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The stability of minimal cones of codimension greater than one in $\mathbb{R}^n$

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THE STABILITY OF MINIMAL CONES OF CODIMENSION GREATER THAN ONE IN $\mathbb{R}^n$

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§ 0 - INTRODUCTION

J. Simons has proved in [3] that if M is a (n-1)-dimensional closed minimal submanifold in S^n, not totally geodesic, with $n \leq 6$, then the cone over M is not stable.

This result is fundamental for regularity of solutions of Plateau's problem and for the Bernstein's problem of codimension one, in $\mathbb{R}^{n+1}$ ([1], [3]).

For higher codimension, the first observation is that Simon's theorem cannot be extended if others hypothesis are not added

Indeed

$$M = \{Z \in \mathbb{C}^n \mid Z \neq 0,\ P(Z) = 0\} \subset \mathbb{R}^{2n}$$

where $P(z) = \sum_{j=1}^{n} z_j^2$, is a 2-codimensional stable minimal cone. It's easy to verify that the immersion $M^{2n-2} \hookrightarrow \mathbb{R}^{2n}$ does not have flat normal bundle.

The existence of such manifolds as well as tecnical motivations have brought us to consider higher codimensional minimal cones over submanifolds of S^n that have flat normal bundle.

The following theorem generalizes Simons' result :

THEOREM 1 - Let $M^n \hookrightarrow \mathbb{R}^{n+m}$ be a minimal cone immersed in $\mathbb{R}^{n+m}$ such that $M \cap S^{n+m-1}$ is a closed manifold of S^{n+m-1} (the unitary ball in $\mathbb{R}^{n+m}$) and suppose that there is a global orthonormal family of parallel fields in the normal bundle of M.

Thus if M is not totally geodesic in $\mathbb{R}^{n+m}$ and $n \leq 6$, M is not stable.

The following shows that this result is sharp.

THEOREM 2 - Let $M = \tilde{M} \times (0, +\infty) \hookrightarrow \mathbb{R}^{n+m}$; $(y,t) \longmapsto ty$ where $\tilde{M} = S^{n_1}(r_1) \times \ldots \times S^{n_s}(r_s)$,

$$\sum_{i=1}^{s} n_i = n - 1 , \quad \sum_{i=1}^{s} r_i^2 = 1 \quad \text{and} \quad m = s-1$$

Then M is a minimal cone in $\mathbb{R}^{n+m}$ having flat normal blundle, and M is stable if, and only if, $n>6$.

§ 1 - BASIC FACTS

1.1 - Let $M^n \hookrightarrow \mathbb{R}^{n+m}$ be an isometric immersion and let $f \in C^1(U)$ where U is an open set of $\mathbb{R}^{n+m}$ such that $U \cap M \neq \emptyset$.

Let δf be the gradient in M of the restriction of f to M, that is

$$\delta f = \operatorname{grad}_M (f|_M)$$

(it is easy to see that δf depends only on the values of f over M)
Then, if $\{\mu_i\}_{1\le i\le n}$ is an orthonormal basis of T_pM , Df is the gradient of f in $\mathbb{R}^{n+m}$, that is,

$$Df = \left(\frac{\partial f}{\partial \mu_1} , \ldots , \frac{\partial f}{\partial \mu_{n+m}} \right) ,$$

and $\{v^{(k)}\}_{1\le k\le m}$ is an orthonormal frame of normal vector fields defined in U∩M, then

$$(1.1.1) \qquad \delta f = Df - \sum_{k=1}^{m} \langle Df, v^{(k)}\rangle \, v^{(k)}$$

Let $\{e_i\}_{1\le i\le n+m}$ be a coordinate system in $\mathbb{R}^{n+m}$, that is , each e_i is a constant field in $\mathbb{R}^{n+m}$ and they constitute an orthonormal basis for $\mathbb{R}^{n+m}$. Thus, if $v_i^{(k)} = \langle v^{(k)} , e_i\rangle$; $D_i f = \langle Df , e_i\rangle$; $\delta_i f = \langle \delta f , e_i\rangle$, we have

$$(1.1.2) \qquad \delta_i f = D_i f - \sum_{k=1}^{m} \langle Df, v^{(k)}\rangle \, v_i^{(k)} \quad ; i=1,2,\ldots,n+m$$

1.2 - If H is the mean curvature field of the immersion, and

$$e_{n+k} = v^{(k)}(p) \quad ; k=1,\ldots,m \ , \text{ at } p\in M,$$

then we have at p,

$$\delta_i = D_i \quad \text{if} \quad i\le n$$

$$\delta_{n+k} = 0 \qquad , \text{ and}$$

$$\langle H,v^{(k)}\rangle_p = -\sum_{i=1}^{n} \langle e_i , \bar\nabla_{e_i} v^{(k)}\rangle_p = -\sum_{i=1}^{n} D_i v_i^{(k)} = -\sum_{i=1}^{n+m} \delta_i v_i^{(k)}(p)$$

where $\bar{\nabla}$ is the riemannian connection of $\mathbb{R}^{n+m}$.

Since $\sum_{i=1}^{n+m} \delta_i V_i^{(k)}$ does not depend on the particular coordinate system, we have

$$< H , V^{(k)} > = - \sum_{i=1}^{n+m} \delta_i V_i^{(k)} \quad , \text{ that is}$$

(1.2.1) $$H = - \sum_{k=1}^{m} \left(\sum_{i=1}^{n+m} \delta_i V_i^{(k)} \right) V^{(k)} \quad ,$$

at any point of M and with respect to any coordinate system of $\mathbb{R}^{n+m}$

Similarly, if $B(x,y) = (\bar{\nabla}_x \; Y)^{\perp}$ where $x,y \in T_pM$ and Y is an arbitrary extension of y that is tangent to M in a neighborhood of p, and

$$<A^v(x),y> \;=\; <B(x,y),v> \; ; \; \forall \; x,y \in T_pM \; ,$$
$$\forall \; v \in (T_pM)^{\perp}$$
$$\forall \; p \in M \; ,$$

(1.2.2) we have, $$<A^{V^{(k)}} , A^{V^{(s)}} > = \sum_{i,j=1}^{n+m} (\delta_i V_j^{(k)})(\delta_j V_i^{(s)})$$

In what follows we set

$$C_{ks} = <A^{V^{(k)}} , A^{V^{(s)}} >$$

(1.2.3) $$C_k^2 = C_{kk}$$

$$C^2 = \sum_{k=1}^{m} C_k^2 = ||A||^2$$

1.3 - Let X be a C^∞ vector field of M and $\text{div}_M X$ be its divergent in M, that is, if $p \in M$,

$$\text{div}_M X\ (p) = \text{trace}\ (\omega \longmapsto \nabla_\omega X),$$

where $\omega \in T_p M$ and ∇ is the riemannian connection of M.

