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Abstract

This paper considers the functional linear calibration model without reference to any specific distributional assumption. Classical and inverse type estimators are proposed. First order approximations are obtained for the asymptotic bias and mean squared errors of the estimators considered. Simulation studies are presented in order to compare the estimators proposed.

1. Introduction

Statistical calibration is an area of practical statistical importance. The calibration model consists of two stages. In the first, the calibration experiment, based on a sample of size n, the relationship between the variables x and y is estimated. In a second stage a sample of size k of the dependent variable y corresponding to an unknown value of x, which we denote by x_0, is observed. The main interest is on estimating the unknown x_0. When the values $x_1, \ldots, x_n$ are fixed, the problem is called controlled calibration. Otherwise, if they are a sample from a random variable, then the problem is called natural calibration. The calibration problem has been extensively studied in the literature (Brown, 1993).The linear calibration model can be represented by the equations

(1.1) $$y_i = \alpha + \beta x_i + \epsilon_i,$$

(1.2) $$y_{0j} = \alpha + \beta x_0 + \epsilon_{0j},$$

$i = 1, \ldots, n$, $j = 1, \ldots, k$, where, $\epsilon_1, \ldots, \epsilon_n, \epsilon_{01}, \ldots, \epsilon_{0k}$ are independent and identically distributed. Typically, the errors are normally distributed with zero mean and constant variance.

Bayesian and classical estimation procedures abound in the literature. Two main estimators are considered in the literature for x_0 in the ordinary regression model. The classical, or maximum likelihood estimator and the inverse estimator, obtained by regressing x on y. A Bayesian justification for the inverse estimator is given in Hoadley (1970). Shukla (1972) derives first order approximation to the asymptotic bias and mean squared error of the two estimators, under the assumption that $|\beta| > 0$. Comparisons between the asymptotic bias and mean squared error conducted by Shukla (1972) show that the inverse estimator is preferable when the observable mean $\bar{x} = \sum_{i=1}^{n} x_i/n$ is close to the unknown x_0. However, Shukla (1972) concluded that the classical estimator seems to present an overall better large sample behavior.

There are many situations, however, in which the covariate x is observed with error. Measurement errors models are studied in Fuller (1987) and Cheng & Van Ness (1999). If the observed value, X_i, is such that

$$X_i = x_i + \delta_i, \tag{1.3}$$

where δ_i, $i = 1, \ldots, n$ are independent and identically distributed, the measurement error is additive. Hwang (1986) considers a multiplicative measurement error model where the observed value is given by

$$X_i = x_i \delta_i,$$

$i = 1, \ldots, n$. When the x_i are considered to be fixed unknown quantities, the functional model follows. Otherwise, if x_i, $i = 1, \ldots, n$, are random quantities then the structural model results. In the functional model with additive measurement errors and under the normality assumption to the errors distribution, Fuller (1987, Section 2.5) discusses properties of the inverse estimator, Bolfarine et al. (1997) and (1999) obtain first order approximation to the asymptotic bias and mean squared errors of the classical and inverse estimators with the additional assumption that $\lambda = \sigma_\epsilon^2/\sigma_\delta^2$ and σ_δ^2, respectively, are known. Bolfarine et al. (1999) consider linear calibration in the functional multiplicative model.

This paper extends Bolfarine et al. (1999) results to a more general class of distribution. It discusses linear calibration in the additive functional model described by equations (1.1)-(1.3) without the normality assumption. The errors ϵ_i, δ_i and ϵ_{0j} are supposed to be independent with $E[\epsilon_i] = E[\epsilon_{0j}] = E[\delta_i] = 0$, $E[\epsilon_i^l] = E[\epsilon_{0j}^l] = m_l^\epsilon$ and $E[\delta_i^l] = m_l^\delta$, where $m_\epsilon^{(l)}$ and $m_\delta^{(l)}$ are finite $1 \leq l \leq 4$, $i = 1, \ldots, n$, $j = 1, \ldots, k$. Classical and inverse type estimators based on the ordinary least squares estimators, which are not consistent under error-in-variables models (Fuller, 1987) and also on consistent estimators are considered.

The outline of the paper is as follows. Section 2 discusses the functional model without the normality assumption. Inverse and classical estimators based on consistent and least squares estimators are considered. Conditions for strong consistency of the estimators considered are studied. First order approximation for the asymptotic bias and mean squared error of the estimators considered for x_0 are obtained. Section 3 is divided in two parts. The first part presents some large sample comparisons of the estimators of x_0 considered in Section 2. The second part presents a simulation study aimed at verifying the accurateness of the expressions derived in Section 2 for the asymptotic bias and mean squared error of the estimators considered and also on the comparisons of the estimators proposed.

2. Estimators and properties

As presented in Section 1, two estimators of x_0 are usually considered under the linear calibration model. The first, usually known as classical estimator, is given by $\hat{x} = \frac{\bar{y}_0 - \hat{\alpha}}{\hat{\beta}}$ and the second estimator, obtained considering the model (1.1) written as $x_i = \gamma + \phi y_i + \epsilon_i^{'}$, $i = 1, \ldots, n$ where $e_i^{'} = -e_i\phi$, $\gamma = -\alpha/\beta$, $\phi = 1/\beta$, usually known as inverse estimator, is given by $\tilde{x} = \hat{\gamma} + \hat{\phi}\bar{y}_0$, where $\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma}$ and $\hat{\phi}$ are estimators of α, β, γ and ϕ, respectively, and $\bar{y}_0 = \sum_{j=1}^{k} y_{0j}/k$.

The least square estimators of α, β, γ and ϕ are defined by $\hat{\alpha}_L = \bar{y} - \bar{X}\hat{\beta}_L$, $\hat{\beta}_L = S_{xy}/S_{XX}$, $\hat{\gamma}_L = \bar{X} - \bar{y}\hat{\phi}_L$ and $\hat{\phi}_L = S_{Xy}/S_{yy}$, where $\bar{y} = \sum_{i=1}^{n} y_i/n$, $\bar{X} = \sum_{i=1}^{n} X_i/n$, $S_{yy} = \sum_{i=1}^{n}(y_i - \bar{y})^2/n$, $S_{XX} = \sum_{i=1}^{n}(X_i - \bar{X})^2/n$, $S_{Xy} = \sum_{i=1}^{n}(X_i - \bar{X})(y_i - \bar{y})/n$. It can be shown that $\hat{\beta}_L$ and $\hat{\phi}_L$ are inconsistent estimators of β and ϕ, respectively. Assuming that the sequence $\{x_i\}_{i\geq 1}$ satisfy the conditions

$$(2.1) \qquad \lim_{n\to\infty} \bar{x} = \mu < \infty, \quad \text{and} \quad \lim_{n\to\infty} \sum_{i=1}^{n} \frac{x_i^2}{n} = \delta^2 < \infty,$$

with finite population moments $m_j^{\delta} = E[\delta_i^j]$ and $m_j^{e} = E[e_i^j]$, $1 \leq j \leq 4$, then consistent estimators of α, β, γ and ϕ can be obtained. The next lemma presents consistent estimators of α and β.

