

Non-existence of sublinear diffusion for a class of torus homeomorphisms

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Abstract. We prove that, if f is a homeomorphism of the 2-torus isotopic to the identity whose rotation set is a non-degenerate segment and f has a periodic point, then it has uniformly bounded deviations in the direction perpendicular to the segment.

Key words: rotation sets, forcing, bounded deviations

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1. Introduction

The study of surface dynamics from a topological viewpoint has been gathering increasing attention in the last decade, in large part because of the development of new tools and techniques that have been proven effective in tackling previously hopeless problems. Many of these developments have been tied to recent improvements in both Brouwer theory and rotation theory. In particular, the search for a greater understanding of the dynamics of torus homeomorphisms in the isotopy class of the identity has been one of the motivating forces behind the developments, due to its connection to relevant physical dynamics like Hamiltonian homeomorphisms in general or specific models such as the kicked Harper model and the Zaslavsky web maps.

Rotation theory for homeomorphisms of the 2-torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ appeared in the early 1990s as an extension of the ideas of the Poincaré rotation number for homeomorphisms of the circle. Given $f : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ a torus homeomorphism isotopic to the identity, and $\tilde{f} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ a lift of f to the universal covering of \mathbb{T}^2 , one can define, following [MZ89], the rotation set $\rho(\tilde{f})$ of \tilde{f} as

$$\rho(\tilde{f}) := \left\{ v \mid \text{there exists } n_k \rightarrow \infty, \tilde{z}_k \in \mathbb{R}^2, \lim_{k \rightarrow \infty} \frac{\tilde{f}^{n_k}(\tilde{z}_k) - \tilde{z}_k}{n_k} = v \right\},$$

which is always compact and convex. This notion, which is invariant to change of coordinates in the isotopy class of the identity, has been shown to be of great utility in describing the dynamics of these maps. For instance, it is in some cases possible to deduce, just by the analysis of the rotation sets, that the dynamics has periodic points of arbitrarily large period [Fra89], that it has positive entropy [LM91], or even that it has a well-defined chaotic region [KT14b].

One relevant feature of rotation sets is that, as per the definition, they describe only linear rates of displacement in the lift. But a question that has appeared in several contexts is to determine if it is possible for sublinear displacements to exist that are not captured by it. For, while it follows directly from the definitions that there exist $M_0, N_0 > 0$ such that $d((\tilde{f}^n(\tilde{z}) - \tilde{z})/n, \rho(\tilde{f})) < M_0$ for all \tilde{z} in \mathbb{R}^2 and all $n > N_0$, one is left to wonder if it is also possible to obtain a better estimate. Specifically, does there also exist some $M_1 > 0$ such that $d((\tilde{f}^n(\tilde{z}) - \tilde{z}), n\rho(\tilde{f})) < M_1$ for all \tilde{z} in \mathbb{R}^2 and all n ? If the latter occurs, we say that f has uniformly bounded deviation from its rotation set. A similar concept is that of bounded deviations in a direction w . Let $w \in \mathbb{R}_*^2$, and define the projection in the w direction as $P_w : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $P_w(x) = \langle x, w/\|w\| \rangle$. One says that f has *bounded w -deviations* if there exists $M_1 > 0$ such that $d(P_w(\tilde{f}^n(\tilde{z}) - \tilde{z}), P_w(n\rho(\tilde{f}))) < M_1$ for all \tilde{z} in \mathbb{R}^2 and all n .

Analysis of bounded deviations for homeomorphisms of \mathbb{T}^2 is a topic increasingly present in the literature. This is due to its intrinsic interest, but also to its use as a fundamental tool in solving some traditional problems in the field. For instance, it has appeared in the proof of Boyland's conjecture in the torus and in the closed annulus [AZ15, CT15, CT19], in determining the existence of Aubry–Mather sets for homeomorphisms of the closed annulus [CT19], in the study of the existence of irrational rotation factors for the dynamics [Jäg09, JT17, JP15, KR17] and in the attempts to solve the remaining case of the Franks–Misiurewicz conjecture [KPS16, PS20, Koc16]. While it is known that bounded deviations do not necessarily hold in all situations, in particular when the rotation set is a singleton (see [KT14a, KK08]), there are several cases where it can be established, such as if $\rho(\tilde{f})$ has non-empty interior [AZ15, CT15]. Furthermore, it is also known (see [Dav18, GKT14]) that if the rotation set is a line segment with two different points with rational coordinates, then f has bounded deviations in the direction that is perpendicular to the segment.

In all these studies, one case which remained unsolved in either direction is to determine, wherever the rotation set was a non-degenerate line segment with at most one rational point, if bounded deviations for a given direction still need to hold. Part of the problem here is the relative lack of examples of these situations. For instance, it was not known until the recent work by Avila (in a manuscript not yet finished) if there existed a homeomorphism whose rotation set was a non-degenerate line segment without rational points. On the other hand, while the existence of examples of rotation sets which are non-degenerate line intervals with a single rational point is a folklore result, these examples always had the rational point as an extremity of the line segment. Indeed, in [CT15] it was shown that this must be the case, that is, that there is no homeomorphism of \mathbb{T}^2 such that its rotation set is a non-degenerate line segment with a single bi-rational point in its relative interior.

In this paper we deal precisely with the latter situation. Let us denote by $(x)_1$ and $(x)_2$ the canonical first and second coordinates, respectively, of a point $x \in \mathbb{R}^2$, and given $\rho_0 = ((\rho_0)_1, (\rho_0)_2)$, let us denote $\rho_0^\perp = (-(\rho_0)_2, (\rho_0)_1)$ and, if $(\rho_0)_1 \neq 0$, denote $\tan(\rho_0) = (\rho_0)_2/(\rho_0)_1$. Our main result is that bounded deviation in the perpendicular direction must exist.

THEOREM A. *Let $f : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ be a torus homeomorphism isotopic to the identity, $\tilde{f} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ a lift of f and $\pi : \mathbb{R}^2 \rightarrow \mathbb{T}^2$ the covering map. Suppose that $\rho(\tilde{f}) = \{t\rho_0 \mid 0 \leq t \leq 1\}$, where $\tan(\rho_0) \notin \mathbb{Q}$. Then there is $M > 0$ such that*

$$|\langle \tilde{f}^n(\tilde{z}) - \tilde{z}, \rho_0^\perp \rangle| < M,$$

for every $\tilde{z} \in \mathbb{R}^2$ and every $n \in \mathbb{Z}$.

Our next result is an immediate corollary, using the results from Dávalos [Dav18] and Le Calvez and the second author [CT15].

COROLLARY B. *Let $f : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ be a torus homeomorphism isotopic to the identity, $\tilde{f} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ a lift of f and $\pi : \mathbb{R}^2 \rightarrow \mathbb{T}^2$ the covering map. Suppose that $\rho(\tilde{f})$ is a non-degenerate line segment and that f has at least one periodic point. Then f has bounded deviations in the direction perpendicular to $\rho(\tilde{f})$.*

Theorem A should have plenty of applications and has already been used in [AZL19] where, among many other considerations on the study of homeomorphisms with rotation sets as in the hypotheses of Theorem A, it is shown that there exist many examples that do not have bounded deviations in the direction of ρ_0 . The main new technical development that made this work possible was the introduction of the new forcing techniques for surface homeomorphisms in [CT15], although in no way is it just a straightforward application. Also, the theory is relatively recent, and this work has as a subproduct some new lemmas that may be useful in its application. The paper is organized as follows: in §2 we describe the main tools necessary for our work, and in §3 we prove the main theorem and its corollary.

2. Preliminaries

2.1. Rotation theory for homeomorphisms of \mathbb{T}^2 . Let $f : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ be a homeomorphism isotopic to the identity, where $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, let $\tilde{f} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a lift of f to the universal covering of \mathbb{T}^2 and let $\pi : \mathbb{R}^2 \rightarrow \mathbb{T}^2$ be the covering map. In the introduction we defined the rotation set of \tilde{f} , also known as the *Misiurewicz–Ziemian rotation set*. We say that a point $x \in \mathbb{T}^2$ has a *rotation vector* v if $\rho(\tilde{f}, x) = \lim_{n \rightarrow \infty} (\tilde{f}^n(\tilde{x}) - \tilde{x})/n = v$, where \tilde{x} is any point in $\pi^{-1}(x)$.

Let $\phi : \mathbb{T}^2 \rightarrow \mathbb{R}^2$ be the displacement function $\phi(x) = \tilde{f}(\tilde{x}) - \tilde{x}$, where \tilde{x} is some point in $\pi^{-1}(x)$. As

$$\frac{1}{n} \sum_{k=0}^{n-1} \phi(f^k(x)) = \frac{1}{n} \sum_{k=0}^{n-1} (\tilde{f}^{k+1}(\tilde{x}) - \tilde{f}^k(\tilde{x})) = \frac{\tilde{f}^n(\tilde{x}) - \tilde{x}}{n},$$

we have, by Birkhoff’s ergodic theorem, that if μ is an ergodic Borel probability measure invariant by f , then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi(f^k(x)) = \int_{\mathbb{T}^2} \phi \, d\mu \quad \text{for } \mu \text{ almost all } x \in \mathbb{T}^2,$$

therefore μ -almost every point has a rotation vector and it is equal to $\int_{\mathbb{T}^2} \phi \, d\mu$, which is called the *rotation vector of the measure* μ . It is well known also that (see [MZ89])

$$\rho(\tilde{f}) = \left\{ \int_{\mathbb{T}^2} \phi \, d\mu \mid \mu \text{ is a Borel probability measure invariant by } f \right\},$$

which bridges the concept of rotation for points and the displacement for invariant measures. As a consequence, one obtains that rotation sets are always compact and convex subsets of the plane, and that if v is an extremal point of the rotation set, then there exists an ergodic f -invariant measure μ whose rotation vector is v .

Of particular importance for this work, we have that, whenever the rotation set of \tilde{f} is a line segment with irrational slope that contains the point $(0, 0)$, one must have that $(0, 0)$ is an extremal point of $\rho(\tilde{f})$ (by [CT15]) and a result due to Franks (see [Fra88]) implies that \tilde{f} must have a fixed point. Furthermore, since in this case $(0, 0)$ is the only point in $\rho(\tilde{f}) \cap \mathbb{Q}^2$, every periodic point of f must be lifted to a periodic point of \tilde{f} .

2.2. *Essential dynamics.* We need the concept of essential points for the dynamics, developed in [KT14b, KT18]. We refer to the papers for a complete exposition, only citing the required results.

An open subset $U \subset \mathbb{T}^2$ is *inessential* if every closed curve contained in U is null homotopic in \mathbb{T}^2 , otherwise U is called *essential*. A general subset $E \subset \mathbb{T}^2$ is inessential if it has a inessential open neighborhood, otherwise it is called essential. Finally, E is said to be *fully essential* if $\mathbb{T}^2 \setminus E$ is inessential.

Definition 1. Let $x \in \mathbb{T}^2$ and $f : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ be a homeomorphism isotopic to the identity. We say that x is an *inessential point of f* if $\cup_{k \in \mathbb{Z}} f^k(U)$ is inessential for some neighborhood U of x , otherwise x is called an *essential point for f* . Furthermore, we say that x is a *fully essential point of f* if $\cup_{k \in \mathbb{Z}} f^k(U)$ is fully essential for any neighborhood U of x .

The following proposition appeared in [GKT15]. $\|x\|_\infty = \max\{|(x)_1|, |(x)_2|\}$ denotes the infinity norm on \mathbb{R}^2 , where $(x)_1$ and $(x)_2$ are the first and second canonical coordinates of a point $x \in \mathbb{R}^2$.

PROPOSITION 2. *Let $O \subset \mathbb{R}^2$ be a connected open set such that $\cup_{n \in \mathbb{Z}} f^n(\pi(O))$ is fully essential and such that $\overline{\pi(O)}$ is inessential. Then there exist $M \in \mathbb{N}$ and $K \subset \mathbb{R}^2$ compact such that $[0, 1]^2$ is contained in a bounded connected component of $\mathbb{R}^2 \setminus K$ and*

$$K \subset \bigcup_{|i| \leq M, \|v\|_\infty \leq M} (\tilde{f}^i(O) + v).$$

We also have the following lemma.

LEMMA 3. Let $f : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ be a homeomorphism isotopic to the identity, $\tilde{f} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ a lift and $z_0 \in \mathbb{T}^2$ a recurrent point such that $\rho(\tilde{f}, z_0)$ does not belong to \mathbb{Q}^2 . If $\rho(\tilde{f})$ is a non-degenerate line segment with irrational slope, then z_0 is a fully essential point for f . In particular, for every $\varepsilon > 0$, the set $U_\varepsilon = \bigcup_{i=0}^\infty f^i(\pi(B(\varepsilon, \tilde{z}_0)))$ is fully essential.

Proof. Assume, for a contradiction, that z_0 is an inessential point. Therefore there exists $\varepsilon > 0$ such that $U_\varepsilon = \bigcup_{i=0}^\infty f^i(\pi(B(\varepsilon, \tilde{z}_0)))$ is inessential. Note that each connected component of U_ε must be contained in a topological open disk. As U_ε is f -invariant, f permutes the connected components of U_ε . Also, since z_0 is recurrent, if U_ε^0 is the connected component of U_ε containing z_0 , there must exist a smallest $N > 0$ such that $f^N(U_\varepsilon^0) = U_\varepsilon^0$. Let $w \in \mathbb{Z}^2$ be the integer vector such that $\tilde{f}^N(\tilde{U}_\varepsilon^0) = \tilde{U}_\varepsilon^0 + w$, where \tilde{U}_ε^0 is the lift of U_ε^0 that contains \tilde{z}_0 . As z_0 is recurrent, there exists a subsequence n_k such that $f^{n_k}(z_0) \rightarrow z_0$. In particular, there exists k_0 such that if $k > k_0$ we have $f^{n_k}(z_0) \in U_\varepsilon^0$. But by the choice of N we have that $f^i(U_\varepsilon^0) \cap U_\varepsilon^0 = \emptyset$ if $1 \leq i < N$, and one deduces that $n_k = p_k N$, for $k > k_0$. Therefore $\tilde{f}^{n_k}(\tilde{z}_0) = \tilde{f}^{p_k N}(\tilde{z}_0) \in \tilde{f}^{p_k N}(\tilde{U}_\varepsilon^0) = \tilde{U}_\varepsilon^0 + p_k w$. So, $\tilde{f}^{p_k N}(\tilde{z}_0) - p_k w \rightarrow \tilde{z}_0$, which implies that $\rho(\tilde{f}, z_0) = w/N$, a contradiction.

Assume now, again for a contradiction, that z_0 is essential but not fully essential. There exists $\varepsilon > 0$ such that $U_\varepsilon = \bigcup_{i=0}^\infty f^i(\pi(B(\varepsilon, \tilde{z}_0)))$ is essential but not fully essential. Note that, as z_0 is recurrent, all connected components of U_ε are periodic, and by [KT14b, Propositions 1.2 and 1.4], there must exist $g = f^l$ a power of f , $\hat{g} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ a lift of g , a vector $W \in \mathbb{Z}_*^2$, and $M > 0$ such that $|\langle \hat{g}^n(\tilde{z}) - \tilde{z}, W \rangle| < M$ for all $\tilde{z} \in \mathbb{R}^2$ and all $n \in \mathbb{Z}$. But this implies that the rotation set of \hat{g} is contained in a segment of rational slope. Since $\rho(\hat{g}) = l(\rho(\tilde{f}) + V)$ for some $V \in \mathbb{Z}^2$, one deduces that $\rho(\tilde{f})$ is also a line segment of rational slope, again a contradiction. □

2.3. *Brouwer homeomorphisms.* We recall that a *Brouwer homeomorphism* is a homeomorphism of the plane preserving orientation without fixed points, and that a *line* in the plane is a continuous, injective and proper map from \mathbb{R} to \mathbb{R}^2 . By the Schönflies theorem, if $\phi : \mathbb{R} \rightarrow \mathbb{R}^2$ is a line, then it can be extended to a homeomorphism $\phi^* : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ preserving orientation, such that $\phi(t) = \phi^*(t, 0)$. We then define canonically the *left and right* of ϕ as $L(\phi) = \phi^*(\mathbb{R} \times (0, +\infty))$ and $R(\phi) = \phi^*(\mathbb{R} \times (-\infty, 0))$ respectively.

The fundamental result on the study of Brouwer Homeomorphisms is the following theorem.

THEOREM 4. [Bro12] Given $h : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ a Brouwer homeomorphism and $x \in \mathbb{R}^2$, there exists a line $\phi : \mathbb{R} \rightarrow \mathbb{R}^2$, with $\phi(0) = x$, such that $h([\phi]) \subset L(\phi)$ and $h^{-1}([\phi]) \subset R(\phi)$, where $[\phi] = \phi(\mathbb{R})$.

A line as in the above result is called a *Brouwer line*. A direct consequence of this theorem is that, if h is a Brouwer homeomorphism, then every point in \mathbb{R}^2 is contained in an open invariant connected and simply connected set, and the dynamics of h in this set is conjugated to a rigid translation. In particular, h only has wandering points.

2.4. *Maximal isotopies.* Let M be an oriented surface, $f : M \rightarrow M$ a homeomorphism isotopic to the identity, and denote by \mathcal{I} the space of all isotopies between f and the identity, that is, if $I \in \mathcal{I}$ then $I = (f_t)_{t \in [0,1]}$, where $f_0 = \text{Id}_M$, $f_1 = f$, for every $t \in [0, 1]$, f_t is a homeomorphism of M , and I is a continuous curve on the space of homeomorphisms of M , using the topology of the uniform convergence over compact subsets. The trajectory of a point $z \in M$ for I is defined as the path $t \mapsto f_t(z)$, which we denote by $I(z)$. By concatenating paths, we can define, for $n \in \mathbb{N}$,

$$I^n(z) = \Pi_{0 \leq k < n} I(f^k(z)), \quad I^{\mathbb{N}}(z) = \Pi_{k \geq 0} I(f^k(z)), \quad I^{\mathbb{Z}}(z) = \Pi_{k \in \mathbb{Z}} I(f^k(z)).$$

Denote by $\text{fix}(I) = \bigcap_{t \in [0,1]} \text{fix}(f_t)$ the set of points whose isotopy path is constant, and let $\text{dom}(I)$ be its complement, which is called the *domain* of I . A closed subset $F \subset \text{fix}(f)$ is said to be *unlinked* for f if there exists $I \in \mathcal{I}$ such that $F \subset \text{fix}(I)$.

