a general context and to obtain existence and multiplicity results, in analogy with the corresponding theory for Riemannian geodesics.

The interest in the existence of (local) length minimizers in a sub-Riemannian manifold comes essentially from Control Theory, where such minimizers represent optimal solutions of a system with linear constraints on the first derivatives of the admissible paths.

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A sub-Riemannian manifold consists of a triple $(\mathcal{M}, \Delta, g)$, where $\mathcal{M}$ is a smooth manifold, $\Delta \subset T\mathcal{M}$ is a smooth distribution in $\mathcal{M}$ and g is a positive definite metric tensor on Δ. The kind of sub-Riemannian geodesics that we are interested in are those curves x which are horizontal with respect to Δ, i.e., $\dot{x} \in \Delta$, and that minimize locally their length. In standard terminology, such geodesics are called *normal*; there is another class of sub-Riemannian geodesics, called *abnormal*, that do not in general satisfy the local minimization property. The abnormal sub-Riemannian geodesics are determined only by the linear structure, that is, by the distribution Δ, and not by the choice of the metric tensor g on Δ. For this reason, the theory of Calculus of Variations is not suited to study such geodesics. Good references for the basics of sub-Riemannian geodesics are [6, 7, 8].

The main obstruction for developing a variational theory for sub-Riemannian geodesics between two points is that, in general, the set of horizontal curves joining two fixed points does not have a differentiable structure, unless one poses strong non-integrability conditions on the distribution Δ. In this paper we do not make such assumptions on Δ, but rather we allow that the final endpoint of a trial path for our variational problem is free to move on a submanifold of $\mathcal{M}$ which is *transversal* to Δ. With such choice we overcome the *rigidity* problem of the fixed endpoint case and we obtain a smooth manifold structure for the set of horizontal curves satisfying suitable regularity conditions.

As a first approach to this technique, we will initially consider the case of a distribution Δ of codimension one in $T\mathcal{M}$, and we will assume that Δ is *transversally oriented* in $T\mathcal{M}$, i.e., that the quotient bundle $T\mathcal{M}/\Delta$ is orientable. One can extend the sub-Riemannian metric defined on Δ to a Riemannian metric in $\mathcal{M}$, such extension is of course non canonical; by the transversal orientation, it is not restrictive to assume that Δ is the orthogonal distribution to a unit vector field on $\mathcal{M}$. Moreover, if the original sub-Riemannian structure is complete, one can assume that the Riemannian extension is also complete. We will therefore consider the following geometric setup.

Let $(\mathcal{M}, g)$ be a complete Riemannian manifold, let Y be a never vanishing smooth vector field on $\mathcal{M}$ and let $\Delta = Y^{\perp}$ denote the orthogonal distribution to Y. For all $q \in \mathcal{M}$, we set $\Delta_q = \Delta \cap T_q\mathcal{M}$; moreover, we will denote by $\langle \cdot, \cdot \rangle$ the positive definite inner product on each tangent space $T_q\mathcal{M}$ given by $g(q)$ and by $|\cdot|$ the corresponding length. We will assume without loss of generality that Y is normalized on $\mathcal{M}$:

$$\langle Y, Y\rangle = 1. \tag{1}$$

Let $\gamma : I\!R \mapsto \mathcal{M}$ be a smooth *inextendible* curve in $\mathcal{M}$, i.e., $\gamma(t)$ is eventually outside every compact subset of $\mathcal{M}$ for $t \mapsto \pm\infty$, which is everywhere *transversal* to Δ, i.e., $\dot\gamma(t) \notin \Delta_{\gamma(t)}$ for all t.

Let ∇ denote the covariant derivative relative to the Levi–Civita connection of g; given a smooth function α on $\mathcal{M}$, we denote by $\nabla\alpha$ the gradient of α with respect to the metric g.

Let p be a fixed point in $\mathcal{M}$ and let $\mathcal{C}^1_{p,\gamma}$ denote the set of all curves of class C^1 in $\mathcal{M}$ parameterized on $[0,1]$ joining p and γ:

$$\mathcal{C}^1_{p,\gamma} = \Big\{ z \in C^1([0,1], \mathcal{M}) : z(0) = p,\ z(1) \in \gamma(I\!R) \Big\};$$

by $\mathcal{C}^1_{p,\gamma}(\Delta)$ we will denote the subset of $\mathcal{C}^1_{p,\gamma}$ consisting of *horizontal* curves:

$$\mathcal{C}^1_{p,\gamma}(\Delta) = \Big\{ z \in \mathcal{C}^1_{p,\gamma} : \dot z(t) \in \Delta_{z(t)} \text{ for all } t \Big\}. \tag{2}$$

The set $\mathcal{C}^1_{p,\gamma}$ has a natural structure of an infinite dimensional Banach differentiable manifold; the differentiable structure of (a suitable completion of) $\mathcal{C}^1_{p,\gamma}(\Delta)$ will be discussed in Section 2. We denote by L and E respectively the Riemannian *length* and *energy* functionals on $\mathcal{C}^1_{p,\gamma}$, defined by:

$$L(z) = \int_0^1 \sqrt{\langle \dot z, \dot z\rangle}\,\mathrm{d}t, \quad E(z) = \frac{1}{2}\int_0^1 \langle \dot z, \dot z\rangle\,\mathrm{d}t. \tag{3}$$

In this paper we will be interested in studying the curves in $\mathcal{C}^1_{p,\gamma}(\Delta)$ that minimize *locally* their length, i.e., in those horizontal curves z between p and γ such that, for $a, b \in [0,1]$ sufficiently close, the restriction $z|_{[a,b]}$ is a horizontal curve of minimal length between $z(a)$ and $z(b)$. Such curves will be called *sub-Riemannian length minimizers* between p and γ. More in particular, we will consider the stationary points of the functional E in $\mathcal{C}^1_{p,\gamma}(\Delta)$, which are geodesics in the sub-Riemannian manifold $(\mathcal{M}, \Delta, g)$. Namely (see Appendix B), the critical points of E in $\mathcal{C}^1_{p,\gamma}(\Delta)$ are normal sub-Riemannian geodesics, and therefore they minimize locally their length. Moreover, the minima in $C^1_{p,\gamma}(\Delta)$ of the functionals E and of L coincide, up to parameterization:

Proposition 1.1. *A curve $x \in C^1_{p,\gamma}(\Delta)$ is a minimal point for E on $C^1_{p,\gamma}(\Delta)$ if and only if it is a sub-Riemannian length minimizer between p and γ satisfying $|\dot{x}(t)| = const.$ on $[0,1]$.*

We have a first existence result concerning the minima of the length functional:

Theorem 1.2. *Let $(\mathcal{M}, g)$ be a complete Riemannian manifold, Y be a never vanishing smooth vector field on $\mathcal{M}$, $\Delta = Y^\perp$ be its orthogonal distribution, and $\gamma : I\!R \mapsto \mathcal{M}$ be an inextendible curve which is transversal to Δ. Then, there exists at least a minimum point x for L in $C^1_{p,\gamma}(\Delta)$, with $|\dot{x}(t)|$ constant on $[0,1]$.*

Some results concerning the characterization of the normal geodesics in a sub-Riemannian manifold, connecting submanifolds of any codimension, as critical points of the action functional can be found, for instance, in Reference [10].

As it will be observed in Section 2, for the proof of Theorem 1.2 it is not restrictive to assume that γ is an integral line of the vector field Y. For the other results of the paper we will explicitly make such assumption.

