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Sônia P. Coelho

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Sônia P. Coelho

Instituto de Matemática e Estatística
Universidade de São Paulo - Brasil

Let KG be the group ring of a group G over a field K, and U its group of units. Given a group H, we shall denote by $\xi(H)$ the center of H and by T(H) the set of all its torsion elements.

The following question appears in [5, p. 231]: When is $U^n \subset \xi(U)$, for some n? It was considered by G. Cliff and S.K. Sehgal in [2], where G is assumed to be a solvable group. A complete answer at characteristic zero is given there. Also they obtain partial results at characteristic $p \neq 0$, with certain restrictions on the exponent n.

In this note, we shall answer the question at characteristic p assuming that G is either a solvable or an FC-group. In fact, we shall need specially the following property which is common to both these families of groups: if H is a finitely generated subgroup of G such that $H/\xi(H)$ is torsion, then both T(H) and H', the derived group of H, are finite groups [1, Lemma 1.5, p. 116 and 2, Lemma 2.1, p. 147].

In § 1, we answer the question for torsion groups assuming only that G is locally finite (Theorem A), and in § 3 we give the answer for non torsion groups that are either solvable or FC(Theorem C).

First, we introduce some notation. We will denote T(G) simply by T, and the integer $p \neq 0$ will always denote the characteristic of K. For an element t in a group, we shall say that t is a p-element if o(t), the order of t, is a power of p, and that t is a p'-element if o(t) is not divisible by p. Similarly, a group H will be called a p'-group if every element of H is a p'-element.

§ 1. The torsion subgroup of G

LEMMA 1.1: If $U^n \subset \xi(U)$ for some n and G has a non central p'-element, then K is finite and the orders of the p'-elements of G are bounded.

PROOF: We shall show first that the orders of the p'-elements of G are bounded.

It is enough to show that if u is a central p'-element of G, then $o(u) \leq n$. If not, take such an u, with $o(u) > n$. Then $K\langle u \rangle = \bigoplus_i K_i$, a direct sum of fields. For every i, denote by $\pi_i : K\langle u \rangle \longrightarrow K_i$ the natural projection (that is, if e_i is the unity element of K_i, then $\pi_i(u) = (u\, e_i)$. Clearly, at least one of the $\pi_i(u)$, say $\pi_1(u)$, has multiplicative order equal to o(u). As a consequence, K_1 has more than n elements.

Now, if t is a non central p'-element of G, we consider $K\langle u, t \rangle = \bigoplus_i K_i[t]$, where $K_i[t]$ denotes the smallest subalgebra of KG that contains K_i and t.

We claim that $K_1[t]$ is not contained in the center of KG. In fact, suppose that $K_1[t]$ is central and

let $e = e_1$ be the unity element of K_1. Then et is central. Now take $x \in G$ such that $xtx^{-1} \neq t$. Then, $xetx^{-1} = extx^{-1} = et$.

By considering supports in the last equality we get: $u^i xtx^{-1} = u^j t$, $0 \leq i, j < o(u)$, $i \neq j$. Hence $xtx^{-1} = u^{j-i} t$.

Again, in the last equality, we obtain : $eu^{j-i} t = et$, or $eu^{j-i} = e$. Then, $(eu)^{j-i} = e$. But $eu = \pi_1(u)$, and hence the multiplicative order of $\pi_1(u)$ divides $|j-i| < o(u)$, a contradiction.

Now, since t is a p'-element, we have: $K_1[t] = \oplus_i Li$, a direct sum of fields which are Galois extensions of K_1. But $K_1[t]$ is not central, hence some of the L_i, say L_1, is not contained in the center of KG. Let $\bar{L}_1$ be the subfield of L_1 consisting of its central elements, and let $\phi \neq 1$ be an $\bar{L}_1$-automorphism of L_1.

Since $L_1 = K(\zeta)$, with $\zeta^{o(t)} = 1$, we have that $\phi(\zeta) = \zeta^i$, for some i.

Now, take an arbitrary element $k \in \bar{L}_1$. Since $U^n \subset \xi(U)$, we get that $(\zeta+k)^n \in \bar{L}_1$. Then, $(\phi(\zeta+k))^n = \phi((\zeta+k)^n) = (\zeta+k)^n$, and $\phi(\zeta+k)$ is a root of $X^n - (\zeta+k)^n$; from this we see that $\phi(\zeta+k) = \alpha(\zeta+k)$, $\alpha^n = 1$, $\alpha \neq 1$.

On the other hand, $\phi(\zeta+k) = \zeta^i + k$, and thus $\zeta^i + k = \alpha(\zeta+k)$.

Solving this equation for k, we have that $k = \dfrac{\zeta^i - \alpha k}{\alpha - 1}$. Here, only α depends on k. Since α can take at most $n-1$ values, we see that $|L_1|$, the number of elements

of L_1, is at most n-1. But $\bar{L}_1 \supset K_1$, and $|K_1| > n$, a contradiction.

It still remains to prove that K is finite. If not, replace K_1 by K in the proof above. Again, we have that $K_1[t] = K[t]$ is not central, and we can repeat the argument to obtain a contradiction. ▲

LEMMA 1.2: Assume that $U^n \subset \xi(U)$ for some n. Then, there exists a positive integer m, which is a power of p, such that x^m is central in KG, for every nilpotent element x in KG.