Then, if $\{e_i\}_{1 \le i \le n+m}$ is a coordinate system for $\mathbb{R}^{n+m}$ such that $e_{n+k} = V^{(k)}(p)$; $k = 1,2,\dots,m$, we have

$$\text{div}_M X\ (p) = \sum_{i=1}^{n+m} \delta_i X_i(p)$$

and since $\sum_{i=1}^{n+m} \delta_i X_i$ does not depend on the coordinate system, we have

(1.3.1) $$\text{div}_M X = \sum_{i=1}^{n+m} \delta_i X_i$$

at any point of M and in relation to any coordinate system of $\mathbb{R}^{n+m}$.

Therefore, the laplacian of $f \in C^2(M)$, Δf, is given by

(1.3.2) $$\Delta f = \sum_{i=1}^{n+m} \delta_i \delta_i f$$

for $\Delta f = \text{div}_M(\text{grad}_M f)$

1.4 - If $\phi \in C^1(M)$ has compact support and $\Psi \in C^2(M)$, we have

(1.4.1) a) $$\int_M \delta\phi\ dM = -\int_M H\phi\ dM$$

(1.4.2) b) $$\int_M \phi\Delta\Psi\ dM = \int_M \Psi\Delta\phi\ dM = -\int_M \langle\delta\phi, \delta\Psi\rangle dM$$

1.5 - Let $M^n \hookrightarrow N^{n+m}$ be an isometric immersion and let $\nabla^\perp$ be the induced connection in the normal bundle of M by the connection of N. Thus the normal curvature tensor of the normal bundle is defined by

$$(1.5.1) \qquad R^\perp(X,Y) = \nabla^\perp_X \nabla^\perp_Y - \nabla^\perp_Y \nabla^\perp_X - \nabla^\perp_{[X,Y]}$$

where X,Y are tangent fields of M.

Suppose that $R^\perp \equiv 0$, then we say that the immersion has flat normal bundle. The following well known results will be useful :

a) $R^\perp \equiv 0$ if, and only if, there is a local orthonormal family $\{V^{(k)}\}_{1\le k\le m}$ of parallel sections in the normal bundle, that is,

$$(1.5.2) \qquad \nabla^\perp_X V^{(k)} = 0 \; ; \; k = 1,2,\ldots,m \; , \; \forall X \in T_pM \; , \; \forall p \in M$$

b) If N has constant curvature, then $R^\perp \equiv 0$ if, and only if, the tensors A^{e_α}, where $\{e_\alpha\}_{1\le\alpha\le m}$ is any orthonormal frame of normal fields, are simultaneously diagonalized

1.6 - Let $M^n \hookrightarrow \mathbb{R}^{n+m}$ be an isometric immersion having flat normal bundle and let $\{V^{(k)}\}_{1\le k\le m}$ be an orthonormal family of parallel sections of the normal bundle of M. Let $p \in M$, and let $\{e_i\}_{1\le i\le n+m}$ be a coordinate system for $\mathbb{R}^{n+m}$ such that

$$V^{(k)}(p) = e_{n+k} \; ; \; k = 1,2,\ldots,m \; .$$

Thus, from

$$<\nabla_{e_i} V^{(k)}, e_j>_p = <\nabla_{e_j} V^{(k)}, e_i>_p \quad \text{if } i,j = 1,2,\dots,n$$
$$k = 1,2,\dots,m$$

and $\delta_{n+s} \equiv 0$ at p ,

and $0 = <\bar{\nabla}_{e_i} V^{(k)}, V^{(s)}>_p = \delta_i V^{(k)}_{n+s}(p)$, if

$$i = 1,2,\dots,n$$
$$k,s = 1,2,\dots,m$$

it follows

$$\delta_i V_j^{(k)}(p) = \delta_j V_i^{(k)}(p) \; ; \quad i,j = 1,2,\dots,n+m$$
$$k = 1,2,\dots,m$$

with respect to $\{e_i\}_{1\le i\le n+m}$.

Considering another system $\{\bar{e}_i\}_{1\le i\le n+m}$ of coordinates, where

$$e_j = \sum_{i=1}^{n+m} \alpha_j^i \; \bar{e}_i$$

if $\bar{V}_i^{(k)} = <V^{(k)}, \bar{e}_l>$ and $\bar{\delta}_i = <\delta, e_i>$

we have

$$(\bar{\delta}_i \bar{V}_j^{(k)} - \bar{\delta}_j \bar{V}_i^{(k)})(p) = \sum_{h,\ell=1}^{n+m} \alpha_h^j \, \alpha_\ell^i \, (\delta_\ell V_h^{(k)} - \delta_h V_\ell^{(k)})(p) = 0$$

Therefore we have

$$\bar{\delta}_i \bar{V}_j^{(k)}(p) = \bar{\delta}_j \bar{V}_i^{(k)}(p)$$

Thus we have, for any system of coordinates of $\mathbb{R}^{n+m}$, at each point of M,

(1.6.1) $$\delta_i V_j^{(k)} = \delta_j V_i^{(k)} \quad ; \quad i,j = 1,2,\ldots,n+m \quad k = 1,2,\ldots,m$$

1.7 - *PROPOSITION* - Let $\{V^{(k)}\}_{1\le k\le m}$ be as in 1.6, let $\{e_i\}_{1\le i\le n+m}$ be any coordinate system of $\mathbb{R}^{n+m}$ and let $\delta_i = \langle\delta, e_i\rangle$.