Given a smooth vector field W on $\mathcal{M}$, we denote by $(\nabla W)^*$ the transpose of the covariant derivative of W, which is the $(1,1)$ tensor field on $\mathcal{M}$ whose value at a point $q \in \mathcal{M}$ is the linear map on $T_q\mathcal{M}$ defined by:

(4)
$$\langle (\nabla W)^*[v_1], v_2 \rangle = \langle \nabla_{v_2} W, v_1 \rangle, \quad \forall\, v_1, v_2 \in T_q\mathcal{M}.$$

For all x in $C^1_{p,\gamma}(\Delta)$, let $\lambda_x : [0,1] \mapsto I\!R$ be the map of class C^1 given by:

(5) $$\lambda_x(t) = e^{\int_0^t \langle \dot{x}, \nabla_Y Y \rangle\, \mathrm{d}s} \cdot \left[\int_t^1 \langle \dot{x}, \nabla_{\dot{x}} Y \rangle e^{-\int_0^s \langle \dot{x}, \nabla_Y Y \rangle\, \mathrm{d}r}\, \mathrm{d}s \right].$$

Theorem 1.3. *Suppose that γ is an integral line of Y. If x is a critical point of E in $C^1_{p,\gamma}(\Delta)$, then x is a curve of class C^2 and it satisfies the equation:*

(6) $$\nabla_{\dot{x}}\dot{x} - \nabla_{\dot{x}}(\lambda_x \cdot Y) + \lambda_x \cdot (\nabla Y)^*[\dot{x}] = 0.$$

Remark 1.4. Observe that the integro-differential equation (5)–(6) is not *local*, in the following sense. Given any subinterval $[a,b]\subseteq[0,1]$, one can consider an alternative integro-differential problem given by the equation (6) and λ_x given by:

$$\lambda_x = e^{\int_a^t \langle \dot x, \nabla_Y Y\rangle\,\mathrm{d}s}\cdot\left[\int_t^b \langle \dot x, \nabla_{\dot x} Y\rangle e^{-\int_0^s \langle \dot x, \nabla_Y Y\rangle\,\mathrm{d}r}\,\mathrm{d}s\right]. \tag{7}$$

Given a solution x of (5)–(6), then the restriction of x to the interval $[a,b]$ is not, in general, a solution of (6)–(7). The interpretation of this fact is that, even though the critical points of E in $\mathcal{C}^1_{p,\gamma}(\Delta)$ locally minimize their length by the results in Appendix B, in general they do *not* minimize locally the distance between a point and an integral line of Y.

Note that if the pair (x,λ_x) satisfies (6)–(7), clearly it satisfies the system of ordinary differential equations:

$$\begin{cases} \nabla_{\dot x}\dot x - \nabla_{\dot x}(\lambda Y) + \lambda\,(\nabla Y)^*[\dot x] = 0,\\ \lambda' - \lambda\,\langle \nabla_Y Y, \dot x\rangle + \langle \dot x, \nabla_{\dot x} Y\rangle = 0. \end{cases} \tag{8}$$

At the end of section 2, we point out that, if $(x(t),\lambda_x(t))$ is a solution of the above system, then $\langle \dot x(t),\dot x(t)\rangle$ and $\langle \dot x(t), Y(x(t))\rangle$ are constant. From this point of view (in analogy with Riemannian geodesics), we could say that the sub-Riemannian geodesics are the solution of (8) with the initial condition $\langle \dot x(0), Y(x(0))\rangle = 0$. Observe also that the pair (x,λ) plays the role of the *Hamiltonian lift* of the sub-Riemannian geodesic x.

Remark 1.5. Suppose that Y is a conformal Killing vector field. Then, for all $v\in Y^\perp$ it is $\langle \nabla_v Y, v\rangle = 0$. Hence, for all $x\in\mathcal{C}^1_{p,\gamma}(\Delta)$ it is $\langle \nabla_{\dot x} Y, \dot x\rangle \equiv 0$ and so $\lambda_x\equiv 0$. From (6), this implies that if x is a critical point of E in $\mathcal{C}^1_{p,\gamma}(\Delta)$, then it is a Riemannian geodesic.

Remark 1.6. Changing the point of view, the stationary paths x for the functional E in $\mathcal{C}^1_{p,\gamma}(\Delta)$ can be thought as *constrained* critical points of E in $\mathcal{C}^1_{p,\gamma}$ subject to the linear constraint on the first derivative $\dot x\in\Delta$. Under this viewpoint, given such a constrained critical point x, then the map λ_x of formula (5) can be interpreted as the corresponding *Lagrange multiplier* (see Appendix A).

Recall that the *Ljusternik–Schnirelman category* $\mathrm{cat}(X)$ of a topological space X is the possibly infinite minimal number of closed contractible subsets of X that form a covering of X. We have a multiplicity result for sub-Riemannian geodesics between p and γ given in terms of the Ljusternik–Schnirelman category of the Banach manifold $C^1_{p,\gamma}(\Delta)$:

Theorem 1.7. *Under the assumptions of Theorem 1.2, if γ is an integral line of Y there are at least $\mathrm{cat}(C^1_{p,\gamma}(\Delta))$ normal geodesics between p and γ. Moreover, if $\mathrm{cat}(C^1_{p,\gamma}(\Delta))$ is infinite, then there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ of normal geodesics between p and γ such that:*

$$\lim_{n\to\infty} E(x_n) = +\infty.$$

If Y does not have closed integral lines, then (using the flow of Y), it is easy to see that the inclusion map $C^1_{p,\gamma}(\Delta) \mapsto C^1_{p,\gamma}$ is a homotopy equivalence, so that $\mathrm{cat}(C^1_{p,\gamma}(\Delta)) = \mathrm{cat}(C^1_{p,\gamma})$. If $\mathcal{M}$ is not contractible, by a well known result of Fadell and Husseini (see [2]), it is $\mathrm{cat}(C^1_{p,\gamma}) = +\infty$, and we have a class of examples where Theorem 1.7 gives the existence of infinite normal geodesics between p and γ.

Let us now look abstractly at the integro-differential equation (6). Observe that it makes perfectly sense to consider solutions of (6) that are not horizontal curves. We will prove in Section 2 that a solution x of (6) is a horizontal curve if and only if $\dot{x}(0) \in \Delta$ (Theorem 2.2). Moreover, we will see that a solution of (6) satisfying $x(0) = v \in T_p\mathcal{M}$ exists and is unique, provided that v is small enough (Proposition 4.1). This fact allows us to introduce a *sub-Riemannian* point-to-line exponential map $\mathfrak{exp}_p$, defined in a neighborhood of $0 \in T_p\mathcal{M}$ by $\mathfrak{exp}_p(v) = x_v(1)$, where x_v is the unique solution of (6) satisfying the initial condition $\dot{x}_v(0) = v$.

In perfect analogy with the well known properties of the Riemannian exponential map, the map $\mathfrak{exp}_p$ is a diffeomorphism between a neighborhood of $0 \in T_p\mathcal{M}$ and a neighborhood of $p \in \mathcal{M}$. As a first important consequence of this fact, we obtain the following local uniqueness result for sub-Riemannian length minimizers between a point and an integral line of Y:

Theorem 1.8. *Under the hypotheses of Theorem 1.2, if γ is sufficiently close to p then there is a unique minimal point x for L in $C^1_{p,\gamma}(\Delta)$ with $|\dot{x}(t)|$ constant on $[0,1]$.*

We conclude with a remark that sub-Riemannian geodesics of the kind considered in this paper have appeared recently in a General Relativistic context (see References [3, 4]). Given a *stationary Lorentzian manifold*, which represents the model for a relativistic spacetime with gravitational field stationary with respect to a distinguished observer field Y, then one has a natural sub-Riemannian metric defined on the orthogonal distribution to Y by taking the restriction of the spacetime metric tensor. In this situation, the sub-Riemannian geodesics (in a suitable conformal perturbation of the metric) joining an event p of $\mathcal{M}$ and an integral line of Y represent the *travel time* brachistochrones between a source and an observer in the spacetime.

2. The variational framework. First variation and the critical points of E

We assume hereafter that $(\mathcal{M}, g)$ is a complete Riemannian manifold, and that Y is a normalized smooth vector field on $\mathcal{M}$; we set $\Delta = Y^{\perp}$. For each $q \in \mathcal{M}$, we set $\Delta_q = \Delta \cap T_q\mathcal{M}$.

Let $I \subset \mathbb{R}$ be any interval and suppose that $\gamma : I \mapsto \mathcal{M}$ is a smooth curve having image in a compact subset of $\mathcal{M}$ which is everywhere transversal to Δ. By the transversality of γ and a partition of unity argument it is easy to prove that we can find a complete Riemannian metric $\tilde{g}$ and a smooth vector field $\tilde{Y}$ on $\mathcal{M}$ such that:

- the orthogonal distribution $\tilde{Y}^{\perp}$ with respect to $\tilde{g}$ coincides with Δ;
- g and $\tilde{g}$ coincide on Δ;
- $\dot{\gamma}(t) = \tilde{Y}(\gamma(t))$ for all $t \in I$.