PROOF: Let $x \in KG$ be a nilpotent element, and let r be such that $x^{p^r} = 0$. Then, $1 + x$ is a p-element of U and by hypothesis $(1 + x)^n \in \xi(U)$. Writing $n = p^t n'$, with $(n',p) = 1$, it is easy to see that x^{p^t} is central in KG. ▲

LEMMA 1.3: Assume that $U^n \subset \xi(U)$ for some n, and let $n = p^t \cdot n'$, with $(n',p) = 1$. If G has a p-element of order greater than $2p^{3t}$, then $G^{p^t} \subset \xi(G)$.

PROOF: From the proof of Lemma 1.2, we see that x^{p^t} is central for every nilpotent element $x \in KG$. So, set $m = p^t$, and take a p-element $h \in G$ such that $o(h) > 2m^3$. Since h-1 is nilpotent, we have that $(h-1)^m = h^m - 1$ is central, hence h^m is central.

Set $h' = h^m$, take $x, y \in G$ and consider the nilpotent element $y(h'-1)$. Again, by Lemma 1.2, we have that $(y(h'-1))^m = y^m(h'^m - 1)$ is central. Hence:

$$xy^m(h'^m - 1) = y^m(h'^m - 1) x,$$

$$xy^m h'^m - xy^m = y^m h'^m x - y^m x.$$

Since $o(h') > 2m^2$, we know that $h'^m - 1 \neq 0$ and hence we have two elements of G in the support of the above element. If $p \neq 2$, we see immediately that $xy^m = y^m x$, thus $y^m \in \xi(G)$. If $p = 2$, we may have:

$$xy^m = y^m h'^m x,$$

$$xy^m h'^m = y^m x.$$

Using the fact that h'^m is central and replacing $y^m x$ in the first equation by its value in the second one, we get that $xy^m = xy^m h'^m h'^m$, or $(h')^{2m} = 1$, which contradicts the fact that $o(h') > 2m^2$. ▲

LEMMA 1.4: Let m be a power of p. If $G^m \subset \xi(G)$ and G contains a normal p-abelian subgroup ϕ such that G/ϕ is a finite p-group, then G is nilpotent.

PROOF: It follows as in [5, 6.6, pp. 157-158]. ▲

LEMMA 1.5: Let S be a commutative ring with identity, I a nil ideal of S, of bounded exponent, and Q a finite group. Let $S(Q,\rho,\sigma)$ be a crossed product of Q over S, with an arbitrary factor system ρ and σ such that $\sigma_t(I) \subset I$, for every $t \in Q$. Then, IQ is a nil ideal of $S(Q,\rho,\sigma)$, of bounded exponent.

PROOF: It is immediate to verify that IQ is an ideal of $S(Q,\rho,\sigma)$.

Let $Q = \{t_1, t_2, \ldots, t_n\}$ and choose m such that $s^m = 0$, for every $s \in I$.

Take $r > m(n+1)^2$ and $x = s_1 t_1 + \ldots + s_n t_n$ an arbitrary element of IQ, $s_i \in I$, $1 \leq i \leq n$.

We want to prove that $x^r = 0$. It is enough to show that any product of r elements from the set $\{s_1t_1,\ldots,s_nt_n\}$ is zero. Then, let $y = s_{i_1}^{'} t_{i_1} \ldots s_{i_r} t_{i_r}$ be such a product.

It is easy to see that there exists an index j such that $s_{i_j} t_{i_j}$ occurs in y k times, with $k > m(n+1)$, and we may suppose without loss of generality that $s_{i_j} t_{i_j} = s_1 t_1$.

As the products $s_i t_i s_j t_j$ still have the form st, s∈I, t∈Q, and $ts_1 = \sigma_t(s_1)t$, for every σ∈T, we can write y in the form $y = (\prod_{i=1}^{k} z_i)\gamma$, with $z_i \in \{\sigma_t(s_1)\,|\,t∈Q\} \cup \{s_1\}$, $\gamma \in IQ$. Since the above set has at most n+1 elements and $k > m(n+1)$, there must exist an index j such that z_j occurs in y more than m times. Now, $z_j ∈ I$, therefore $z_j^m = 0$, and hence $y = 0$. ▲

LEMMA 1.6: Let G = T, a locally finite group. If $U^n \subset \xi(U)$ for some n, then KT satisfies a polynomial identity.

PROOF: Let m be as in Lemma 1.2, and consider 2m arbitrary elements of KT, say $x_1, x_2, \ldots, x_{2m}$. By considering the subgroup generated by the supports of these elements, we may suppose that T is finite.

Denote by J(KT) the Jacobson radical of KT. Then $KT/J(KT) = \bigoplus_i M_{n_i}(D_i)$, a direct sum of full matrix rings over division rings D_i.

Set x′ for the image of an element x ∈ KT through the natural epimorphism KT ⟶ KT/J(KT). For a given index i, take x_i an arbitrary nilpotent element in $M_{n_i}(D_i)$,

and choose any element $y_i \in KT$ such that $(y_i)' = x_i$. Then y_i is nilpotent, provided $J(KT)$ is nilpotent because T is finite. By Lemma 1.2, y_i^m is central in KT. Hence $x_i^m = (y_i^m)'$ is a central nilpotent element of $KT/J(KT)$, so it must be zero.

Now it is easy to see that the size of the matrices is bounded by m, that is, $n_i \leq m$, for every i.