We can define a pre-order in \mathcal{I} as follows.

Definition 5. Let $I_1, I_2 \in \mathcal{I}$. We say that $I_1 \leq I_2$ if:

- (i) $\text{fix}(I_1) \subset \text{fix}(I_2)$;
- (ii) I_2 is homotopic to I_1 relative to $\text{fix}(I_1)$.

We say that $I \in \mathcal{I}$ is a *maximal isotopy* if it is maximal for the pre-order defined above. Note that this is equivalent to the property that, for all $z \in \text{fix}(f) \setminus \text{fix}(I)$, the trajectory of z , which is a closed loop, is not null homotopic in $\text{dom}(I)$ (see [BCR20]).

Now, if I is a maximal isotopy, denoting by $\tilde{I} = (\tilde{f}_t)_{t \in [0,1]}$ the lift of $I|_{\text{dom}(I)}$ to the universal covering space $\widetilde{\text{dom}(I)}$ of $\text{dom}(I)$, we have that $\tilde{f}_1 = \tilde{f}$ has no fixed points and $\tilde{f}_0 = \text{Id}_{\widetilde{\text{dom}(I)}}$, and therefore the restriction of \tilde{f}_1 to each of the connected components of its domain is a Brouwer homeomorphism.

THEOREM 6. [BCR20] *For all $I \in \mathcal{I}$, there exists $I' \in \mathcal{I}$ such that $I \leq I'$ and I' is maximal.*

Therefore, given an unlinked subset F of fixed points for f , one can always find a maximal isotopy I' such that $F \subset \text{fix}(I')$.

2.5. *Paths transverse to oriented foliations.* If M is an oriented surface, we define an *oriented topological foliation with singularities* of M as a topological oriented foliation \mathcal{F} defined on an open subset of M . This subset will be called the *domain* of \mathcal{F} and denoted $\text{dom}(\mathcal{F})$, while its complement is the set of *singularities* of \mathcal{F} , and is denoted $\text{sing}(\mathcal{F})$.

Let us fix \mathcal{F} an oriented singular foliation of M and, for each $z \in \text{dom}(\mathcal{F})$, denote by ϕ_z the leaf of \mathcal{F} passing through z . A *path* in M is a continuous function $\gamma : J \rightarrow M$ where J is a non-degenerate interval of the line. We denote by $[\gamma]$ the image of γ .

A path $\gamma : J \rightarrow \text{dom}(\mathcal{F})$ is said to be *transverse* to \mathcal{F} if, for all $t \in J$, there exists a homeomorphism $c : W \rightarrow (0, 1)^2$, where W is a neighborhood of $\gamma(t)$, such that c sends the restriction of \mathcal{F} to the foliation by downward oriented vertical lines in $(0, 1)^2$, and such that $\pi_1 \circ c \circ \gamma$ is strictly increasing in a neighborhood of t , where π_1 is the projection onto the first coordinate. Intuitively, a path is transverse to \mathcal{F} if it always locally crosses transversally leaves from right to left.

Definition 7. If $M = \mathbb{R}^2$ and \mathcal{F} does not have singularities, we say that two transverse paths are *equivalent* if they intersect the same leaves. In the general case we say that two transverse paths $\gamma : J \rightarrow \text{dom}(\mathcal{F})$ and $\gamma' : J' \rightarrow \text{dom}(\mathcal{F})$ are *equivalent* if there exist $H : J \times [0, 1] \rightarrow \text{dom}(\mathcal{F})$ continuous, and $h : J \rightarrow J'$ an increasing homeomorphism such that:

- (i) $H(t, 0) = \gamma(t), H(t, 1) = \gamma'(h(t))$;
- (ii) for all $t \in J$ and for all $s_1, s_2 \in [0, 1], \phi_{H(t,s_1)} = \phi_{H(t,s_2)}$.

In this case, we will denote $\gamma \sim_{\mathcal{F}} \gamma'$.

The previous definition is equivalent to showing that, if $\tilde{\mathcal{F}}$ is the lift of \mathcal{F} to the universal covering of $\text{dom}(\mathcal{F})$, then there exist lifts $\tilde{\gamma}$ and $\tilde{\gamma}'$ of γ and γ' , respectively, to the universal covering $\text{dom}(\mathcal{F})$ such that $\tilde{\gamma} \sim_{\tilde{\mathcal{F}}} \tilde{\gamma}'$.

We remark that, by a version of the Poincaré–Bendixson theorem, if \mathcal{F} is an oriented non-singular foliation of \mathbb{R}^2 and $\gamma : \mathbb{R} \rightarrow \mathbb{R}^2$ is a line transverse to \mathcal{F} , then every leaf ϕ intersecting γ does so at exactly one point. Also, since γ locally crosses every leaf from right to left, and the leaf ϕ crosses γ from left to right, that is, if $t' \in \mathbb{R}$ is such that $\phi(t') \in [\gamma]$, then $\phi(t) \in L(\gamma)$ if $t < t'$, and $\phi(t) \in R(\gamma)$ if $t > t'$.

Given three lines $\gamma_i : \mathbb{R} \rightarrow \mathbb{R}^2, i \in \{0, 1, 2\}$, we say that γ_0 *separates* γ_1 and γ_2 if $[\gamma_1]$ and $[\gamma_2]$ lie in different connected components of the complement of $[\gamma_0]$. We say that γ_2 *is above* γ_1 *with respect to* γ_0 (see Figure 1) if:

- (i) the lines are pairwise disjoint;
- (ii) no line separates the other two;
- (iii) if λ_1, λ_2 are two disjoint paths joining $z_1 = \gamma_0(t_1), z_2 = \gamma_0(t_2)$ to $z'_1 \in [\gamma_1], z'_2 \in [\gamma_2]$, respectively, and not intersecting the lines but at the extremal points, then $t_2 > t_1$.

We need the fundamental definition of \mathcal{F} -transversal intersection.

Definition 8. Let $M = \mathbb{R}^2$ and let \mathcal{F} be non-singular. Given two transverse paths $\gamma_i : J_i \rightarrow \mathbb{R}^2, i \in \{1, 2\}$, such that $\phi_{\gamma_1(t_1)} = \phi_{\gamma_2(t_2)} = \phi$ for some t_1, t_2 in the interior of J_1, J_2 respectively, we say that γ_1 *intersects* γ_2 *\mathcal{F} -transversally at* ϕ if there exist $a_1, b_1 \in J_1$ with $a_1 < t_1 < b_1$ and $a_2, b_2 \in J_2$ with $a_2 < t_2 < b_2$ such that

- (i) $\phi_{\gamma_2(a_2)}$ is below $\phi_{\gamma_1(a_1)}$ with respect to ϕ ;
- (ii) $\phi_{\gamma_2(b_2)}$ is above $\phi_{\gamma_1(b_1)}$ with respect to ϕ .

See Figure 2. In general, where M is any oriented surface and \mathcal{F} is a singular foliation, we say that two transverse paths $\gamma_i : J_i \rightarrow \text{dom}(\mathcal{F}), i \in \{1, 2\}$, such that $\phi_{\gamma_1(t_1)} = \phi_{\gamma_2(t_2)} = \phi$, *intersects* γ_2 *\mathcal{F} -transversally at* ϕ , if there are lifts $\tilde{\gamma}_i$ to the universal covering $\text{dom}(\mathcal{F})$ that intersect $\tilde{\mathcal{F}}$ -transversally, where $\tilde{\mathcal{F}}$ is the lift of \mathcal{F} .

In both cases, we denote $\gamma_1|_{[a_1,b_1]} \pitchfork_{\mathcal{F}} \gamma_2|_{[a_2,b_2]}$.

Whenever the context is clear, we will just say that γ_1 and γ_2 intersect transversally. Also, if $\gamma_1(t_1) = \gamma_2(t_2)$ and γ_1 intersects γ_2 \mathcal{F} -transversally at $\phi_{\gamma_1(t_1)} = \phi_{\gamma_2(t_2)}$, we will just say that γ_1 and γ_2 intersect \mathcal{F} -transversally at $\gamma_1(t_1)$. If a path has a transversal intersection with itself, we say that γ has a *transversal self-intersection*.

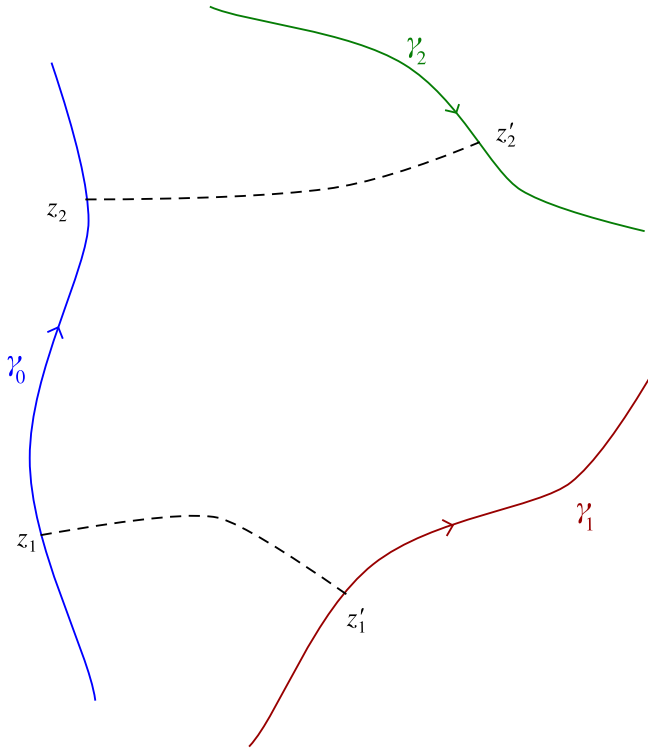


FIGURE 1. γ_2 is above γ_1 with respect to γ_0 .

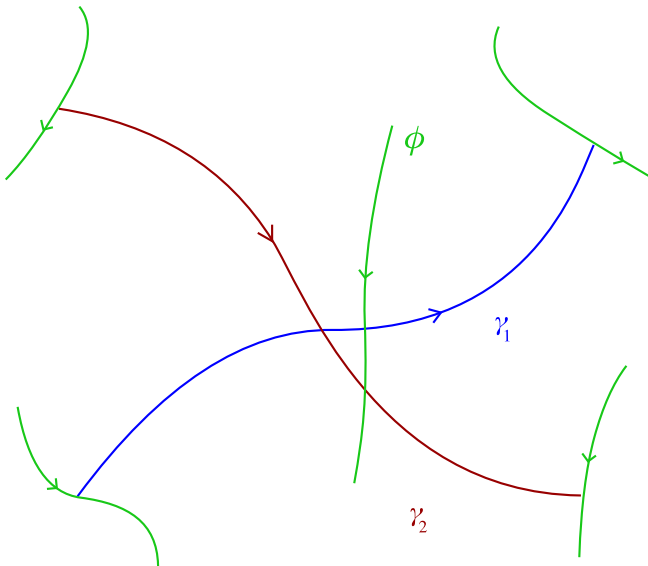


FIGURE 2. γ_1 and γ_2 with an \mathcal{F} -transversal intersection.

Note that, if $\gamma_1|_{I_1} \pitchfork_{\mathcal{F}} \gamma_2|_{I_2}$ and $\gamma_2|_{I_2} \sim_{\mathcal{F}} \gamma_3|_{I_3}$, then $\gamma_1|_{I_1} \pitchfork_{\mathcal{F}} \gamma_3|_{I_3}$. Note also that if $\gamma_1|_{I_1} \pitchfork_{\mathcal{F}} \gamma_2|_{I_2}$ at ϕ , this does not imply that γ_1 and γ_2 actually intersect at a point of ϕ . However, $\gamma_1|_{I_1}$ and $\gamma_2|_{I_2}$ have at least a common point.

2.6. *Brouwer–Le Calvez foliations.* The following is one of the most useful tools in the study of homeomorphisms of surfaces.

THEOREM 9. [Cal05] *Let $h : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a Brouwer homeomorphism and G a discrete group of orientation-preserving homeomorphisms of the plane that acts freely and properly. If h commutes with the elements of G , then there exists a foliation \mathcal{F} of \mathbb{R}^2 by Brouwer lines for h . Furthermore, \mathcal{F} is G -invariant.*

A foliation as in Theorem 9 will be called a *Brouwer–Le Calvez foliation*.

Note that, by combining the previous result with Theorem 6, one can obtain that, given a homeomorphism f of M isotopic to identity and an unlinked subset F of $\text{fix}(f)$, one can always find a maximal isotopy I such that $F \subset \text{fix}(I)$. Since the lift of the restriction of f to $\text{dom}(I)$ is a Brouwer homeomorphism commuting with all covering automorphisms, Theorem 9 tells us that one may find a foliation $\tilde{\mathcal{F}}$ by Brouwer lines that descends to an oriented non-singular foliation of $\text{dom}(I)$. It is therefore possible to define a singular foliation \mathcal{F} of M such that $\text{dom}(\mathcal{F}) = \text{dom}(I)$, and such that F is contained in $\text{sing}(\mathcal{F})$. Furthermore, one can show [Cal05] the following theorem.

THEOREM 10. *Given a maximal isotopy I , there exists \mathcal{F} a oriented topological foliation with singularities of M , with $\text{dom}(\mathcal{F}) = \text{dom}(I)$, such that for all $z \in \text{dom}(I)$ the trajectory $I(z)$ is homotopic with fixed endpoints in $\text{dom}(I)$ to a path transverse to \mathcal{F} , and this path is unique up to equivalence.*

We denote by $I_{\mathcal{F}}(z)$ the class of paths described in the previous theorem, as well as its representatives whenever the context is clear. We further denote

$$I_{\mathcal{F}}^n(z) = \Pi_{0 \leq k < n} I_{\mathcal{F}}(f^k(z)), \quad I_{\mathcal{F}}^{\mathbb{N}}(z) = \Pi_{k \geq 0} I_{\mathcal{F}}(f^k(z)), \quad I_{\mathcal{F}}^{\mathbb{Z}}(z) = \Pi_{k \in \mathbb{Z}} I_{\mathcal{F}}(f^k(z)),$$

where the product here denotes the concatenation of paths.

Definition 11. A transversal path $\gamma : [a, b] \rightarrow \text{dom}(I)$ is called *admissible of order n* if there exists $z \in \text{dom}(I)$ such that γ is equivalent to a path in $I_{\mathcal{F}}^n(z)$. A path that is admissible of some order is just called *admissible*.

LEMMA 12. *Let $\beta, \gamma : [a, b] \rightarrow \text{dom}(I)$ be paths transversal to \mathcal{F} , and suppose γ is admissible of order n . If $J = [c, d] \subset [a, b]$ is such that $\beta \pitchfork_{\mathcal{F}} \gamma|_J$, then $\gamma|_J$ is also admissible of order n .*

Proof. The result is a direct consequence of [CT15, Proposition 19], because if $\tilde{\mathcal{F}}, \tilde{\gamma}$ and $\tilde{\beta}$ are respectively the lifts of \mathcal{F}, γ and β to $\widetilde{\text{dom}(I)}$ such that $\tilde{\beta} \pitchfork_{\tilde{\mathcal{F}}} \tilde{\gamma}|_J$, then the leaf $\phi_{\tilde{\beta}(a)}$ lies in the left of $\phi_{\tilde{\gamma}(c)}$ and in the right of $\phi_{\tilde{\gamma}(d)}$ but does not intersect $[\tilde{\gamma}|_J]$. \square

2.7. *Forcing.* Let us now present some of the results from the forcing theory developed by Le Calvez and the second author. As before, we assume fixed a homeomorphism f of the surface M , I a maximal isotopy joining the identity and f , and \mathcal{F} a Brouwer–Le Calvez foliation for I . We first state the Fundamental Lemma.

PROPOSITION 13. [CT15] *Suppose $\gamma_i : [a_i, b_i] \rightarrow M$, $i = 1, 2$ are two transverse paths that intersect \mathcal{F} -transversally at $\gamma_1(t_1) = \gamma_2(t_2)$. If γ_1 is admissible of order n_1 and γ_2 is admissible of order n_2 , then both the transverse paths $\gamma_1|_{[a_1, t_1]}\gamma_2|_{[t_2, b_2]}$ and $\gamma_2|_{[a_2, t_2]}\gamma_1|_{[t_1, b_1]}$ are admissible of order $n_1 + n_2$. Furthermore, either one of these paths is admissible of order $\min(n_1, n_2)$ or both are admissible of order $\max(n_1, n_2)$.*

Definition 14. Let $\gamma : \mathbb{R} \rightarrow \mathbb{R}^2$ be a proper path and let $\rho \in \mathbb{R}^2$, with $\|\rho\| = 1$. We say that γ is directed by ρ if

$$\lim_{t \rightarrow \pm\infty} \|\dot{\gamma}(t)\| = +\infty, \quad \lim_{t \rightarrow +\infty} \dot{\gamma}(t)/\|\dot{\gamma}(t)\| = \rho, \quad \lim_{t \rightarrow -\infty} \dot{\gamma}(t)/\|\dot{\gamma}(t)\| = -\rho.$$

We note that whenever $f : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ is a homeomorphism homotopic to the identity and $z \in \mathbb{T}^2$ is a point whose rotation vector is well defined and equal to $\rho \neq 0$, then we can find γ a representative of $I_{\mathcal{F}}^{\mathbb{Z}}(z)$, such that every lift of γ to \mathbb{R}^2 (the universal covering of \mathbb{T}^2) is a path directed by $\rho/\|\rho\|$.