Thus,

(1.7.1) $$[\delta_i,\delta_j] = \delta_i\delta_j - \delta_j\delta_i = \sum_{k=1}^{m}\sum_{h=1}^{n+m} (V_i^{(k)}\ \delta_i V_h^{(k)} - V_j^{(k)}\ \delta_i V_h^{(k)})\,\delta_h$$

PROOF. Having in mind that $R \equiv 0$ together with (1.1.2), (1.6.1) and the fact that

(1.7.2) $$\sum_{h=1}^{n+m} V_h^{(r)}\ \delta_j V_h^{(k)} = 0 \quad ; \quad j = 1,2,\ldots,n+m \quad k,r = 1,2,\ldots,m$$

the proof is a straightforward computation.

To prove (1.7.2) we choose a coordinate system $\{e_i\}_{1\le i\le n+m}$ of $\mathbb{R}^{n+m}$ such that $V^{(k)}(p) = e_{n+k}$ for all k. Then,

$$\sum_{h=1}^{n+m} V_h^{(r)}\ \delta_j V_h^{(k)}(p) = \langle\bar{\nabla}_{e_j} V^{(k)}\ ,\ V^{(r)}\rangle_p = 0$$

because $R^{\perp} \equiv 0$. Now we argue as in the end of 1.6 for any system of coordinates.

1.8 - *PROPOSITION* - With the hypothesis of 1.6 and supposing that the immersion is minimal, we have :

$$(1.8.1) \qquad \Delta V^{(k)} = -C_k^2 \, V^{(k)} - \sum_{\substack{k,r=1 \\ k \neq r}}^{m} C_{kr} \, V^{(r)} ,$$

where $\Delta V^{(k)} = (\Delta V_1^{(k)}, \ldots, \Delta V_{n+m}^{(k)})$ and the components of $V^{(k)}$ are taken with respect to an arbitrary coordinate system of $\mathbb{R}^{n+m}$.

PROOF. The flatness of the normal bundle implies

$$\sum_{h=1}^{n+m} V_h^{(r)} \, \delta_j V_h^{(k)} = 0 \quad ; \quad j=1,2,\ldots,n+m \quad k,r=1,2,\ldots,m$$

Thus

$$\Delta V_j^{(k)} = \sum_{i=1}^{n+m} \delta_i \delta_i \, V_j^{(k)} = \sum_{i=1}^{n+m} \delta_i \delta_j \, V_i^{(k)} =$$

$$= \sum_{i=1}^{n+m} \delta_j \delta_i V_i^{(k)} + \sum_{r=1}^{m} \sum_{ih=1}^{n+m} (V_i^{(r)} \, \delta_j V_h^{(r)} - V_j^{(r)} \delta_i V_h^{(r)}) \delta_h V_i^{(k)}$$

$$(1.8.2) \quad \Delta V_j^{(k)} = - \sum_{r=1}^{m} V_j^{(r)} \, [\sum_{ih=1}^{n+m} (\delta_i V_h^{(r)}) \, (\delta_h V_i^{(k)})]$$

Therefore (1.8.1) follows from (1.2.2) and (1.2.3) .

§ 2 - *THE LAPLACIAN OF THE NORM OF THE SECOND FUNDAMENTAL FORM*

2.1 - *THEOREM* - Let $M^n \hookrightarrow \mathbb{R}^{n+m}$ be a minimal isometric immersion having flat normal bundle. Let $\{V^{(k)}\}_{1\le k\le m}$ be a local orthonormal family of parallel sections of the normal bundle defined in an open subset U of M. If $\{e_i\}_{1\le i\le n+m}$ is a coordinate system of $\mathbb{R}^{n+m}$ such that $V^{(k)}(p) = e_{n+k}$; $k=1,\dots,m$, $p\in M$, we have

$$(2.1.1)\quad \Big(\frac{1}{2}\Delta C_k^2 + C_k^4 + \sum_{\substack{k,r=1\\ k\neq r}}^{m} C_{kr}\Big)(p) = \sum_{hij=1}^{n} (\delta_i\delta_j V_h^{(k)})^2(p)$$

for all $k=1,2,\dots,m$.

PROOF. Since $R^{\perp}\equiv 0$, $C_k^2 = \sum_{ij=1}^{n+m} (\delta_i V_j^{(k)})^2$

Therefore ,

$$\frac{1}{2}\Delta C_k^2 = \frac{1}{2}\sum_{i=1}^{n+m} \delta_i\delta_i\Big(\sum_{hj=1}^{n+m} (\delta_h V_j^{(k)})^2\Big) =$$

$$= \sum_{hij=1}^{n+m} (\delta_i\delta_h V_j^{(k)})^2 + \sum_{hij=1}^{n+m} (\delta_h V_j^{(k)})(\delta_i\delta_i\delta_h V_j^{(k)})$$

Using $[\delta_i,\delta_h]$ twice we have

$$\frac{1}{2}\Delta C_k^2 = \sum_{hij=1}^{n+m} (\delta_i\delta_h V_j^{(k)})^2 + \sum_{hij=1}^{n+m} (\delta_h V_j^{(k)})\,\delta_h(\Delta V_j^{(k)}) +$$

$$+ \sum_{\ell=1}^{m} \sum_{hijs} V_i^{(\ell)} (\delta_h V_j^{(k)}) (\delta_h V_s^{(\ell)}) (\delta_s \delta_i V_j^{(k)}) - \sum_{\ell=1}^{m} \sum_{hijs} (\delta_h V_j^{(k)}) (\delta_i V_h^{(\ell)}) (\delta_i V_s^{(\ell)}) (\delta_s V_j^{(k)})$$