On the other hand, given i and $d_i \neq 0$ in D_i, we can choose $u \in U$ such that $u' = d_i$ (see [5, Lemma 3.3, p. 179]). As $u^n \in \xi(U)$, d_i^n is central in D_i, and hence D_i is a field, by [3, Theorem 3.22, p.79].

Therefore, $KT/J(KT)$ satisfies $S_{2m}(X_1, X_2, \ldots, X_{2m})$, the standard polynomial of degree $2m$ in the non commuting variables $X_1, X_2, \ldots, X_{2m}$. Again, since $J(KT)$ is nilpotent, we can use Lemma 1.2 to obtain, for every $z \in KT$:
$$(S_{2m}(x_1, \ldots, x_{2m}))^m \, z = z \, (S_{2m}(x_1, \ldots, x_{2m}))^m. \qquad \blacktriangle$$

We may now obtain a characterization of the fact of U being of bounded exponent over the center when G is a locally finite group.

Theorem A: Let $G = T$, a locally finite group. Then, $U^n \subset \xi(U)$ for some n if and only if the following conditions hold:

    (i) $T^\ell \subset \xi(U)$ for some ℓ.

    (ii) T contains a normal p-abelian subgroup of finite index.

    (iii) either every p-element of T is central or T is of bounded exponent and K is finite.

PROOF: Suppose $U^n \subset \xi(U)$ for some n. Then (i) is trivial and (ii) follows from Lemma 1.6 and a Theorem by Passman [1, Corollary 3.10, p. 197].

To prove (iii), assume that not every p'-element of T is central. By Lemma 1.1, K is finite and $t^r = 1$, for a suitable r and for every p'-element $t \in T$. Now, if $n = p^t \cdot n'$, with $(n', p) = 1$, and T has a p-element of order greater than $2p^{3t}$, then $T^{p^t} \subset \xi(T)$ by Lemma 1.3, and hence every p'-element is central, a contradiction.

So, $t^s = 1$ for a suitable s and for every p-element $t \in T$.

Now take $x \in T$ and let T_0 be a normal p-abelian subgroup of index u, as in (ii). Then, $x^u \in T_0$, and we may write: $x^u = yz$, where $y, z \in T_0$, y is a p-element and z is a p'-element. Since T_0/T_0' is abelian, taking $(rs)^{th}$-powers, we get: $x^{urs} \equiv y^{rs} z^{rs} \pmod{T_0'} \equiv 1 \pmod{T_0'}$. But T_0' is finite, so we have that $x^{urs|T_0'|} = 1$, and (iii) is proved.

Assume now that conditions (i), (ii) and (iii) hold, and let T_0 be a normal p-abelian subgroup of finite index in T, as in (ii), and A the set: $\{t \in T \mid t \text{ is a p'-element}\}$.

Suppose first that A is a central subgroup of T. Then, as T is locally finite, it is easy to see that $T = P \times A$, where $P = \{t \in T \mid t \text{ is a p-element}\}$ is a subgroup of T.

Considering the subgroup $\phi = T_0 \cdot A$, it is easy to see that T satisfies the conditions of Lemma 1.4 and

hence it is nilpotent. Furthermore, T contains a normal p-abelian subgroup ϕ such that T/ϕ is a finite p-group, and we conclude from [5, Theorem 6.1, p. 155] that KT is Lie m-Engel for a suitable m. Hence, $U^n \subset \xi(U)$ for some n by [5, Lemma 4.3, p. 151].

Suppose now that A is a non central subset. By (iii), $T^S = 1$, for some s, and K is finite. Because T_0' is a finite p-group, it is easy to see that $P_0 = \{t \in T_0 \mid t$ is a p-element$\}$ is a normal subgroup of T.

We claim that $\Delta(T, P_0)$, the kernel of the natural epimorphism $KT \longrightarrow KT/P_0$, is nil of bounded exponent. Indeed, $\Delta(T_0/T_0', P_0/T_0')$ is nil of bounded exponent because T_0/T_0' is abelian and P_0/T_0' is of bounded exponent. Setting $S = KT_0/T_0'$, $Q = (T/T_0')/(T_0/T_0') \simeq T/T_0$, we see that KT/T_0' is the crossed product $S(Q, \rho, \sigma)$, with ρ and σ as usual. If $I = \Delta(T_0/T_0', P_0/T_0')$, by Lemma 1.5 we conclude that $IQ = \Delta(T/T_0', P_0/T_0')$ is nil of bounded exponent. Since T_0' is a finite p-group, we see that $\Delta(T_0, T_0')$ is nilpotent and hence it is easy to see that $\Delta(T, P_0)$ is nil of bounded exponent by considering the natural epimorphism $KT \longrightarrow KT/T_0'$.

Now, T_0/P_0 is a normal subgroup of T/P_0, of finite index, say, r. By [1, Lemma 1.10, p. 176], we get that $KT/P_0 \subset M_r(KT_0/P_0)$.