Assume M is \mathbb{R}^2 , and let $\gamma : \mathbb{R} \rightarrow \mathbb{R}^2$ be a transverse line. Denote, as before, the two connected components of its complement by $R(\gamma)$ and $L(\gamma)$. We define the *foliated right* (respectively, *foliated left*) of γ , denoted by $r(\gamma)$ (respectively, $l(\gamma)$) as the set of leaves and singularities of \mathcal{F} strictly contained in $R(\gamma)$ (respectively, $L(\gamma)$). Thus $l(\gamma) \cup r(\gamma)$ contains all leaves and singularities not intersected by γ . The following proposition gives a useful criterion to detect transversal intersections.

PROPOSITION 15. *Let \mathcal{F} be an oriented topological foliation with singularities of \mathbb{R}^2 , $\gamma : \mathbb{R} \rightarrow \mathbb{R}^2$ be a transverse path where γ is a transverse line, and let $\gamma' : [a, b] \rightarrow \mathbb{R}^2$ be a transverse path. If $[\gamma'] \cap l(\gamma) \neq \emptyset$, $[\gamma'] \cap r(\gamma) \neq \emptyset$ and if $J^* = \{t \in \mathbb{R} \mid \phi_{\gamma(t)} \text{ crosses } \gamma'\}$ is bounded, then there exist intervals J and J' such that $\gamma'|_{J'} \pitchfork_{\mathcal{F}} \gamma|_J$.*

Proof. Let t_0, s_0 in $[a, b]$ be such that $\gamma'(t_0) \in l(\gamma)$ and $\gamma'(s_0) \in r(\gamma)$, and we assume, without loss of generality, that $t_0 < s_0$. Since $l(\gamma)$ and $r(\gamma)$ are closed, let $t_1 = \max_{t_0 \leq t < s_0} \{t \mid \gamma'(t) \in l(\gamma)\}$ and let $s_1 = \min_{t_1 < t \leq s_0} \{t \mid \gamma'(t) \in r(\gamma)\}$. Let $J' = [t_1, s_1]$, and note that $[\gamma']_{J'}$ must intersect γ , so let t' be such that $\gamma'(t') \in [\gamma]$. Also, for $t_1 < t < s_1$, $\gamma'(t)$ is in a leaf that crosses γ and so there exist a_0, b_0 such that $\gamma'|_{(t_1, s_1)}$ is equivalent to $\gamma|_{(a_0, b_0)}$, where a_0 and b_0 are finite since both are contained in J^* . Set $J = [a_0, b_0]$ and let s be such that $\gamma(s) = \gamma'(t')$. Note that, by lifting $\gamma'|_{J'}$ and $\gamma|_J$ to paths $\tilde{\gamma}'$ and $\tilde{\gamma}$ in the universal covering of $\text{dom}(\mathcal{F})$ such that $\tilde{\gamma}(s) = \tilde{\gamma}'(t')$, and lifting \mathcal{F} to $\tilde{\mathcal{F}}$, one has that the leaf $\phi_0 = \phi_{\tilde{\gamma}'(t')}$ is such that $\tilde{\gamma}(a_0)$ is above $\tilde{\gamma}'(t_1)$ and $\tilde{\gamma}(b_0)$ is below $\tilde{\gamma}'(s_1)$. Therefore $\tilde{\gamma}' \pitchfork_{\tilde{\mathcal{F}}} \tilde{\gamma}$ and so $\gamma'|_{J'} \pitchfork_{\mathcal{F}} \gamma|_J$. □

The next result shows that admissible paths obey some sort of continuity.

LEMMA 16. [CT15] Fix $z \in \text{dom}(I)$, $n \geq 1$, and let us parameterize $I_{\mathcal{F}}^n(z)$ by $[0, 1]$. For each $0 < a < b < 1$, there exists a neighborhood V of z such that, for all $z' \in V$, $I_{\mathcal{F}}^n(z)|_{[a,b]}$ is equivalent to a subpath of $I_{\mathcal{F}}^n(z')$. Furthermore, there exists W a neighborhood of z such that, for all $z', z'' \in W$, the path $I_{\mathcal{F}}^n(z')$ is equivalent to a subpath of $I_{\mathcal{F}}^{n+2}(f^{-1}(z''))$.

One key fact used on the proof of the main theorem of this work is that typical points for ergodic measures are recurrent. We present a similar definition for transverse paths.

Definition 17. A transverse path $\gamma : \mathbb{R} \rightarrow M$ is called \mathcal{F} -recurrent if for every compact $J \subset \mathbb{R}$ and all $t \in \mathbb{R}$ there exists segment $J' \subset (-\infty, t]$ and $J'' \subset [t, +\infty)$ such that $\gamma|_{J'} \sim_{\mathcal{F}} \gamma|_J$ and $\gamma|_{J''} \sim_{\mathcal{F}} \gamma|_J$.

Lemma 16 has the following corollary.

COROLLARY 18. If $z \in \text{dom}(I)$ is a recurrent point for f , then $I_{\mathcal{F}}^{\mathbb{Z}}(z)$ is \mathcal{F} -recurrent.

We also need the following technical proposition.

PROPOSITION 19. [CT15] Let $f : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ be a homeomorphism isotopic to the identity, $\hat{f} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ a lift of f , and assume that $\rho(\hat{f}) = \{t\rho_0 \mid 0 \leq t \leq 1\}$, where $\tan(\rho_0) \notin \mathbb{Q}$. Then there exists $\hat{y}_0 \in \text{dom}(\hat{F})$ such that for every $\epsilon \in \{-1, 1\}$ there exists a sequence $(p_l, q_l)_{l \geq 0}$ in $\mathbb{Z}^2 \times \mathbb{N}$ satisfying

$$\lim_{l \rightarrow +\infty} q_l = +\infty, \quad \lim_{l \rightarrow +\infty} \hat{f}^{q_l}(\tilde{y}_0) - \tilde{y}_0 - p_l = 0, \quad \epsilon \langle p_l, \rho_0^\perp \rangle > 0,$$

and a sequence $(p'_l, q'_l)_{l \geq 0}$ in $\mathbb{Z}^2 \times \mathbb{N}$ satisfying

$$\lim_{l \rightarrow +\infty} q'_l = +\infty, \quad \lim_{l \rightarrow +\infty} \hat{f}^{-q'_l}(\tilde{y}_0) - \tilde{y}_0 - p'_l = 0, \quad \epsilon \langle p'_l, \rho_0^\perp \rangle > 0.$$

Furthermore, we can take $y_0 = \pi(\tilde{y}_0)$ such that $\rho(\tilde{f}, y_0) = \rho_0$.

We finish with one of the main results from [CT18].

THEOREM 20. Let M be an oriented surface, f a homeomorphism of M isotopic to the identity, I a maximal isotopy for f and \mathcal{F} a Brouwer–Le Calvez foliation transverse to I . Assume that $\gamma : [a, b] \rightarrow \text{dom}(I)$ is an admissible path of order r with a transverse self-intersection at $\gamma(s) = \gamma(t)$, where $s < t$. Let $\tilde{\gamma}$ be a lift of γ to the universal covering $\widehat{\text{dom}}(I)$ of $\text{dom}(I)$ and let T be a covering transformation such that $\tilde{\gamma}$ and $T(\tilde{\gamma})$ have an $\tilde{\mathcal{F}}$ -transverse intersection at $\tilde{\gamma}(t) = T(\tilde{\gamma})(s)$. Let \tilde{f} be the lift of $f|_{\text{dom}(I)}$ to $\widehat{\text{dom}}(I)$, and \hat{f} be the homeomorphism of the annular covering space $\widehat{\text{dom}}(I) = \widehat{\text{dom}}(I)/T$ that is lifted by \tilde{f} . Then the following statements hold.

- (i) For every rational number $p/q \in (0, 1]$, written in its irreducible form, there exists $\tilde{z} \in \widehat{\text{dom}}(I)$ such that $\tilde{f}^{q_r}(\tilde{z}) = T^p(\tilde{z})$ and $\tilde{I}_{\mathcal{F}}^{\mathbb{Z}}(\tilde{z})$ is equivalent to $\Pi_{k \in \mathbb{Z}} T^k(\tilde{\gamma}|_{[s,t]})$.
- (ii) For every irrational number $\lambda \in [0, 1/r]$, there exists a compact \hat{f} invariant set $\hat{Z}_\rho \subset \widehat{\text{dom}}(I)$ such that every point $\hat{z} \in \hat{Z}_\rho$ has a rotation number $\rho(\tilde{f}, \hat{z}) = \lambda$. Furthermore, if \tilde{z} is a lift of \hat{z} , then $\tilde{I}_{\mathcal{F}}^{\mathbb{Z}}(\tilde{z})$ is equivalent to $\Pi_{k \in \mathbb{Z}} T^k(\tilde{\gamma}|_{[s,t]})$.

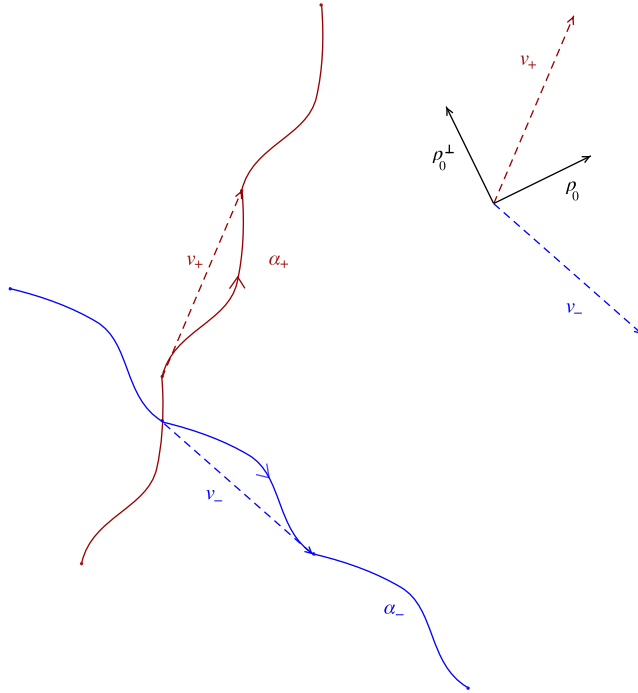


FIGURE 3. Illustration of Lemma 21.

3. Proofs of the main results

3.1. *Proof of Theorem A.* Let us fix an isotopy $I : \mathbb{T}^2 \times [0, 1] \rightarrow \mathbb{T}^2$ between f and the identity, such that $\text{fix}(I) \neq \emptyset$, and $\tilde{I} : \mathbb{R}^2 \times [0, 1] \rightarrow \mathbb{R}^2$ is a lift of I such that $\text{fix}(\tilde{I}) \neq \emptyset$. By Theorem 6, we may assume that I is maximal. Let \mathcal{F} be the singular foliation of \mathbb{T}^2 given by Theorem 10, and $\tilde{\mathcal{F}}$ its lift for \mathbb{R}^2 . Let also μ_0 be an ergodic measure whose the rotation vector is ρ_0 . Let us further assume that ρ_0 is in the first quadrant, that is, that both $(\rho_0)_1$ and $(\rho_0)_2$ are positive; the other cases are analogous.

The following lemma is illustrated in Figure 3.

LEMMA 21. *There are $\tilde{\mathcal{F}}$ -transverse lines $\alpha_-, \alpha_+ : \mathbb{R} \rightarrow \mathbb{R}^2$ and $v_-, v_+ \in \mathbb{Z}^2$ such that $\alpha_-(t + 1) = \alpha_-(t) + v_-$, $\alpha_+(t + 1) = \alpha_+(t) + v_+$, for every $t \in \mathbb{R}$, and such that $\langle v_-, \rho_0^\perp \rangle < 0$, $\langle v_-, \rho_0 \rangle > 0$, $\langle v_+, \rho_0^\perp \rangle > 0$ and $\langle v_+, \rho_0 \rangle > 0$.*

Proof. Let us start by building α_+ and v_+ . Let there be \tilde{y}_0 and $(p_l, q_l)_{l \geq 0}$ given by Proposition 19 such that $\lim_{l \rightarrow +\infty} q_l = +\infty$, $\lim_{l \rightarrow +\infty} (\tilde{f}^{q_l}(\tilde{y}_0) - \tilde{y}_0 - p_l) = 0$ and $\langle p_l, \rho_0^\perp \rangle > 0$. Let us fix $\alpha_0 \in \tilde{\mathcal{F}}(\tilde{y}_0)$. Let $V_0 \subset \mathbb{R}^2$ be a small neighborhood of \tilde{y}_0 such that there exists $h_0 : V_0 \rightarrow (0, 1)^2$ a homeomorphism mapping the restriction of $\tilde{\mathcal{F}}$ to V_0 into the vertical foliation oriented downwards in $(0, 1)^2$ and such that the projection of $h_0([\alpha_0] \cap V_0)$ in the first coordinate axis is the interval $[(h_0(\tilde{y}_0))_1, 1)$ (i.e. $h(\alpha_0 \cap V_0)$ crosses all the leaves on the left of the leaf that passes through $h_0(\tilde{y}_0)$). Let also $\varepsilon > 0$ such that $B(\varepsilon, \tilde{y}_0) \subset V_0$. As noted in Proposition 19, we can assume that $\rho(\tilde{f}, \tilde{y}_0) = \rho_0$,

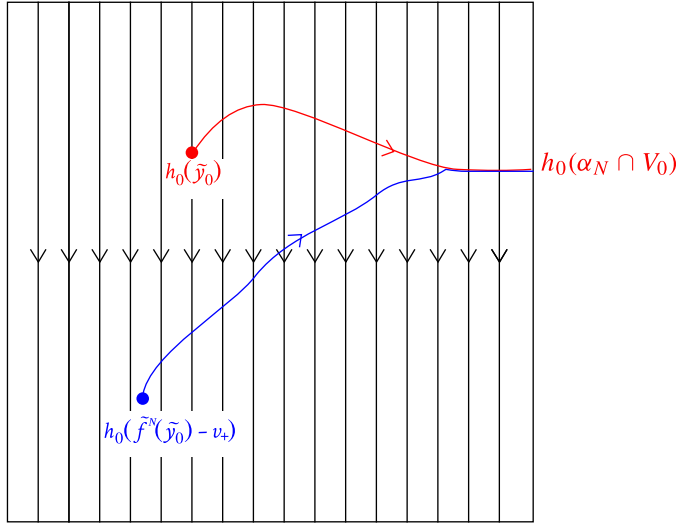


FIGURE 4. Construction of α'_N .

so there is $l_0 \in \mathbb{N}$ such that for $l \geq l_0$ we have $\langle p_l, \rho_0 \rangle > 0$. By Proposition 19 we have $\lim_{l \rightarrow +\infty} (\tilde{f}^{ql}(\tilde{y}_0) - \tilde{y}_0 - p_l) = 0$, so there is $l_1 > 0$ such that $\tilde{f}^{ql}(\tilde{y}_0) - p_l \in B(\varepsilon, \tilde{y}_0)$ and $\langle p_l, \rho_0^\perp \rangle > 0$, for all $l \geq l_1$. Setting $l' = \max\{l_0, l_1\}$, let us denote $N = ql'$ and $v_+ = p_{l'}$.

Consider now $\alpha_N \in \tilde{I}_{\tilde{F}}^N(\tilde{y}_0)$ such that $[\alpha_0] \subset [\alpha_N]$ and α_N is parameterized by $[0, 1]$. Since $\tilde{f}^N(\tilde{y}_0) - v_+ \in B(\varepsilon, \tilde{y}_0)$, we have that $\tilde{f}^N(\tilde{y}_0) - v_+ \in V_0$, therefore we can modify α_N inside V_0 in order to obtain a transverse path $\alpha'_N : [0, 1] \rightarrow \mathbb{R}^2$ so that α_N and α'_N are equal outside of V_0 and $\alpha'_N(0) = \tilde{f}^N(\tilde{y}_0) - v_+$ (to modify α_N in V_0 , it is enough to modify $h_0(\alpha_N \cap V_0)$ in $(0, 1)^2$ and take it back to V_0 using h_0^{-1} ; see Figure 4). Now, let us define $\alpha'_+ = \Pi_{k \in \mathbb{Z}}(\alpha'_N + kv_+)$. Since α'_+ is a transverse path, if α'_+ has self-intersection, that is, if $\alpha'_+(s) = \alpha'_+(t)$, with $s < t$, we can remove the arc $\alpha'_+|_{(s,t]}$ and reparameterize in a suitable way, thus obtaining a new path, which we denote by α_+ . Therefore we can assume that α_+ is a simple path and so, by the construction, we get that α_+ is a line satisfying the conditions of the statement.

The construction of α_- and v_- is analogous, using Proposition 19 with $\varepsilon = -1$. □

LEMMA 22. *There is $L_0 > 0$ such that, for every $\tilde{x} \in \mathbb{R}^2 \setminus \text{sing}(\tilde{F})$, there is a transverse path $\tilde{\gamma}_{\tilde{x}} \in \tilde{I}_{\tilde{F}}(\tilde{x})$, with $\text{diam}(\tilde{\gamma}_{\tilde{x}}) < L_0$.*

Proof. First, let us note that it is enough to prove the result for points in $[0, 1]^2$, since $\tilde{I}_{\tilde{F}}(\tilde{x} + w) = \tilde{I}_{\tilde{F}}(\tilde{x}) + w$ for every $\tilde{x} \in \mathbb{R}^2$ and every $w \in \mathbb{Z}^2$. Now, since \tilde{I} is continuous, there is $L > 0$ such that $\tilde{I}([0, 1]^2 \times [0, 1]) \subset B(L, 0)$, that is, for every point $\tilde{x} \in [0, 1]^2$ the isotopy path $\tilde{I}(\tilde{x})$ is contained in $B(L, 0)$.