Lemma 2.1. *Under the calibration model (1.1)-(1.3) with the assumption (2.1), and with m_j^{δ} and m_j^{e}, finite $1 \leq j \leq 4$, it follows that*

$$(2.2) \qquad \hat{\beta}_c = \frac{S_{Xy}}{S_{XX} - m_2^{\delta}} \quad \text{and} \quad \hat{\alpha}_c = \bar{y} - \hat{\beta}_c\bar{X}$$

or

$$(2.3) \qquad \hat{\beta}_v = \frac{(S_{yy} - \lambda S_{XX}) + \sqrt{(S_{yy} - \lambda S_{XX})^2 + 4\lambda S_{Xy}^2}}{2S_{Xy}} \quad \text{and} \quad \hat{\alpha}_v = \bar{y} - \hat{\beta}_v\bar{X}$$

with the additional assumption that m_2^{δ} or $\lambda = \frac{m_2^{e}}{m_2^{\delta}}$ is known, respectively, are strongly consistent estimators of α and β.

Proof. See Appendix B.

We call attention to the fact that these estimadors coincide with the consistent estimators obtained in Bolfarine et al. (1997 and 1999) under the normality assumption.

Consistent estimators of γ and ϕ are presented in the next lemma.

Lemma 2.2. *Under the calibration model (1.1)-(1.3) with the assumption (2.1), and m_j^δ and m_j^ε, finite $1 \le j \le 4$, it follows that*

$$\widehat{\phi}_v = \frac{S_{Xy}}{S_{yy} - m_2^\varepsilon} \quad \text{and} \quad \widehat{\gamma}_v = \bar{X} - \widehat{\phi}_v \bar{y} \tag{2.4}$$

or

$$\widehat{\phi}_c = \frac{S_{XX} - m_2^\delta}{S_{Xy}} \quad \text{and} \quad \widehat{\gamma}_c = \bar{X} - \widehat{\phi}_c \bar{y} \tag{2.5}$$

with the additional assumption that m_2^ε or m_2^δ is known, respectively, are strongly consistent estimators of ϕ and γ.

The proof is similar to the one in Lemma 2.2 and thus not provided.

Assuming that only m_2^δ is known, the consistent estimators considered are given by (2.2) and (2.5). Moreover, notice that $\widehat{\phi}_c = 1/\widehat{\beta}_c$, so that

$$\widehat{X}_c = -\frac{\widehat{\alpha}_c}{\widehat{\beta}_c} + \frac{\bar{y}_0}{\widehat{\beta}_c} = \widehat{\gamma}_c + \widehat{\phi}_c \bar{y}_0 = \tilde{X}_c, \tag{2.6}$$

meaning that in this case the classical and inverse estimators coincide. By considering the estimators of α, β, γ and ϕ defined above the following estimators of x_0 are considered:

$$\widehat{x}_c = \frac{\bar{y}_0 - \widehat{\alpha}_c}{\widehat{\beta}_c}, \qquad \widehat{x}_L = \frac{\bar{y}_0 - \widehat{\alpha}_L}{\widehat{\beta}_L},$$

and

$$\tilde{x}_L = \widehat{\gamma}_L + \widehat{\phi}_L \bar{y}_0.$$

According to (2.6), there is no need to consider $\tilde{x}_c = \widehat{\gamma}_c + \widehat{\phi}_c \bar{y}_0$ because it coincides with $\widehat{x}_c$. Notice that $\widehat{x}_c$ and $\tilde{x}_c$ are the classical and inverse estimators of x_0 combined with the consistent estimators of (α, β) and (γ, ϕ), respectively. On the other hand, $\widehat{x}_L$ and $\tilde{x}_L$ are the classical and inverse estimators of x_0 combined with the least squares estimators of (α, β) and (γ, ϕ), respectively. In Lemma 2.3 and 2.4, first order approximations are obtained for the expected value and variance of the least squares and consistent estimators of β, respectively.

Lemma 2.3. *Under the calibration model (1.1)-(1.3), it follows that*

$$E(\widehat{\beta}_L) = \frac{\beta l}{l + m_2^\delta} \left(1 + \frac{3l m_2^\delta + m_4^\delta - 2(m_2^\delta)^2}{n(l + m_2^\delta)^2} \right) + O(n^{-2}),$$

and

$$Var(\widehat{\beta}_L) = \frac{m_2^\varepsilon}{n(l + m_2^\delta)} + \frac{l\beta^2 \left(l^2 m_2^\delta - 3l(m_2^\delta)^2 + (m_2^\delta)^3 + l m_4^\delta \right)}{n(l + m_2^\delta)^4} + O(n^{-2}),$$

where $l = \sum_{i=1}^{n} \frac{(x_i-\bar{x})^2}{n}$.

Proof. See Appendix B.

Notice that $\widehat{\beta}_c$ and $\widehat{\phi}_c$ are continuous function of $(S_{Xy}, S_{XX}-m_2^\delta)$ and (S_{Xy}, S_{yy}) respectively, so the proofs of Lemma 2.4 and 2.5 are similar to that of Lemma 2.3 and they are not presented.

Lemma 2.4. *Under the calibration model (1.1)-(1.3) with the assumption that m_2^ε is known, it follows that*

$$E(\widehat{\beta}_c) = \beta + \frac{\beta(3lm_2^\delta + m_4^\delta - (m_2^\delta)^2)}{nl^2} + O(n^{-2})$$

and

$$Var(\widehat{\beta}_c) = \frac{\beta^2 m_2^\delta + m_2^\varepsilon}{nl} + \frac{\beta^2\left(m_4^\delta - (m_2^\delta)^2\right) + m_2^\varepsilon m_2^\delta}{nl^2} + O(n^{-2}),$$

where l is given in Lemma 2.3.

Results about the least square estimator of ϕ are presented next.