Pick now $u \in U$. Considering the subgroup generated by the support of u, we may suppose that T is finite. Hence T_0/P_0 is a finite abelian p'-group, such that $(T_0/P_0)^S = 1$

Therefore, $KT_0/P_0 = \underset{i}{\oplus} K_i$, a direct sum of fields, all of then contained in $K(\zeta)$, with $\zeta^S = 1$. Hence, $M_r(KT_0/P_0) = M_r(\underset{i}{\oplus} K_i) = \underset{i}{\oplus} M_r(K_i)$, and we have that $KT/P_0 \subset \underset{i}{\oplus} M_r(F_i)$, with $F_i = K(\zeta)$, for every i. Set S for $\underset{i}{\oplus} M_r(F_i)$ and u' for the image of u by the composition map of the natural epimorphism f: $KT \longrightarrow KT/P_0$ followed by the inclusion g: $KT/P_0 \longrightarrow S$. As K is finite, the group of non-singular matrices of $M_r(K(\zeta))$ is finite, say of order q, depending on r and s only. So, $u'^q = 1$ and we get:

$$u^q = 1 + \delta, \quad \delta \in \Delta(T,P_0).$$

As $\Delta(T,P_0)$ is nil of bounded exponent, we can take m a power of p such that $x^m = 0$ for all $x \in \Delta(T,P_0)$. Now we can conclude that $u^{qm} = 1 + \delta^m = 1 \in \xi(U)$. ▲

COROLLARY; Let G = T, a locally finite group, and assume that the set of all p-elements of T is not of bounded exponent. Then the following conditions are equivalent:

(i) $U^n \subset \xi(U)$ for some n.

(ii) KT is Lie m-Engel for some m.

PROOF: First suppose that $U^n \subset \xi(\Pi)$ for some n. By the preceeding theorem, we get that (i), (ii) and (iii) hold.

Furthermore, every p'-element of T is central by Lemma 1.3. Follow now the "only if" - part of the proof of the theorem to conclude that KT is Lie m-Engel for some m.

By [5, Lemma 4.3, p. 151], the converse is obvious. ▲

§ 2. A certain nil ideal of KG

In this section, G will be either a solvable or an FC-group. As we mentioned in the introduction, if $G/\xi(G)$ is torsion, than we can conclude that T is a locally finite subgroup of G and G' is contained in T.

We shall denote by A the set of all p'-elements of G and by P the set of all p'-elements of G.

LEMMA 2.1: Suppose that $U^n \subset \xi(U)$ for some n and G has an element of infinite order. Then, every idempotent of KG is central. ▲

PROOF: See [2, Lemma 2.4, p. 148].

LEMMA 2.2: Suppose that $U^n \subset \xi(U)$ for some n and G has an element of infinite order. Then:

(i) A is an abelian subgroup of G.

(ii) If A is non central, then K is finite and for every $x \in G$ and every $t \in A$ there exists an integer r such that $xtx^{-1} = t^{p^r}$, and $(K:\mathbb{F}_p) \mid r$.

(iii) P is a subgroup of G.

(IV) $T = P \times A$.

PROOF: For the proof of (i) see [2, Corollary 2.5, p. 148].

To prove (ii) we notice that if A is non central, then K is finite by Lemma 1.1.

Now, take $x \in G$ and $t \in A$ such that $xtx^{-1} \neq t$. We have that $K <t> = \oplus K_i$, a direct sum of fields such that at least one of them, say K_1, is of the form $K_i = K(\zeta)$, where

ζ is a root of unity whose order is equal to the order of t, and the natural projection $K<t> \longrightarrow K_1$ maps t on ζ.

Since, by Lemma 2.1, every idempotent is central, we must have $xtx^{-1} = t^i$ for some i (this can be seen by considering the idempotent $e = (o(t))^{-1} (1+t+\ldots+t^{o(t)-1})$. Hence, conjugation by x defines an automorphism ϕ of K_1. By the above, $\phi(\zeta) = \zeta^i$.

On the other hand, since K_1 is finite, ϕ is a power of the Frobenius automorphism of K_1, F, given by: $F(k) = k^p$, for all $k \in K_1$. If $\phi = F^r$, we have that $\phi(\zeta) = \zeta^{p^r} = \zeta^i$, from which we conclude that $o(t) = o(\zeta)$ divides $p^r - i$. Then, $p^r \equiv i \pmod{o(t)}$, and $xtx^{-1} = t^i = t^{p^r}$.

Furthermore, as every element of K is fixed by ϕ, we have that $k^{p^r} = k$ for every $k \in K$, and hence K is contained in a field with p^r elements, that is: $(K:\mathbb{F}_p)|r$.

For (iii), we observe first that every p-element commutes with every p'-element. If not, then by Lemma 1.1 K is finite. Now take $\pi \in P$ and $t \in A$ such that $\pi t \pi^{-1} \neq t$ and proceed like in [2, 3.2, p. 152] to conclude that this implies the existence of a non central idempotent, which contradicts Lemma 2.1.

As T is locally finite, the proof of (iv) is now trivial. ▲

LEMMA 2.3: Let A_1 be an abelian p'-subgroup of G, and K a finite field. If, for every $t \in A_1$ and every $x \in G$, there exists an integer r such that $xtx^{-1} = t^{p^r}$, and $(K:\mathbb{F}_p)|r$, then every idempotent of KA_1 is central in KG.

PROOF: Let $e \in KA_1$ be an idempotent, and let $x \in G$. By considering the subgroup generated by the support of e, we may suppose that A_1 is finite.

Let $A_1 = \langle t_1 \rangle \times \ldots \times \langle t_s \rangle$, a direct product of cyclic groups. It is easy to see that we may choose an integer r such that $x t_i x^{-1} = t_i^{p^r}$, for every i.