Now, let α_+ and α_- be the transverse lines given by Lemma 21, and also let $v_1, v_2, v_3, v_4 \in \mathbb{Z}^2$ be such that $B(L, 0)$ is contained in a bounded connected component

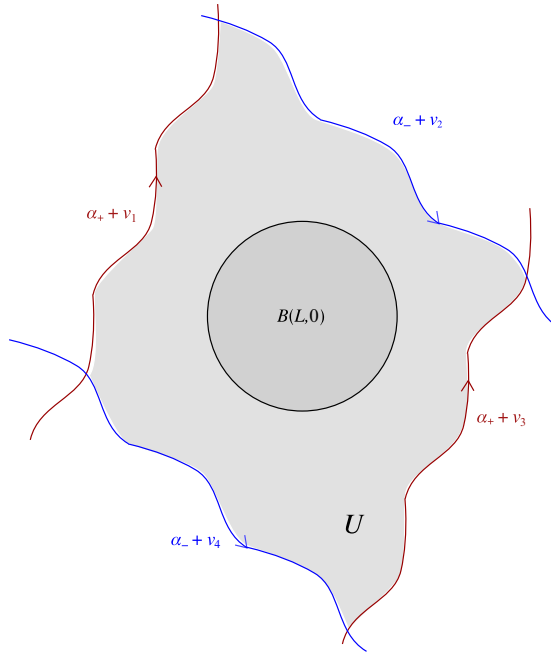


FIGURE 5. Illustration of the set U .

of $\mathbb{R}^2 \setminus ([\alpha_+ + v_1] \cup [\alpha_- + v_2] \cup [\alpha_+ + v_3] \cup [\alpha_- + v_4])$, which we shall denote by U , and $U \subset R(\alpha_+ + v_1) \cap R(\alpha_- + v_2) \cap L(\alpha_+ + v_3) \cap L(\alpha_- + v_4)$ (see Figure 5).

We claim that if ϕ is a leaf of $\tilde{\mathcal{F}}$ that intersects U , then $[\phi] \cap U$ is connected (i.e. a segment of ϕ). In fact, as remarked right after Definition 7, since α_+ and α_- (as well as their translations) are transverse lines, the leaf ϕ crosses each line at most once, and always from left to right of the line. Now let $t_1 < t_2$ be such that $\phi(t_i) \in U$, $i \in \{1, 2\}$. Note that to prove that $[\phi] \cap U$ is connected, it is enough to prove that $\phi(t) \in U$, for every $t_1 < t < t_2$. Since $\phi(t_1) \in U$ and $t > t_1$, we have that $\phi(t) \in R(\alpha_+ + v_1)$ and $\phi(t) \in R(\alpha_- + v_2)$. Analogously, since $\phi(t_2) \in U$ and $t < t_2$, $\phi(t) \in L(\alpha_+ + v_3)$ and $\phi(t) \in L(\alpha_- + v_4)$. So, $\phi(t) \in R(\alpha_+ + v_1) \cap R(\alpha_- + v_2) \cap L(\alpha_+ + v_3) \cap L(\alpha_- + v_4)$. Furthermore, as $\phi(t_1)$ and $\phi(t_2)$ belong to U , no point of $\phi|_{[t_1, t_2]}$ intersects $[\alpha_+ + v_1] \cup [\alpha_- + v_2] \cup [\alpha_+ + v_3] \cup [\alpha_- + v_4]$. Therefore $\phi(t) \in U$, proving that $[\phi] \cap U$ is connected.

Let $\widehat{\text{dom}}(\tilde{I})$ be the universal covering of $\text{dom}(\tilde{I})$, $\hat{\pi} : \widehat{\text{dom}}(\tilde{I}) \rightarrow \text{dom}(\tilde{I})$ the covering map, \hat{I} a lift of $\tilde{I}|_{\text{dom}(\tilde{I})}$ to $\widehat{\text{dom}}(\tilde{I})$ and $\hat{\mathcal{F}}$ the lift of $\tilde{\mathcal{F}}$ to $\widehat{\text{dom}}(\tilde{I})$. Since I is maximal, \tilde{I} is also maximal, and therefore $\hat{f} : \widehat{\text{dom}}(\tilde{I}) \rightarrow \widehat{\text{dom}}(\tilde{I})$ is a Brouwer homeomorphism, where \hat{f} is a lift of $\tilde{f}|_{\text{dom}(\tilde{I})}$. Let us now fix $\tilde{x} \in [0, 1]^2$ such that $\tilde{x} \notin \text{sing}(\tilde{\mathcal{F}})$ and $\hat{x} \in \widehat{\text{dom}}(\tilde{I})$ a lift of \tilde{x} . Since $\Phi(\hat{x}) = \{\phi \in \hat{\mathcal{F}} \mid \hat{x} \in R(\phi) \text{ and } \hat{f}(\hat{x}) \in L(\phi)\} \cup \{\phi_{\hat{x}}, \phi_{\hat{f}(\hat{x})}\}$, note that $\Phi(\hat{x})$ is the set of leaves that intersect the transverse trajectory $\hat{I}_{\hat{\mathcal{F}}}(\hat{x})$, and beyond that, $\Phi(\hat{x})$ is totally ordered by the relation $\phi_1 < \phi_2$ if $R(\phi_1) \subset R(\phi_2)$, because \hat{f} is a Brouwer homeomorphism and the leaves of $\hat{\mathcal{F}}$ are Brouwer lines. Using this

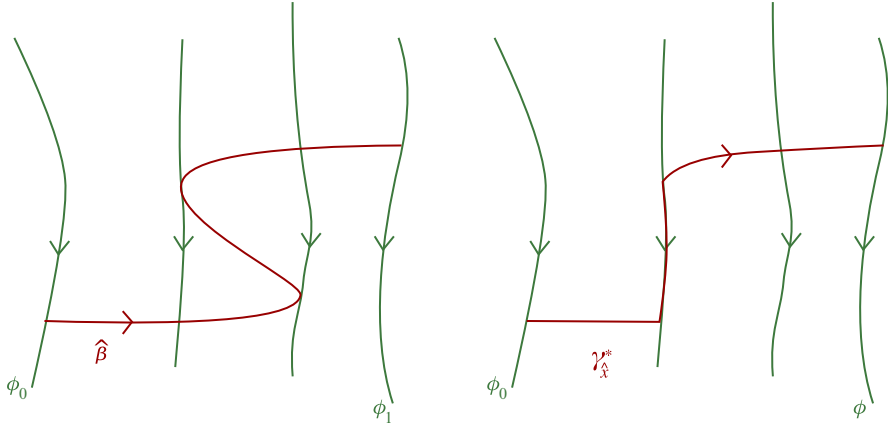


FIGURE 6. Illustration of the curves $\hat{\beta}$ and $\gamma_{\hat{x}}^*$.

order, we can parameterize $\Phi(\hat{x})$ by a parameter $s \in [0, 1]$ in such a way that $\phi_0 = \phi_{\hat{x}}$ and $\phi_1 = \phi_{\hat{f}(\hat{x})}$.

Denoting the isotopy path $\hat{I}(\hat{x}) : [0, 1] \rightarrow \widehat{\text{dom}}(\tilde{I})$ by $\hat{\beta}$, we can define the following functions of the parameter s :

$$t_-(s) = \begin{cases} 0 & \text{if } s = 0, \\ \inf\{t \in [0, 1] \mid \hat{\beta}([t, 1]) \cap R(\phi_s) = \emptyset\} & \text{if } s \in (0, 1), \end{cases}$$

and

$$t_+(s) = \begin{cases} \inf\{t \in [0, 1] \mid \hat{\beta}([t, 1]) \subset L(\phi_s)\} & \text{if } s \in [0, 1), \\ 1 & \text{if } s = 1. \end{cases}$$

Intuitively, $t_-(s)$ is the moment when $\hat{\beta}$ was on the right of ϕ_s for the last time, and $t_+(s)$ is the first moment in which $\hat{\beta}$ is always on the left of ϕ_s . Note that, if $s_1 < s_2$, then $t_-(s_1) \leq t_+(s_1) < t_-(s_2) \leq t_+(s_2)$. So t_- and t_+ coincide, except possibly in a countable set of discontinuities. However, note that if we list the discontinuity points as $(s_i)_{i \in \mathbb{N}}$, we have that $\sum d_i \leq 1$, where $d_i = t_+(s_i) - t_-(s_i)$ (because $t_{\pm}([0, 1]) \subset [0, 1]$).

Now let us define a path $\gamma_{\hat{x}}^* : [0, 1] \rightarrow \widehat{\text{dom}}(\tilde{I})$ which will be transverse except in the discontinuities of t_- and t_+ as follows: at the points where $t_-(s) = t_+(s)$ we make $\gamma_{\hat{x}}^*$ equal to $\hat{\beta}$, and where $t_-(s) \neq t_+(s)$ we make $\gamma_{\hat{x}}^*$ equal to the leaf segment ϕ_s which connects $\hat{\beta}(t_-(s))$ to $\hat{\beta}(t_+(s))$ (see Figure 6).

Now, since $\gamma_{\hat{x}}^*$ is made of leaves points or leaves segments, we have that for each $s \in [0, 1]$ we can find V_s a tubular neighborhood of the point (or segment) of $\gamma_{\hat{x}}^*$ which intersects the leaf ϕ_s and $\varepsilon_s \in (0, 1)$ so that $V_s \subset B(\varepsilon_s, [\gamma_{\hat{x}}^*] \cap [\phi_s])$. Then $[\gamma_{\hat{x}}^*] \subset \bigcup_{s \in [0, 1]} V_s$, and by compactness we have $[\gamma_{\hat{x}}^*] \subset \bigcup_{j=0}^k V_{s_j} \subset \bigcup_{j=0}^k B(\varepsilon_{s_j}, [\gamma_{\hat{x}}^*] \cap [\phi_{s_j}])$, for some $k \in \mathbb{N}$ (note that the diameter of $B(\varepsilon_{s_j}, [\gamma_{\hat{x}}^*] \cap [\phi_{s_j}])$ is uniformly bounded, even for the values of s_j such that $[\gamma_{\hat{x}}^*] \cap [\phi_{s_j}]$ is a leaf segment, because $\sum d_i \leq 1$). Therefore, we can partition the interval $[0, 1]$ into a finite number of closed subintervals and modify

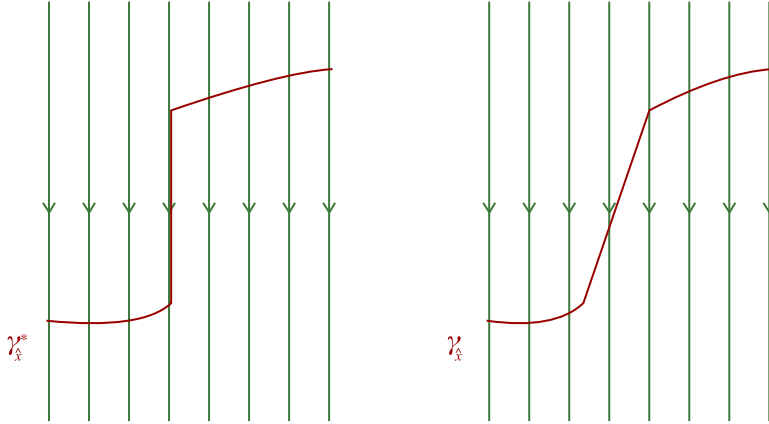


FIGURE 7. Modification of $\gamma_{\hat{x}}^*$ in $\gamma_{\hat{x}}$.

$\gamma_{\hat{x}}^*$ in each interval, inside the tubular neighborhoods, keeping the ends of the intervals fixed, in order to obtain a transverse path $\gamma_{\hat{x}}$ such that $[\gamma_{\hat{x}}] \subset \bigcup_{j=0}^k B(\varepsilon_{s_j}, [\gamma_{\hat{x}}^*] \cap [\phi_{s_j}])$ (see Figure 7).

Note now that, since $\hat{\pi}(\hat{\beta}) = \tilde{I}(\tilde{x})$ is contained in U and, if ϕ is a leaf of $\tilde{\mathcal{F}}$, then $[\phi] \cap U$ is connected, it follows that $\hat{\pi}(\gamma_{\hat{x}}^*)$ is also contained in U . So, $\hat{\pi}(\bigcup_{j=0}^k B(\varepsilon_{s_j}, [\gamma_{\hat{x}}^*] \cap [\phi_{s_j}])) \subset B(\max\{\varepsilon_{s_j}\}, U) \subset B(1, U)$, and then, denoting $\hat{\pi}(\gamma_{\hat{x}}) = \tilde{\gamma}_{\tilde{x}}$, we have that $\tilde{\gamma}_{\tilde{x}} \subset B(1, U)$ and $\tilde{\gamma}_{\tilde{x}} \in \tilde{I}_{\tilde{\mathcal{F}}}(\tilde{x})$. Since U is bounded, there is $L_0 > 0$ such that $\text{diam}(\tilde{\gamma}_{\tilde{x}}) < L_0$, proving the result. \square

From now on, let us take \tilde{z}_0 to be the point given by Proposition 19 and let $z_0 = \pi(\tilde{z}_0)$. Note that z_0 is recurrent, and has rotation vector ρ_0 . We will denote by $\tilde{\gamma}_0$ an element of $\tilde{I}_{\tilde{\mathcal{F}}}^{\mathbb{Z}}(\tilde{z}_0)$ which passes through \tilde{z}_0 and by $[\tilde{\gamma}_0]$ its image. Using Lemma 22, we can assume that for each $n \in \mathbb{Z}$, the segment of $\tilde{\gamma}_0$ between $\tilde{f}^n(\tilde{z}_0)$ and $\tilde{f}^{n+1}(\tilde{z}_0)$ has diameter less than L_0 . We can also assume that $\tilde{\gamma}_0$ is parameterized so that $\tilde{\gamma}_0(n) = \tilde{f}^n(\tilde{z}_0)$, for every $n \in \mathbb{Z}$.

Definition 23. Let $v \in \mathbb{R}^2$ be a unit vector such that $v \neq \rho_0$ and $v \neq \rho_0^\perp$, and denote by v_s the vector symmetrical to v with respect to the direction of ρ_0 (i.e. $\langle v_s, \rho_0 \rangle = \langle v, \rho_0 \rangle$ and $\langle v_s, \rho_0^\perp \rangle = -\langle v, \rho_0^\perp \rangle$). Let us now denote the straight lines generated by v and v_s passing through \tilde{z}_0 by r_v and r_{v_s} (i.e. $r_v(t) = \tilde{z}_0 + tv$, for $t \in \mathbb{R}$). Note that $\mathbb{R}^2 \setminus ([r_v] \cup [r_{v_s}])$ has four connected components, and denote by C_1 and C_2 the components that intersect the straight line generated by ρ_0 passing through \tilde{z}_0 . We will call the closure of $C_1 \cup C_2$ the cone generated by ρ_0 with inclination v and origin \tilde{z}_0 , and we will denote such a set by $C_{\tilde{z}_0}^{\rho_0}(v)$ (see Figure 8).

Note that in Definition 23 we have $\partial(C_{\tilde{z}_0}^{\rho_0}(v)) = [r_v] \cup [r_{v_s}]$

In the next lemma we will denote, for $L > 0$, $B(L, A) = \bigcup_{\tilde{x} \in A} B(L, \tilde{x})$, where $A \subset \mathbb{R}^2$.

LEMMA 24. *Let $\tilde{\gamma}_0$ be as defined after Lemma 22. Given $v \in \mathbb{R}^2$ as in Definition 23, there is $L_1 > 0$ such that $[\tilde{\gamma}_0] \subset B(L_0, C_{\tilde{z}_0}^{\rho_0}(v)) \cup B(L_1, \tilde{z}_0)$, where L_0 is given by Lemma 22.*

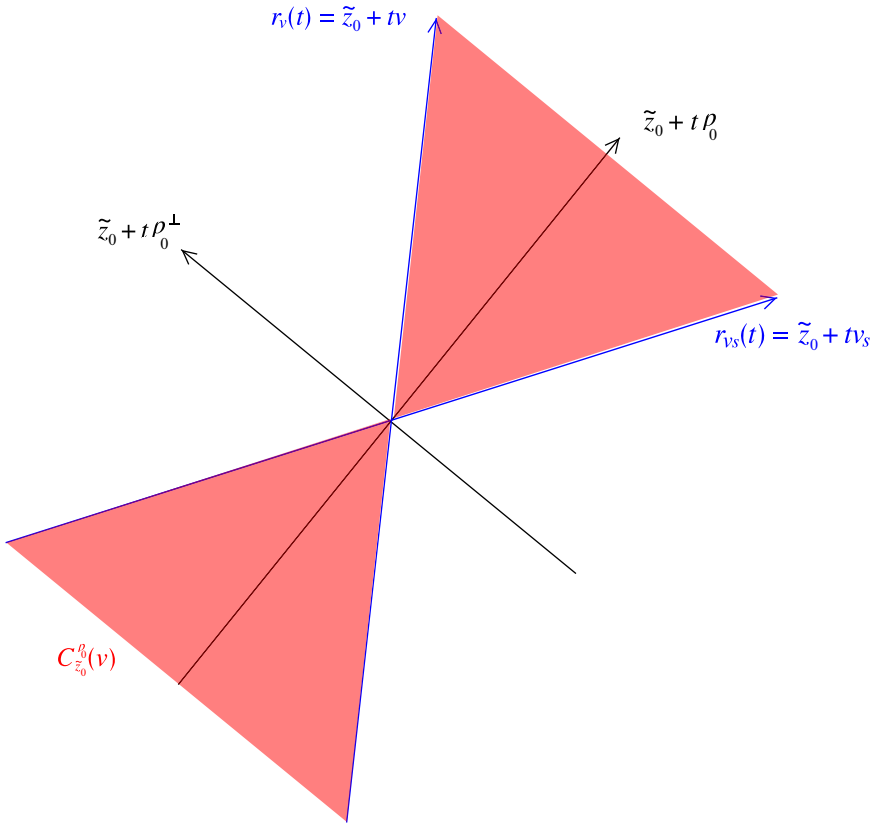


FIGURE 8. Illustration of $C_{\tilde{z}_0}^{\rho_0}(v)$.

Proof. First, let us consider the straight lines r_v and r_{v_s} , as in Definition 23. Denoting by $d(r_v, \tilde{x})$ the distance between the straight line r_v and the point \tilde{x} , we have that $d_n = d(r_v, n\rho_0 + \tilde{z}_0) = d(r_{v_s}, n\rho_0 + \tilde{z}_0) = n\|\langle \rho_0, v \rangle v - \rho_0\|$, for $n \in \mathbb{Z}$. Note that $B(d_n, n\rho_0 + \tilde{z}_0) \subset C_{\tilde{z}_0}^{\rho_0}(v)$.