Lemma 2.5. *Under the calibration model (1.1)-(1.3) it follows that*

$$E(\widehat{\phi}_L) = \frac{Z}{\beta(1+Z)} + \frac{Z\left[3Z - 2 + \left(m_4^\varepsilon/(m_2^\delta)^2\right)\right]}{n\beta(1+Z)^3} + O(n^{-2})$$

and

$$Var(\widehat{\phi}_L) = \frac{m_2^\delta}{nm_2^\varepsilon(1+Z)} + \frac{Z\left[Z^2 - 3Z + 1 + Z\left(m_4^\varepsilon/(m_2^\delta)^2\right)\right]}{n\beta^2(1+Z)^4} + O(n^{-2}),$$

where

$$Z = \beta^2 \frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{nm_2^\varepsilon} = \frac{\beta^2 l}{m_2^\varepsilon}.$$

The following three theorems present the main results of the section. They provide first order approximation for the asymptotic bias, variance and mean square error of the estimators considered for x_0.

Theorem 2.1. *Under model (1.1)-(1.3) with the assumptions that m_i^ε and m_i^δ, $1 \le i \le 4$, are finite with m_2^δ known, it follows that*

$$E(\widehat{x}_c) = x_0 + \frac{(x_0-\bar{x})}{nl\beta^2}\left(m_2^\varepsilon - 2\beta^2 m_2^\delta + \frac{m_2^\varepsilon m_2^\delta}{l}\right) + O(n^{-2}),$$

$$
\begin{aligned}
Var(\widehat{x}_c) &= \left(\frac{\beta^2 m_2^\delta + m_2^\varepsilon}{n\beta^2}\right) + \frac{m_2^\varepsilon}{k\beta^2} - \frac{m_2^\varepsilon}{nkl^2\beta^2}\left(3lm_2^\delta - m_4^\delta + (m_2^\delta)^2\right) + \frac{(m_2^\varepsilon)^2}{nkl^2\beta^4}\left(3l + 3m_2^\delta\right) \\
&+ \frac{(x_0 - \bar{x})^2}{nl^2\beta^2}\left(l\beta^2 m_2^\delta + lm_2^\varepsilon + \beta^2 m_4^\delta - \beta^2 (m_2^\delta)^2 + m_2^\varepsilon m_2^\delta\right) + O(n^{-2})
\end{aligned}
$$

and

$$
MSE(\widehat{x}_c) = Var(\widehat{x}_c),
$$

where l is given in Lemma 2.3.

Proof. See Appendix B.

The proofs of Theorem 2.2 and 2.3 are similar to that of Theorem 2.1 and so they are presented.

Theorem 2.2. *Under model (1.1)-(1.3) with the assumptions that m_i^ε and m_i^δ, $1 \le i \le 4$, are finite, it follows that*

$$
\begin{aligned}
E(\widehat{x}_L) &= x_0 + \frac{m_2^\delta (x_0 - \bar{x})}{l} + \frac{(x_0 - \bar{x})}{nl}\left\{\frac{m_2^\delta (l^2 - 3lm_2^\delta + (m_2^\delta)^2) + lm_4^\delta}{l(l + m_2^\delta)} + \frac{m_2^\varepsilon (l + m_2^\delta)^2}{\beta^2 l^2}\right. \\
&- \left.\frac{\left(3lm_2^\delta + m_4^\delta - 2(m_2^\delta)^2\right)}{(l + m_2^\delta)}\right\} + O(n^{-2}),
\end{aligned}
$$

$$
\begin{aligned}
Var(\widehat{x}_L) &= \frac{m_2^\delta}{n} + \frac{(x_0 - \bar{x})^2}{nl^3}\left(\frac{m_2^\varepsilon (l + m_2^\delta)^3}{l\beta^2} + l^2 m_2^\delta - 3l(m_2^\delta)^2 + (m_2^\delta)^3 + lm_4^\delta\right) \\
&+ \frac{m_2^\varepsilon (l + m_2^\delta)^2}{l^2\beta^2}\left(\frac{3m_2^\varepsilon (l + m_2^\delta)}{nkl^2\beta^2} + \frac{1}{k} + \frac{1}{n}\right) \\
&- \frac{m_2^\varepsilon}{nkl^2\beta^2}\left(\frac{3l^2 m_2^\delta + 5l(m_2^\delta)^2 - 3(m_2^\delta)^3 - lm_4^\delta}{l}\right) + O(n^{-2})
\end{aligned}
$$

and

$$
\begin{aligned}
MSE(\widehat{x}_L) &= \frac{m_2^\delta}{n} + \frac{m_2^\varepsilon (l + m_2^\delta)^2}{l^2\beta^2}\left(\frac{3m_2^\varepsilon (l + m_2^\delta)}{nkl^2\beta^2} + \frac{1}{k} + \frac{1}{n}\right) - \frac{m_2^\varepsilon}{nkl^2\beta^2} \times \\
&\left(\frac{3l^2 m_2^\delta + 5l(m_2^\delta)^2 - 3(m_2^\delta)^3 - lm_4^\delta}{l}\right) + \frac{(x_0 - \bar{x})^2}{nl^2}\left\{\frac{m_2^\varepsilon (3m_2^\delta + l)(l + m_2^\delta)^2}{l^2\beta^2}\right. \\
&+ \left.\frac{(3m_2^\delta + l)(l^2 m_2^\delta - 3l(m_2^\delta)^2 + (m_2^\delta)^3 + lm_4^\delta) - 2lm_2^\delta (3lm_2^\delta - 2(m_2^\delta)^2 + m_4^\delta)}{l(l + m_2^\delta)}\right\} \\
&+ \left(\frac{m_2^\delta (x_0 - \bar{x})}{l}\right)^2 + O(n^{-2}),
\end{aligned}
$$

where l is given in Lemma 2.3.

Theorem 2.3. *Under model (1.1)-(1.3) with the assumptions that m_i^ξ and m_i^δ, $1 \le i \le 4$, are finite, it follows that*

$$E(\tilde{x}_L) = x_0 - \frac{(x_0 - \bar{x})}{Z+1} + \frac{(x_0 - \bar{x})Z(3Z - 2 + [m_4^\xi/(m_2^\xi)^2])}{n(Z+1)^3} + O(n^{-2}),$$

$$\begin{aligned} Var(\tilde{x}_L) &= \frac{m_2^\delta}{n} + \frac{Z^2 m_2^\xi}{n\beta^2(Z+1)^2} + \frac{Z^2 m_2^\xi}{k\beta^2(Z+1)^2} + \frac{(x_0-\bar{x})^2}{n(Z+1)} \left\{ \frac{\beta^2 m_2^\delta}{m_2^\xi} \right. \\ &+ \left. \frac{Z(Z^2 - 3Z + 1 + Z[m_4^\delta/(m_2^\xi)^2])}{(Z+1)^3} \right\} + \frac{m_2^\delta}{nk(Z+1)} \\ &+ \frac{Z}{nk\beta^2(Z+1)^4} \left\{ m_2^\xi(7Z^2 - 7Z + 1) + \frac{3Zm_4^\delta}{m_2^\xi} \right\} + O(n^{-2}) \end{aligned}$$