We have that $e = f(t_1, \ldots, t_s)$, where $f(X_1, \ldots, X_s)$, is a polynomial in the commuting variables $X_1, \ldots, X_s$, with coeficients in K. Conjugating by x, we have: $x e x^{-1} = f(t_1^{p^r}, \ldots, t_s^{p^r})$. But by hypothesis every element $k \in K$ satisfies $k^{p^r} = k$, therefore this is true for the coeficients of f. Hence $f(t_1^{p^r}, \ldots, t_s^{p^r}) = (f(t_1, \ldots, t_1))^{p^r}$ and $x e x^{-1} = e^{p^r} = e$, as we wished to prove. ▲

We can now give a partial characterization of the fact of U being of bounded exponent over the center, when G is non torsion.

Theorem B: Suppose that G has an element of infinite order. Then, $U^n \subset \xi(U)$ for some n if and only if the following conditions hold:

(i) $G^\ell \subset \xi(G)$ for some ℓ.

(ii) A is an abelian subgroup of G and, if A is non central, then K is finite, A is of bounded exponent and for every $t \in A$ and every $x \in G$ there exists an integer r such that $x t x^{-1} = t^{p^r}$, where $(K : \mathbb{F}_p) \mid r$.

(iii) P is a subgroup of G contained in the centralizer of A.

(iv) There exists are integer m, which is a

power of p, such that x^m is central in KG, for every $x \in \Delta(G,P)$.

PROOF: Suppose first that $U^n \subset \xi(U)$ for some n. (i) is trivial, (ii) follows from Lemma 1.1 and Lemma 2.2, and (iii) follows from Lemma 2.2.

To prove (iv), let $x \in \Delta(G,P)$. We may suppose that G is finitely generated and hence P is a finite normal subgroup of G. Therefore, x is nilpotent and we can apply Lemma 1.2 to obtain the thesis.

Suppose now conditions (i) to (iv) hold, and pick $u \in U$. Again we may suppose that G is finitely generated and hence T is finite.

We observe that if $KA = \bigoplus_i K_i$, a direct sum of fields, then $K(A \times P) = \bigoplus_i K_i P$.

Consider now the natural epimorphism $KG \to KG/P$, with kernel $\Delta(G,P)$.

Setting S' for the image of a subset S of KG through this epimorphism, we have: $(K(A \times P))' = \bigoplus K_i'$ where each K_i' is a field. Furthermore, $T(G/P) = (A \times P)/P$, and hence $KT(G/P) = \bigoplus_i K_i'$.

Since, by condition (ii) and Lemma 2.3, every idempotent of KA is central in KG, and $G' \subset T$, we may apply [5, Lemma 3.22, p. 194], and u can be written in the form:

(*) $u = \sum_i f_i g_i + \delta$, $f_i \in K_i$, $g_i \in G$, $\delta \in \Delta(G,P)$.

Now we consider both cases separately.

Suppose first that A is central. Taking the

$(\ell)^{th}$ - power in $(*)$, we have: $u^\ell = \sum_i f_i{}^\ell g_i{}^\ell + \delta'$, $\delta' \in \Delta(G,P)$. Now, $\sum_i f_i{}^\ell g_i{}^\ell$ is central, and it is sufficient to apply condition (iv).

Suppose now that K is finite and A is of bounded exponent s. Then, $K_i \subset K(\zeta)$, with $\zeta^s = 1$, for all i.

Computing $(\ell)^{th}$-powers in $(*)$, we obtain:

$u^\ell = \sum_i f_i' \, g_i{}^\ell + \delta'$, $f_i' \in K_i$, $g_i{}^\ell \in \xi(G)$, $\delta' \in \Delta(G,P)$.

Since $K_i \subset K(\zeta)$, for all i, we have that $f_i'{}^r = 1$ for every $f_i' \neq 0$ and a suitable r which depends on K and s only.

Taking $(r)^{th}$-powers above, we have: $u^{\ell r} = \sum_i g_i{}^{\ell r} + \delta''$, $\delta'' \in \Delta(G,P)$.

Now, $\sum_i g_i{}^{\ell r}$ is central and it suffices to use condition (iv). ▲

COROLLARY: Suppose that the set of all p-elements of G is not of bounded exponent. Then the following conditions are equivalent:

(i) $U^n \subset \xi(U)$ for some n.

(ii) KG is Lie ℓ-Engel for some ℓ.

PROOF: Suppose first $U^n \subset \xi(U)$ for some n. By the corollary to theorem A, KT is Lie m-Engel for some m and hence T is nilpotent.

As $G' \subset T$, we can conclude that G is solvable (even if it were an FC-Group).

By Lemma 1.3, $G^{p^t} \subset \xi(G)$ for some t and by

[2, Lemma 2.2, p. 148] G' is a p-group. Hence, as we noted before, $\Delta(G,G')$ is a nil ideal.

Given $x,y \in KG$, $xy-yx \in \Delta(G,G')$, By (iv) of Theorem B, for every $z \in KG$ we have that $(xy-yx)^m z = z (xy-yx)^m$.

Hence, KG satisfies a polynomial identity. By Passman's theorem [4, Theorem 1.1, p. 142], setting ϕ for the FC-subgroup of G, we have: $|G/\phi|<\infty$, $|\phi'|<\infty$.

Furthermore, it is easy to see that both are p-groups. By Lemma 1.4, G is nilpotent. We conclude from [5, Theorem 6.1, p. 155] that KG is Lie ℓ-Engel for some ℓ.