Remember that $\lim_{n \rightarrow \infty} (\tilde{f}^n(\tilde{z}_0) - \tilde{z}_0)/n = \rho_0$. So, setting $\varepsilon = \|\langle \rho_0, v \rangle v - \rho_0\|$, there is $n_1 > 0$ such that

$$\begin{aligned} d(\tilde{f}^n(\tilde{z}_0), n\rho_0 + \tilde{z}_0) &= \|\tilde{f}^n(\tilde{z}_0) - \tilde{z}_0 - n\rho_0\| \\ &< n\varepsilon = n\|\langle \rho_0, v \rangle v - \rho_0\| = d(r_v, n\rho_0 + \tilde{z}_0) \quad \text{for all } n \geq n_1. \end{aligned}$$

Proceeding analogously for \tilde{f}^{-1} , we obtain

$$\begin{aligned} d(\tilde{f}^{-n}(\tilde{z}_0), -n\rho_0 + \tilde{z}_0) &= \|\tilde{f}^{-n}(\tilde{z}_0) - \tilde{z}_0 - n(-\rho_0)\| \\ &< n\varepsilon = n\|\langle \rho_0, v \rangle v - \rho_0\| = d(r_{v_s}, n\rho_0 + \tilde{z}_0) \quad \text{for all } n \geq n_2. \end{aligned}$$

Therefore, setting $n_0 = \max\{n_1, n_2\}$, we have that $d(\tilde{f}^n(\tilde{z}_0), n\rho_0 + \tilde{z}_0) < d_n = d(r_v, n\rho_0 + \tilde{z}_0)$, and since $B(d_n, n\rho_0 + \tilde{z}_0) \subset C_{\tilde{z}_0}^{\rho_0}(v)$, we have $\tilde{f}^n(\tilde{z}_0) \in C_{\tilde{z}_0}^{\rho_0}(v)$, for $|n| \geq n_0$ (see Figure 9).

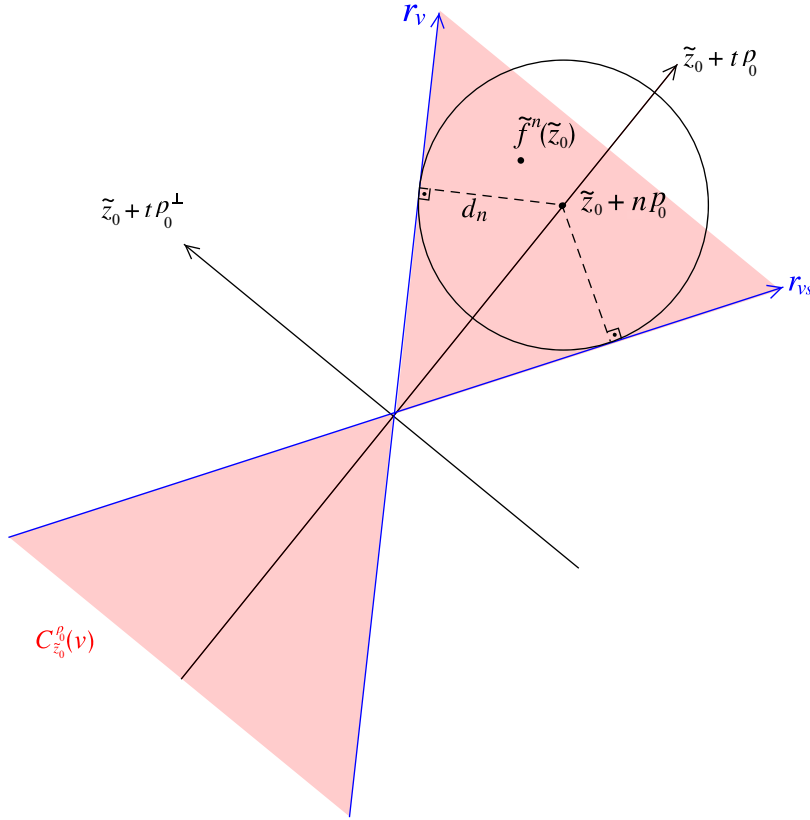


FIGURE 9. Illustration of $\tilde{f}^n(\tilde{z}_0) \in C_{\tilde{z}_0}^{\rho_0}(v)$.

By construction, the diameter of the segment $\tilde{\gamma}_0$ between $\tilde{f}^n(\tilde{z}_0)$ and $\tilde{f}^{n+1}(\tilde{z}_0)$ is smaller than L_0 , so for $|n| \geq n_0$ such segment is contained in $B(L_0, C_{\tilde{z}_0}^{\rho_0}(v))$.

For n such that $|n| < n_0$, $\tilde{f}^n(\tilde{z}_0)$ may be out of the cone but, since such points exist only in finite quantity, there is $L' > 0$ such that $\|\tilde{f}^n(\tilde{z}_0) - \tilde{z}_0\| < L'$, for $|n| < n_0$. Again, since each segment $\tilde{\gamma}_0$ has diameter bounded by L_0 , the segments of $\tilde{\gamma}_0$ between $\tilde{f}^n(\tilde{z}_0)$ and $\tilde{f}^{n+1}(\tilde{z}_0)$, with $|n| < n_0$, are contained in $B(L_1, \tilde{z}_0)$, where $L_1 = L' + L_0$. \square

LEMMA 25. *If $\tilde{\alpha} : [a, b] \rightarrow \mathbb{R}^2$ is an admissible path for \tilde{f} , there is no $w \in \mathbb{Z}_*^2$ such that $\tilde{\alpha} \pitchfork_{\tilde{f}} (\tilde{\alpha} + w)$.*

Proof. Suppose there are $w \in \mathbb{Z}_*^2$ and $\tilde{\alpha} : [a, b] \rightarrow \mathbb{R}^2$ a path r -admissible such that $\tilde{\alpha} \pitchfork_{\tilde{f}} (\tilde{\alpha} + w)$. Thus, we have by Theorem 20 that given $p/q \in (0, 1]$ written in an irreducible way, f will have a periodic point with rotation vector equal to $(p/qr) \cdot w$, which is absurd, because $\rho(\tilde{f}) \cap \mathbb{Q}^2 = \{(0, 0)\}$. \square

LEMMA 26. *There is no $w \in \mathbb{Z}^2$ such that $\tilde{\gamma}_0 \pitchfork_{\tilde{f}} (\tilde{\gamma}_0 + w)$.*

Proof. If $w \neq 0$, the result is a direct consequence of Lemma 25. For $w = 0$, let us suppose by contradiction that $\tilde{\gamma}_0$ has a transverse self-intersection. So, there are intervals $I, J \subset \mathbb{R}$ such that $\tilde{\gamma}_0|_I \pitchfork_{\tilde{\mathcal{F}}} \tilde{\gamma}_0|_J$. We can suppose that $I \subset (-N, N)$, for some $N \in \mathbb{N}$. Now let us note that since z_0 is recurrent, there are $n_k \in \mathbb{N}$ and $w_k \in \mathbb{Z}^2$ such that $\tilde{f}^{n_k}(\tilde{z}_0) - w_k \rightarrow \tilde{z}_0$, and then $\tilde{f}^{-N+n_k}(\tilde{z}_0) - w_k \rightarrow \tilde{f}^{-N}(\tilde{z}_0)$. Also, since $\rho(f, z_0) = \rho_0$, we have $w_k/n_k \rightarrow \rho_0$ and then there is k_0 such that $w_k \neq 0$ if $k > k_0$.

Note that, by the way $\tilde{\gamma}_0$ has been parameterized, $\tilde{\gamma}_0(-N) = \tilde{f}^{-N}(\tilde{z}_0)$ and $\tilde{\gamma}_0(N) = \tilde{f}^N(\tilde{z}_0)$, and then $\tilde{\gamma}_0|_{[-N, N]} = \tilde{I}_{\tilde{\mathcal{F}}}^{2N}(\tilde{f}^{-N}(\tilde{z}_0))$. Since $\tilde{f}^{-N+n_k}(\tilde{z}_0) - w_k \rightarrow \tilde{f}^{-N}(\tilde{z}_0)$, by Lemma 16 if k' , is large enough, $\tilde{I}_{\tilde{\mathcal{F}}}^{2N}(\tilde{f}^{-N}(\tilde{z}_0))$ is equivalent to a subpath of $\tilde{I}_{\tilde{\mathcal{F}}}^{2N+2}(\tilde{f}^{-1}(\tilde{f}^{-N+n_{k'}}(\tilde{z}_0) - w_{k'}))$. But

$$\begin{aligned} \tilde{I}_{\tilde{\mathcal{F}}}^{2N+2}(\tilde{f}^{-1}(\tilde{f}^{-N+n_{k'}}(\tilde{z}_0) - w_{k'})) &= \tilde{I}_{\tilde{\mathcal{F}}}^{2N+2}(\tilde{f}^{-N-1+n_{k'}}(\tilde{z}_0) - w_{k'}) \\ &= (\tilde{\gamma}_0 - w_{k'})|_{[-N-1+n_{k'}, N+1+n_{k'}]} \end{aligned}$$

and we can also assume that $k' > k_0$. So, there is $I' \subset [-N-1+n_{k'}, N+1+n_{k'}]$ such that $(\tilde{\gamma}_0 - w_{k'})|_{I'} \sim_{\tilde{\mathcal{F}}} \tilde{\gamma}_0|_{[-N, N]}$. Since $I \subset [-N, N]$, there is $I'' \subset I'$ such that $(\tilde{\gamma}_0 - w_{k'})|_{I''} \sim_{\tilde{\mathcal{F}}} \tilde{\gamma}_0|_I$, but $\tilde{\gamma}_0|_I \pitchfork_{\tilde{\mathcal{F}}} \tilde{\gamma}_0|_J$, therefore $(\tilde{\gamma}_0 - w_{k'})|_{I'} \pitchfork_{\tilde{\mathcal{F}}} \tilde{\gamma}_0|_J$, and given that $k' > k_0$, we have $w_{k'} \neq 0$, and thus we get a contradiction, from Lemma 25. \square

LEMMA 27. $\tilde{\gamma}_0$ intersects each leaf at most once.

Proof. Suppose, by contradiction, that there are $t' < t''$ such that $\phi_{\tilde{\gamma}_0(t')} = \phi_{\tilde{\gamma}_0(t'')}$. So $\tilde{\gamma}_0|_{[t', t'']}$ is $\tilde{\mathcal{F}}$ -equivalent to a transverse closed curve Γ . But since Γ has some subpath $\Gamma_0 : J \rightarrow \mathbb{R}^2$ which is transverse, closed and simple, there exists some $t' \leq a' < b' \leq t''$ such that $\tilde{\gamma}_0|_{[a', b']}$ is $\tilde{\mathcal{F}}$ -equivalent to Γ_0 . Since $\tilde{\gamma}_0$ does not have a transverse self-intersection, it follows from [CT18, Proposition 20] that

$$\bigcup_{s \in J} [\phi_{\Gamma_0(s)}] = A_{\Gamma_0}$$

is an open topological annulus. Let $J_0 = (a, b)$ with $a < a' < b' < b$ be such that, for all $t \in J_0$, $\tilde{\gamma}_0(t) \in A_{\Gamma_0}$, and such that if $-\infty < a$ (or if $b < \infty$) then $\tilde{\gamma}_0(a) \notin A_{\Gamma_0}$ (respectively, $\tilde{\gamma}_0(b) \notin A_{\Gamma_0}$). In the language of [CT18], J_0 is a drawing component for Γ_0 .

We claim that if $-\infty < c < a < b < d < +\infty$, then $\tilde{\gamma}_0(c)$ and $\tilde{\gamma}_0(d)$ cannot both belong to unbounded connected components of $\mathbb{R}^2 \setminus A_{\Gamma_0}$. Indeed, also using the terminology from [CT18], we call an interval $I_0 = (a_0, b_0) \subset \mathbb{R}$ a *crossing component* for Γ_0 if $\tilde{\gamma}_0(t) \in A_{\Gamma_0}$ for all $t \in (a_0, b_0)$, and such that $\tilde{\gamma}_0(a_0)$ and $\tilde{\gamma}_0(b_0)$ are in different connected components of the complement of A_{Γ_0} and in different connected components of Γ_0 . By [CT18, Proposition 20], if the interval J_0 is not a crossing component of A_{Γ_0} , then the interval (a, b) must be unbounded, and the claim is vacuously true. So assume that the statement is false, and that the interval J_0 is a crossing component. Then one of $\tilde{\gamma}_0(a)$ and $\tilde{\gamma}_0(b)$ must be in the bounded connected component of the complement of A_{Γ_0} , and the other must be in an unbounded connected component of the complement of A_{Γ_0} , and we assume without loss of generality that $\tilde{\gamma}_0(a)$ is in the unbounded connected component, the other case being similar. But since the claim is false, there must exist some $d > b$ such that

$\tilde{\gamma}_0(d)$ is an unbounded connected component of the complement of A_{Γ_0} . This implies that there must exist another crossing interval of A_{Γ_0} contained in $[b, d]$, which violates the property (2) of [CT18, Proposition 20], a contradiction establishing the claim.

Since z_0 is recurrent and has rotation vector ρ , we have, as in the proof of Lemma 26, that there are $t_k \rightarrow +\infty$, $w_k^+ \in \mathbb{Z}^2$, with $\|w_k^+\| \rightarrow +\infty$, and $s_k \rightarrow -\infty$, $v_k^- \in \mathbb{Z}^2$, with $\|v_k^-\| \rightarrow +\infty$, such that $\tilde{\gamma}_0(t_k) \in A_{\Gamma_0} + w_k^+$ and $\tilde{\gamma}_0(s_k) \in A_{\Gamma_0} + v_k^-$. First, note that, as Γ_0 is a transverse simple closed curve, every leaf ϕ that intersects Γ_0 does so at only one point. From this it follows that either every leaf that intersects Γ_0 has its ω -limit contained in the bounded connected component of its complement, or any leaf intersecting Γ_0 has its α -limit contained in the bounded connected component of its complement. We will assume, without loss of generality, that the first situation occurs, and the second case is treated in the same way. But, since the foliation is invariant by integer translations, it follows that an analogous property holds for the leaves intersecting $\Gamma_0 + w$, for integer vectors w . So, if $[\phi] \subset A_{\Gamma_0}$ and $[\Gamma_0 + w] \cap [\Gamma_0] = \emptyset$, then we have that $[\phi] \cap A_{\Gamma_0} + w = \emptyset$, because the ω -limit of a leaf at the intersection would be contained in two disjoint sets.

So, for k large enough, since $[\Gamma_0 + w_k^+] \cap [\Gamma_0] = \emptyset$, we have that $\phi_{\tilde{\gamma}_0(t_k)}$ is contained in an unbounded connected component of $\mathbb{R}^2 \setminus A_{\Gamma_0}$, and the same holds for $\phi_{\tilde{\gamma}_0(s_k)}$, which is impossible since, when k is sufficiently large, $s_k < a < b < t_k$. □

COROLLARY 28. *$\tilde{\gamma}_0$ is a line.*

Proof. By Lemma 27, $\tilde{\gamma}_0$ is a simple path, and since $\|\tilde{f}^n(\tilde{z}_0)\| \rightarrow +\infty$ when $n \rightarrow \pm\infty$ and $\tilde{\gamma}_0([n, n + 1])$ has diameter smaller than L_0 and contains $\tilde{f}^n(\tilde{z}_0)$, it follows that $\tilde{\gamma}_0$ must also be proper. □

Therefore, since $\tilde{\gamma}_0$ is a line, the sets $L(\tilde{\gamma}_0)$ and $R(\tilde{\gamma}_0)$ are well defined, and recall that $l(\tilde{\gamma}_0)$ is the foliated left of $\tilde{\gamma}_0$, that is, the union of all leaves and singularities contained in $L(\tilde{\gamma}_0)$. Likewise $r(\tilde{\gamma}_0)$ is the foliated right of $\tilde{\gamma}_0$.

LEMMA 29. *Every connected component of $r(\tilde{\gamma}_0)$ is unbounded. Likewise, every connected component of $l(\tilde{\gamma}_0)$ is unbounded.*

Proof. Note that $r(\tilde{\gamma}_0)$ and $l(\tilde{\gamma}_0)$ are disjoint, so it suffices to show that every connected component of $r(\tilde{\gamma}_0) \cup l(\tilde{\gamma}_0)$ is unbounded. The complement of $r(\tilde{\gamma}_0) \cup l(\tilde{\gamma}_0)$ is the union of all leaves that are crossed by $\tilde{\gamma}_0$. By Lemma 27, $\tilde{\gamma}_0$ is a line and it crosses each leaf at most once. Let $O = \bigcup_{t \in \mathbb{R}} [\phi_{\tilde{\gamma}_0(t)}]$ be the set of leaves that intersect $\tilde{\gamma}_0$, whose complement is precisely $r(\tilde{\gamma}_0) \cup l(\tilde{\gamma}_0)$. If we associate to the singular oriented foliation $\tilde{\mathcal{F}}$ a continuous flow $\Phi : \mathbb{R}^2 \times \mathbb{R}$ such that the flowlines are the leaves of $\tilde{\mathcal{F}}$, then the function $G : \mathbb{R}^2 \rightarrow O$ given by $G(t, s) = \Phi(\tilde{\gamma}_0(t), s)$ is a homeomorphism between the plane and O , which implies that O is simply connected. Therefore all connected components of the complement of O are unbounded. □

LEMMA 30. *If $K \subset \mathbb{R}^2$ is compact, then the set $I_K = \{t \in \mathbb{R} \mid [\phi_{\tilde{\gamma}_0(t)}] \cap K \neq \emptyset\}$ is also compact.*

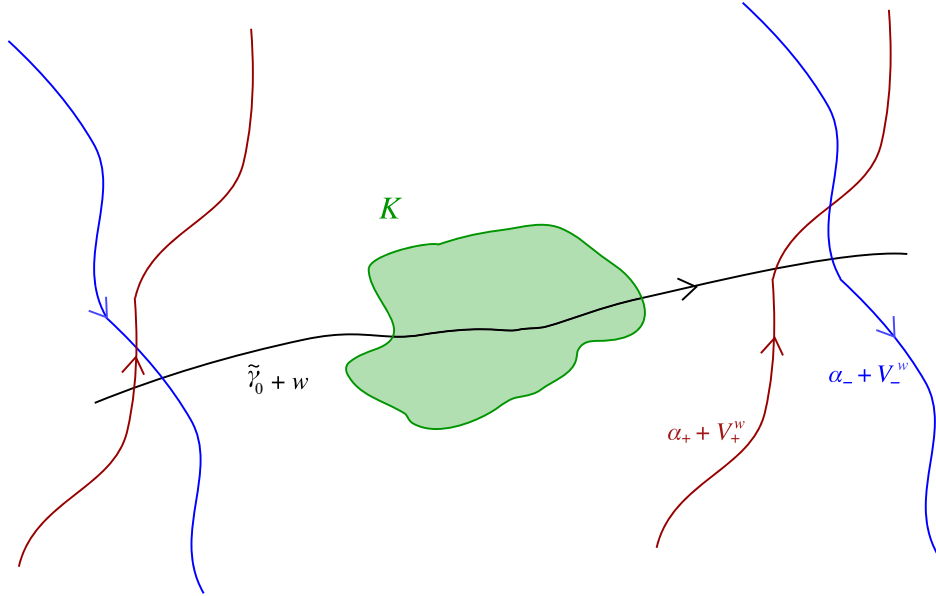


FIGURE 10. Proof of Lemma 30.