and

$$\begin{aligned} MSE(\tilde{x}_L) &= \frac{m_2^\delta}{n} + \frac{Z^2 m_2^\xi}{n\beta^2(Z+1)^2} + \frac{Z^2 m_2^\xi}{k\beta^2(Z+1)^2} + \frac{m_2^\delta}{nk(Z+1)} \\ &+ \frac{Z}{nk\beta^2(Z+1)^4} \left\{ m_2^\xi(7Z^2 - 7Z + 1) + \frac{3Zm_4^\xi}{m_2^\xi} \right\} + \frac{(x_0-\bar{x})^2}{n(Z+1)} \left\{ \frac{\beta^2 m_2^\delta}{m_2^\xi} \right. \\ &+ \left. \frac{Z(Z^2 - 9Z + 5 + (Z-2)(m_4^\xi/(m_2^\xi)^2))}{(Z+1)^3} \right\} + \frac{(x_0-\bar{x})^2}{(Z+1)^2} + O(n^{-2}), \end{aligned}$$

where Z is given in Lemma 2.5.

We call attention to the fact that considering the calibration model (1.1)-(1.3) with the additional assumption that the errors are independent and normally distributed, the expressions obtained for the asymptotic bias and mean squared error of the estimators proposed coincide with the ones given in Bolfarine et al.(1999). Notice also that, if $m_2^\delta = \sigma_\delta^2 = 0$, that is, the covariate x is observed without error and under the normality assumption the results presented coincides with the corresponding expression given in Shukla (1972). Lwin (1981) reports also some studies under the same model considered in Shukla (1972) but without the normality assumption. The errors are considered to be in a class of distribution with finite fourth moment. However, his asymptotic square error of the inverse estimator ($\tilde{x}_L$) is different from the one obtained in this article. We conclude that the asymptotic mean squared error of the inverse estimador is not dependent on the skewness coefficient reported in Lwin (1981).

3. Comparing the estimators

In this section, large sample results and a simulation study are used to compare the estimators considered for x_0. The large sample comparisons, which are based on the leading terms of the expansions obtained in Theorems 2.1, 2.2 and 2.3, are presented next. The simulation study is based on the expansions derived, including the first order terms for the asymptotic bias and mean squared errors obtained in Section 2.

3.1. Large sample comparisons

From Theorems 2.1, 2.2 and 2.3, it follows that

$$\lim_{n\to\infty} Bias(\widehat{x}_c) = 0,$$

$$\lim_{n\to\infty} Bias(\widehat{x}_L) = \frac{m_2^\delta(x_0 - \bar{x})}{l}$$

and

$$\lim_{n\to\infty} Bias(\tilde{x}_L) = -\frac{(x_0 - \bar{x})}{Z+1}.$$

Although all the estimators are biased, the consistent classical estimator is asymptotically unbiased. Notice that the three estimators are asymptotically unbiased when $x_0 = \bar{x}$. Furthermore, as $n \to \infty$ and $k \to \infty$, the behavior of the biases are the same.

Comparing the asymptotic mean squared errors of the estimators considered we can notice that

$$\lim_{n\to\infty} MSE(\widehat{x}_c) = \frac{m_2^\varepsilon}{k\beta^2},$$

$$\lim_{n\to\infty} MSE(\widehat{x}_L) = \frac{m_2^\varepsilon(l + m_2^\delta)^2}{kl^2\beta^2} + \left(\frac{m_2^\delta(x_0 - \bar{x})}{l}\right)^2$$

and

$$\lim_{n\to\infty} MSE(\tilde{x}_L) = \frac{Z^2 m_2^\varepsilon}{k\beta^2(Z+1)^2} + \frac{(x_0 - \bar{x})^2}{(Z+1)^2}.$$

Thus, as $n \to \infty$,

$$MSE(\widehat{x}_c) < MSE(\widehat{x}_L)$$

and

$$MSE(\widehat{x}_c) - MSE(\tilde{x}_L) = \frac{m_2^\varepsilon}{k\beta^2}\left(1 - \frac{Z^2}{(Z+1)^2}\right) - \frac{(x_0 - \bar{x})^2}{(Z+1)^2}.$$

Thus, for large n, when $x_0 = \bar{x}$,

$$MSE(\tilde{x}_L) < MSE(\widehat{x}_c) < MSE(\widehat{x}_L),$$

that is, with respect to the mean squared error the inverse estimator is better than the classical consistent estimator and this is better than the classical with mean squared when x_0 is closed to $\overline{x}$.

Further, as $n \to \infty$ and $k \to \infty$, it follows that

$$\lim_{k\to\infty} \lim_{n\to\infty} MSE(\widehat{x}_c) = 0,$$

$$\lim_{k\to\infty} \lim_{n\to\infty} MSE(\widehat{x}_L) = \left(\frac{m_2^\delta(x_0 - \overline{x})}{l}\right)^2$$

and

$$\lim_{k\to\infty} \lim_{n\to\infty} MSE(\tilde{x}_L) = \frac{(x_0 - \overline{x})^2}{(Z+1)^2} = \frac{(m_2^\varepsilon)^2(x_0 - \overline{x})^2}{(\beta^2 l + m_2^\varepsilon)^2}.$$

Thus,

$$MSE(\widehat{x}_L) - MSE(\tilde{x}_L) = \frac{(x_0 - \overline{x})^2}{l^2(\beta^2 l + m_2^\varepsilon)^2}[(m_2^\delta)^2(\beta^2 l + m_2^\varepsilon)^2 - l^2(m_2^\varepsilon)^2].$$

The above results imply that the classical estimator combined with consistent estimators of α and β is consistent, while classical and inverse estimator combined with the mean squared estimators of α e β are not. But all estimators are consistent when $x_0 = \bar{x}$. Comparisons between $\widehat{x}_L$ and $\tilde{x}_L$ will depend on the model parameters, and can be studied numerically. If $0 < l < 1$, that is, the values of x are closed, it follows that $MSE(\tilde{x}_L) < MSE(\widehat{x}_L)$. When $l > 1$, the choose of one of the estimators is not well defined by mean squared error.

