On indecomposable Bernstein Algebras*

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The main purpose of this talk is to prove that there are "many" indecomposable exceptional Bernstein algebras, an indication of the difficulty of finding all them. The next proposition shows us how to construct exceptional algebras. We recall that a Bernstein algebra is non trivial (or induced by a projection operator, in the language of [1]) if $N^2 = 0$.

Proposition 1: 1) If A is non trivial exceptional Bernstein algebra then $N = ker\ w$ is solvable of index 3.

2) Every commutative solvable algebra of index 3 is the kernel of some nontrivial exceptional Bernstein algebra.

Proof: Take a Peirce decomposition $A = Fe \oplus U_e \oplus Z_e$ of A. Then $N^2 = (U_e \oplus Z_e)^2 = = U_e Z_e + Z_e^2 \subseteq U_e$ so $(N^2)^2 \subseteq U_e^2 = 0$. Conversely, take N solvable of index 3 so $N^2 \neq 0$. Let Z be a complementary subspace of N^2 in $N: N = N^2 \oplus Z$. Define $\tau : N \to N$ by $\tau\mid_{N^2} = \frac{1}{2} id_{N^2}$ and $\tau \mid Z = 0$. Then equations $2\tau^2 = \tau$, $a^2\tau(a) = 0$, $a^2 = 4\tau(a)^2 + 2\tau(a^2)$, all $a \in N$, are satisfied. In fact if $n = u + z \in N^2 \oplus Z$ then $2\tau^2(u+z) = 2\tau(\frac{1}{2}u) = \tau(u) = \tau(u+z)$ so $2\tau^2 = \tau$. Moreover $4\tau(a)^2 + 2\tau(a^2) = 4(\frac{1}{2}u)^2 + 2\tau(u^2 + uz + z^2) = uz + z^2 = a^2$ and $a^2\tau(a) = (uz + z^2)u = 0$ as uz, z^2 and $u \in N^2$. Then $F \oplus N$ is a Bernstein algebra where, if $e = (1,0)$, then $U_e = N^2$ and $Z_e = Z$.

Proposition 2: Given integers $r \geq s \geq 1$ there exists an exceptional Bernstein algebra (A, w) of type $(1 + r, s)$ such that $N = \ker w$ is indecomposable in the variety defined by the monomial $(x^2)^2$ (hence A itself is indecomposable).

Proof: First of all, we construct a commutative solvable algebra N of dimension $r + s$, such that $dim\ N^2 = r$ and N is indecomposable. Take vector spaces U and Z freely generated by symbols $u_1, ..., u_r$ and $z_1, ..., z_s$ respectively. In the vector space $N = U \oplus Z$ define a commutative multiplication by:
$u_i z_j = u_i$ $(i = 1, ..., r;\ j = 1, ..., s)$; $z_j^2 = u_j$ $(j = 1, ..., s)$; other products are zero.

From the table, $dim\ N^2 = dim\ U = r$. For $x = u + z$, $x' = u' + z'$ $(u, u' \in U, z, z' \in Z)$ we have $xx' = uz' + zu' + zz' \in U$ so $0 \neq N^2 = U$ and $(N^2)^2 = U^2 = 0$ and N is solvable of index 3. Denote by φ the linear form on N defined by $\varphi(u_i) = 0$ $(i = 1, ..., r)$ and $\varphi(z_j) = 1$ $(j = 1, ..., s)$ so that $\varphi(x) = \varphi(u + z) = \varphi(z)$ and $ker\ \varphi = U \oplus ker(\varphi \mid Z)$. Now $xu_i = (u + z)u_i = zu_i = \varphi(z)u_i$