Proof. Let ϕ be a leaf such that $\phi(\bar{s}) \in K$, for some $\bar{s} \in \mathbb{R}$, and $V_-^w, V_+^w \in \mathbb{Z}^2$ such that $K \subset R(\alpha_- + V_-^w)$ and $K \subset L(\alpha_+ + V_+^w)$ (see Figure 10), where α_- and α_+ are the transverse lines given by Lemma 21 (note that we can find such lines since K is compact, and therefore bounded). Then $\phi(\bar{s}) \in R(\alpha_- + V_-^w) \cap L(\alpha_+ + V_+^w)$, and so, as remarked after Definition 7, $\phi(s) \in R(\alpha_- + V_-^w)$, for every $s > \bar{s}$, and $\phi(s) \in L(\alpha_+ + V_+^w)$, for every $s < \bar{s}$, that is, $[\phi] \subset R(\alpha_- + V_-^w) \cup L(\alpha_+ + V_+^w)$. However, since α_- is directed by v_- such that $\langle v_-, \rho_0^\perp \rangle < 0$ and $\langle v_-, \rho_0 \rangle > 0$ (and α_+ is directed by v_+ such that $\langle v_+, \rho_0^\perp \rangle > 0$ and $\langle v_+, \rho_0 \rangle > 0$) and, by Lemma 24, $[\tilde{\gamma}_0 + w]$ is contained in $B(L_0, C_{\tilde{z}_0}^{\rho_0}(v) + w) \cup B(L_1, \tilde{z}_0 + w)$, where the cone $C_{\tilde{z}_0}^{\rho_0}(v)$ is generated by ρ_0 and v is a unit vector such that $\langle v, \rho_0 \rangle > 0$ and $0 < \langle v, \rho_0^\perp \rangle < \langle v_+, \rho_0^\perp \rangle$, we have that there is $\bar{t} = \max\{t \in \mathbb{R} \mid (\tilde{\gamma}_0 + w)(t) \in R(\alpha_- + V_-^w) \cup L(\alpha_+ + V_+^w)\}$, and therefore ϕ can only cross $\tilde{\gamma}_0 + w$ before \bar{t} . Proceeding symmetrically, we can obtain a lower bound for the instant in which ϕ can cross $\tilde{\gamma}_0 + w$, thus proving the statement. \square

COROLLARY 31. *Let $\zeta : [a, b] \rightarrow \mathbb{R}^2$ be a transverse path and $w \in \mathbb{Z}^2$. If $[\zeta] \cap l(\tilde{\gamma}_0 + w) \neq \emptyset$ and $[\zeta] \cap r(\tilde{\gamma}_0 + w) \neq \emptyset$, then $\zeta \pitchfork_{\tilde{F}}(\tilde{\gamma}_0 + w)$.*

Proof. By Proposition 15, it is sufficient to prove that the set

$$I_\zeta = \{t \in \mathbb{R} \mid \phi_{(\tilde{\gamma}_0 + w)(t)} \text{ crosses } \zeta\}$$

is compact. But this follows directly from Lemma 30. \square

LEMMA 32. *If $\tilde{\gamma}' : \mathbb{R} \rightarrow \mathbb{R}^2$ is such that $\tilde{\gamma}' \sim_{\tilde{F}} \tilde{\gamma}_0$, then $\tilde{\gamma}'$ is a line.*

Proof. By Lemma 27, since $\tilde{\gamma}' \sim_{\tilde{\mathcal{F}}} \tilde{\gamma}_0$, $\tilde{\gamma}'$ intersects each leaf at most once, therefore $\tilde{\gamma}'$ is a simple path. Now let $K \subset \mathbb{R}^2$ be a compact set. Since $\tilde{\gamma}' \sim_{\tilde{\mathcal{F}}} \tilde{\gamma}_0$, there is reparameterization of $\tilde{\gamma}'$ so that $\phi_{\tilde{\gamma}_0(t)} = \phi_{\tilde{\gamma}'(t)}$, for every $t \in \mathbb{R}$. Note that $\tilde{\gamma}'$ being proper is a property that does not depend on its parameterization. So, by Lemma 30, there is $M > 0$ such that $\phi_{\tilde{\gamma}'(t)} \cap K = \emptyset$, for every $|t| > M$. Thus, if $t' \in (\tilde{\gamma}')^{-1}(K)$, we have that $|t'| < M$, and so $\tilde{\gamma}'$ is a proper path, proving the lemma. \square

LEMMA 33. *Given $w \in \mathbb{Z}_*^2$, if $\tilde{\gamma}_0$ and $\tilde{\gamma}_0 + w$ intersect the same leaf ϕ , then there are sequences $t_k^+, s_k^+ \nearrow +\infty$, $t_k^-, s_k^- \searrow -\infty$ and transverse paths $\tilde{\gamma}'_i : \mathbb{R} \rightarrow \mathbb{R}^2$, $i = 1, 2$, such that:*

- (i) $\tilde{\gamma}'_1 \sim_{\tilde{\mathcal{F}}} \tilde{\gamma}_0$ and $\tilde{\gamma}'_2 \sim_{\tilde{\mathcal{F}}} \tilde{\gamma}_0 + w$;
- (ii) $\tilde{\gamma}'_1(t_k^+) = \tilde{\gamma}'_2(s_k^+)$ and $\tilde{\gamma}'_1(t_k^-) = \tilde{\gamma}'_2(s_k^-)$, for every $k \in \mathbb{N}$.

Proof. Since $\tilde{\gamma}_0$ is recurrent, we can find sequences $t_k^+, s_k^+ \nearrow +\infty$, such that $\tilde{\gamma}_0(t_k^+)$ and $(\tilde{\gamma}_0 + w)(s_k^+)$ belong to the leaf $\phi_k^+ = \phi + w_k^+$, for some $w_k^+ \in \mathbb{Z}^2$ (and in the same way, sequences $t_k^-, s_k^- \searrow -\infty$ with the same property). Let, for each $k \in \mathbb{N}$, W_k^+ be a tubular neighborhood of the leaf $\phi + w_k^+$ such that $\tilde{\gamma}_0(t_k^+), (\tilde{\gamma}_0 + w)(s_k^+) \in W_k^+$ and W_k^- a tubular neighborhood of the leaf $\phi + w_k^-$ such that $\tilde{\gamma}_0(t_k^-), (\tilde{\gamma}_0 + w)(s_k^-) \in W_k^-$. Since $\|w_k^\pm\| \rightarrow \infty$, we can assume that the neighborhoods are mutually disjoint. So, making in each neighborhood a modification similarly to that made in Lemma 21 (see Figure 4) we can obtain transverse paths $\tilde{\gamma}'_1$ and $\tilde{\gamma}'_2$ which satisfy properties (i) and (ii) of the statement. \square

Remark 34. Note that, by Lemma 32, $\tilde{\gamma}'_1$ and $\tilde{\gamma}'_2$ are lines.

LEMMA 35. *If $\tilde{\gamma}_0$ and $\tilde{\gamma}_0 + w$ cross the same leaf, then every connected component of $L(\tilde{\gamma}'_1) \cap R(\tilde{\gamma}'_2)$ and of $R(\tilde{\gamma}'_1) \cap L(\tilde{\gamma}'_2)$ is bounded, where $\tilde{\gamma}'_1$ and $\tilde{\gamma}'_2$ are the paths given by Lemma 33.*

Proof. Let us assume, by contradiction, that the result is not true. We can, without loss of generality, assume that there is an unbounded connected component O of $R(\tilde{\gamma}'_1) \cap L(\tilde{\gamma}'_2)$. The case where there is an unbounded connected component of $L(\tilde{\gamma}'_1) \cap R(\tilde{\gamma}'_2)$ is similar. Since $\tilde{\gamma}'_1$ and $\tilde{\gamma}'_2$ are lines that cross each other, any connected component of the complement of $[\tilde{\gamma}'_1] \cup [\tilde{\gamma}'_2]$ has on its boundary points that are in $[\tilde{\gamma}'_1]$ but not in $[\tilde{\gamma}'_2]$, and also points that are in $[\tilde{\gamma}'_2]$ but not in $[\tilde{\gamma}'_1]$. Let P_1 and P_2 be points in $\partial O \cap [\tilde{\gamma}'_1] \cap (\mathbb{R}^2 \setminus [\tilde{\gamma}'_2])$ and $\partial O \cap (\mathbb{R}^2 \setminus [\tilde{\gamma}'_1]) \cap [\tilde{\gamma}'_2]$, respectively. Let $\phi_1, \phi_2 : \mathbb{R} \rightarrow \text{dom}(\tilde{\mathcal{I}})$ be the leaves of $\tilde{\mathcal{F}}$ passing through P_1 and P_2 respectively, and let us assume that $\phi_1(0) = P_1$ and $\phi_2(0) = P_2$. Note that $\phi_1(0) \in L(\tilde{\gamma}'_2)$, and since every leaf of $\tilde{\mathcal{F}}$ that intersects $\tilde{\gamma}'_1$ or $\tilde{\gamma}'_2$ must cross that path from left to right, we have that $\phi_1((-\infty, 0))$ is contained in $L(\tilde{\gamma}'_1) \cup L(\tilde{\gamma}'_2)$. Moreover, if $\phi_1((-\infty, 0))$ is bounded, then the α -limit set of ϕ_1 will be contained in $l(\tilde{\gamma}'_1) \cup l(\tilde{\gamma}'_2)$. Furthermore, if $\varepsilon > 0$ is sufficiently small, then $\phi_1(\varepsilon)$ belongs to O . Analogously, it is possible to show that $\phi_2((0, +\infty))$ is contained in $R(\tilde{\gamma}'_1) \cup R(\tilde{\gamma}'_2)$, its ω -limit set is contained in $r(\gamma_1) \cup r(\gamma_2)$ and, if $\varepsilon > 0$ is sufficiently small, then $\phi_2(-\varepsilon)$ belongs to O .

We know that, since $\tilde{\gamma}_0$ and $\tilde{\gamma}_0 + w$ have no $\tilde{\mathcal{F}}$ -transverse intersection, the same holds for $\tilde{\gamma}'_1$ and $\tilde{\gamma}'_2$, since these paths are equivalent to $\tilde{\gamma}_0$ and $\tilde{\gamma}_0 + w$, respectively. Therefore,

by Corollary 31, $\tilde{\gamma}'_1$ cannot intersect both $r(\tilde{\gamma}'_2)$ and $l(\tilde{\gamma}'_2)$. Let us assume initially that $\tilde{\gamma}'_1$ does not intersect $l(\tilde{\gamma}'_2)$.

Let t_0 be such that $\tilde{\gamma}'_1(t_0) = P_1$. By Lemma 33 there are $t_- < t_0 < t_+$ and $s_- < s_+$ such that $\tilde{\gamma}'_1(t_-) = \tilde{\gamma}'_2(s_-)$ and $\tilde{\gamma}'_1(t_+) = \tilde{\gamma}'_2(s_+)$. Let s'_- be the largest real number smaller than s_+ such that $\tilde{\gamma}'_2(s'_-) = \tilde{\gamma}'_1(t'_-)$, with $t'_- < t_0$, and let s'_+ be the smallest real number larger than s'_- such that $\tilde{\gamma}'_2(s'_+) = \tilde{\gamma}'_1(t'_+)$, with $t'_+ > t_0$. It follows that if $C_1 = \tilde{\gamma}'_1([t'_-, t'_+]) \cup \tilde{\gamma}'_2([s'_-, s'_+])$, then C_1 is the image of a simple closed curve separating the plane into two disjoint components, one of which is bounded. Also, P_1 belongs to C_1 and if ε is sufficiently small, then $\phi_1(-\varepsilon)$ and $\phi_1(\varepsilon)$ belong to distinct components of the complement of the curve. But if H_1 is a connected component of $l(\tilde{\gamma}'_2)$ which contains the α -limit set of ϕ_1 , then $F_1 = H_1 \cup \phi_1((-\infty, -\varepsilon))$ does not intersect $[\tilde{\gamma}'_1] \cup [\tilde{\gamma}'_2]$. Furthermore, by Lemma 29, H_1 is an unbounded set, therefore F_1 is also unbounded, and thus is contained in an unbounded connected component of the complement of C_1 . Note also that $\phi_1(-\varepsilon)$ is in the same connected component of the complement of C_1 as F_1 . However, O is contained in the complement of $[\tilde{\gamma}'_1] \cup [\tilde{\gamma}'_2]$, and therefore in the complement of C_1 , and O is also unbounded. Note that $\phi_1(\varepsilon) \in O$ and that $\phi_1(-\varepsilon)$ and $\phi_1(\varepsilon)$ are in separate components of the complement of C_1 . Then O and F_1 are contained in distinct components of the complement of C_1 and both are unbounded, which is absurd.

Let us now assume that $\tilde{\gamma}'_1$ does not intersect $r(\tilde{\gamma}'_2)$. Let s_0 be such that $\tilde{\gamma}'_2(s_0) = P_2$. We can, as before, find $t''_- < t''_+$ and $s''_- < s_0 < s''_+$ such that $\tilde{\gamma}'_1([t''_-, t''_+]) \cup \tilde{\gamma}'_2([s''_-, s''_+])$ is the image of a simple closed curve C_2 , and such that if ε is sufficiently small, $\phi_2(-\varepsilon)$ and $\phi_2(\varepsilon)$ belong to distinct components of the complement of the curve C_2 . Note that, if H_2 is the connected component of $r(\tilde{\gamma}'_2)$ which contains the ω -limit set of ϕ_2 , then $F_2 = H_2 \cup \phi_2((\varepsilon, +\infty))$ does not intersect $[\tilde{\gamma}'_1] \cup [\tilde{\gamma}'_2]$. Proceeding as before, we show that the two connected components of the complement of C_2 are unbounded, thus obtaining a contradiction. □

LEMMA 36. *If $r(\tilde{\gamma}_0) \cap l(\tilde{\gamma}_0 + w) \neq \emptyset$, then:*

- (i) $[\tilde{\gamma}_0] \cap l(\tilde{\gamma}_0 + w) \neq \emptyset, [\tilde{\gamma}_0] \cap r(\tilde{\gamma}_0 + w) = \emptyset;$
- (ii) $[\tilde{\gamma}_0 + w] \cap r(\tilde{\gamma}_0) \neq \emptyset, [\tilde{\gamma}_0 + w] \cap l(\tilde{\gamma}_0) = \emptyset.$

Proof. Note that if $\tilde{\gamma}_0$ and $\tilde{\gamma}_0 + w$ do not cross a common leaf the result is trivial. Suppose then that $\tilde{\gamma}_0$ and $\tilde{\gamma}_0 + w$ cross the same leaf, and let $\tilde{\gamma}'_1$ and $\tilde{\gamma}'_2$ be the lines given by Lemma 33. Note that, since $\tilde{\gamma}'_1 \sim_{\tilde{F}} \tilde{\gamma}_0$ and $\tilde{\gamma}'_2 \sim_{\tilde{F}} \tilde{\gamma}_0 + w$, we have $r(\tilde{\gamma}_0) = r(\tilde{\gamma}'_1)$ and $l(\tilde{\gamma}_0 + w) = l(\tilde{\gamma}'_2)$. Let $\tilde{p}' \in r(\tilde{\gamma}'_1) \cap l(\tilde{\gamma}'_2)$. Since $r(\tilde{\gamma}'_1) \subset R(\tilde{\gamma}'_1)$ and $l(\tilde{\gamma}'_2) \subset L(\tilde{\gamma}'_2)$, we have that \tilde{p}' belongs to $R(\tilde{\gamma}'_1) \cap L(\tilde{\gamma}'_2)$ and so, by Lemma 35, it also belongs to a bounded connected component of $R(\tilde{\gamma}'_1) \cap L(\tilde{\gamma}'_2)$. But the connected component C of $r(\tilde{\gamma}'_1)$ which contains \tilde{p}' is unbounded, by Lemma 29, and therefore needs to intersect $[\tilde{\gamma}'_1] \cup [\tilde{\gamma}'_2]$. Since C is disjoint from $[\tilde{\gamma}'_1]$, it intersects $[\tilde{\gamma}'_2]$, which implies that there is \bar{t} such that $[\phi_{\tilde{\gamma}'_2(\bar{t})}] \subset r(\tilde{\gamma}_0)$, and since $\tilde{\gamma}'_2 \sim_{\tilde{F}} \tilde{\gamma}_0 + w$, this implies that some point of $\tilde{\gamma}_0 + w$ is on the leaf $\phi_{\tilde{\gamma}'_2(\bar{t})}$, and therefore is also in $r(\tilde{\gamma}_0)$. Since $\tilde{\gamma}_0$ and $\tilde{\gamma}_0 + w$ do not have a transverse intersection, we deduce by Corollary 31 that $[\tilde{\gamma}_0 + w] \cap l(\tilde{\gamma}_0) = \emptyset$. The proof that $l(\tilde{\gamma}_0 + w)$ intersects $\tilde{\gamma}_0$ and therefore $[\tilde{\gamma}_0] \cap r(\tilde{\gamma}_0 + w) = \emptyset$ is analogous. □

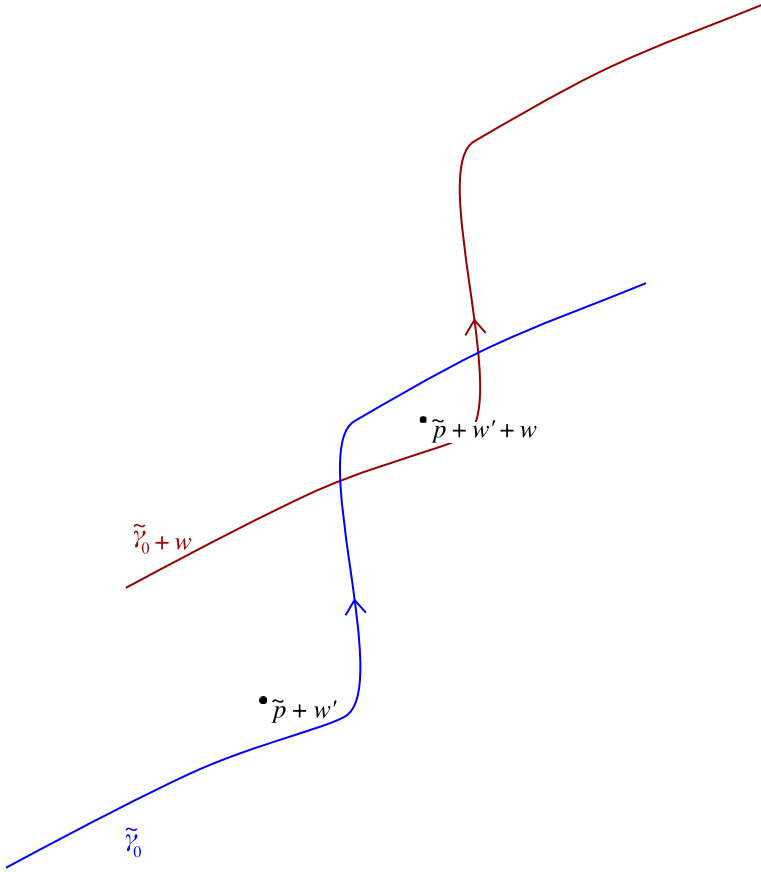


FIGURE 11. Illustration of Lemma 38.