3.2. A simulation study

In this section a simulation study is presented to compare the estimators proposed and to illustrate the behavior of the asymptotic bias and mean squared error obtained in Section 2. The simulation study was made using MATLAB, version 4.0. In the study, 1,000 samples of size $n = 10, 20, 30, 50, 100$ were generated according to the model (1.1) - (1.3) considering $\alpha = 1$, $\beta = 5$, $x_0 = 10.0$ and $k = 2$. The parameters x_i, $i = 1, \ldots, n$, are fixed but independently drawn from a $U(10, 30)$. Moreover, the errors were generated from the distributions normal, t-Student and extreme value, as follows. The approximated (including first order term) asymptotic bias and mean squared error of the estimators $\widehat{x}_c$, $\widehat{x}_L$, $\tilde{x}_L$ given in Theorems 2.1-2.3 are obtained for the parameters given above. For each selected sample, the estimators $\widehat{x}_c$, $\widehat{x}_L$ and $\tilde{x}_L$ were computed and their simulated bias and mean squared error evaluated by $\sum(\widehat{x}_G - x_0)/1000$ and $\sum(\widehat{x}_G - x_0)^2/1000$, respectively, where $\widehat{x}_G$ is any one of the above estimators and $\sum$ is extended to the 1,000 generated samples.

Model with t-Student Error Distribution

The errors $\varepsilon = (\varepsilon_1, ..., \varepsilon_n, \varepsilon_{0n+1}, ..., \varepsilon_{0n+k})$ were generated according to the t-Student distribution with $\nu = 5$ and 7 degrees of freedom that correspond to $m_2^\varepsilon = 5/3$ e $7/5$, respectively. The errors $\delta_1, ..., \delta_n$ were also generated from the t-Student, supposing that $m_2^\delta = 5/3$. The approximated and empirical bias and mean squared errors for each estimator proposed are presented in Tables 3.1 and 3.2.

Table 3.1. Bias of $\widehat{x}_G$ with $m_2^\delta = 5/3$

		$Bias\,(\widehat{x}_c)$		$Bias\,(\widehat{x}_L)$		$Bias\,(\tilde{x}_L)$	
ε	n	EMPIR.	APPROX.	EMPIR.	APPROX.	EMPIR.	APPROX.
$\upsilon = 5$	10	0.244	0.169	-0.626	-0.701	0.012	0.024
	20	0.100	0.073	-0.648	-0.675	0.031	0.025
	30	0.060	0.035	-0.479	-0.503	0.034	0.019
	50	0.035	0.019	-0.450	-0.465	0.030	0.018
	100	0.018	0.009	-0.439	-0.448	0.023	0.018
$\upsilon = 7$	10	0.243	0.169	-0.627	-0.701	0.007	0.020
	20	0.102	0.073	-0.646	-0.675	0.027	0.021
	30	0.060	0.035	-0.479	-0.503	0.031	0.016
	50	0.034	0.019	-0.451	0.465	0.026	0.015
	100	0.020	0.009	-0.437	-0.450	0.023	0.015

Table 3.2. Mean Squared Error of $\widehat{x}_G$ with $m_2^\delta = 5/3$

		$MSE\,(\widehat{x}_c)$		$MSE\,(\widehat{x}_L)$		$MSE\,(\tilde{x}_L)$	
ε	n	EMPIR.	APPROX.	EMPIR.	APPROX.	EMPIR.	APPROX.
$\upsilon = 5$	10	1.720	1.662	1.892	2.000	1.315	1.064
	20	0.742	0.737	1.096	1.142	0.607	0.514
	30	0.366	0.354	0.570	0.592	0.322	0.277
	50	0.227	0.201	0.418	0.412	0.212	0.163
	100	0.075	0.110	0.268	0.311	0.069	0.094
$\upsilon = 7$	10	1.700	1.650	1.871	1.985	1.301	1.052
	20	0.729	0.728	1.080	1.132	0.595	0.505
	30	0.363	0.347	0.567	0.584	0.318	0.270
	50	0.218	0.194	0.410	0.405	0.203	0.157
	100	0.071	0.105	0.262	0.304	0.066	0.088

With the above specifications, and from the results reported in Theorems 2.1, 2.2 and 2.3, it follows that the asymptotic bias and mean squared error of the estimators considered, as $n \to \infty$ and $\upsilon = 5$, are such that

$$Bias\,(\widehat{x}_c) \to 0 \quad \text{and} \quad MSE\,(\widehat{x}_c) \to 0.033,$$
$$Bias\,(\widehat{x}_L) \to -0.125 \quad \text{and} \quad MSE\,(\widehat{x}_L) \to 0.050,$$
$$Bias\,(\tilde{x}_L) \to 0.005 \quad \text{and} \quad MSE\,(\tilde{x}_L) \to 0.033,$$

and when $n \to \infty$ and $v = 7$, are

$$Bias\,(\widehat{x}_c) \to 0 \quad \text{and} \quad MSE\,(\widehat{x}_c) \to 0.028,$$
$$Bias\,(\widehat{x}_L) \to -0.125 \quad \text{and} \quad MSE\,(\widehat{x}_L) \to 0.044,$$
$$Bias\,(\tilde{x}_L) \to 0.004 \quad \text{and} \quad MSE\,(\tilde{x}_L) \to 0.028.$$

Model with Extreme Value Error Distribution

The errors $\varepsilon = (\varepsilon_1, ..., \varepsilon_n, \varepsilon_{0n+1}, ..., \varepsilon_{0n+k})$ were generated from a extreme value distribution with zero mean and variance $m_2^\varepsilon = 1.64493\phi^2$, where $\phi = 1$ and 2. The errors $\delta_1, ..., \delta_n$ were also generated from a extreme value distribution with zero mean and supposing variance $m_2^\delta = 5/3$. Tables 3.3 and 3.4 present, respectively, the approximated and empirical bias and mean squared error of all the estimators proposed.