From (1.8.1) we have

$$\delta_h (\Delta V_j^{(k)}) = - (\delta_h C_k^2) V_j^{(k)} - (\delta_h V_j^{(k)}) C_k^2 -$$

$$- \sum_{\substack{r=1 \\ r\neq k}}^{m} \delta_h (C_{kr}) V_j^{(r)} - \sum_{\substack{r=1 \\ r\neq k}}^{m} C_{kr} \delta_h V_j^{(r)}$$

Therefore (1.2.3) and (1.7.2) imply

$$(\delta_h V_j^{(k)}) \delta_h (\Delta V_j^{(k)}) = - C_k^4 - \sum_{\substack{r=1 \\ r\neq k}}^{m} C_{kr}^2$$

Thus

$$\frac{1}{2} \Delta C_k^2 + C_k^4 + \sum_{\substack{r=1 \\ r\neq k}}^{m} C_{kr}^2 =$$

$$= \sum_{hij=1}^{n+m} (\delta_i \delta_h V_j^{(k)})^2 - 2 \sum_{\ell=1}^{m} \sum_{hijs=1}^{n+m} (\delta_h V_j^{(k)}) (\delta_h V_s^{(\ell)}) (\delta_s V_i^{(\ell)}) (\delta_i V_j^{(k)})$$

Using $[\delta_h, \delta_{n+r}]$ we have

$$\delta_h \delta_{n+r} V_j^{(k)} (p) = - \sum_{s=1}^{n+m} (\delta_h V_s^{(r)}) (\delta_s V_j^{(k)}) (p)$$

So at p we have

$$\left(\frac{1}{2} \Delta C_k^2 + C_k^4 + \sum_{\substack{r=1 \\ r \neq k}}^{m} C_{kr}{}^2 \right)(p) =$$

$$= \sum_{hij=1}^{n+m} (\delta_i \delta_h V_j^{(k)})^2 (p) - 2 \sum_{\ell=1}^{m} \sum_{sj=1}^{n+m} (\delta_s \delta_{n+\ell} V_j{}^{(k)})^2 (p)$$

Using again $[\delta_i , \delta_{n+\ell}]$ (p) and

$$\delta_s V_{n+j}^{(k)} (p) = \langle \nabla^{\perp}_{e_s} V^{(k)} , e_{n+j} \rangle_p = 0$$

we obtain

$$\left(\frac{1}{2} \Delta C_k^2 + C_k^4 + \sum_{\substack{r=1 \\ r \neq k}}^{m} C_{kr}{}^2 \right)(p) = \sum_{hij=1}^{n} (\delta_i \delta_h V_j^{(k)})^2 (p) +$$

$$+ \sum_{\ell=1}^{m} \sum_{ih=1}^{n} \delta_i \delta_h V_{n+\ell}^{(k)})^2 (p) + \sum_{\ell=1}^{m} \sum_{ij=1}^{n} (\delta_i \delta_{n+\ell} V_j^{(k)})^2 (p) -$$

$$- 2 \sum_{\ell=1}^{m} \sum_{ij=1}^{n} (\delta_i \delta_{n+\ell} V_j^{(k)})^2 (p) =$$

$$= \sum_{hij=1}^{n} (\delta_i \delta_h V_j^{(k)})^2 (p) .$$

2.2 - *OBSERVATION* - Let $C^2 = \sum_{k=1}^{m} C_k^2$,

$$\Delta C^2 = \sum_{k=1}^{m} \Delta C_k^2 \quad \text{and} \quad C_{kr} = \langle A^{V^{(k)}} , A^{V^{(r)}} \rangle$$

Thus,

(2.2.1) $$\sum_{k=1}^{m} C_k^4 + \sum_{\substack{kr=1\\ r\neq k}}^{m} C_{kr}^2 \leq \sum_{k} C_k^4 + \sum_{k\neq r} C_k^2 C_r^2 = C^4$$

Therefore, with the same hipothesis of Theorem 2.1. , (2.2.1) implies

(2.2.2) $$\left(\frac{1}{2}\Delta C^2 + C^4\right)(p) \geq \sum_{k=1}^{m} \sum_{hij=1}^{n+m} (\delta_i \delta_h V_j^{(k)})^2 \ (p)$$

2.3 - *THEOREM* - Let $M^n \hookrightarrow \mathbb{R}^{n+m}$ be a minimal cone isometrically immersed and having flat normal bundle.

Let $\{V^{(k)}\}_{1\leq k\leq m}$ be a local orthonormal family of sections of the normal bundle defined in an open subset U of M .

Then, if $||x||$ is the distance from $x\in M$ to the vertex of M, we have

(2.3.1) $$\frac{1}{2}\Delta C^2 + C^4 \geq |\delta C|^2 + \frac{2C^2}{||x||^2}$$

(2.3.2) $$\frac{1}{2}\Delta C_k^2 + C_k^4 + \sum_{\substack{r=1\\ r\neq k}}^{m} C_{kr}{}^2 \geq |\delta C_k|^2 + \frac{2C_k{}^2}{||x||^2} \ ; \ k=1,2,\dots,m$$

PROOF. We may consider the vertex of the cone as the origin of $\mathbb{R}^{n+m}$,

$$U = \{t\,y \ / \ y\in \tilde{U} \ , \ t\in (0,+\infty)\}$$

where $\tilde{U}$ is an open subset of S^{n+m-1} (the euclidean unitary sphere), and that $V^{(k)}$ is constant through the rays of M.

Given $p \in M$, let $\{e_i\}_{1 \le i \le n+m}$ be a coordinate system of $\mathbb{R}^{n+m}$ such that $V^{(k)}(p) = e_{n+k}$ for all k, and $p = \|p\| e_n$.