Now, given any inextendible curve γ in $\mathcal{M}$ which is everywhere transversal to Δ, clearly all the curves joining a fixed point p and γ, whose length is bounded above by a given constant, remain inside a compact subset of $\mathcal{M}$. This implies that the sub-Riemannian length minimizers between p and γ are to be found inside a compact subset of $\mathcal{M}$. By the above argument, to prove the results announced in the Introduction it suffices to treat the case that γ is a maximal integral line of the vector field Y.

We will assume hereafter that $\gamma : \mathbb{R} \mapsto \mathcal{M}$ is an integral curve of Y.

As customary, if $I \subseteq \mathbb{R}$ is any interval, we will denote by $H^1(I, \mathbb{R}^n)$ the Sobolev space of absolutely continuous curves $z : I \longmapsto \mathbb{R}^n$ such that the integral $\int_I |\dot{z}|^2 \, dt$ is finite, where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n$.

Given *any* differentiable manifold N, the set $H^1([0,1], N)$ is defined as the set of all absolutely continuous curves $z : [0,1] \longmapsto N$ such that, for every local chart (V, φ) on N, with $\varphi : U \longmapsto \mathbb{R}^n$ a diffeomorphism, and for every closed subinterval $I \subseteq [0,1]$ such that $z(I) \subset V$, it is $\varphi \circ z \in H^1(I, \mathbb{R}^n)$. For all differentiable manifold N, with $\dim(N) = n$, the set $H^1([0,1], N)$ has the structure of an infinite dimensional manifold, modeled on the Hilbert space $H^1([0,1], \mathbb{R}^n)$. We will denote by TN the tangent bundle of N and by $\pi : TN \mapsto N$ the canonical projection; for $p \in N$, $T_pN = \pi^{-1}(p)$ denotes the tangent space of N at p. A vector field along a curve $z : [0,1] \mapsto N$ is a map $\zeta : [0,1] \mapsto TN$ with $\pi(\zeta(t)) = z(t)$ for all t. Given any $z \in H^1([0,1], N)$, the tangent space $T_zH^1([0,1], N)$ is identified with the set:

$$T_zH^1([0,1], N) = \Big\{\zeta \in H^1([0,1], TN) : \zeta \text{ vector field along } z\Big\},$$

which is an infinite dimensional vector space, with a topology that makes it into a *Hilbertable* space.

We introduce the sets:

$$\Omega_{p,\gamma} = \Big\{z \in H^1([0,1], \mathcal{M}) : z(0) = p, \ z(1) \in \gamma(\mathbb{R})\Big\}, \tag{9}$$

and

$$\Omega_{p,\gamma}(\Delta) = \Big\{z \in \Omega_{p,\gamma} : \langle \dot{z}, Y \rangle = 0 \text{ a.e. in } [0,1]\Big\}. \tag{10}$$

It is well known that $\Omega_{p,\gamma}$ is a smooth, infinite dimensional Hilbert submanifold of $H^1([0,1], \mathcal{M})$; for all $z \in \Omega_{p,\gamma}$ the tangent space $T_z\Omega_{p,\gamma}$ is given by:

$$T_z\Omega_{p,\gamma} = \Big\{V \in T_zH^1([0,1], \mathcal{M}) : V(0) = 0, \ V(1) \parallel Y(z(1))\Big\}.$$

We endow $T_z\Omega_{p,\gamma}$ with the Hilbert space structure induced by the inner product:

$$\langle\!\langle V, V \rangle\!\rangle = \int_0^1 \langle \nabla_{\dot{z}} V, \nabla_{\dot{z}} V \rangle \, dt; \tag{11}$$

then, $\Omega_{p,\gamma}$ becomes an infinite dimensional Riemannian manifold with the metric defined by (11).

The functionals E and L defined in formula (3) have a continuous extension to $H^1([0,1],\mathcal{M})$. The energy functional E is smooth, and so is its restriction to $\Omega_{p,\gamma}$; the length functional L is only Lipschitz continuous. For $z \in H^1([0,1],\mathcal{M})$, the Gateaux derivative $\mathrm{d}E(z)$ is given by the bounded linear map on $T_zH^1([0,1],\mathcal{M})$:

$$\mathrm{d}E(z)[V] = \int_0^1 \langle \nabla_{\dot z} V, \dot z\rangle \,\mathrm{d}t. \tag{12}$$

Proposition 2.1. *$\Omega_{p,\gamma}(\Delta)$ is a smooth Hilbert submanifold of $\Omega_{p,\gamma}$. For all $z \in \Omega_{p,\gamma}(\Delta)$, the tangent space $T_z\Omega_{p,\gamma}(\Delta)$ is given by the Hilbert subspace of $T_z\Omega_{p,\gamma}$:*

$$T_z\Omega_{p,\gamma}(\Delta) = \Big\{V \in T_z\Omega_{p,\gamma} : \langle \nabla_{\dot z} V, Y\rangle + \langle \dot z, \nabla_V Y\rangle = 0\Big\}. \tag{13}$$

The restriction of E to $\Omega_{p,\gamma}(\Delta)$ is smooth.

Proof. We consider the map $F : \Omega_{p,\gamma} \mapsto L^2([0,1],\mathbb{R})$ defined by:

$$F(z) = \langle \dot z, Y\rangle. \tag{14}$$

It is easy to see that F is smooth, $\Omega_{p,\gamma}(\Delta) = F^{-1}(0)$, and that the Gateaux derivative $\mathrm{d}F(z)$ of F at z is given by:

$$\mathrm{d}F(z)[V] = \langle \nabla_{\dot z} V, Y\rangle + \langle \dot z, \nabla_V Y\rangle. \tag{15}$$

By the Implicit Function Theorem (see [5]), to prove the Proposition we need to show that, for all $z \in \Omega_{p,\gamma}(\Delta)$, the differential $\mathrm{d}F(z) : T_z\Omega_{p,\gamma} \mapsto L^2([0,1],\mathbb{R})$ is surjective. To this aim, let $h \in L^2([0,1],\mathcal{M})$ be fixed; consider the vector field $V_h = \phi_h \cdot Y$ along z, where ϕ_h is the function:

$$\phi_h(t) = e^{-\int_0^t \langle \nabla_Y Y, \dot z\rangle \,\mathrm{d}s} \cdot \left[\int_0^t h(s)\cdot e^{\int_0^s \langle \nabla_Y Y, \dot z\rangle \,\mathrm{d}r}\,\mathrm{d}s\right]. \tag{16}$$

It is easily checked that $\phi_h \in H^1([0,1],\mathbb{R})$ and $\phi_h(0) = 0$, which implies that $V_h \in T_z\Omega_{p,\gamma}$. Moreover, $\mathrm{d}F(z)[V_h] = h$, which proves that $\mathrm{d}F(z)$ is surjective. Finally, for $z \in \Omega_{p,\gamma}(\Delta)$ the tangent space $T_z\Omega_{p,\gamma}(\Delta)$ is given by the kernel of $\mathrm{d}F(z)$, and (13) is proven. □

Theorem 2.2. *The critical points of E in $\Omega_{p,\gamma}(\Delta)$ are curves of class C^2. They are characterized as the solutions on the interval $[0,1]$ of the following integro–differential equation:*

$$\nabla_{\dot x}\dot x - \nabla_{\dot x}(\lambda_x \cdot Y) + \lambda_x \cdot (\nabla Y)^*[\dot x] = 0, \tag{17}$$

where

$$\lambda_x(t) = e^{\int_0^t \langle \nabla_Y Y, \dot x\rangle \,\mathrm{d}s} \cdot \left[\int_t^1 \langle \dot x, \nabla_{\dot x} Y\rangle e^{-\int_0^s \langle \nabla_Y Y, \dot x\rangle \,\mathrm{d}r}\,\mathrm{d}s\right], \tag{18}$$

and $\dot x(0) \in \Delta_p$.