* Joint work with A.Suazo and M.Neuburg

so that $xu_i = 0$ if and only if $x \in ker\ \varphi$. Then $xU = 0$ if and only if $x \in ker\ \varphi$. Suppose now that N is decomposed as a direct sum of two nontrivial ideals I and J. If $IU = JU = 0$ then $I \subseteq ker\ \varphi$, $J \subseteq ker\ \varphi$ and so $N = I \oplus J \subseteq ker\ \varphi$, which is clearly impossible. So in at least one of the two ideals there is an element x such that $x\ U \neq 0$. We may assume, by symmetry, that $x \in I$. Then $\varphi(x) \neq 0$ and from $xu_i = \varphi(x)u_i \in I$ we get $u_i \in I$ and $U \subseteq I$. Let now z_j be anyone of the basis vectors of Z and decompose $z_j = a_j + b_j$, $a_j \in I$ and $b_j \in J$. We have $u_j = z_j^2 = (a_j + b_j)^2 = a_j^2 + b_j^2$ and so $b_j^2 = 0$ as $u_i \in I$. Then $z_j b_j = (a_j + b_j)b_j = a_j b_j + b_j^2 = 0$. Moreover $b_j U \subseteq I$ and also $b_j U \subseteq J$ so that $b_j U = 0$ and $\varphi(b_j) = 0$. Write $b_j = \sum_{k=1}^{r} \alpha_{jk}u_k + \sum_{l=1}^{s} \beta_{jl}z_l$, $\alpha_{jk}, \beta_{jl} \in F$. Then $b_j z_j = \sum_{k=1}^{r} \alpha_{jk}u_k z_j + \sum_{l=1}^{s} \beta_{jl}z_l z_j = \sum_{k=1}^{r} \alpha_{jk}u_k + \beta_{jj}u_j = 0$ means that $b_j = \beta_{jj}(z_j - u_j) + \sum_{l=1, l\neq j}^{s} \beta_{jl}z_l$. Now $\varphi(b_j) = \sum_{l=1}^{s} \beta_{jl} = 0$ means that $b_j = \left(\sum_{l=1,l\neq j}^{s} \beta_{jl}\right)(u_j - z_i) + \sum_{l=1,l\neq j}^{s} \beta_{jl}z_l$. For convenience of notation we denote by $b'_j = \sum_{l=1,l\neq j}^{s} \beta_{jl}z_l$ so that $b_j = \varphi(b'_j)(u_j - z_j) + b'_j$. Now $b_j^2 = \varphi(b'_j)^2(u_j - z_j)^2 + 2\varphi(b'_j)(u_j - z_i)b'_j + b'^2_j = -\varphi(b'_j)^2 u_j + 2\varphi(b'_j)^2 u_j + \sum_{l=1,l\neq j}^{s} \beta_{jl}^2 u_l = \varphi(b'_j)^2 u_j + \sum_{l=1,l\neq j}^{s} \beta_{jl}^2 u_l = 0$ implies $\beta_{jl} = 0$ $(l \neq j)$ so $b'_j = 0$ and $\varphi(b'_j) = 0$, that is $b_j = 0$. This proves that $z_j = a_j \in I$ so that $Z \subseteq I$ and N=I. This contradiction proves that N must be indecomposable in the variety defined by the monomial $(x^2)^2$. Having proved that N is solvable of index 3, we simply apply the above Proposition 1 to get an exceptional Bernstein algebra of type $(1 + r, s)$, whose kernel is indecomposable.

The above Proposition 2 exhibits examples of exceptional Bernstein algebras of type $(1 + r, s)$, $r \geq s \geq 1$, whose kernel is indecomposable. We can exhibit also examples of indecomposable exceptional Bernstein algebras of every type $(1 + r, s)$ in the following way, by modifying slightly the preceding example.

Proposition 3: Given integers $r, s \geq 1$ there exists an indecomposable exceptional Bernstein algebra of type $(1 + r, s)$.

Proof: Take a vector space U (resp. Z) freely generated by symbols $u_1, ..., u_r$ (resp. $z_1, ..., z_s$). In the vector space $U \oplus Z = N$ define a commutative multiplication by:

$$u_i z_j = u_i (i = 1, ..., r;\ j = 1, ..., s)\ ;\ z_j^2 = u_1 (j = 1, ..., s);\ \text{other products are zero}$$

As in the case of Proposition 2, we prove that N is solvable of index 3 (but now $dim Z^2 = 1$). The linear operator $\tau : N \to N$ defined by $\tau(u) = \frac{1}{2}u$, $u \in U$ and $\tau(z) = 0$, $z \in Z$ satisfies the three equations appearing in Prop. 1 so we get an exceptional Bernstein algebra of type $(1 + r, s)$, which will be indecomposable. Suppose to the contrary we have two nontrivial ideals I and J of A such that $N = I \oplus J$. Then $Z = (Z \cap I) \oplus (Z \cap J)$ so that every z_j can be decomposed as $z_j = a_j + b_j$, $a_j \in Z \cap I$ and $b_j \in Z \cap J$. Write