Thus, for all $t \in (0,+\infty)$ and all k, we have

$$D_n V^{(k)}(tp) = 0$$

$$V_n^{(k)}(tp) = V_n^{(k)}(p) = 0$$

$$\delta_i V_n^{(k)}(tp) = \delta_n V_i^{(k)}(tp) = (D_n V_i^{(k)} - \sum_{\ell=1}^{m} \langle V^{(\ell)}, DV_i^{(k)} \rangle V_n^{(\ell)})(tp) = 0$$

Therefore ,

(2.3.3) $$\delta_n(\delta_n V_i^{(k)})(tp) = \delta_n(\delta_i V_n^{(k)})(tp) = 0$$

From (1.6.1) and (1.7.1) we have

(2.3.4) $$\delta_i \delta_j V_n^{(k)} = \delta_i \delta_n V_j^{(k)}$$

(2.3.5) $$\delta_i \delta_n V_j^{(k)}(p) = \delta_n \delta_i V_j^{(k)}(p)$$

So,

$$[\sum_{k=1}^{m} \sum_{ihr=1}^{n} (\delta_i \delta_h V_r^{(k)})^2 - |\delta c|^2](p) =$$

$$= \sum_{k=1}^{m} \sum_{ihr=1}^{n} (\delta_i \delta_h V_r^{(k)})^2 - \frac{1}{C^2} \sum_{k\ell=1}^{m} \sum_{ihrsj=1}^{n} (\delta_h V_r^{(k)})(\delta_i \delta_h V_r^{(k)})(\delta_s V_j^{(\ell)})(\delta_i \delta_s V_j^{(\ell)})$$

$$= \frac{1}{2C^2} \sum_{k\ell=1}^{m} \sum_{ihrsj=1}^{n} \Big[(\delta_h V_r^{(k)})^2 (\delta_i \delta_s V_j^{(\ell)})^2 + (\delta_s V_j^{(\ell)})^2 (\delta_i \delta_h V_r^{(k)})^2 -$$

$$- 2(\delta_h V_r^{(k)})(\delta_i \delta_h V_r^{(k)})(\delta s V_j^{(\ell)})(\delta_i \delta_s V_j^{(\ell)}) \Big] =$$

$$= \frac{1}{2C^2} \sum_{k\ell=1}^{m} \sum_{ihrsj=1}^{n} \left[(\delta_h V_r^{(k)})(\delta_i \delta_s V_j^{(\ell)}) - (\delta_s V_j^{(\ell)})(\delta_i \delta_h V_r^{(k)}) \right]^2 =$$

$$= \frac{1}{2C^2} \sum_{k\ell=1}^{m} \left\{ \sum_{i=1}^{n} \sum_{hjsr=1}^{n-1} \left[(\delta_h V_r^{(k)})(\delta_i \delta_s V_j^{(\ell)}) - (\delta_s V_j^{(\ell)})(\delta_i \delta_h V_r^{(k)}) \right]^2 + \right.$$

$$\left. + \; 4\, C_k^2 \sum_{is=1}^{n} (\delta_n \delta_i V_s^{(\ell)})^2 \right\}$$

Therefore ,

$$(2.3.6) \qquad \left[\sum_{k=1}^{m} \sum_{ihr=1}^{n} (\delta_i \delta_h V_r^{(k)})^2 - |\delta c|^2 \right] (p) \geq$$

$$\geq \; 2 \sum_{k=1}^{m} \sum_{ih=1}^{n} (\delta_n \delta_i V_h^{(k)})^2 \; (p)$$

Now, given any $\emptyset \in C^1(M)$, we have

$$\delta_n \emptyset \; (p) = \langle D\emptyset, e_n \rangle_p = \frac{1}{||p||} \langle D \emptyset , p \rangle = \frac{1}{||p||} \sum_{j=1}^{n+m} p_j \, D_j \emptyset(p)$$

Thus we have

$$(2.3.7) \qquad \delta_n (\delta_i V_h^{(k)})(p) = \frac{1}{||p||} \sum_{j=1}^{n+m} p_j \, D_j (\delta_i V_h^{(k)}) \; (p)$$

Since $V_h^{(k)}(x) = V_h^{(k)}(tx)$; $\forall x \in M$,

$\forall t \in (0,+\infty)$

$$\delta_i V_h^{(k)}(tx) = \frac{1}{t}\,\delta_i V_h^{(k)}(x)$$

which gives, by differentiation with respect to t,

$$< D(\delta_i V_h^{(k)})(tx) \;,\; x > \;=\; -\frac{1}{t^2}\,(\delta_i V_h^{(k)})(x)$$

In particular for $t = 1$,

$$\sum_{j=1}^{n+m} x_j\, D_j(\delta_i V_h^{(k)}) = < D(\delta_i V_h^{(k)})(x) \;,\; x > \;=\; -(\delta_i V_h^{(k)})(x) \;, \forall x \in M$$

This together with (2.3.7) gives

$$\sum_{ih=1}^{n} (\delta_n \delta_i V_h^{(k)})^2 (p) = \frac{1}{||p||^2} \sum_{ih=1}^{n} (\delta_i V_h^{(k)})^2 (p) = \frac{C_k^2(p)}{||p||^2}$$

Thus , (2.3.6) implies

$$(2.3.8) \quad \left[\, \sum_{k=1}^{m} \sum_{ihr=1}^{n} (\delta_i \delta_n V_r^{(k)})^2 - |\delta c|^2 \right](p) \geq \frac{2c^2(p)}{||p||^2}$$

From (2.2.2) and (2.3.8) we get (2.3.1).

By a similar computation, we get for each k,

$$\left[\, \sum_{ihr=1}^{n} (\delta_i \delta_h V_r^{(k)})^2 - |\delta C_k|^2 \right](p) \geq \frac{2C_k^2(p)}{||p||^2}$$

which together (2.2.1) give (2.3.2).

§ 3 - *MINIMAL CONES*

3.1 - The formula of the second variation of the area.