As we noted before, the converse is immediate by [5, Lemma 4.3, p. 151].

▲

§ 3. Augmentation ideals of bounded exponent

We shall now discuss condition (iv) of Theorem B.

In all this section, except in Theorem C, G will be either a solvable or an FC-group, such that T is a locally finite subgroup of G, and $G' \subset T$.

Furthermore, we shall assume that T has the form: $T = P \times A$, where $A = \{t \in G \mid t$ is a p'-element$\}$ is an abelian subgroup of G and $P = \{t \in G \mid t$ is a p-element$\}$ is a subgroup of G, of bounded exponent.

We introduce some notation. Given a group H, $\phi(H)$ will denote the FC-subgroup of H. The group $\phi(G)$ will

be denoted simply by ϕ.

LEMMA 3.1: Suppose that there exists an integer m, which is a power of p, such that x^m is central in KG for all $x \in \Delta(G,P)$. Then G contains a normal subgroup H, of finite index, containing ϕ, such that $H' \cap P$ is finite.

PROOF: By Passman [4, Theorem 1.1, p. 142], if KG satisfies a polynomial identity, then $|G/\phi| < \infty$, $|\phi'| < \infty$. Since $G' \subset P \times A$, and $(\phi.A)/A \subset \phi(G/A)$, it is enough to prove that $K(G/A)$ satisfies a polynomial identity.

In the group ring $K(G/A)$, the ideal $\Delta(G/A, (P \times A)/A)$ is the image of $\Delta(G,P)$ by the natural epimorphism $KG \to K(G/A)$ and hence it also satisfies the hypothesis.

As $(G/A)' \subset (P \times A)/A$, we have that $\Delta(G/A, (G/A)') \subset \Delta(G/A, (P \times A)/A)$ and it follows easily, as in the proof of the corollary to Theorem B, that $K(G/A)$ satisfies the polynomial $(XY-YZ)^m Z - Z(XY-YX)^m$. ▲

In the following lemmas, H will be a normal subgroup of G, of finite index, containing ϕ, as in the previous lemma, P_0 will be $H' \cap P$ (hence a finite group) and P_1 will be $H \cap P$.

LEMMA 3.2: Suppose that there exists a positive integer m, which is a power of p, such that x^m is central in KG, for all x in $\Delta(G,P)$: Then, $\Delta(G,P)$ is nil of bounded exponent.

PROOF: We observe first that it is enough to prove that $\Delta(H,P_1)$ is nil of bounded exponent. In fact, suppose that this were proved. Given $x \in \Delta(G,P)$, then x^m is a central element. But every central element of $\Delta(G,P)$ is contained in $\Delta(G,P) \cap K\phi$, which is contained in $\Delta(H,P_1)$. Then $\Delta(G,P)$

is nil of bounded exponent.

Let us prove now that $\Delta(H,P_1)$ is nil of bounded exponent. As P_0 is finite, $\Delta(H,P_0)$ is nilpotent and hence it suffices to prove that $\Delta(H/P_0, P_1/P_0)$ is nil of bounded exponent.

We have that $(H/P_1)'$ is a p'-group, P_1/P_0 is a central subgroup of H/P_0, and for all $x \in \Delta(H/P_0, P_1/P_0)$, x^m is central in KH/P_0. Furthermore, all hypothesis on G carry on to H/P_0. Therefore, in order to prove that $\Delta(H/P_0, P_1/P_0)$ is nil of bounded exponent, we may replace H/P_0 by H and P_1/P_0 by P_1 and assume in addition that H' is a p'-group and P_1 is central.

Now, we shall see which is the form of a central element of KH that belongs to $\Delta(H,P_1)$.

Let S' be a transversal of T(H) in H. Observing that $T(H) = P_1 \times A_1$, where $A_1 = \{t \in H | t \text{ is a p'-element}\}$, it is easy to see that $S = \{ab | a \in A_1, b \in S'\}$ is a transversal of P_1 in H.

Now, let $x \in \phi(H)$, $y \in H$, $x = hab$, $h \in P_1$, $ab \in S$. Since $y(ab)y^{-1}(ab)^{-1} \in H' \subset A_1$ and P_1 is central, yxy^{-1} has the form: $yxy^{-1} = ha'b$, $a' \in A_1$. So denoting by $\gamma(x)$ the sum of the elements of the conjugacy class of x, we have that $\gamma(hab) = h(a_1+...+a_r)b$, $a_j \in A_1$, $1 \le j \le r$. Therefore, we may write every central element $z \in KH$ in the form:

$$(1) \quad z = \sum_i \alpha_i b_i,$$

where $\alpha_i \in KT(H)$, $\alpha_i b_i$ is central in KH for all i, and the b_i are distinct elements of S'.

We claim that $\alpha_i b_i = b_i \alpha_i$, for all i. In fact, as $\alpha_i b_i$ is central, we have that $\alpha_i b_i = b_i (\alpha_i b_i) b_i^{-1} = b_i \alpha_i$.

If in addition $z \in \Delta(H, P_1)$, then $\alpha_i \in \Delta(T(H), P_i)$, for all i. In fact, denote by x' the image of an element $x \in KH$ by the natural epimorphism $KH \longrightarrow KH/P_1$. Since $z \in \Delta(H, P_1)$, $z' = \sum\limits_i \alpha_i' b_i' = 0$. But the b_i' are distinct elements of a basis of the $K(T(H)/P_1)$-module KH/P_1. Hence $\alpha_i' = 0$, for all i, which means that $\alpha_i \in \Delta(H, P_1) \cap KT(H) = \Delta(T(H), P_1)$.