Table 3.3. Bias of $\widehat{x}_G$ with $m_2^\delta = 5/3$

		$Bias\,(\widehat{x}_c)$		$Bias\,(\widehat{x}_L)$		$Bias\,(\tilde{x}_L)$	
ε	n	EMPIR.	APPROX.	EMPIR.	APPROX.	EMPIR.	APPROX.
$\phi = 1$	10	0.168	0.169	-0.700	-0.701	0.018	0.024
	20	0.061	0.073	-0.688	-0.675	0.012	0.025
	30	0.030	0.035	-0.509	-0.503	0.031	0.020
	50	0.021	0.019	-0.463	-0.465	0.023	0.018
	100	0.011	0.009	-0.447	-0.448	0.022	0.018
$\phi = 2$	10	0.164	0.158	-0.706	-0.713	0.094	0.094
	20	0.053	0.068	-0.697	-0.680	0.082	0.099
	30	0.030	0.033	-0.509	-0.505	0.088	0.076
	50	0.023	0.018	-0.461	-0.467	0.079	0.071
	100	0.007	0.008	-0.451	-0.449	0.070	0.069

Table 3.4. Mean Squared Error of $\widehat{x}_G$ with $m_2^\delta = 5/3$

		$MSE\,(\widehat{x}_c)$		$MSE\,(\widehat{x}_L)$		$MSE\,(\tilde{x}_L)$	
ε	n	EMPIR.	APPROX.	EMPIR.	APPROX.	EMPIR.	APPROX.
$\phi = 1$	10	1.165	1.394	1.490	1.732	1.067	1.062
	20	0.486	0.636	0.914	1.041	0.494	0.513
	30	0.264	0.319	0.507	0.557	0.274	0.276
	50	0.152	0.183	0.361	0.394	0.151	0.163
	100	0.089	0.102	0.288	0.303	0.091	0.093
$\phi = 2$	10	1.360	1.617	1.733	2.018	1.267	1.272
	20	0.642	0.792	1.107	1.229	0.646	0.669
	30	0.381	0.446	0.639	0.700	0.394	0.407
	50	0.267	0.297	0.486	0.521	0.268	0.278
	100	0.187	0.208	0.400	0.420	0.190	0.201

From the results reported in Theorems 2.1-2.3 and with the above specifications, it follows that the asymptotic bias and mean squared error of the estimators considered, as $n \to \infty$ and $\phi = 1$, are such that

$$\begin{array}{lcl} Bias\,(\widehat{x}_c) \to 0 & \text{and} & MSE\,(\widehat{x}_c) \to 0.033, \\ Bias\,(\widehat{x}_L) \to -0.125 & \text{and} & MSE\,(\widehat{x}_L) \to 0.049, \\ Bias\,(\tilde{x}_L) \to 0.005 & \text{and} & MSE\,(\tilde{x}_L) \to 0.033, \end{array}$$

and when $n \to \infty$, and $\phi = 2$, are

$$\begin{array}{lcl} Bias\,(\widehat{x}_c) \to 0 & \text{and} & MSE\,(\widehat{x}_c) \to 0.132, \\ Bias\,(\widehat{x}_L) \to -0.125 & \text{and} & MSE\,(\widehat{x}_L) \to 0.151, \\ Bias\,(\tilde{x}_L) \to 0.020 & \text{and} & MSE\,(\tilde{x}_L) \to 0.132. \end{array}$$

Note from the tables that the bias and MSE of all the estimators tend toward the limiting values presented above. The tables also show that, in general, the approximations are satisfactory, that is, the approximated and empirical values are closer, even for small samples sizes.

Notice that independent of the distribution considered to the errors (t-Student, extreme value), the behavior of the bias and mean squared error of the estimators of x_0, are, in general, similar. For large values of n, note that the approximated and empirical bias of the estimator $\widehat{x}_c$ are less than the ones for the other estimators, and for small values of n, the inverse estimator seems to performs best. With respect to the mean squared error we can note that,

$$MSE(\tilde{x}_L) < MSE(\widehat{x}_c) < MSE(\widehat{x}_L),$$

that is, the least mean squared error is given by the inverse estimator, that is better than the classical with consistent estimator and this is better than the classical with least squared estimator. When the sample size increase, the difference between MSE of $\tilde{x}_L$ and $\widehat{x}_c$ is reduced, that is, the MSE of both estimators are closer. In accordance with the comparisons (Section 2), the behavior of $\widehat{x}_L$ is worse than the others estimators, only for $n = 20$ it seems to be similar. It shows that the bias and mean squared error of the estimators of x_0 do not depend on the skewness of the distribution, as we comment in Section 2.

The study conducted by the authors includes considering $\varepsilon = (\varepsilon_1, ..., \varepsilon_n, \varepsilon_{0n+1}, ..., \varepsilon_{0n+k})$ generated from a t-student distribution and a extreme value distribution and the errors $\delta_1, ..., \delta_n$ generated from a Normal distribution $N(0,1)$. The overall conclusions seem to be similar to the ones shown in Table 3.1 - 3.4 and thus are not reported.

Appendix A

It can be show that

$$E(S_{Xy}) = \beta l,$$

$$E(S_{XX}) = l + m_2^\delta - \frac{m_2^\delta}{n} = l + m_2^\delta + O(n^{-1}),$$

$$E(S_{yy}) = \beta^2 l + m_2^\varepsilon - \frac{m_2^\varepsilon}{n} = \beta^2 l + m_2^\varepsilon + O(n^{-1}),$$

$$Var(S_{Xy}) = \beta^2 l\frac{m_2^\delta}{n} + l\frac{m_2^\varepsilon}{n} + \frac{m_2^\delta m_2^\varepsilon}{n} - \frac{m_2^\delta m_2^\varepsilon}{n^2} = l\left(\frac{\beta^2 m_2^\delta + m_2^\varepsilon}{n}\right) + \frac{m_2^\delta + m_2^\varepsilon}{n} + O(n^{-2}),$$

$$\begin{aligned} Var(S_{XX}) &= 4l\frac{m_2^\delta}{n} - \frac{(m_2^\delta)^2}{n} + \frac{m_4^\delta}{n} - \frac{2m_4^\delta}{n^2} + \frac{4(m_2^\delta)^2}{n^2} + \frac{5(m_2^\delta)^2}{n^2} + \frac{m_4^\delta}{n^3} - \frac{6(m_2^\delta)^2}{n^3} \\ &= 4l\frac{m_2^\delta}{n} + \frac{(m_4^\delta - (m_2^\delta)^2)}{n} + O(n^{-2}), \end{aligned}$$

$$\begin{aligned} Var(S_{yy}) &= 4\beta^2 l\frac{m_2^\varepsilon}{n} + \frac{m_4^\varepsilon}{n} - \frac{(m_2^\varepsilon)^2}{n} - \frac{2m_4^\varepsilon}{n^2} + \frac{7(m_2^\varepsilon)^2}{n^2} + \frac{m_4^\varepsilon}{n^3} - \frac{6(m_2^\varepsilon)^2}{n^3} \\ &= 4\beta^2 l\frac{m_2^\varepsilon}{n} + \frac{m_4^\varepsilon}{n} - \frac{(m_2^\varepsilon)^2}{n} + O(n^{-2}), \end{aligned}$$

$$E(S_{Xy}S_{XX}) = \beta l\left(l + m_2^\delta + \frac{m_2^\delta}{n}\right) = \beta l\left(l + m_2^\delta\right) + O(n^{-1}),$$