Remark 37. Note that we can obtain a result symmetrical to the previous one, with $r(\tilde{\gamma}_0 + w) \cap l(\tilde{\gamma}_0) \neq \emptyset$.

Let p be a singularity of \mathcal{F} and let us fix a lift \tilde{p} of p . We have that \tilde{p} is a singularity of $\tilde{\mathcal{F}}$ and suppose that $\tilde{p} \in l(\tilde{\gamma}_0)$.

LEMMA 38. *Let $w, w' \in \mathbb{Z}^2$ be such that $\langle w, \rho_0^\perp \rangle > 0$ and $\tilde{p} + w' \in l(\tilde{\gamma}_0)$. Then $(\tilde{p} + w') + w \in l(\tilde{\gamma}_0)$.*

Proof. Suppose, by contradiction, that $(\tilde{p} + w') + w \in r(\tilde{\gamma}_0)$. Note that $(\tilde{p} + w') + w \in l(\tilde{\gamma}_0 + w)$, so $l(\tilde{\gamma}_0 + w) \cap r(\tilde{\gamma}_0) \neq \emptyset$ (see Figure 11). Therefore, by Lemma 36, $[\tilde{\gamma}_0 + w] \cap r(\tilde{\gamma}_0) \neq \emptyset$.

We claim that this implies that $r(\tilde{\gamma}_0 + w) \subset \mathbb{R}^2 \setminus l(\tilde{\gamma}_0)$: indeed, if $r(\tilde{\gamma}_0 + w) \cap l(\tilde{\gamma}_0) \neq \emptyset$, we have again by Lemma 36 that $[\tilde{\gamma}_0 + w] \cap l(\tilde{\gamma}_0) \neq \emptyset$. So, we get by Proposition 15 that $\tilde{\gamma}_0 \pitchfork_{\mathcal{F}} (\tilde{\gamma}_0 + w)$, which is a contradiction by Lemma 26, proving the claim. Therefore, since $(\tilde{p} + w') + 2w$ is a singularity and so must be contained

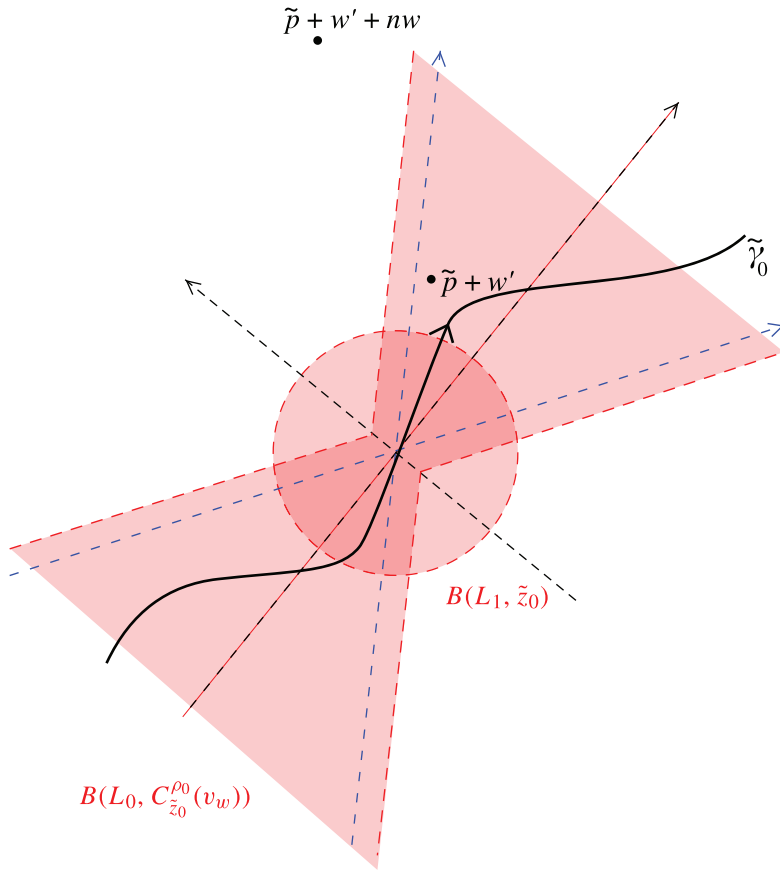


FIGURE 12. Illustration of Lemma 38.

in $r(\tilde{\gamma}_0) \cup l(\tilde{\gamma}_0)$, and since $(\tilde{p} + w') + 2w \in r(\tilde{\gamma}_0 + w)$ and $r(\tilde{\gamma}_0 + w) \subset \mathbb{R}^2 \setminus l(\tilde{\gamma}_0)$, we have that $(\tilde{p} + w') + 2w \in r(\tilde{\gamma}_0)$. So, by induction, $(\tilde{p} + w') + nw \in r(\tilde{\gamma}_0)$, for all $n \in \mathbb{N}$.

Let us now denote $v'_w = w/\|w\|$. If θ is the smallest angle between v'_w and the line generated by ρ_0 , let v_w be a unit vector such that its angle to the line generated by ρ_0 is $\theta/2$. Thus, for n sufficiently large, $(\tilde{p} + w') + nw$ is in the connected component of $\mathbb{R}^2 \setminus (B(L_0, C_{\tilde{z}_0}^{\rho_0}(v_w)) \cup B(L_1, \tilde{z}_0))$ contained in the left of the line generated by ρ_0 passing through \tilde{z}_0 (see Figure 12). But, by Lemma 24, $[\tilde{\gamma}_0] \subset B(L_0, C_{\tilde{z}_0}^{\rho_0}(v_w)) \cup B(L_1, \tilde{z}_0)$, therefore $(\tilde{p} + w') + nw \in l(\tilde{\gamma}_0)$, which is a contradiction. \square

Note that with an analogous proof we can obtain the following lemma.

LEMMA 39. *Let $w, w' \in \mathbb{Z}^2$ be such that $\langle w, \rho_0^\perp \rangle < 0$ and $\tilde{p} + w' \in r(\tilde{\gamma}_0)$. Then $(\tilde{p} + w') + w \in r(\tilde{\gamma}_0)$.*

Since the slope of ρ_0^\perp is irrational, we can define the following order in \mathbb{Z}^2 :

Definition 40. $w \succ w' \Leftrightarrow \langle w - w', \rho_0^\perp \rangle > 0$.

Remark 41. Note that such order is defined simply by projecting \mathbb{Z}^2 on the line generated by ρ_0^\perp and using the natural order of such a line. In addition, such projection is dense on the line.

We will denote $Z_r = \{w \in \mathbb{Z}^2 \mid \tilde{p} + w \in r(\tilde{\gamma}_0)\}$ and $Z_l = \{w \in \mathbb{Z}^2 \mid \tilde{p} + w \in l(\tilde{\gamma}_0)\}$. Note that, since $(\tilde{p} + \mathbb{Z}^2) \cap [\tilde{\gamma}_0] = \emptyset$, we have that $\mathbb{Z}^2 = Z_l \cup Z_r$. The following lemma will show that the projections of Z_r and Z_l on the line generated by ρ_0^\perp are contained in two disjoint half-lines.

LEMMA 42. *The sets Z_r and Z_l defined above satisfy the following properties:*

- (i) *if $w \in Z_l$ and $w' \succ w$, then $w' \in Z_l$;*
- (ii) *if $w \in Z_r$ and $w' \prec w$, then $w' \in Z_r$.*

Proof. To prove (i), let us take $w \in Z_l$ and $w' \succ w$. So we have $\langle w' - w, \rho_0^\perp \rangle > 0$. Since $w \in Z_l$, we have by definition that $\tilde{p} + w \in l(\tilde{\gamma}_0)$ and then, by Lemma 38, $\tilde{p} + w + (w' - w) \in l(\tilde{\gamma}_0)$, and so $w' \in Z_l$.

The proof of (ii) is analogous. □

LEMMA 43. *Let $w \in \mathbb{Z}^2$ be such that $w \prec 0$. Then there is $w^* \in \mathbb{Z}^2$ such that $w^* \in Z_l$ and $w^* + w \in Z_r$.*

Proof. Let us denote $\langle A, \rho_0^\perp \rangle = \{\langle w, \rho_0^\perp \rangle \mid w \in A\}$. Using such notation, we have that $\langle \mathbb{Z}^2, \rho_0^\perp \rangle = \langle Z_l, \rho_0^\perp \rangle \cup \langle Z_r, \rho_0^\perp \rangle$ and $\langle \mathbb{Z}^2, \rho_0^\perp \rangle$ is dense in \mathbb{R} . Let us prove that $\langle Z_l, \rho_0^\perp \rangle$ is bounded from below. By contradiction, if $\langle Z_l, \rho_0^\perp \rangle$ is unbounded from below, there are $w_n \in \mathbb{Z}^2, n \in \mathbb{N}$, such that $\langle w_n, \rho_0^\perp \rangle \rightarrow -\infty$. But, by Lemma 42, if $w_n \in Z_l$, then $w' \in Z_l$, if $w' \succ w_n$, and so $\langle Z_l, \rho_0^\perp \rangle = \langle \mathbb{Z}^2, \rho_0^\perp \rangle$, which is a contradiction. Therefore $\langle Z_l, \rho_0^\perp \rangle$ is bounded from below. Analogously, we have that $\langle Z_r, \rho_0^\perp \rangle$ is bounded from above. Let us denote $l^* = \inf \langle Z_l, \rho_0^\perp \rangle$ and $r^* = \sup \langle Z_r, \rho_0^\perp \rangle$. We claim that $l^* = r^*$. In fact, if $r^* < l^*$, we have a contradiction, because we would have $(r^*, l^*) \cap \langle \mathbb{Z}^2, \rho_0^\perp \rangle = \emptyset$, and $\langle \mathbb{Z}^2, \rho_0^\perp \rangle$ is dense in \mathbb{R} . If $l^* < r^*$, there are $w_l \in Z_l$ and $w_r \in Z_r$ such that $w_r - w_l \succ 0$. Thus, by Lemma 38, $w_l + (w_r - w_l) \in l(\tilde{\gamma}_0)$, which is a contradiction. Therefore, $l^* = r^*$. Since $l^* = \inf \langle Z_l, \rho_0^\perp \rangle$, we have that there is a sequence $w_n \in Z_l$ such that $\langle w_n, \rho_0^\perp \rangle \rightarrow l^*$. So, since $\langle w, \rho_0^\perp \rangle < 0$, there is $n' \in \mathbb{N}$ such that $\langle w_{n'} + w, \rho_0^\perp \rangle < l^*$. Thus, setting $w^* = w_{n'}$, we have that $w^* \in Z_l$ and $w^* + w \in Z_r$, proving the lemma. □

Remark 44. Analogously to the previous lemma, given $w \succ 0$, we can obtain $w^{**} \in \mathbb{Z}^2$ such that $w^{**} \in Z_r$ and $w^{**} + w \in Z_l$.

LEMMA 45. *Let $w \in \mathbb{Z}^2_*$. If $w \succ 0$, then for every $M > 0$ there exist $t_M^+ > M$ and $t_M^- < -M$ such that both $\tilde{\gamma}_0(t_M^-)$ and $\tilde{\gamma}_0(t_M^+)$ lie in $r(\tilde{\gamma}_0 + w)$. Likewise, there exist $s_M^+ > M$ and $s_M^- < -M$ such that both $\tilde{\gamma}_0 + w(s_M^-)$ and $\tilde{\gamma}_0 + w(s_M^+)$ lie in $l(\tilde{\gamma}_0)$. In particular, $\tilde{\gamma}_0$ and $\tilde{\gamma}_0 + w$ are not \tilde{F} -equivalent.*

Proof. Let us look at the case where $w \prec 0$. By Lemma 43, there is $w^* \in \mathbb{Z}^2$ such that $\tilde{p} + w^* \in l(\tilde{\gamma}_0)$ and $\tilde{p} + w^* + w \in r(\tilde{\gamma}_0)$, and therefore $\tilde{p} + w^* + w \in r(\tilde{\gamma}_0) \cap l(\tilde{\gamma}_0 + w)$. By Lemma 36, $[\tilde{\gamma}_0] \cap l(\tilde{\gamma}_0 + w) \neq \emptyset$ and $[\tilde{\gamma}_0 + w] \cap r(\tilde{\gamma}_0) \neq \emptyset$, so one can find some

t_0, s_0 such that $\tilde{\gamma}_0(t_0) \in l(\tilde{\gamma}_0 + w)$ and $\tilde{\gamma}_0 + w(s_0) \in r(\tilde{\gamma}_0)$. Fix $M > 0$ and let us show the existence of t_M^+ ; the other cases are similar. Let ϕ_0 be the leaf that passes through $\tilde{\gamma}_0(t_0)$ and note that, if $w^* \succ 0$ is in \mathbb{Z}^2 , then $\phi_0 \subset l(\tilde{\gamma}_0 + w + w^*) \subset l(\tilde{\gamma}_0 + w)$. Let $N > t_0$ be an integer. Using Proposition 19, one can find a sequence (p_l, q_l) in $\mathbb{Z}^2 \times \mathbb{N}$ with q_l going to infinity and $p_l \succ 0$ such that $\tilde{f}^{q_l}(\tilde{z}_0) - p_l$ converges to \tilde{z}_0 . This implies, by Lemma 16, that for sufficiently large l the path $\tilde{\gamma}_0 - p_l|_{[q_l-N-1, q_l+N+1]}$ contains a subpath equivalent to $\tilde{\gamma}_0|_{[-N, N]}$. But since $\tilde{\gamma}_0|_{[-N, N]}$ intersects ϕ_0 , $\tilde{\gamma}_0 - p_l|_{[q_l-N-1, q_l+N+1]}$ must also do so, and therefore $\tilde{\gamma}_0|_{[q_l-N-1, q_l+N+1]}$ intersects $\phi_0 + p_l$, which lies in $l(\tilde{\gamma}_0 + w)$. It suffices then to take l such that $q_l > M - N - 1$.

The proof for $w \succ 0$ is analogous, using Remark 44. □

LEMMA 46. *Given $t_1 > 0$, there is $0 < \varepsilon < \frac{1}{2}$ such that, if $\tilde{z}' \in B(\varepsilon, \tilde{z}_0)$, then every element of $\tilde{I}_{\tilde{F}}^{\mathbb{Z}}(\tilde{z}')$ contains a subpath \tilde{F} -equivalent to $\tilde{\gamma}_0|_{[0, t_1]}$.*

Proof. This follows directly by Lemma 16. □

In what follows, we will say that f does not have bounded deviation in the positive direction of ρ_0^\perp if there are $\tilde{x}_k \in \mathbb{R}^2$ and $(n_k)_{k \in \mathbb{N}}$ an increasing sequence such that $\lim_{k \rightarrow \infty} \langle \tilde{f}^{n_k}(\tilde{x}_k) - \tilde{x}_k, \rho_0^\perp \rangle = +\infty$. Analogously, we will say that f does not have bounded deviation in the negative direction of ρ_0^\perp if there are \tilde{x}_k and $(n_k)_{k \in \mathbb{N}}$ as before such that the previous limit is equal to $-\infty$. Note that if f does not have bounded deviation in the direction of ρ_0^\perp then either f does not have bounded deviation in the positive direction of ρ_0^\perp or f does not have bounded deviation in the negative direction of ρ_0^\perp .

LEMMA 47. *If f does not have bounded deviation in the positive direction of ρ_0^\perp , then, given $0 < \varepsilon < \frac{1}{2}$, there are $\tilde{x} \in \mathbb{R}^2, N \in \mathbb{N}$ and $P \in \mathbb{Z}^2$ such that:*

- (i) $\tilde{x} \in B(\varepsilon, \tilde{z}_0)$;
- (ii) $P \succ (-2, 0)$;
- (iii) $\tilde{f}^N(\tilde{x}) \in B(\varepsilon, \tilde{z}_0 + P)$.