Proof. To determine the integro-differential equation (17)–(18) we argue as follows. Let x any point in $\Omega_{p,\gamma}(\Delta)$; for all $W \in T_x\Omega_{p,\gamma}$, we define a *projection* V_W of W onto $T_x\Omega_{p,\gamma}(\Delta)$ by setting:

$$V_W = W - \psi_W \cdot Y, \tag{19}$$

where

$$\psi_W(t) = e^{-\int_0^t \langle \nabla_Y Y, \dot x\rangle \,\mathrm{d}s} \cdot \left[\int_0^t C_W(s) \cdot e^{\int_0^s \langle \nabla_Y Y, \dot x\rangle \,\mathrm{d}r}\,\mathrm{d}s\right], \tag{20}$$

and

$$C_W = \langle \dot x, \nabla_W Y\rangle + \langle \nabla_{\dot x} W, Y\rangle = \langle W, (\nabla Y)^*[\dot x]\rangle + \langle \nabla_{\dot x} W, Y\rangle. \tag{21}$$

Observe that, since $\langle Y, Y\rangle = 1$, then $\langle \nabla_{\dot x} Y, Y\rangle = 0$, and

$$C_Y = \langle \dot x, \nabla_Y Y\rangle.$$

Checking that, with the above definitions, V_W is in $T_x\Omega_{p,\gamma}(\Delta)$ is straightforward and the details are omitted.

Now, if x is a critical point of E in $\Omega_{p,\gamma}(\Delta)$, it is $\mathrm{d}E(x)[V_W] = 0$ for all $W \in T_x\Omega_{p,\gamma}$, and, since $\langle \dot x, Y\rangle = 0$, (12) gives:

$$0 = \mathrm{d}E(x)[W - \psi_W \cdot Y] = \int_0^1 \left[\langle \nabla_{\dot x} W, \dot x\rangle - \psi_W \cdot \langle \nabla_{\dot x} Y, \dot x\rangle\right]\,\mathrm{d}t. \tag{22}$$

Reversing the order of integration with Fubini's Theorem, recalling (18) and (21) we get:

(23)

$$\int_0^1 \psi_W \cdot \langle \nabla_{\dot{x}} Y, \dot{x} \rangle \, \mathrm{d}t =$$

$$= \int_0^1 \langle \nabla_{\dot{x}} Y, \dot{x} \rangle e^{-\int_0^t \langle \nabla_Y Y, \dot{x} \rangle \, \mathrm{d}r} \cdot \left[\int_0^t C_W(s) e^{\int_0^s \langle \nabla_Y Y, \dot{x} \rangle \, \mathrm{d}r} \right] \mathrm{d}t =$$

$$= \int_0^1 C_W(s) e^{\int_0^s \langle \nabla_Y Y, \dot{x} \rangle \, \mathrm{d}r} \cdot \left[\int_s^1 \langle \nabla_{\dot{x}} Y, \dot{x} \rangle e^{-\int_0^t \langle \nabla_Y Y, \dot{x} \rangle \, \mathrm{d}r} \, \mathrm{d}t \right] \mathrm{d}s =$$

$$= \int_0^1 C_W(s) \cdot \lambda_x(s) \, \mathrm{d}s = \int_0^1 \left[\langle W, (\nabla Y)^*[\dot{x}] \rangle + \langle \nabla_{\dot{x}} W, Y \rangle \right] \cdot \lambda_x(s) \, \mathrm{d}s.$$

Suppose that x is of class C^2; integrating by parts the terms containing the covariant derivative $\nabla_{\dot{x}} W$, from (22) and (23) we obtain:

$$\int_0^1 \langle W, \nabla_{\dot{x}}(\dot{x} - \lambda_x \cdot Y) + \lambda_x \cdot (\nabla Y)^*[\dot{x}] \rangle \, \mathrm{d}t = 0, \tag{24}$$

for all $W \in T_x\Omega_{p,\gamma}$. Observe that there is no boundary term arising from the integration by parts, because $W(0) = 0$, $\lambda_x(1) = 0$ and $\langle W(1), \dot{x}(1) \rangle = 0$. This last equality follows from the fact that $W(1)$ is parallel to $\dot{\gamma}$, hence to Y, and $\langle \dot{x}, Y \rangle = 0$.

Since in (24) W is arbitrary, the Fundamental Lemma of Calculus of Variation tells us that x satisfies (17), and we have proven that the critical points of E in $\Omega_{p,\gamma}(\Delta)$ satisfy the integro-differential problem (17)–(18).

The C^2-regularity of the critical points of E is obtained by a bootstrap argument. Considering an arbitrary smooth vector field W defined around the image of the curve x and such that $W(x(0)) = W(x(1)) = 0$, we substitute (23) in (22) obtaining:

$$\begin{aligned} 0 &= \int_0^1 \langle \nabla_{\dot{x}} W, \dot{x} - \lambda_x(s) Y \rangle - \langle W, \lambda_x(s)(\nabla Y)^*[\dot{x}] \rangle \mathrm{d}s = \\ &= \int_0^1 \langle \nabla_{\dot{x}} W, \dot{x} - \lambda_x(s) Y + \left[\int_0^s \lambda_x(s)(\nabla Y)^*[\dot{x}] \, \mathrm{d}t \right] \rangle \, \mathrm{d}s, \end{aligned} \tag{25}$$

because W vanishes at both endpoints of x.

Observe that the integral between square brackets appearing in formula (25) is a *covariant integral* along x, which is defined as the unique absolutely continuous vector field A along x such that $A(0) = 0$ and:

$$\nabla_{\dot{x}} A = \lambda_x \cdot (\nabla Y)^*[\dot{x}].$$

From now on, all the integrals of vector fields along curves will be understood in this sense.

From (25) we obtain the existence of a function χ such that

$$\nabla_{\dot{x}} \chi = 0 \tag{26}$$

and

$$\dot{x} = \mathcal{H} + \chi, \tag{27}$$

where

$$\mathcal{H}(s) = \int_0^s \lambda_x(s)(\nabla Y)^*[\dot{x}]\, \mathrm{d}t - \lambda_x(s) Y(x(s)). \tag{28}$$

Using (26) in a fixed local coordinate system and Gronwall's lemma, it follows that $\dot{\chi}$ is L^1 and then χ is continuous. Since $\lambda_x(s)$ is C^0, $\mathcal{H}$ is C^0 and then $\dot{x}$ is C^0, then x is C^1. Using this fact again into a coordinate expression of (26) we have that $\dot{\chi}$ is C^0 and then χ is C^1, whereas from continuity of $\dot{x}$ in (28) $\mathcal{H}$ is C^1. Hence $\dot{x}$ is C^1, obtaining the desired regularity.

Conversely, if x is a solution of (17)–(18) such that $\dot{x}(0) \in \Delta$, then x is a horizontal curve. Namely, using (17), we compute:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\langle \dot{x}, Y \rangle &= \langle \nabla_{\dot{x}} \dot{x}, Y \rangle + \langle \dot{x}, \nabla_{\dot{x}} Y \rangle = \\ &= \lambda_x' - \lambda_x \langle \nabla_Y Y, \dot{x} \rangle + \langle \dot{x}, \nabla_{\dot{x}} Y \rangle = 0, \end{aligned} \tag{29}$$

where the last equality is due to (18). Hence, if $\langle \dot{x}(0), Y \rangle = 0$, then $\langle \dot{x}, Y \rangle \equiv 0$. Moreover, it is easy to see that all elements $V \in T_x\Omega_{p,\gamma}(\Delta)$ are of the form V_W for some $W \in T_x\Omega_{p,\gamma}$ (formula (19)), and so every solution x of (17)–(18) such that $x(0) = p$ and $\dot{x}(0) \in \Delta_p$ is a critical point of E in $\Omega_{p,\gamma}(\Delta)$. This concludes the proof. □

Remark 2.3. From (17) we have that a solution $(x(t), \lambda_x(t))$ of the system (8) satisfies the conservation law $\langle \dot{x}(t), Y(x(t)) \rangle = 0$.

We remark that the map λ_x in Theorem 2.2 can be interpreted as the *Lagrangian multiplier* of the constraint critical point x in $\Omega_{p,\gamma}(\Delta)$ (see Lemma A.1).