Let $f:M^n \hookrightarrow \mathbb{R}^{n+m}$ be an isometric immersion of a compact orientable manifold M and let $F:M\times(-\varepsilon,\varepsilon)\longrightarrow \mathbb{R}^{n+m}$ be a C^∞ variation of f such that

$$F_t(\partial M) = f(\partial M)\ ,\ \text{for all } t$$

Let A(t) be the n-dimensional area of $F_t(M)$, $E(p) = \frac{\partial}{\partial t} F(p,0)$ the variational field, dM the volume form of M, $\tilde{A}={}^tA\circ A$ and

$$\nabla^2 E^\perp = \sum_{i=1}^{n} \nabla^\perp_{\mu_i} \nabla^\perp_{\mu_i} E^\perp - \sum_{i=1}^{n} \nabla^\perp_{\nabla_{\mu_i} \mu_i} E^\perp$$

where $\{\mu_i\}_{1\le i\le n}$ is an orthonormal tangent frame. Thus, if f is a minimal immersion having normal bundle and F is a variation such that $E^\perp = \sum_{k=1}^{m} h_k V^{(k)}$, where $\{V^{(k)}\}_{1\le k\le m}$ is an orthonormal family of local parallel sections of the normal bundle of M, defined in an open set $U\subset M$, and each $h_k\in C^1_0(U)$, we have

$$\text{(3.1.1)} \qquad A''(0) = \int_M \langle -\nabla^2 E^\perp - \tilde{A}(E^\perp), E^\perp \rangle \, dM$$

(see [2])

3.2 - *PROPOSITION* - With the hipothesis of 3.1, we have

(3.2.1) $$A''(0)\int_M \left[\sum_{k=1}^{m}(|\delta h_k|^2 - h_k^2 C_k^2) - \sum_{\substack{kr=1\\ k\neq r}}^{m} h_k h_r C_{kr}\right] dM$$

PROOF.

(3.2.2) $$\int_M \langle -\nabla^2 E^\perp, E^\perp\rangle dM = -\int_M \langle \nabla^\perp E^\perp, \nabla^\perp E^\perp\rangle\, dM ,$$

where $\langle\nabla^\perp E^\perp, \nabla^\perp E^\perp\rangle = \sum_{i=1}^{n} \langle \nabla^\perp_{\mu_i} E^\perp, \nabla^\perp_{\mu_i} E^\perp\rangle$

for any orthonormal tangent frame $\{\mu_i\}_{1\le i\le n}$. Moreover $R^\perp \equiv 0$ implies that

$$\langle\nabla^\perp E^\perp,\nabla^\perp E^\perp\rangle = \sum_{i=1}^{n}\sum_{k=1}^{m} (\mu_i[h_k])^2$$

Thus, if $\{e_i\}_{1\le i\le n+m}$ is a coordinate system of $\mathbb{R}^{n+m}$ such that, for $p\in U$, $V^{(k)}(p) = e_{n+k}$; $k = 1,2,\dots,m$, we have

$$\langle\nabla^\perp E^\perp, \nabla^\perp E^\perp\rangle_p = \sum_{i=1}^{n}\sum_{k=1}^{m} (D_i h_k)^2(p) =$$

$$= \sum_{i=1}^{n}\sum_{k=1}^{m} (\delta_i h_k)^2(p) = \sum_{k=1}^{m}\sum_{i=1}^{n+m} (\delta_i h_k)^2(p) = \sum_{k=1}^{m} |\delta h_k|^2(p)$$

Therefore, in M, we have

(3.2.3) $$\langle\nabla^\perp E^\perp, \nabla^\perp E^\perp\rangle = \sum_{k=1}^{m} |\delta h_k|^2$$

Since

$$-\langle\tilde{A}(E^{\perp}), E^{\perp}\rangle = -\langle A^{E^{\perp}}, A^{E^{\perp}}\rangle$$

$$(3.2.4) \qquad -\langle\tilde{A}(E^{\perp}), E^{\perp}\rangle = -\sum_{k=1}^{m} h_k^2 C_k^2 - \sum_{\substack{kr=1\\ k\neq r}}^{m} h_k h_r C_{kr} ,$$

we have (3.2.1) from (3.1.1), (3.2.2), (3.2.3) and (3.2.4)

3.3 - *PROOF OF THEOREM 1*

Let $\{V^{(k)}\}_{1\le k\le m}$ be an orthonormal family of parallel fields in the normal bundle of M. Thus, $V^{(k)}(y) = V^{(k)}(ty)$ for all t and $y\in M\cap S^{n+m-1}$, $k = 1,2,\dots,m$.

Supposing that M is stable and that E is the variational field of a variation of M such that

$$E^{\perp} = \sum_{k=1}^{m} h_k V^{(k)} \quad ; \ h_k\in C_0^1(M) ,$$

we have, from (3.2.1) ,

$$\int_M \Big(\sum_{k=1}^{m} |\delta h_k|^2 - \sum_{k=1}^{m} h_k C_k^2 - \sum_{\substack{kr=1\\ k\neq r}}^{m} h_k h_r C_{kr}\Big) DM \ge 0$$

In particular, if $h_k = \emptyset\, C_k$, $\emptyset\in C_0^1(M)$,

$$\delta h_k = C_k^2|\delta\emptyset|^2 + \emptyset^2|\delta C_k|^2 + \frac{1}{2}\langle\delta\emptyset^2 , \delta C_k^2\rangle , \quad \text{and}$$

$$(3.3.1) \quad \int_M \emptyset^2\Big(\sum_{k=1}^{m} C_k^4 + \sum_{\substack{kr=1\\ k\neq r}}^{m} C_k C_r C_{kr}\Big) dM \le \int_M \Big(C^2|\delta\emptyset|^2 + \emptyset^2 \sum_{k=1}^{m} |\delta C_k|^2 + \frac{1}{2}\langle\delta\emptyset^2, \delta C^2\rangle\Big) dM$$

which , together with (1.4.1) and (2.3.2), implies

$$(3.3.2) \quad \int_M \phi^2 \sum_{\substack{kr=1 \\ k\neq r}}^{m} (C_k C_r - C_{kr}) C_{kr} dM \leq \int_M c^2 |\delta\phi|^2 dM - \int_M \phi^2 \frac{2c^2}{||x||^2} dM$$

(we observe that for $m=1$, the first side of (3.3.2) is zero)

Now let ϕ such that $\phi(x) = \phi(ty) = f(t)$ where $x = ty$, $t = ||x||$, $y \in M \cap S^{n+m-1}$ and $f \in C_0^1(0,+\infty)$.