Therefore every central element z of $\Delta(H, P_1)$ has the form (1), and furthermore $\alpha_i \in \Delta(T(H), P_1)$, for all i.

Call s the exponent of P_1. Since $T(H)$ is abelian, computing the $(s)^{th}$-power of z in (1) we have that:
$$z^s = \sum\nolimits_i (\alpha_i b_i)^s = \sum\limits_i \alpha_i^s b_i^s = 0.$$

We have proved that $z^s = 0$, for a fixed s and every central element z of $\Delta(H, P_1)$. As every $x \in \Delta(H, P_1)$ is such that x^m is central, we can conclude that $\Delta(H, P_1)$ is nil of bounded exponent. ▲

LEMMA 3.3: If $\Delta(H, P_1)$ is nil of bounded exponent, then either P_1 is finite or H contains a characteristic p-abelian subgroup of finite index.

PROOF: Suppose that P_1 is infinite. By Passman [1, Corollary 3.10, p. 197], it is enough to prove that KH satis fies a polynomial identity. Since $\Delta(H, P_0)$ is nilpotent it

suffices to prove that KH/P_0 satisfies a polynomial identity.

We observe that all the hypothesis of the lemma carry on to KH/P_0. In fact, $\Delta(H/P_0, P_1/P_0)$ is nil of bounded exponent and P_1/P_0 is infinite (because P_0 is finite). Furthermore, $(H/P_0)'$ is a p'-group and P_1/P_0 is central in H/P_0.

Therefore, to prove that KH/P_0 satisfies a polynomial identity we may replace H/P_0 by H and assume in addition that H' is a p'-group and P_1 is a central subgroup of H.

Now, pick $n>0$ such that $x^n=0$ for all $x \in \Delta(H,P_1)$. We want to show that there exist elements $x_1, x_2, \ldots, x_n$ in $\Delta(P_1) = \Delta(P_1,P_1)$ such that $x_i^2=0$, for all i, but $x_1 x_2 \ldots x_n \neq 0$. In fact, as P_1 is abelian of bounded exponent, it is a direct product of cyclic groups [4, Theorem 11.2, p. 44]. P_1 is infinite and hence the number of cyclic groups is infinite. Choose $h_1, \ldots, h_n$ generators in different cyclic subgroups and set $m_i = o(h_i) - 1$. The elements $x_i = (h_i - 1)^{m_i}$ are easily verified to satisfy the required conditions.

Take now $S = \{g_i\}_{i \in I}$ a transversal of P_1 in H, and elements $g_1, \ldots, g_n \in S$. Then, the element $\alpha = g_1 x_1 + \ldots + g_n x_n$ belongs to $\Delta(H,P)$, hence $\alpha^n=0$.

Since the x_i are central and $x_i^2=0$, computing α^n we get: $\alpha^n = F(g_1, \ldots, g_n) x_1 x_2 \ldots x_n$, where $F(X_1, \ldots X_n)$ is a polynomial in the non-commuting variables $X_1, X_2, \ldots, X_n$, namely: $F(X_1, \ldots, X_n) = \Sigma X_{\sigma(1)} \ldots X_{\sigma(n)}$, the sum running over

all $\sigma \in S_n$, the symmetric group on n elements.

We want to show that $F(g_1, \ldots, g_n) = 0$.

First we make some observations. If
$g_{\sigma(1)} \, g_{\sigma(2)} \cdots g_{\sigma(n)} \equiv g_{\tau(1)} \, g_{\tau(2)} \cdots g_{\tau(n)}$ (mod P_1) for σ,
$\tau \in S_n$, then $g_{\sigma(1)} \, g_{\sigma(2)} \cdots g_{\sigma(n)} = a \, g_{\tau(1)} \, g_{\tau(2)} \cdots g_{\tau(n)}$, for
some $a \in P_1$.

On the other other hand, since $g_i \, g_j = \alpha(i,j) g_j \, g_i$,
where $\alpha(i,j) \in H'$, we can write the product $g_{\sigma(1)} g_{\sigma(2)} \cdots g_{\sigma(n)}$
in the form $b \, g_{\tau(1)} \, g_{\tau(2)} \cdots g_{\tau(n)}$ for some $b \in H'$.

Therefore, we obtain: $a = b$, with $a \in P_1$ and
$b \in H'$, which is a p'-group. Thus $a = b = 1$.

We have shown that $g_{\sigma(1)} \, g_{\sigma(2)} \cdots g_{\sigma(n)} \equiv$
$g_{\tau(1)} \, g_{\tau(2)} \cdots g_{\tau(n)}$ (mod P_1) if and only if $g_{\sigma(1)} g_{\sigma(2)} \cdots g_{\sigma(n)} =$
$g_{\tau(1)} \, g_{\tau(2)} \cdots g_{\tau(n)}$.