$$E(S_{Xy}(S_{XX} - m_2^\delta)) = \beta l\left(l + \frac{m_2^\delta}{n}\right) = \beta l^2 + O(n^{-1}),$$

$$E(S_{Xy}S_{yy}) = \beta l\left(\beta^2 l + m_2^\varepsilon + \frac{m_2^\varepsilon}{n}\right) = \beta l\left(\beta^2 l + m_2^\varepsilon\right) + O(n^{-1}),$$

$$Cov(S_{Xy}, S_{yy}) = 2\beta l\frac{m_2^\varepsilon}{n},$$

$$Cov(S_{Xy}, S_{XX}) = 2\beta l\frac{m_2^\delta}{n},$$

where

$$l = \frac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{n}.$$

Appendix B

Proof of Lemma 2.1. Notice that $X_1y_1, X_2y_2, \ldots$ are independent random variables with $E(X_iy_i) = \alpha x_i + \beta x_i^2$ and $Var(X_iy_i) = m_2^\delta(\alpha^2 + 2\alpha\beta x_i + \beta^2 x_i^2 + m_2^\varepsilon) + x_i^2 m_2^\varepsilon$, $i = 1, 2, \ldots$, which is finite according to the assumption and implies that $\sum_{n=1}^{\infty} \frac{Var(X_nY_n)}{n^2} < \infty$. By using the Kolmogorov's strong law of large numbers it follows that

$$\sum_{i=1}^{n} \frac{X_iy_i}{n} - \alpha \sum_{i=1}^{n} \frac{x_i}{n} - \beta \sum_{i=1}^{n} \frac{x_i^2}{n} \xrightarrow{q.c} 0.$$

Assumption (2.1) implies that

$$\sum_{i=1}^{n} \frac{X_iy_i}{n} \xrightarrow{q.c} \alpha\mu + \beta\delta^2.$$

Similarly, it can be shown that

$$\overline{y} = \sum_{i=1}^{n} \frac{y_i}{n} \xrightarrow{q.c} \alpha + \beta\mu, \quad \overline{X} = \sum_{i=1}^{n} \frac{X_i}{n} \xrightarrow{q.c} \mu,$$

$$\sum_{i=1}^{n} \frac{X_i^2}{n} \xrightarrow{q.c} \delta^2 + m_2^\delta, \quad \sum_{i=1}^{n} \frac{y_i^2}{n} \xrightarrow{q.c} \alpha^2 + 2\alpha\beta\mu + \beta^2\delta^2 + m_\varepsilon^2,$$

$$S_{XX} \xrightarrow{q.c} \delta^2 + m_2^\delta - \mu^2, \qquad S_{yy} \xrightarrow{q.c} \beta^2(\delta^2 - \mu^2) + m_2^\varepsilon,$$

and

$$S_{Xy} \xrightarrow{q.c} \beta(\delta^2 - \mu^2).$$

Since the estimators $\widehat{\beta}_c$, $\widehat{\beta}_v$, $\widehat{\alpha}_v$ and $\widehat{\alpha}_c$ are continuous function of the sample moments $\overline{X}, \overline{y}, S_{XX}, S_{Xy}$ and S_{yy}, except when $S_{Xy} = 0$ and $S_{XX} = m_2^\delta$, which has probability zero, it is easy to show that

$$\lim_{n\to\infty} \widehat{\beta}_c = \beta, \quad \lim_{n\to\infty} \widehat{\beta}_v = \beta \quad \lim_{n\to\infty} \widehat{\alpha}_c = \alpha \quad \text{and} \quad \lim_{n\to\infty} \widehat{\alpha}_v = \alpha,$$

with probability one, concluding the strong consistency of these estimators.

Proof of Lemma 2.3. Notice that the estimator $\widehat{\beta}_L$ is a continuous function of (S_{Xy}, S_{XX}).

A Taylor series expansion of $\widehat{\beta}_L$, at $a = (E(S_{Xy}), E(S_{XX}))$, leads, up to first order terms,to

$$\begin{aligned}
\widehat{\beta}_L &= \frac{\beta l}{l + m_2^\delta - \frac{m_2^\delta}{n}} + \frac{(S_{Xy} - E(S_{Xy}))}{1!}\left(\frac{1}{l + m_2^\delta} + O(n^{-1})\right) \\
&+ \frac{(S_{XX} - E(S_{XX}))}{1!}\left(-\frac{\beta l}{(l + m_2^\delta)^2} + O(n^{-1})\right) \\
&+ \frac{(S_{XX} - E(S_{XX}))^2}{2!}\left(\frac{2\beta l}{(l + m_2^\delta)^3} + O(n^{-1})\right) \\
&+ (S_{Xy} - E(S_{Xy}))(S_{XX} - E(S_{XX}))\left(-\frac{1}{(l + m_2^\delta)^2} + O(n^{-1})\right) + O_p(n^{-2}).
\end{aligned}$$

Taking the expectation and the variance of this expansion, it follows that

$$E(\widehat{\beta}_L) = \frac{\beta l}{l + m_2^\delta - \frac{m_2^\delta}{n}} + \frac{\beta l}{(l + m_2^\delta)^3} Var(S_{XX}) - \frac{Cov(S_{Xy}, S_{XX})}{(l + m_2^\delta)^2} + O(n^{-2})$$

and

$$Var(\widehat{\beta}_L) = \frac{Var(S_{Xy})}{(l + m_2^\delta)^2} + \frac{\beta^2 l^2}{(l + m_2^\delta)^4} Var(S_{XX}) - \frac{2\beta l Cov(S_{Xy}, S_{XX})}{(l + m_2^\delta)^3} + O(n^{-2}).$$

The results follow by replacing the expressions for variances and covariances of the sample moments obtained in Appendix A.