Thus, given $p \in M$, if $\{e_i\}_{1\leq i\leq n+m}$ is a coordinate system of $\mathbb{R}^{n+m}$ with origin at the vertex of M, and such that $V^{(k)}(p) = e_{n+k}$ for all $k = 1,2,\ldots,m$ and $||p|| e_n = p$, we have

$$\delta_i \phi(p) = D_i \phi(p) \ ; \ i = 1,2,\ldots,n \ , \ k = 1,2,\ldots,m$$

and so, $\delta_i \phi(p) = 0$ if $i \neq n$

and $\delta_n \phi(p) = f'(||p||)$

and $|\delta\phi|^2(p) = (f'(||p||))^2$

Thus,

$$(3.3.3) \quad |\delta\phi|^2(x) = |\delta\phi|^2(ty) = (f'(t))^2, \ \forall x \in M$$

Since

$$(3.3.4)\quad \begin{cases} C^2(ty) = \frac{1}{t^2}\, C^2(y) \\ \sum\limits_{\substack{kr=1\\ k\neq r}}^{m} (C_kC_r - C_{kr})C_{kr}\,(ty) = \frac{1}{t^4} \sum\limits_{\substack{kr=1\\ k\neq r}}^{m} (C_kC_r - C_{kr})C_{kr}(y) \\ dM(ty) = t^{n-1}\, d\tilde{M}(y)\wedge dt\ ;\ \tilde{M} = M\cap S^{n+m-1} \end{cases}$$

we have, from (3.3.2), (3.3.3) and (3.3.4),

$$(3.3.5)\quad \int_{\tilde{M}} \Big(\int_0^{+\infty} f^2(t)\, \frac{t^{n-1}}{t^4}\, dt \Big) \Big(\sum_{\substack{kr=1\\ k\neq r}}^{m} (C_kC_r - C_{kr})C_{kr}(y) \Big) d\tilde{M} \le$$

$$\le \int_{\tilde{M}} \Big(\int_0^{+\infty} (f'(t))^2\, \frac{t^{n-1}}{t^2} - f^2(t)\, \frac{2t^{n-1}}{t^4}\, dt \Big) C^2(y)\, d\tilde{M}\ ,\ \forall f\ C_0^1(0,+\infty)$$

Since $\int_{\tilde{M}} d\tilde{M} > 0$ and we may

suppose that $K = \int_{\tilde{M}} \sum\limits_{\substack{kr=1\\ k\neq r}}^{m} (C_kC_r - C_{kr})C_{kr}\, d\tilde{M} \ge 0$

(see 3.4), we have, from (3.3.5),

$$(3.3.6)\quad \int_0^{+\infty} [(f'(t))^2\, t^{n-3} - f^2(t)\, 2t^{n-5}]\,dt \ge 0,\ \forall f\in C_0^1(0,+\infty)$$

(3.3.6) is still true for $f(t) = t^{\alpha}\, g(t)^{\beta}$, $t\in(0,+\infty)$ where $\alpha>0$, $\alpha+\beta<0$ and $g(t) = \max\{t,1\}$, if we have

(3.3.7) $$\int_0^{+\infty} f(t)\ t^{n-5}\ dt < +\infty$$

(see 3.5)

But

$$\int_0^{+\infty} f^2(t)t^{n-1}dt = \int_0^1 t^{n-5+2\alpha}dt + \int_1^{+\infty} t^{n-1+2(\alpha+\beta)}dt < +\infty$$

only if

(3.3.8) $$\alpha > \frac{4+n}{2}$$ $$\alpha+\beta < \frac{4+n}{2}$$

Therefore, for these values of α and $\alpha+\beta$, (3.3.6) implies

(3.3.9) $$(\alpha^2-2)\int_0^1 t^{2\alpha+n-5}dt + ((\alpha+\beta)^2-2)\int_1^{+\infty} t^{2(\alpha+\beta)+n-5}dt \geq 0$$

But if $(\frac{4+n}{2})^2 < 2$, there is

$\alpha > 0$, $\frac{4+n}{2} < \alpha < \sqrt{2}$, and β

such that $\alpha+\beta > 0$ and $-\sqrt{2} < \alpha+\beta < \frac{4+n}{2} < \sqrt{2}$,

which satisfy (3.3.8). For these values of α and β we have

$$\alpha^2 - 2<0 \quad \text{and} \quad (\alpha+\beta)^2 - 2<0 .$$

and this is incompactible with (3.3.9)

Therefore for those values of n such that $\left(\frac{4+n}{2} \right)^2 < 2$, M cannot be stable. That is, if $2 \le n \le 6$, M is not stable.

3.4 - *LEMMA* - If $m \neq 1$, $K \ge 0$.

PROOF. Let $a_{kr} = C_k C_r - C_{kr}$ and

$$S_i = \sum_{\substack{kr=1 \\ k\neq r}}^{m} \int_{\tilde{M}} a_{kr} < A^{V^{(k)}} , A^{V^{(r)}} > d\tilde{M} , \ i \ge 2$$

By Schwarz inequality we have $a_{kr} \ge 0$

Thus changing, eventually, $V^{(2)}$ by $-V^{(2)}$, we have

$$S_2 = \int_{\tilde{M}} a_{12} < A^{V^{(1)}} , A^{V^{(2)}} > d\tilde{M} \ge 0$$

Supposing that for a given $i<m$,

$$S_i = \sum_{\substack{kr=1 \\ k\neq r}}^{m} \int_{\tilde{M}} a_{kr} < A^{V^{(k)}} , A^{V^{(r)}} > d\tilde{M} \ge 0 ,$$

we have

$$S_{i+1} = S_i + \sum_{k=1}^{i} \int_{\tilde{M}} a_{k(i+1)} < A^{V^{(k)}} , A^{V^{(i+1)}} > d\tilde{M}$$

and changing, eventually, $V^{(i+1)}$ by $-V^{(i+1)}$ we have $S_{i+1} \geq 0$.

The result follows by induction.