$a_j = \sum_{k=1}^{s} \lambda_{jk} z_k$, $b_j = \sum_{k=1}^{s} \mu_{jk} z_k$, λ_{jk} , $\mu_{jk} \in F$. Then $\lambda_{jj} + \mu_{jj} = 1$ and $\lambda_{jk} + \mu_{jk} = 0$ for $k \neq j$. Multiply a_j and b_j by any z_t $(1 \leq t \leq s)$ to get:

$$a_j z_t = \left(\sum_{k=1}^{s} \lambda_{jk} z_k \right) z_t = \lambda_{jt} u_1 \in I$$

$$b_j z_t = \left(\sum_{k=1}^{s} \mu_{jk} z_k \right) z_t = \mu_{jt} u_1 \in J$$

If both λ_{jt} and μ_{jt} were nonzero then u_1 would belong to $I \cap J$, which is impossible. Then one of then is zero, at least. This implies that $\lambda_{jk} = \mu_{jk} = 0$ for $k \neq j$ and also that $(\lambda_{jj}, \mu_{jj}) = (1,0)$ or $(\lambda_{jj}, \mu_{jj}) = (0,1)$. Suppose for instance $(\lambda_{jj}, \mu_{jj}) = (1,0)$. Then $b_j = 0$ and $a_j = z_j$, that is, $z_j \in I$. But now $u_i z_j = u_i \in I$ and $U \subseteq I$. (If we had assumed $(\lambda_{jj}, \mu_{jj}) = (0,1)$ then z_j would be in J). Suppose now some basis vector $z_l \in J$. Then $u_i z_l = u_i \in J$ so $U \subseteq J$, a contradiction to $I \cap J = 0$. Hence all z_i belong to the same ideal, say I. This means $U \oplus Z \subseteq I$, a final contradiction.

Remark: In [2] the authors have given an example of a indecomposable Bernstein algebra A such A^2 is decomposable. Incidentally we have found many more examples now. Let A be an exceptional Bernstein algebra of type $(1+r, s)$ where $r \geq 2$. If $A = Fe \oplus U_e \oplus Z_e$ then $A^2 = Fe \oplus U_e$ is isomorphic to the gametic algebra $G(r+1, 2)$ which is decomposed as $G(2,2) \vee G(2,2) \vee ... \vee G(2,2)$. If A is exceptional of type $(2, s)$ the A^2 is isomorphic to $G(2,2)$, which is indecomposable. The existence of indecomposable exceptional algebras of every type is ensured by Proposition 3. So in the class of exceptional algebras all the four possibilities concerning decomposability of A and A^2 effectively happen. In this connection, we have the following general result. Recall that for $A = Fe \oplus U_e \oplus Z_e$, $A^2 = Fe \oplus U_e \oplus U_e^2$ so ker $(\omega \mid_{A^2}) = U_e \oplus U_e^2$.

Proposition 4: Let $A = Fe \oplus N = Fe \oplus U_e \oplus Z_e$ be a decomposable Bernstein algebra such that A^2 is indecomposable. If I and J are nontrivial ideals of A such that $I \oplus J = N$ then necessarily ker $(\omega \mid_{A^2}) \subseteq I$ or ker $(\omega \mid_{A^2}) \subseteq J$.

Proof: As $U_e = (U_e \cap I) \oplus (U_e \cap J)$ we have $ker(\omega \mid_{A^2}) = U_e \oplus U_e^2 = ((U_e \cap I) \oplus (U_e \cap I)^2) \oplus ((U_e \cap J) \oplus (U_e \cap J)^2)$. It is easily seen that both direct summands are ideals of A^2. Then necessarily one of them is zero and the other equals ker $(\omega \mid_{A^2})$. If say $(U_e \cap J) \oplus (U_e \cap J)^2 =$ ker $(\omega \mid_{A^2})$ then ker $(\omega \mid_{A^2}) \subseteq J$.

References

[1] Yuri I. Ljubich: Mathematical Structures in Populations Genetics, Biomathematics, vol. 22, Springer-Verlag, 1992.

[2] R.Costa and H.Guzzo Jr.: Indecomposable baric algebras, Linear Algebra and its Applications, 183:223-236 (1993)