Proof. We have by Lemma 3 that the set $U_\varepsilon = \bigcup_{i=0}^\infty f^i(\pi(B(\varepsilon, \tilde{z}_0)))$ is fully essential, and since $\varepsilon < \frac{1}{2}$, we have that $\pi(B(\varepsilon, \tilde{z}_0))$ is inessential. So, applying Proposition 2, we get a compact set of the plane K and $M \in \mathbb{N}$ such that $[0, 1]^2$ is contained in a bounded connected component of $\mathbb{R}^2 \setminus K$, which we shall denote by A , and $K \subset \bigcup_{|i| \leq M, \|v\|_\infty \leq M} (\tilde{f}^i(B(\varepsilon, \tilde{z}_0)) + v)$. Since f does not have bounded deviation in the positive direction of ρ_0^\perp , there are $P' \in \mathbb{Z}^2$ and $l \in \mathbb{N}$ such that $l > 2M$, $\langle P', \rho_0^\perp \rangle > \langle (-2, 0), \rho_0^\perp \rangle + 2M$ and $\tilde{f}^l([0, 1]^2) \cap ([0, 1]^2 + P') \neq \emptyset$. Since $[0, 1]^2 \subset A$ and $\tilde{f}^l([0, 1]^2) \cap ([0, 1]^2 + P') \neq \emptyset$, we have that $\tilde{f}^l(A)$ intersects $A + P'$. Since A is bounded, we have that $\tilde{f}^l(\partial A) \cap (\partial A + P') \neq \emptyset$, and since $\partial A \subset K$, we get $\tilde{f}^l(K) \cap (K + P') \neq \emptyset$. Now, let $\tilde{y} \in \tilde{f}^l(K) \cap (K + P')$. So there are $n_i \in \mathbb{Z}, |n_i| < M$ and $v_i \in \mathbb{Z}^2, \|v_i\|_\infty < M$, for $i = 1, 2$, such that

$$\begin{aligned} \tilde{y} \in \tilde{f}^l(K) &\Rightarrow \tilde{y} \in \tilde{f}^l(\tilde{f}^{n_1}(B(\varepsilon, \tilde{z}_0)) + v_1) = \tilde{f}^{l+n_1}(B(\varepsilon, \tilde{z}_0)) + v_1, \\ \tilde{y} \in K + P' &\Rightarrow \tilde{y} \in \tilde{f}^{n_2}(B(\varepsilon, \tilde{z}_0)) + v_2 + P'. \end{aligned}$$

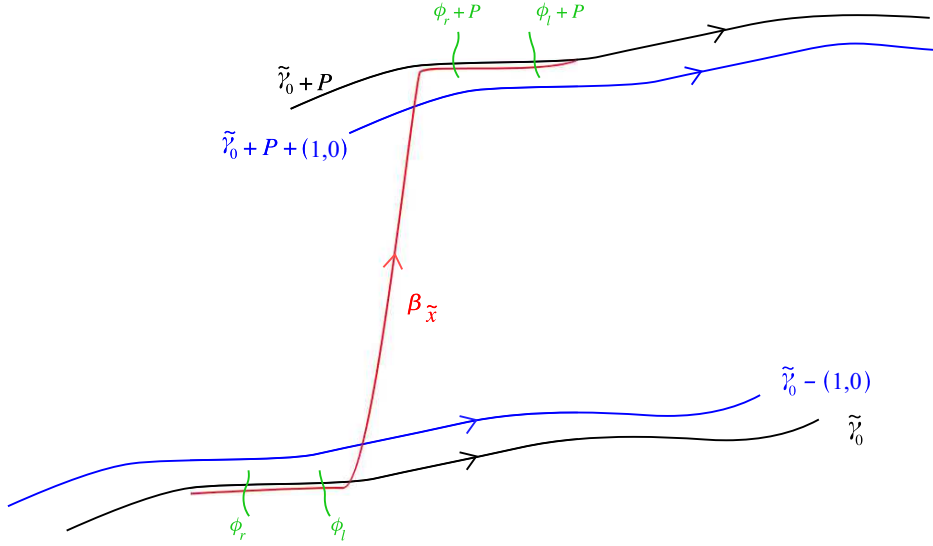


FIGURE 13. Construction of $\beta_{\tilde{x}}$.

Then we get

$$\begin{aligned} \tilde{f}^{-n_2}(\tilde{y}) &\in (\tilde{f}^{l+n_1-n_2}(B(\varepsilon, \tilde{z}_0)) + v_1) \cap B(\varepsilon, \tilde{z}_0 + v_2 + P'), \\ \tilde{f}^{-n_2}(\tilde{y}) - v_1 &\in \tilde{f}^{l+n_1-n_2}(B(\varepsilon, \tilde{z}_0)) \cap B(\varepsilon, \tilde{z}_0 + v_2 - v_1 + P'). \end{aligned}$$

Thus, setting $N = l + n_1 - n_2$ and $P = v_2 - v_1 + P'$, we get the result. □

Note that we can prove an analogous result for the case where f does not have bounded deviation in the negative direction of ρ_0^\perp .

With all the results proven so far we can complete the proof of Theorem A.

Proof of Theorem A. Suppose by contradiction that f does not have bounded deviation in the direction of ρ_0^\perp . Let us assume that f does not have bounded deviation in the positive direction of ρ_0^\perp (the other case is analogous).

Since $(1, 0) \prec 0$, we have by Lemma 45 that $\tilde{\gamma}_0$ and $\tilde{\gamma}_0 + (1, 0)$ are not $\tilde{\mathcal{F}}$ -equivalent, and beyond that, $\tilde{\gamma}_0$ intersects a leaf in $l(\tilde{\gamma}_0 + (1, 0))$. Let t_0 be a moment in which such intersection occurs, that is, $\tilde{\gamma}_0(t_0)$ belongs to a leaf, which we shall denote by ϕ_l , which is contained in $l(\tilde{\gamma}_0 + (1, 0))$. Analogously, since $-(1, 0) \succ 0$ we have by Lemma 45 that $\tilde{\gamma}_0$ and $\tilde{\gamma}_0 - (1, 0)$ are not $\tilde{\mathcal{F}}$ -equivalent, and beyond that, $\tilde{\gamma}_0$ intersects a leaf in $r(\tilde{\gamma}_0 - (1, 0))$. Let t_1 be such that $\tilde{\gamma}_0(t_1)$ belongs to a leaf, which we shall denote by ϕ_r , which is contained in $r(\tilde{\gamma}_0 - (1, 0))$. Note that, by Lemma 45, we can assume that both t_0 and t_1 are positive, and that $0 < t_0 < t_1$. Now let $0 < \varepsilon < \frac{1}{2}$ be given by Lemma 46, and let $\tilde{x} \in \mathbb{R}^2$, $N \in \mathbb{N}$ and $P \in \mathbb{Z}^2$ be given by Lemma 47. We will denote a fixed element of $\tilde{I}_{\tilde{\mathcal{F}}}^{\mathbb{Z}}(\tilde{x})$ by $\beta_{\tilde{x}}$ (see Figure 13).

Let us prove that $\beta_{\tilde{x}}$ intersects $r(\tilde{\gamma}_0 - (1, 0))$ and $l(\tilde{\gamma}_0 + P + (1, 0))$. Since $\tilde{x} \in B(\varepsilon, \tilde{z}_0)$, we have by Lemma 46 that $\tilde{\gamma}_0|_{[0, t_1]}$ is equivalent to a subpath of $\beta_{\tilde{x}}$, thus $\beta_{\tilde{x}}$

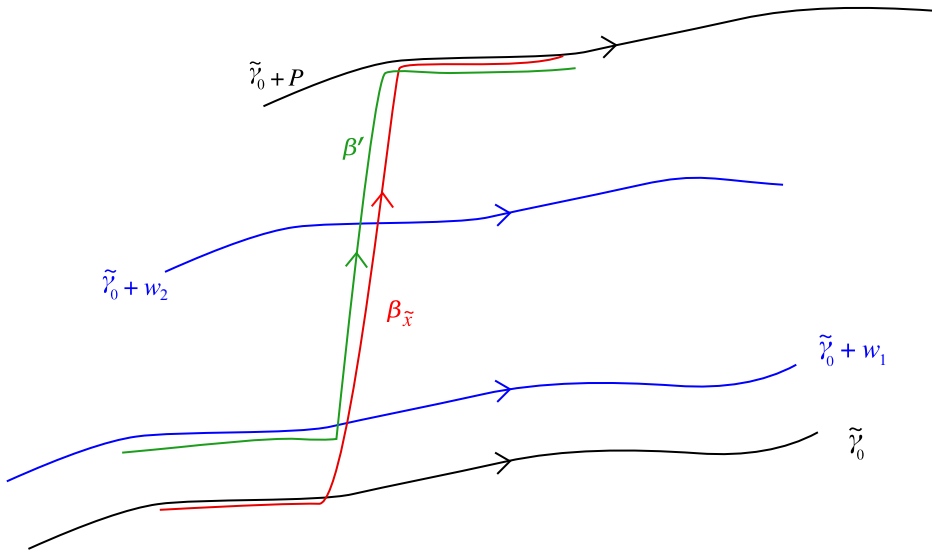


FIGURE 14. Construction of β' .

crosses ϕ_r . Similarly, since $\tilde{f}^N(\tilde{x}) \in B(\varepsilon, \tilde{z}_0 + P)$, we have that $\tilde{\gamma}_0|_{[0,t_1]} + P$ is equivalent to a subpath of $\beta_{\tilde{x}}$, and so $\beta_{\tilde{x}}$ intersects $\phi_l + P$. Let us denote by $I = [a, b]$ an interval such that $\beta_{\tilde{x}}(a)$ belongs to ϕ_r , $\beta_{\tilde{x}}(b)$ belongs to $\phi_l + P$.

We claim that, for every $w \in \mathbb{Z}^2$ such that $-(1, 0) \prec w \prec P + (1, 0)$, $(\tilde{\gamma}_0 + w)$ intersects $\beta_{\tilde{x}}|_I$ $\tilde{\mathcal{F}}$ -transversally. Let us first prove that $[\phi_l + P] \subset l(\tilde{\gamma}_0 + w)$ and $[\phi_r] \subset r(\tilde{\gamma}_0 + w)$. Let us suppose by contradiction that $[\phi_r] \not\subset r(\tilde{\gamma}_0 + w)$. We have two possibilities: $[\phi_r] \subset l(\tilde{\gamma}_0 + w)$ or $\tilde{\gamma}_0 + w$ crosses ϕ_r . If $[\phi_r] \subset l(\tilde{\gamma}_0 + w)$, we have that $[\tilde{\gamma}_0] \cap l(\tilde{\gamma}_0 + w) \neq \emptyset$. In addition, since $0 \prec -(1, 0) \prec w$, we have by Lemma 45 that $[\tilde{\gamma}_0] \cap r(\tilde{\gamma}_0 + w) \neq \emptyset$. Therefore, by Lemma 31, $\tilde{\gamma}_0 \pitchfork_{\tilde{\mathcal{F}}} \tilde{\gamma}_0 + w$, which is a contradiction, by Lemma 25. If $\tilde{\gamma}_0 + w$ crosses ϕ_r , since $[\phi_r] \subset r(\tilde{\gamma}_0 - (1, 0))$, we have $[\tilde{\gamma}_0 + w] \cap r(\tilde{\gamma}_0 - (1, 0)) \neq \emptyset$. In addition, since $-(1, 0) \prec w$, we have by Lemma 45 that $[\tilde{\gamma}_0] \cap l(\tilde{\gamma}_0 - (1, 0) - w) \neq \emptyset$, and thus $[\tilde{\gamma}_0 + w] \cap l(\tilde{\gamma}_0 - (1, 0)) \neq \emptyset$. Again, by Lemma 31, $(\tilde{\gamma}_0 - (1, 0)) \pitchfork_{\tilde{\mathcal{F}}} (\tilde{\gamma}_0 + w)$, which is a contradiction, by Lemma 25. Therefore $[\phi_r] \subset r(\tilde{\gamma}_0 + w)$. Symmetrically, using the fact that $w \prec P + (1, 0) \prec P$, we can prove that $[\phi_l + P] \subset l(\tilde{\gamma}_0 + w)$. So, by Lemma 31, $(\tilde{\gamma}_0 + w) \pitchfork_{\tilde{\mathcal{F}}} \beta_{\tilde{x}}|_I$, proving the claim. Note that, by Lemma 12, both $\tilde{\gamma}_0 + w$ and $\beta_{\tilde{x}}|_I$ are admissible.

Now let $w_1, w_2 \in \mathbb{Z}^2$ be such that $-(1, 0) \prec w_1 \prec w_2 \prec P + (1, 0)$. Since $(\tilde{\gamma}_0 + w_1) \pitchfork_{\tilde{\mathcal{F}}} \beta_{\tilde{x}}|_I$, we have that there are $t', s' \in \mathbb{R}$ such that $\tilde{\gamma}_0 + w_1$ and $\beta_{\tilde{x}}$ intersect $\tilde{\mathcal{F}}$ -transversally at $(\tilde{\gamma}_0 + w_1)(t') = \beta_{\tilde{x}}(s')$. In particular, one can find an interval $J_1 = [a_1, b_1]$ containing t' such that $(\tilde{\gamma}_0 + w_1)|_{J_1} \pitchfork_{\tilde{\mathcal{F}}} \beta_{\tilde{x}}$, and Lemma 12 and Proposition 13 show that, for any $c < a_1$, the path $\beta'_c = (\tilde{\gamma}_0 + w_1)|_{[c,t']} \beta_{\tilde{x}}|_{[s',b]}$ is admissible, see Figure 14. Note that, by Lemma 45, there exists some $c_0 < a_1$ such that $\phi_0 = \phi_{(\tilde{\gamma}_0 + w_1)(c_0)}$ is in $r(\tilde{\gamma}_0 + w_2)$. This implies that, for any $c \leq c_0$, $\beta'_c \pitchfork_{\tilde{\mathcal{F}}} \tilde{\gamma}_0 + w_2$, as it intersects both $\phi_0 \subset r(\tilde{\gamma}_0 + w_2)$ and $\phi_l + P \subset l(\tilde{\gamma}_0 + w_2)$. Let $J_2 = [a_2, b_2]$ be an interval such that $\beta'_c \pitchfork_{\tilde{\mathcal{F}}} (\tilde{\gamma}_0 + w_2)|_{J_2}$. We can, as in the proof of Lemma 45, find w_3 such that there

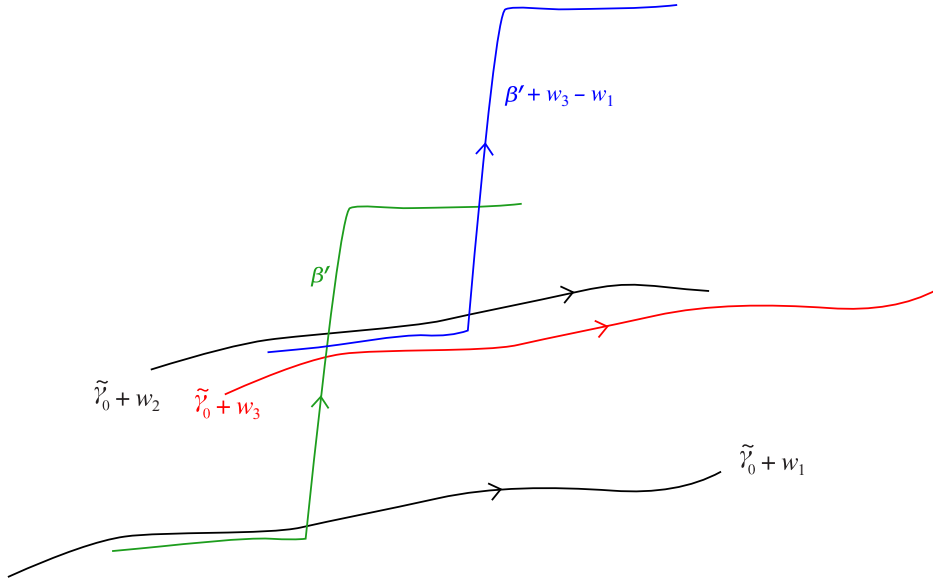


FIGURE 15. Intersection of β' with $\beta' + w_3 - w_1$.

exists some interval $J_3 = [a_3, b_3]$ with $b_3 < c_0$ such that $(\tilde{\gamma}_0 + w_3) |_{J_3}$ is equivalent to $(\tilde{\gamma}_0 + w_2) |_{J_2}$. But this implies that the path β'_{a_3} has a transverse intersection with $\beta'_{a_3} + w_3 - w_1$ (see Figure 15) since the latter has a subpath equivalent to $(\tilde{\gamma}_0 + w_1) |_{J_3} + (w_3 - w_1) = (\tilde{\gamma}_0 + w_3) |_{J_3}$, contradicting Lemma 25, and concluding the proof. \square

3.2. *Proof of Corollary B.* Assume $f : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ is isotopic to the identity, has a periodic point, let $\tilde{f} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a lift of f and assume $\rho(\tilde{f})$ is a non-degenerate line segment. Note that, as f has a periodic point, $\rho(\tilde{f})$ has at least one point in \mathbb{Q}^2 . If $\rho(\tilde{f})$ has two distinct points in \mathbb{Q}^2 , then the result follows directly from [Dav18]. So we may assume that $\rho(\tilde{f})$ has a single point with rational coordinates, and then Theorem C from [CT15] implies that there exist integers p_1, p_2, q and some vector ρ_0 such that $\rho(\tilde{f}) = \{(p_1/q, p_2/q) + t\rho_0 \mid 0 \leq t \leq q\}$. But then we can take $g = f^q$ and its lift $\tilde{g} = \tilde{f}^q - (p_1, p_2)$ and apply Theorem A to them, deducing that g has uniformly bounded deviations in the direction ρ_0^\perp . But if a power of f has uniformly bounded deviations in a given direction, f itself must also have this property, which concludes the proof of the corollary.

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