3.5 - *LEMMA* - (3.3.6) is true for $f(t) = t^{\alpha} g(t)^{\beta}$ where $t \in (0,+\infty)$, and $\alpha>0$ and $\alpha+\beta<0$ satisfy (3.3.8).

PROOF. Let

$$f_j(t) = \begin{cases} t^{\alpha} & , \text{ if } \frac{1}{j} \leq t \leq 1 \\ t^{\alpha+\beta} & , \text{ if } 1 \leq t \leq j \\ 0 & , \text{ if } t \notin [\frac{1}{j}, j], \ j = 1,2,\ldots \end{cases}$$

Given $\varepsilon>0$, let

$$\rho_{\varepsilon}(t) = \varepsilon \rho\left(\frac{t}{\varepsilon}\right)$$

where $\rho \in C_0^{\infty}(\mathbb{R})$, $\rho(t) \geq 0$, $\rho(t) = 0$ if $|t| \geq 1$ and $\int_{\mathbb{R}} \rho(t)dt = 1$

Making the convolution of f_j with ρ_{ε} we get

$$f_{j,\varepsilon}(t) = \int_{\mathbb{R}} \rho_{\varepsilon}(t-\mu) f_j(\mu)\, d\mu$$

Moreover, $\frac{d}{dt} f_{j,\varepsilon}(t) = f'_{j\varepsilon}(t) = \int_{\mathbb{R}} \rho_{\varepsilon}(t-\mu) f'_j(\mu) d\mu$

Since $f_{j,\varepsilon}$ and $f'_{j,\varepsilon}$ converge to f_j and f'_j in $L^1(\mathbb{R})$, there is a subsequence $\{f_{j,\varepsilon_k}\}$ of $\{f_j\}$ such that $\{f_{j,\varepsilon_k}\}$ and

$\{f'_{j,\varepsilon_k}\}$ converge almost everywhere to f_j and f'_j , respectively.

From the boundness eveywhere of f_j and f'_j , we have , as a consequence of dominated convergence theorem,

$$\lim_{\varepsilon_k \to 0} \int_0^{+\infty} (f_{j,\varepsilon_k}(t))^2 \, t^{n-5} \, dt = \int_0^{+\infty} (f_j(t))^2 \, t^{n-5} \, dt$$

$$\lim_{\varepsilon_k \to 0} \int_0^{+\infty} (f'_{j,\varepsilon_k}(t))^2 \, t^{n-3} \, dt = \int_0^{+\infty} (f'_j(t))^2 \, t^{n-3} \, dt$$

Therefore, since (3.3.6) is true fot any f_{j,ε_k} we have that (3.3.6) is also true for f_j . Now the result follows from the monotonic convergence theorem.

3.6 - *PROOF OF THEOREM 2*

Let $\mathbb{R}^{n+m} = \mathbb{R}^{n_1+1} \times \dots \times \mathbb{R}^{n_s+1}$. It is easy to show that $\tilde{M} = S^{n_1}(r_1) \times \dots \times S^{n_s}(r_s) \subset S^{n+m-1}$ is a minimial submanifold of S^{n+m-1} and so that M is a minimial cone in $\mathbb{R}^{n+m}$.

Moreover, if $x = (x_1,\dots,x_s) \in \tilde{M}$,

$$e_0^i = \frac{1}{r_i} (0,\dots,0 , x_i , 0,\dots,0) \; ; \; i = 1,2,\dots,s$$

$$R_k = \sum_{j=k+1}^{s} r_j^2 \quad ; \quad R_s = 0 \quad , \quad \text{and}$$

(3.6.1) $$\tilde{v}^{(k)} = \frac{1}{\sqrt{1-R_k}\,\sqrt{1-R_{k+1}}} (r_1 r_{k+1} e_0^1 + \dots + r_k r_{k+1} e_0^k - (1-R_k) e_0^{k+1}) \; ;$$

$k=1,2,\dots,m$

is straightforward to show that

a) $\{\tilde{V}^{(k)}\}_{1\le k\le m}$ is an orthonormal family of parallel sections in the normal bundle of $\tilde{M}$ in S^{n+m-1} ;

b) With respect to it, $\tilde{C}_{k\ell} \equiv 0$ if $k \neq \ell$, $\tilde{C}_k^2 \equiv n-1$ and $\tilde{C}^2 \equiv m(n-1)$.

Thus setting $V^{(k)}(ty) = \tilde{V}^{(k)}(y)$; $y\in\tilde{M}$, $t\in(0,+\infty)$, we get a family of parallel orthonormal sections in the normal bundle of M in $\mathbb{R}^{n+m}$, such that with respect to it we have ,

$$C_k^2(ty) = \frac{1}{t^2}\cdot\tilde{C}_k^2(y) = \frac{n-1}{t^2}$$

(3.6.2)
$$C^2(ty) = \frac{\tilde{C}^2(y)}{t^2} = \frac{m(n-1)}{t^2}$$

$$C_{k\ell}(ty) = \frac{1}{t^2}\tilde{C}_{k\ell}(y) = 0, \text{ if } k \neq \ell$$

Now let E be the variational field of a variation of M in $\mathbb{R}^{n+m}$ such that $E^{\perp} = \sum_{k=1}^{m} h_k V^{(k)}$, where $h_k \in C_0^{\infty}(M)$.

From (3.2.1) we have :

$$A''(0) = \int_M \sum_{k=1}^{m} (|\delta h_k|^2 - \frac{C^2}{m} h_k^2)\,dM = \sum_{k=1}^{m}\int_M (-h_k\Delta h_k - \frac{C^2}{m} h_k^2)\,dM$$

The remainder of the proof follows step by step the one made by J. Simons [3, theorems 6.1.1 and 6.1.2, pag 97]

R E F E R E N C E S

[1] - H. Federer - "The singular sets of area minimizing"
Bull. A.M.S. (1970)

[2] - J.B. Lawson - "Lectures on minimal submanifolds" (IMPA)

[3] - J. Simons - "Minimal varieties in riemannian manifolds".
Ann. of Math., 88 (1968).

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