Corollary 2.4. *If x is a critical point of E in $\Omega_{p,\gamma}(\Delta)$, then it satisfies the conservation law:*

$$|\dot{x}| \equiv \mathit{const}. \tag{30}$$

Proof. Taking the product of (17) by $\dot{x}$ we obtain:

$$\begin{aligned} 0 &= \langle \nabla_{\dot{x}}\dot{x}, \dot{x}\rangle - \lambda_x' \cdot \langle Y, \dot{x}\rangle - \lambda_x \cdot \langle \nabla_{\dot{x}} Y, \dot{x}\rangle + \lambda_x \cdot \langle \nabla_{\dot{x}} Y, \dot{x}\rangle = \\ &= \langle \nabla_{\dot{x}}\dot{x}, \dot{x}\rangle = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle \dot{x}, \dot{x}\rangle, \end{aligned} \tag{31}$$

which proves the claim. □

3. Minimal curves for L

In this section we show the existence of a minimum for the energy functional E in $\Omega_{p,\gamma}(\Delta)$, whose regularity has already been proven in Theorem 2.2. For this aim, we show that E satisfies a good enough compactness property, namely the Palais–Smale condition. Later on, Proposition 1.1, which establishes the relation between minimal points for E and sub-Riemannian length minimizers, will be proven.

We recall that, given a C^1 functional $F : X \to \mathbb{R}$ on a Hilbert manifold (X, h), then F satisfies the *Palais–Smale condition* at level $c \in \mathbb{R}$ if every sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ such that

$$\begin{aligned} &\lim_{n\to\infty} F(x_n) = c, \\ &\lim_{n\to\infty} \|\mathrm{d}F(x_n)\| = 0, \end{aligned} \tag{32}$$

(where $\|\cdot\|$ denotes the norm in the Hilbert space $T_{x_n}X$), has a subsequence converging in X.

Proposition 3.1. *The functional E satisfies the Palais–Smale condition at every level $c \in \mathbb{R}$.*

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in $\Omega_{p,\gamma}(\Delta)$ satisfying (32). Since $\mathcal{M}$ is complete and $\int_0^1 \langle \dot{x}_n, \dot{x}_n\rangle \mathrm{d}t \le$ const., up to subsequences we can suppose that x_n is convergent to x uniformly and $\dot{x}_n$ is weakly convergent to $\dot{x}$ in L^2. Then (32) yields

$$\int_0^1 \langle \dot{x}_n, \nabla_{\dot{x}_n}\zeta \rangle \, dt = \int_0^1 \langle a_n, \nabla_{\dot{x}_n}\zeta \rangle \, dt \tag{33}$$

for every admissible variation ζ and for some sequence a_n converging to 0 in L^2. Then, with a similar argument as in Theorem 2.2, (33) gives the existence of a sequence b_n converging to 0 in L^2 such that

$$\dot{x}_n - b_n - \varphi_{x_n} Y(x_n) + \int_0^s \varphi_{x_n} \cdot (\nabla Y)^*[\dot{x}_n] \, dt = z_n, \tag{34}$$

where

$$\varphi_{x_n}(\tau) = \int_\tau^1 \langle \dot{x}_n, \nabla_{\dot{x}_n} Y(x_n) \rangle \, e^{\int_\sigma^\tau \langle \dot{x}_n, \nabla_{Y(x_n)} Y(x_n) \rangle \, d\rho} \, d\sigma, \tag{35}$$

and z_n is a sequence in L^2 such that $\nabla_{\dot{x}_n} z_n = 0$. From (35), φ_{x_n} is uniformly bounded in L^∞ and it has uniformly bounded derivative in L^1. Moreover, the covariant integral V_n

$$V_n = \int_0^s A_n \, dt, \quad A_n = \varphi_{x_n} \cdot (\nabla Y)^*[\dot{x}_n], \tag{36}$$

solves the equation

$$\begin{cases} \nabla_{\dot{x}_n} V_n = A_n, \\ V_n(0) = 0; \end{cases} \tag{37}$$

since $\dot{x}_n$ is uniformly bounded in L^2, using the coordinate expression of (37) we have that V_n is uniformly bounded in L^∞ and $\dot{V}_n$ is uniformly bounded in L^2, then V_n is uniformly bounded in H^1.

The uniform boundedness of z_n in L^2 implies the existence of a sequence $s_n \in [0,1]$ such that $z_n(s_n)$ is bounded. Using again the coordinate expression for the equation

$$\begin{cases} \nabla_{\dot{x}_n} z_n = 0, \\ z_n(s_n) = B_n, \end{cases} \tag{38}$$

with B_n bounded, we obtain that z_n is uniformly bounded in H^1.

In conclusion, (34) yields the existence of a uniformly c_n which is bounded in L^∞ and has uniformly bounded derivative in L^1 such that

$$\dot{x}_n - b_n = c_n. \tag{39}$$

But c_n has a converging subsequence in L^2 (see [1]), and b_n converges to 0 in L^2; these two facts imply that $\dot{x}_n$ has a converging subsequence in L^2, and then E satisfies the Palais–Smale condition at every level $c \in \mathbb{R}$. ☐

Since $(\mathcal{M}, g)$ is complete, $\Omega_{p,\gamma}(\Delta)$ is complete, and moreover E is clearly bounded from below. Then, using classical results (see [4]), Proposition 3.1 yields the following

Corollary 3.2. *The functional E attains its minimum in $\Omega_{p,\gamma}(\Delta)$.*

Proof of Proposition 1.1. Let x be a sub-Riemannian length minimizer satisfying

$$|\dot{x}(t)| \equiv c \text{ (constant).} \tag{40}$$

Then, for any other curve y in $C^1_{p,\gamma}(\Delta)$,

$$E(x) = \frac{1}{2}[L(x)]^2 \leq \frac{1}{2}[L(y)]^2 \leq E(y). \tag{41}$$

Conversely, let x be a minimizer for E in $C^1_{p,\gamma}(\Delta)$, and let us suppose that there exists a curve $y \in C^1_{p,\gamma}(\Delta)$ such that $L(y) < L(x)$. Thus we can reparameterize $y(t)$ proportionally to arc length, obtaining a curve $z : [0,1] \to \mathcal{M}$ such that:

$$z(\rho) = y(\sigma) \quad \text{where } \rho = \frac{1}{L(y)} \int_0^\sigma \sqrt{\langle \dot{y}(\tau), \dot{y}(\tau) \rangle}\, d\tau. \tag{42}$$

It is not difficult to see that z is of class H^1, and it can also be seen that

$$E(z) = \frac{1}{2}[L(z)]^2 = \frac{1}{2}[L(y)]^2,$$

therefore

$$E(z) = \frac{1}{2}[L(z)]^2 < \frac{1}{2}[L(x)]^2 = E(x).$$

Hence previous Corollary 3.2 yields the existence of a minimum $w \in \Omega_{p,\gamma}(\Delta)$ for E, and $w \neq x$. By Theorem 2.2, w is C^2 and therefore it is a minimum for E in $C^1_{p,\gamma}(\Delta)$, that is a contradiction. ☐

Theorem 1.2 and Theorem 1.3 now follow from Proposition 1.1, together with Theorem 2.2 and Corollary 3.2 in a straighforward way.

Note that the proof of Theorem 1.7 follows by the classical theory of Ljusternik and Schnirelman [9] and by Theorem 2.2. Indeed, E satisfies the Palais–Smale condition on $\Omega_{p,\gamma}(\Delta)$ (see Proposition 3.1).

4. LOCAL THEORY: THE EXPONENTIAL MAP

In this section we study the flow on $\mathcal{M}$ defined by the integro-differential equation (6), with the aim of proving the local uniqueness of sub-Riemannian length minimizers between a point and an integral line of y.

We start by proving an existence and uniqueness result for local solutions of (6):

Proposition 4.1. *Let $p \in \mathcal{M}$ and $v_0 \in T_p\mathcal{M}$; suppose that $|v_0|$ is sufficiently small. Then there exists a unique solution of the integro-differential equation* (17)–(18) *satisfying the initial conditions $x(0) = p$ and $\dot{x}(0) = v_0$.*

Proof. As we have observed in Corollary 2.4, the solutions of the integro differential problem satisfy $|\dot{x}| = \text{const.}$, hence all the solutions of our initial value problem remain inside a ball of radius b around the point p. Using local coordinates, we may therefore assume that $\mathcal{M} = I\!R^n$, $p = 0$.