We now define an equivalent relation $\sim$ in S_n
setting: for σ, $\tau \in S_n$, $\sigma \sim \tau$ if and only if $g_{\sigma(1)} \cdots g_{\sigma(n)} =$
$g_{\tau(1)} \cdots g_{\tau(n)}$. We denote by $S_1, S_2, \ldots, S_t$ the equivalence
classes of this relation. Choosing, for each i, an element
$\sigma_i \in S_i$, it follows from the above that we may write:
$$F(g_1, \ldots, g_n) = g_{\sigma_1(1)} \cdots g_{\sigma_1(n)} |S_1| + \ldots + g_{\sigma_t(1)} \cdots g_{\sigma_t(n)} |S_t| \, .$$

Now, since $g_{\sigma_i(1)} \cdots g_{\sigma_i(n)} \not\equiv g_{\sigma_j(1)} \cdots g_{\sigma_j(n)}$
(mod P_1) if $i \neq j$, from $F(g_1, \ldots, g_n) x_1 x_2 \ldots x_n = 0$, we get:
$g_{\sigma_i(1)} \cdots g_{\sigma_i(n)} |S_i| x_1 x_2 \ldots x_n = 0$, for all i.

But $g_{\sigma_i(1)} \cdots g_{\sigma_i(n)}$ is invertible and
$x_1 x_2 \ldots x_n \neq 0$; hence this can happen only if $|S_i| = 0$, for
all i (that is, $p \mid |S_i|$, for all i).

Therefore, $F(g_1, \ldots, g_n) = 0$, for arbitrary elements $g_1, \ldots, g_n \in S$.

Now, KH is a left module over the central subalgebra KP having S as a basis, and $F(X_1, \ldots, X_n)$ is a multilineal polynomial. By $[1, \text{Lemma } 1.2, \text{ p. } 171]$, F is a polynomial identity for KH. ▲

COROLLARY: If $\Delta(G,P)$ is nil of bounded exponent, then either P is finite or G contains a normal p-abelian subgroup of finite index.

PROOF: Suppose that P is infinite, and take H as in Lemma 3.1. Since $|G/H| < \infty$, we have that $|PH/H| < \infty$ and hence $|P/P_1| < \infty$. Therefore, P_1 must be infinite. By Lemma 3.3, H contains a characteristic p-abelian subgroup of finite index, and thus the thesis follows. ▲

LEMMA 3.4: If G contains a normal p-abelian subgroup of finite index, then $\Delta(G,P)$ is nil of bounded exponent.

PROOF: Let L be such a subgroup of G. We have that L/L' is abelian; hence $\Delta(L/L', PL/L')$ is nil of bounded exponent.

Setting $S = KL/L'$, $Q = (G/L')/(L/L') \cong {}^G/_L$, we see that $K(G/L')$ is the crossed product $S(Q, \rho, \sigma)$, with ρ and σ as usual.

If $I = \Delta(L/L', PL/L')$, applying Lemma 1.5 we conclude that $IQ = \Delta(G/L', (P \cap L)/L')$ is nil of bounded exponent. Since L' is a finite p-group, $\Delta(G,L')$ is nilpotent and hence $\Delta(G, P \cap L)$ is nil of bounded exponent.

Now, let us consider the natural epimorphism $\Phi: KG \longrightarrow K(G/(P \cap L))$. We have that $\Phi(\Delta(G,P)) = \Delta(G/(P \cap L), P/(P \cap L))$. But $P/(P \cap L) \cong PL/L$, and PL/L is a finite p-group since it is contained in G/L. Therefore, there exists an integer in such that $\Delta(G,P)^n \subset \Delta(G, P \cap L)$. Since we have shown that this ideal is nil of bounded exponent, the Lemma is proved. ▲

We can now give a complete answer to the initial question.

<u>Theorem C</u>: Suppose that G is non-torsion and either solvable or FC. Then, $U^n \subset \xi(U)$ for some n if and only if either KG is Lie m-Engel for some m or the following conditions hold:

(i) $G^{\ell} \subset \xi(G)$ for some ℓ.

(ii) A is an abelian subgroup of G and, if A is non central, then K is finite, A is of bounded exponent and for every $x \in G$ and every $t \in A$ there exists an integer r such that $xtx^{-1} = t^{p^r}$, where $(K : \mathbb{F}_p) \mid r$.

(iii) P is a subgroup of G of bounded exponent, contained in the centralizer of A and, if P is not finite, then G contains a normal p-abelian subgroup of finite index.

<u>PROOF</u>: Suppose that $U^n \subset \xi(U)$ for some n and that KG is not Lie m-Engel.

By theorem B, the conditions (i) and (ii) hold, and by (iii) of theorem B, P is a subgroup of G contained in the centralizer of A. If P is not of bounded ex-

ponent, by the corollary to Theorem B, KG is Lie m-Engel, for some m, a contradiction. Hence, P is of bounded expo nent.

Also, since condition (iv) of theorem B holds, we have that $\Delta(G,P)$ is nil of bounded exponent by lemma 3.2. By the corollary to lemma 3.3, the remainder of condition (iii) holds.

Suppose now KG is Lie m-Engel for some m. Then, as we have noted before, $U^n \subset \xi(U)$ for a suitable n [see 5, Lemma 4.3, p. 151].

Finally, suppose that conditions (i), (ii) and (iii) hold. If P is finite, then $\Delta(G,P)$ is nilpotent and condition (iv) of theorem B holds. If G contains a normal p-abelian subgroup of finite index, then using lemma 3.4, again condition (iv) of theorem B holds. Since conditions (i), (ii) and (iii) of this theorem are verified, then there exists an n such that $U^n \subset \xi(U)$.

▲

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