Proof of Theorem 2.1. Notice that we can write

$$\widehat{x}_c = \frac{\overline{y}_0 - \widehat{\alpha}_c}{\widehat{\beta}_c} = \overline{X} + \frac{\beta(x_0 - \overline{x}) + \overline{\varepsilon}_0 - \overline{\varepsilon}}{\widehat{\beta}_c},$$

where $\overline{\varepsilon} = \frac{\sum_{i=1}^{n} \varepsilon_i}{n}$ and $\overline{\varepsilon}_0 = \frac{\sum_{i=n+1}^{n+k} \varepsilon_{0i}}{k}$. The mean and variance of $\widehat{x}_c$ are given by

$$E(\widehat{x}_c) = E(\overline{X}) + \beta(x_0 - \overline{x}) E\left[\frac{1}{\widehat{\beta}_c}\right] \tag{B.1}$$

and

$$Var(\widehat{x}_c) = Var(\overline{X}) + \beta^2 (x_0 - \overline{x})^2 Var\left[\frac{1}{\widehat{\beta}_c}\right] + Var\left[\frac{\overline{\varepsilon}_0}{\widehat{\beta}_c}\right] + Var\left[\frac{\overline{\varepsilon}}{\widehat{\beta}_c}\right]$$

$$+ \; 2Cov\left(\overline{X}, \frac{\beta(x_0-\overline{x})}{\widehat{\beta}_c}\right) + 2Cov\left(\overline{X}, \frac{\overline{\varepsilon}_0}{\widehat{\beta}_c}\right) - 2Cov\left(\overline{X}, \frac{\overline{\varepsilon}}{\widehat{\beta}_c}\right)$$

$$+ \; 2Cov\left(\frac{\beta(x_0-\overline{x})}{\widehat{\beta}_c}, \frac{\overline{\varepsilon}_0}{\widehat{\beta}_c}\right) - 2Cov\left(\frac{\beta(x_0-\overline{x})}{\widehat{\beta}_c}, \frac{\overline{\varepsilon}}{\widehat{\beta}_c}\right) - 2Cov\left(\frac{\overline{\varepsilon}_0}{\widehat{\beta}_c}, \frac{\overline{\varepsilon}}{\widehat{\beta}_c}\right).$$

It can be shown that the covariances are all zero, so

$$(B.2) \qquad Var(\widehat{x}_c) = Var(\overline{X}) + \beta^2(x_0-\overline{x})^2 Var\left[\frac{1}{\widehat{\beta}_c}\right] + Var\left[\frac{\overline{\varepsilon}_0}{\widehat{\beta}_c}\right] + Var\left[\frac{\overline{\varepsilon}}{\widehat{\beta}_c}\right].$$

The mean and variance depend on $\frac{1}{\widehat{\beta}_c}$, which can be written, after a Taylor series expansion, as

$$(B.3) \qquad \frac{1}{\widehat{\beta}_c} = \frac{1}{E(\widehat{\beta}_c)} - \frac{(\widehat{\beta}_c - E(\widehat{\beta}_c))}{[E(\widehat{\beta}_c)]^2} + \frac{(\widehat{\beta}_c - E(\widehat{\beta}_c))^2}{[E(\widehat{\beta}_c)]^3} + O_p(n^{-2})$$

A Taylor series expanding of $\frac{1}{E(\widehat{\beta}_c)}$, $\frac{1}{[E(\widehat{\beta}_c)]^2}$ e $\frac{1}{[E(\widehat{\beta}_c)]^3}$ at $a=\beta$, leads

$$(B.4) \qquad \frac{1}{E(\widehat{\beta}_c)} = \frac{1}{\beta} - \frac{Bias(\widehat{\beta}_c)}{\beta^2} + O(n^{-2}) = \frac{1}{\beta} + O(n^{-1}),$$

$$(B.5) \qquad \frac{1}{[E(\widehat{\beta}_c)]^2} = \frac{1}{\beta^2} - \frac{2Bias(\widehat{\beta}_c)}{\beta^3} + O(n^{-2}) = \frac{1}{\beta^2} + O(n^{-1})$$

and

$$(B.6) \qquad \frac{1}{[E(\widehat{\beta}_c)]^3} = \frac{1}{\beta^3} - \frac{3Bias(\widehat{\beta}_c)}{\beta^4} + O(n^{-2}) = \frac{1}{\beta^3} + O(n^{-1}),$$

where $Bias\left(\widehat{\beta}_c\right) = E\left(\widehat{\beta}_c\right) - \beta + O(n^{-2})$.

Replacing $(B.4)$ – $(B.6)$ in $(B.3)$ and taking the mean and the variance it follows that

$$(B.7) \qquad E\left(\frac{1}{\widehat{\beta}_c}\right) = \frac{1}{\beta} - \frac{Bias(\widehat{\beta}_c)}{\beta^2} + \frac{Var(\widehat{\beta}_c)}{\beta^3} + O(n^{-2})$$

and

$$(B.8) \qquad Var\left(\frac{1}{\widehat{\beta}_c}\right) = \frac{Var(\widehat{\beta}_c)}{\beta^4} + O(n^{-2}).$$

Similarly, it can be shown that

$$(B.9) \qquad Var\left(\frac{\bar{\varepsilon}_0}{\widehat{\beta}_c}\right) = \frac{Var(\bar{\varepsilon}_0)}{\beta^2} + \frac{3Var(\bar{\varepsilon}_0)Var(\widehat{\beta}_c)}{\beta^4} - \frac{2Var(\bar{\varepsilon}_0)Bias(\widehat{\beta}_c)}{\beta^3} + O(n^{-2})$$

and

$$(B.10) \qquad Var\left(\frac{\bar{\varepsilon}}{\widehat{\beta}_c}\right) = \frac{Var(\bar{\varepsilon})}{\beta^2} + O(n^{-2}),$$

where $Var(\bar{\varepsilon}_0) = \frac{m_2^\varepsilon}{k}$ and $Var(\bar{\varepsilon}) = \frac{m_2^\varepsilon}{n}$.

Replacing $(B.7) - (B.10)$ in (B.1) and (B.2) it follows that

$$E(\widehat{x}_c) = x_0 + (x_0 - \bar{x})\left(\frac{Var(\widehat{\beta}_c)}{\beta^2} - \frac{Bias(\widehat{\beta}_c)}{\beta}\right) + O(n^{-2})$$

and

$$\begin{aligned} Var(\widehat{x}_c) &= \frac{m_2^\delta}{n} + \frac{m_2^\varepsilon}{k\beta^2} + \frac{m_2^\varepsilon}{n\beta^2} + (x_0 - \bar{x})^2\frac{Var(\widehat{\beta}_c)}{\beta^2} \\ &+ \frac{3m_2^\varepsilon Var(\widehat{\beta}_c)}{k\beta^4} - \frac{2m_2^\varepsilon Bias(\widehat{\beta}_c)}{k\beta^3} + O(n^{-2}). \end{aligned}$$

The results follow replacing $E(\widehat{\beta}_c)$ and $Var(\widehat{\beta}_c)$ given in Lemma 2.4.

The asymptotic mean squared error of the estimators can be obtained by adding the asymptotic variance of the estimator to the square of the asymptotic bias. Because the asymptotic bias of the estimator $\widehat{x}_c$ is of the order n^{-1}, the asymptotic mean squared error of the estimator coincides with its asymptotic variance.

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