For all $v_1, v_2, v_3 \in I\!R^n$, we set:

$$A(v_1)[v_2] = (\nabla_{v_2} Y)(v_1) - (\nabla Y)^*[z](v_1) + \langle z, \nabla_{Y(v_1)} Y \rangle, \tag{43}$$

and

$$B(v_1)[v_2, v_3] = -\Gamma(v_1)[v_2, v_3] - \langle v_2, \nabla_{v_3} Y \rangle_{v_1}, \tag{44}$$

where $\Gamma(q)[\cdot,\cdot] : I\!R^n \times I\!R^n \mapsto I\!R^n$ is the bilinear map given by the Christoffel symbols at the point q of the metric g in local coordinates. Observe that $B(x)[z, z]$ is continuous in x and bilinear in z, while $A(x)[z]$ is continuous in x and linear in z. We now consider the map:

(45)
$$G : C^0([0,1], I\!R^n) \times C^0([0,1], I\!R^n) \times I\!R^n \mapsto C^0([0,1], I\!R^n) \times C^0([0,1], I\!R^n)$$

given by $G = (G_1, G_2)$, where:

$$G_1(z, x, v)(t) = z(t) - v - \int_0^t \Big[B(x)[z, z] + \Lambda(z, x)\, A(x)[z] \Big] \,\mathrm{d}t, \tag{46}$$

$$G_2(z, x, v)(t) = x(t) - \int_0^t z \,\mathrm{d}s, \tag{47}$$

the maps A and B are defined in (43) and (44), and, finally

$$\Lambda(z,x)(t) = e^{\int_0^t \langle \nabla_Y Y, z\rangle \,\mathrm{d}s} \cdot \left[\int_t^1 \langle z, \nabla_z Y\rangle e^{-\int_0^s \langle \nabla_Y Y, z\rangle \,\mathrm{d}r} \,\mathrm{d}s\right]. \tag{48}$$

Clearly, G is of class C^1, and $G(0,0,0) = 0$.

An elementary calculation shows that $(z, x, w_0) \in G^{-1}(0)$ if and only if x is of class C^2, $z = \dot{x}$, and x is a solution for the integro-differential problem (17)–(18) satisfying $x(0) = p$ and $\dot{x}(0) = z(0) = w_0$. Once we prove that the jacobian:

$$\frac{\partial G}{\partial(z,x)}(0,0,0) : C^0([0,1], I\!R^{2n}) \mapsto C^0([0,1], I\!R^{2n}) \tag{49}$$

is invertible, there exists, for v sufficiently small, a C^1 map

$$v \mapsto (z_v, x_v) \tag{50}$$

such that

$$G(z_v, x_v, v) = 0. \tag{51}$$

To prove this, we differentiate formally the maps (46) and (47) at a generic point (z, x, v) in the direction (ω, ξ), obtaining:

(52)

$$\begin{aligned}
&\frac{\partial G_1}{\partial z}(z,x,v)[\omega] + \frac{\partial G_1}{\partial x}(z,x,v)[\xi] = \\
&= \omega(t) - \int_0^t \Big[\frac{\mathrm{d}B}{\mathrm{d}x}[\xi][z,z] + B(x)[\omega,z] + B(x)[z,\omega] + \frac{\partial \Lambda}{\partial z}[\omega]\, A(x)[z]\Big] \,\mathrm{d}s \\
&\qquad - \int_0^t \Big[\frac{\partial \Lambda}{\partial x}[\xi]\, A(x)[z] + \Lambda(z,x)\, \frac{\mathrm{d}A}{\mathrm{d}x}[\xi][z] + \Lambda(z,x)\, A(x)[\omega]\Big] \,\mathrm{d}s,
\end{aligned}$$

and

$$\frac{\partial G_2}{\partial z}(z,x,v)[\omega] + \frac{\partial G_2}{\partial x}(z,x,v)[\xi] = \xi(t) - \int_0^t \omega \,\mathrm{d}s. \tag{53}$$

When we evaluate (52) and (53) at $(z, x, v) = (0,0,0)$, considering the linearity of the above functions in the variables between brackets and the fact that $\Lambda(0,0) = 0$, we obtain the following simple expression for the Jacobian (49):

$$\frac{\partial G}{\partial(z,x)}(0,0,0)[\omega,\xi] = \left(\omega, \xi - \int_0^t \omega \,\mathrm{d}s\right). \tag{54}$$

Clearly, this is an invertible map, whose inverse is easily computed as:

$$\left(\frac{\partial G}{\partial(z,x)}(0,0,0)\right)^{-1}[\omega,\xi]=\left(\omega,\xi+\int_0^t\omega\,\mathrm{d}s\right). \tag{55}$$

This proves that the pair (z_v, x_v), and in particular the curve x_v depends regularly on v for v small enough such that (50) is well defined. □

By Proposition 4.1, for all $p\in\mathcal{M}$ there exists a neighborhood $\mathcal{U}_p$ of 0 in $T_p\mathcal{M}$ such that for all $v\in\mathcal{U}_p$ the integro-differential (17)–(18) admits a unique solution x_v satisfying $x_v(0)=p$ and $\dot{x}_v(0)=v$. We can therefore define the following *exponential map* $\mathfrak{exp}_p:\mathcal{U}_p\mapsto\mathcal{M}$ by:

$$\mathfrak{exp}_p(v)=x_v(1). \tag{56}$$

Proposition 4.2. *For all $p\in\mathcal{M}$, $\mathfrak{exp}_p$ is a local diffeomorphism between an open neighborhood of 0 in $T_p\mathcal{M}$ and an open neighborhood of p in $\mathcal{M}$. In particular, $\mathfrak{exp}_p$ gives a local diffeomorphism between a neighborhood of 0 in Δ_p and a hypersurface Σ_p of $\mathcal{M}$ through p, with $T_p\Sigma_p=\Delta_p$, which is transversal to Y.*

Proof. From Proposition 4.1 we know that $\mathfrak{exp}_p$ is a map of class C^1 around $v=0$; we now show that its differential $\mathrm{d}\mathfrak{exp}_p(0)$ is the *identity map* on $T_p\mathcal{M}$.

From the Implicit Function Theorem, the differential at $v=0$ of the map $v\mapsto(z_v,x_v)$ is given by:

$$-\left(\frac{\partial G}{\partial(z,x)}(0,0,0)\right)^{-1}\circ\frac{\partial G}{\partial v}(0,0,0),$$

which is easily computed from (46), (47) and (55) as:

$$-\left(\frac{\partial G}{\partial(z,x)}(0,0,0)\right)^{-1}\left[\frac{\partial G}{\partial v}(0,0,0)[w]\right]=(w,w\cdot t),\quad\forall w\in I\!R^n. \tag{57}$$

The derivative of $\mathfrak{exp}_p$ at $v=0$ in the direction w is given by the evaluation at $t=1$ of the second component of (57). Hence, $\mathrm{d}\mathfrak{exp}_p(0)[w]=w$ and $\mathfrak{exp}_p$ is a local diffeomorphism in a neighborhood $\mathcal{U}_p$ of $0\in T_p\mathcal{M}$, which proves the first part of the statement.

Since Δ_p is a vector subspace of codimension 1 in $T_p\mathcal{M}$, it follows that the image Σ_p of $\Delta_p\cap\mathcal{U}_p$ through $\mathfrak{exp}_p$ is a codimension 1 submanifold of $\mathcal{M}$. Since $\mathrm{d}\mathfrak{exp}_p(0)$ is the identity map, it follows that

$T_p\Sigma_p = \Delta_p$; moreover, since γ is transversal to Δ_p, by continuity every flow line of Y will be transversal to Σ_p around the point p, and this concludes the proof. □

Corollary 4.3. *If $p \in \mathcal{M}$ and γ is an integral line of Y which is sufficiently close to p, then there exists a unique sub-Riemannian length minimizer joining p and γ, up to reparameterizations (namely Theorem 1.8 holds).*

Proof. If γ is sufficiently close to p, then, by the transversality, γ intercepts exactly once the hypersurface Σ_p of Proposition 4.2. The conclusion follows immediately from Proposition 1.1 and Proposition 4.1. □

Appendix A. The method of Lagrange multipliers

In this short appendix we use the method of Lagrange multipliers to study the solutions of our variational problem and we show that the map λ_x of formula (5) and the integro-differential (6) appear naturally also in this context.

We recall that $x \in \Omega_{p,\gamma}(\Delta)$ is a *constrained* critical point of E if and only if there exists a function $\lambda_x \in L^2([0,1], I\!R)$ such that x is a *free* critical point in $\Omega_{p,\gamma}$ for the functional E_{λ_x} defined by:

$$E_{\lambda_x}(z) = E(z) - \int_0^1 \lambda_x \cdot \langle \dot{z}, Y \rangle \, dt. \tag{58}$$

In this case, the function λ_x is necessarily unique, and it is called the *Lagrange multiplier* associated to x.

We have the following:

Lemma A.1. *Let $x \in \Omega_{p,\gamma}(\Delta)$ be a critical point for E. Then, the corresponding Lagrange multiplier λ_x is a C^2-function given by:*

$$\lambda_x(t) = e^{\int_0^t \langle \dot{x}, \nabla_Y Y \rangle \, ds} \cdot \left[\int_t^1 \langle \dot{x}, \nabla_{\dot{x}} Y \rangle e^{-\int_0^s \langle \dot{x}, \nabla_Y Y \rangle \, dr} \, ds \right]. \tag{59}$$

Proof. The condition that x is a constrained critical point for E is that the following equation is satisfied for all $V \in T_x\Omega_{p,\gamma}$:

$$\int_0^1 \left[\langle \dot{x}, \nabla_{\dot{x}} V \rangle - \lambda_x \left(\langle \nabla_{\dot{x}} V, Y \rangle + \langle \dot{x}, \nabla_V Y \rangle \right) \right] dt = 0. \tag{60}$$

Taking the covariant integral of $\lambda_x(\nabla Y)^*[(|x)[\dot{x}]$, vanishing at $s = 1$, we see that $\dot{x} - \lambda_x Y$ is of class C^0. Taking its scalar product with Y we obtain that λ_x is of class C^0, therefore $\dot{x}$ is C^0. Repeating the above argument we have also that $\dot{x}$ is C^1.

At this point, integration by parts in (60) of the terms containing the covariant derivative $\nabla_{\dot{x}} V$ gives:

(61)

$$\int_0^1 \langle -\nabla_{\dot{x}}\dot{x} + \nabla_{\dot{x}}(\lambda_x \cdot Y) - \lambda_x(\nabla Y)^*[\dot{x}], V\rangle \,\mathrm{d}t - \lambda_x(1)\langle V(1), Y(x(1))\rangle = 0.$$

Since $V(1)$ is arbitrary, it is easy to see that (61) is satisfied for all $V \in T_x\Omega_{p,\gamma}$ if and only if the following two equations are satisfied:

$$-\nabla_{\dot{x}}\dot{x} + \nabla_{\dot{x}}(\lambda_x \cdot Y) - \lambda_x \cdot (\nabla Y)^*[\dot{x}] = 0, \quad \lambda_x(1) = 0. \tag{62}$$

Taking the product of the differential equation in (62) by Y, and considering that, since $\langle \dot{x}, Y\rangle \equiv 0$, it is:

$$-\langle \nabla_{\dot{x}}\dot{x}, Y\rangle = \langle \dot{x}, \nabla_{\dot{x}} Y\rangle,$$

we obtain the following Cauchy problem for λ_x (recall $\langle Y, Y\rangle = 1$):

$$\begin{cases} \lambda_x' - \lambda_x \cdot \langle \dot{x}, \nabla_Y Y\rangle + \langle \dot{x}, \nabla_{\dot{x}} Y\rangle = 0, \\ \lambda_x(1) = 0. \end{cases} \tag{63}$$

The unique solution of (63) is (59), whose regularity is deduced with the same bootstrap argument used in Theorem 2.2. This concludes the proof. □

Appendix B. Local minimality of normal geodesics

In this Appendix we prove that sufficiently small portions of a normal geodesic obtained as solutions of our variational problem are length minimizers. To this aim, let x be a normal geodesic between a point $p \in \mathcal{M}$ and an integral line γ of the vector field Y; we consider the system of differential equations satisfied by x and by its Lagrangian multiplier λ obtained from (6) and (63):

$$\begin{cases} \nabla_{\dot{x}}\dot{x} - \nabla_{\dot{x}}(\lambda Y) + \lambda\,(\nabla Y)^*[\dot{x}] = 0, \\ \lambda' - \lambda\langle \nabla_Y Y, \dot{x}\rangle + \langle \dot{x}, \nabla_{\dot{x}} Y\rangle = 0. \end{cases} \tag{64}$$

We look at (64) from an abstract point of view; it is easy to prove the following homogeneity property for its solutions:

Lemma B.1. *Suppose that* $(x, \lambda) : [a, b] \mapsto \mathcal{M} \times I\!R$ *is a solution of* (64). *Then, for all* $c \in I\!R$, $t \mapsto \big(x(ct), c\,\lambda(ct)\big)$ *is also a solution of* (64). *Moreover, the following quantities are constant in* $[a, b]$*:*

$$\big\langle \dot{x}, Y \big\rangle \quad \big\langle \dot{x}, \dot{x} \big\rangle.$$

Proof. The proof of the first part of the thesis is obtained by direct substitution of the pair $\big(x(ct), c\,\lambda(ct)\big)$ into (64).

For the second part, see Remark 2.3 and Corollary 2.4. □

We now prove that sufficiently small portions of the normal geodesic obtained as solutions of our variational problem are length minimizers between their endpoints. The following proposition is the Lagrangian counterpart of the result contained in Appendix C of [6].

Proposition B.2. *Let* $(x, \lambda) : [a, b] \mapsto \mathcal{M} \times I\!R$ *be a solution of* (64) *with* $\big\langle \dot{x}(a), Y(x(a)) \big\rangle = 0$ *and* $\dot{x}(a) \neq 0$. *Then, for* $\varepsilon > 0$ *small enough, the restriction* $x|_{[a,a+\varepsilon]}$ *is a sub-Riemannian length minimizers between* $x(a)$ *and* $x(a + \varepsilon)$.

Proof. By Lemma B.1, x is a horizontal curve with $\big\langle \dot{x}, \dot{x} \big\rangle$ constant; by an affine reparameterization we can assume that $\big\langle \dot{x}, \dot{x} \big\rangle = 1$.

Let $\mathcal{S} \subset \mathcal{M}$ be a codimension one submanifold of $\mathcal{M}$, with $x(a) \in \mathcal{S}$ and $T_{x(a)}\mathcal{S} = \Big[\dot{x}(a) - \lambda(a)\,Y(x(a))\Big]^{\perp}$.

Observe that the vector $v = \dot{x}(a) - \lambda(a)\,Y(x(a))$ is non zero, because $\big\langle v, \dot{x}(a) \big\rangle = \big\langle \dot{x}(a), \dot{x}(a) \big\rangle = 1$; moreover, clearly $\dot{x}(a) \notin T_{x(a)}\mathcal{S}$, hence $\dot{x}(a)$ and $T_{x(a)}\mathcal{S}$ generate $T_{x(a)}\mathcal{M}$.

We now choose a smooth vector field X along $\mathcal{S}$ and a smooth function on $\mathcal{S}$, denoted again by $\lambda : \mathcal{S} \mapsto I\!R$, satisfying the following properties:

(1) $\big\langle X, X \big\rangle = 1$ and $X \in \mathcal{D}$, i.e., $\big\langle X, Y \big\rangle = 0$;
(2) $X(u) - \lambda(u)\,Y(u) \in T_u\mathcal{S}^{\perp}$ for all $u \in \mathcal{S}$;
(3) $X(x(a)) = \dot{x}(a)$ and $\lambda(x(a)) = \lambda(a)$.

For such a choice, it suffices to consider any smooth vector field n which is everywhere orthogonal to $\mathcal{S}$, with $n_{\mathcal{D}}(x(a))$ a positive multiple of $\dot{x}(a)$ and $n_{\mathcal{D}} \neq 0$, where $n_{\mathcal{D}}$ denotes the orthogonal projection of n onto $\mathcal{D}$.

Then one sets:

$$X = \frac{n_{\mathcal{D}}}{\langle n_{\mathcal{D}}, n_{\mathcal{D}} \rangle^{\frac{1}{2}}}, \quad \lambda = -\frac{\langle n, Y \rangle}{\langle n_{\mathcal{D}}, n_{\mathcal{D}} \rangle^{\frac{1}{2}}}.$$

For u in a sufficiently small neighborhood of $x(a)$ in $\mathcal{S}$, $\varepsilon > 0$ small enough and $t \in]a - \varepsilon, a + \varepsilon[$, let $t \mapsto \big(x(t,u), \lambda(t,u)\big)$ be the unique solution of (64) satisfying the initial conditions

$$x(a,u) = u, \quad \frac{\mathrm{d}}{\mathrm{d}t} x(a,u) = X(u), \quad \lambda(a,u) = \lambda(u).$$

We claim that the map $(u,t) \mapsto x(u,t)$ is a diffeomorphism between a sufficiently small neighborhood of $(a, x(a))$ in $]a - \varepsilon, a + \varepsilon[\times \mathcal{S}$ and a neighborhood V of $x(a)$ in $\mathcal{M}$. Note that $\mathcal{S}$ is the set where the initial value u of x lies, while the initial velocity is in $T_u\mathcal{S}$, so if $t \neq a$ then $x(t,u) \notin \mathcal{S}$. For, using the definition of $x(u,t)$, it is easy to calculate its differential at the point $(a, x(a))$ as follows:

$$\frac{\partial x}{\partial t}(a, x(a)) = \dot{x}(a), \quad \frac{\partial x}{\partial u}(a, x(a)) = \text{inclusion of } T_{x(a)}\mathcal{S} \text{ in } T_{x(a)}\mathcal{M}.$$

The claim follows from the Inverse Function Theorem, keeping in mind that $\dot{x}(a)$ and $T_{x(a)}\mathcal{S}$ generate $T_{x(a)}\mathcal{M}$.

We define a smooth function τ on V by:

$$\tau(x(t,u)) = t, \tag{65}$$

moreover we *extend* the vector field X and the function λ to V by setting:

$$X(x(t,u)) = \frac{\mathrm{d}}{\mathrm{d}t} x(t,u), \quad \lambda(x(t,u)) = \lambda(t,u).$$

Observe that $\langle X, X \rangle = 1$ and $\langle X, Y \rangle = 0$.

We want to prove that:

$$\mathrm{d}\tau = \langle X - \lambda Y, \cdot \rangle. \tag{66}$$

We shall first prove that $\mathcal{L}_X(\langle X - \lambda Y, \cdot \rangle) = 0$, where $\mathcal{L}$ denotes the *Lie derivative* of tensor fields. For this aim it is sufficient to take a vector field $v(t)$ along $x(t,u)$ which is invariant by the flow of X, and to prove that $\langle X - \lambda Y, v \rangle$ is constant.

Given such an invariant vector field v(which satisfies $\nabla_X v = \nabla_v X$), we differentiate the expression $\langle X - \lambda Y, v\rangle$ using the first equation in (64) and the fact that $\langle X, X\rangle$ and $\langle X, Y\rangle$ are constant:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle X-\lambda Y, v\rangle &= \langle \nabla_X(X-\lambda Y), v\rangle + \langle X-\lambda Y, \nabla_X v\rangle\\ &= \langle \nabla_X X, v\rangle - \langle \nabla_X X, v\rangle - \lambda\langle (\nabla Y)^*[X], v\rangle + \langle X-\lambda Y, \nabla_v X\rangle\\ &= -\lambda\langle \nabla_v Y, X\rangle + \langle X-\lambda Y, \nabla_v X\rangle = -\lambda\Big(\langle \nabla_v Y, X\rangle + \langle Y, \nabla_v X\rangle\Big)\\ &= -\lambda\Big(v\langle X, Y\rangle\Big) = 0.\end{aligned}$$

Now let us denote by Φ_X the flow of X and by $\Phi_X^t = \Phi_X(t,\cdot)$. Since $\tau \circ \Phi_X^s = \tau + s$ for all s, differentiating this expression we obtain, for any fixed s,

$$\mathrm{d}\tau \circ \mathrm{d}\Phi_X^s = \mathrm{d}\tau,$$

i.e., τ is invariant by the flow of X, namely $\mathcal{L}_X \mathrm{d}\tau = 0$. Since $\mathcal{L}_X(\langle X - \lambda Y, \cdot\,\rangle) = \mathcal{L}_X \mathrm{d}\tau$, to conclude the proof of (66), we now show that $\mathrm{d}\tau$ and $\langle X - \lambda Y, \cdot\rangle$ coincide on $\mathcal{S}$. To this aim, first observe that $T\mathcal{S}$ is in the kernel of both $\mathrm{d}\tau$ ($\tau = a$ on $\mathcal{S}$) and $\langle X - \lambda Y, \cdot\rangle$. Moreover, we clearly have:

$$\langle X - \lambda Y, X\rangle = \mathrm{d}\tau(X) = 1,$$

where the last equality follows by differentiating 65 with respect to t.

Since $T\mathcal{S}$ and X generate $T\mathcal{M}$ along $\mathcal{S}$, we conclude that $\mathrm{d}\tau$ and $\langle X - \lambda Y, \cdot\rangle$ coincide on $\mathcal{S}$ and (66) is proven.

Since $\langle X, X\rangle = 1$ and Y is orthogonal to $\mathcal{D}$, it follows immediately:

$$\|\mathrm{d}\tau|_{\mathcal{D}}\| = 1, \tag{67}$$

where $\|\cdot\|$ denotes the norm in $\mathcal{D}^*$ induced by $g|_{\mathcal{D}}$, and as a consequence of this, we have that for all horizontal curve $z : [c, d] \mapsto \mathcal{M}$ having image in the domain of τ the following estimate on the length of z holds:

$$L(z) \geq \left|\int_c^d \mathrm{d}\tau(\dot z)\,\mathrm{d}s\right| = |\tau(z(d)) - \tau(z(c))|. \tag{68}$$

Let now $\mu : [a, a+\varepsilon] \mapsto V$ be a horizontal curve with $\mu(a) = x(a)$ and $\mu(a+\varepsilon) = x(a+\varepsilon)$. Using (68), the length of μ is estimated as follows:

$$L(\mu) \geq \tau(\mu(a+\varepsilon)) - \tau(\mu(a)) = \varepsilon = L\big(x|_{[a,a+\varepsilon]}\big).$$

This implies that $x|_{[a,a+\varepsilon]}$ is a length minimizer between $x(a)$ and $x(a+\varepsilon)$ among all the horizontal curves with image in V. The conclusion of the proof will follow from the next Lemma, by possibly considering a smaller ε. □

The following Lemma appears in [10]; we repeat its proof here for the reader's convenience:

Lemma B.3. *Let $(\mathcal{M}, \Delta, g)$ be a sub-Riemannian manifold and let $V \subset \mathcal{M}$ be an open subset. Given $x_0 \in U$ there exists $r > 0$ such that every horizontal curve $\mu : [a, b] \mapsto \mathcal{M}$ with $\mu(a) = x_0$ and $L(\mu) < r$ satisfies $\mu([a, b]) \subset V$.*

Proof. We compare the sub-Riemannian metric g with the Euclidean metric relative to an arbitrary coordinate system around x_0. Let $\varphi : W \mapsto \widetilde{W}$ be a coordinate system in $\mathcal{M}$ with $x_0 \in W$, $W \subset V$ and $\widetilde{W}$ is an open neighborhood of 0 in $\mathbb{R}^n$. Let $B \subset W$ be the inverse image through φ of a closed ball of radius s, $B[\varphi(x_0); s] \subset \widetilde{W}$. For $m \in W$ and $v \in T_m\mathcal{M}$, denote by $\|v\|_e$ the Euclidean norm of the vector $\mathrm{d}\phi(m)[v]$. The set of vectors $v \in \Delta$ that are tangent to the points of B with $\|v\|_e = 1$ form a compact subset of $T\mathcal{M}$, in which the continuous function $v \mapsto \langle v, v\rangle^{\frac{1}{2}} = \|v\|$ attains a positive minimum k. Observe that for all $v \in \Delta$ tangent to some point of B, it is $\|v\| \geq k \cdot \|v\|_e$.

Take $r = ks > 0$. If $\mu : [a, b] \mapsto \mathcal{M}$ is a horizontal curve with $\mu(a) = x_0$ and $\mu([a, b]) \not\subset V$, then there exists $c \in \,]a, b[$ with $\mu([a, c] \subset B$ and $\gamma(c) \in \partial B$. Therefore,

$$L(\mu) \geq L(\mu|_{[a,c]}) \geq kL_e\big(\varphi \circ \mu|_{[a,c]}\big) \geq ks = r,$$

where L_e denotes the Euclidean length of a curve. This concludes the proof. □

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