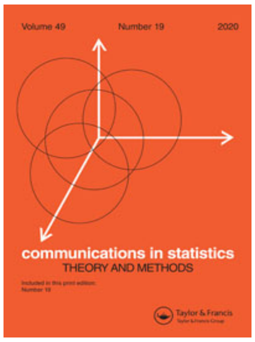

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Juan M. Astorga[a], Yuri A. Iriarte[b], Héctor W. Gómez[b], and Heleno Bolfarine[c]

[a]Departamento de Tecnologías de la Energía, Facultad Tecnológica, Universidad de Atacama, Copiapó, Chile; [b]Departamento de Matemáticas, Facultad de Ciencias Básicas, Universidad de Antofagasta, Antofagasta, Chile; [c]Departamento de Estatística, IME, Universidad de Sao Paulo, Sao Paulo, Brasil

ABSTRACT

In this article, we introduce a new distribution for modeling positive data sets with high kurtosis, the modified slashed generalized exponential distribution. The new model can be seen as a modified version of the slashed generalized exponential distribution. It arises as a quotient of two independent random variables, one being a generalized exponential distribution in the numerator and a power of the exponential distribution in the denominator. We studied various structural properties (such as the stochastic representation, density function, hazard rate function and moments) and discuss moment and maximum likelihood estimating approaches. Two real data sets are considered in which the utility of the new model in the analysis with high kurtosis is illustrated.

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Generalized exponential distribution; kurtosis; maximum likelihood estimator; slashed generalized exponential distribution

1. Introduction

The generalized exponential (GE) distribution has proved useful in many applications, including reliability, life testing, and survival analysis. This model was studied in Gupta, Gupta, and Gupta (1998), which is a particular case of the exponentiated Weibull distribution (Mudholkar and Srivastava 1993; Mudholkar, Srivastava, and Freimer 1995) with zero location. It is said that a random variable X follows a GE distribution if its probability density function (pdf) is given by

$$f(x;\lambda,\alpha) = \alpha\lambda\left(1-e^{-\lambda x}\right)^{\alpha-1}e^{-\lambda x}, \qquad x>0 \tag{1}$$

where $\lambda>0$ is a scale parameter and $\alpha>0$ is a shape parameter, hereafter denoted $X\sim\mathrm{GE}(\lambda,\alpha)$. When $\alpha=1$, the GE distribution reduces to the classical exponential distribution. For more details on the mathematical properties of the GE distribution, see Gupta and Kundu (1999, 2001a, 2001b, 2002, 2007).

Recently, and following the approach proposed in Gómez, Quintana, and Torres (2007) and Gómez and Venegas (2008), Astorga, Gómez, and Bolfarine (2017) introduced a three-parameter extension of the GE distribution named the slashed generalized exponential (SGE) distribution. A random variable X follows a SGE distribution, denoted as $X\sim\mathrm{SGE}(\lambda,\alpha,\mathrm{q})$, if it can be represented as

CONTACT Héctor W. Gómez hector.gomez@uantof.cl Departamento de Matemáticas, Facultad de Ciencias Básicas, Universidad de Antofagasta, Antofagasta, Chile.

$$X = \frac{Y}{U} \tag{2}$$

where $Y \sim \mathrm{GE}(\lambda, \alpha)$ and $U \sim \mathrm{Beta}(q, 1)$ are independent.

The pdf associated with Equation (2) is given by

$$f_X(x; \lambda, \alpha, q) = \frac{\alpha q}{\lambda^q x^{q+1}} J_{(\alpha,q)}\left(1 - e^{-\lambda x}\right), \qquad x>0$$

where $\lambda>0$ is a scale parameter, $\alpha>0$ is a shape parameter, $q>0$ is a kurtosis parameter and $J_{(\alpha,q)}(t) = \int_0^t \log^q\left(\frac{1}{1-u}\right) u^{\alpha-1} du$. If $q \to \infty$, the SGE distribution tends to the ordinary GE distribution. The SGE distribution has a wider kurtosis range than the GE distribution, being appropriate to fit data sets with atypical observations.

In this work, we propose a modified version of the SGE distributions which can be used as an alternative model to those where GE and SGE distributions have been used. This new distribution arises as a quotient of two independent random variables, one being a GE distribution in the numerator and a power of the exponential (Exp) distribution in the denominator. The resulting model allows accommodating data with higher kurtosis, among other advantages. We follow the idea proposed by Reyes, Gómez, and Bolfarine (2013), who presented an extension of the normal distribution, which they called modified slashed (MS) distribution. A random variable X follows a MS distribution, denoted as $X \sim \mathrm{MS}(q)$, if it can be represented as

$$X = \frac{Y}{U^{1/q}}, \qquad q>0 \tag{3}$$

where $Y \sim \mathrm{N}(0, 1)$ and $U \sim \mathrm{Exp}\,(2)$ are independent. When $q \to \infty$, the MS distribution tends to the normal distribution. This MS distribution presents heavier tails than the classical slash distribution, that is, more kurtosis.

The paper is organized as follows. Section 2 deals with the stochastic representation for the new distribution and studies its moments and asymmetry and kurtosis coefficients. In Section 3, we discuss moment and maximum likelihood estimators for the parameters of the new model. In addition, we perform a small scale simulation study for illustrating the performance of the maximum likelihood estimator. The main conclusion is that the approach yields good parameter recovery. Section 4 is dedicated to presenting the analysis of two real data set illustrating the performance of the proposed methodology. Final remarks and conclusions are referred to Section 5.

2. The model

In this section, we present some structural properties of the new model such as the stochastic representation, density function, hazard rate function and distributional moments.

2.1. Stochastic representation

A random variable X follows a modified slashed generalized exponential (MSGE) distribution, denoted as $X \sim \mathrm{MSGE}(\lambda, \alpha, q)$, if it can be represented as

$$X = \frac{Y}{U^{1/q}}, \qquad q>0 \tag{4}$$

where $Y \sim \mathrm{GE}(\lambda, \alpha)$ and $U \sim \mathrm{Exp}\,(2)$ are independent.

Remark 1.

i. *Note that if $q \to \infty$ then $X \to Y$.*
ii. *When $q = 1$ we say that X follows a canonical modified slashed generalized exponential distribution, represented as $X = YU^{-1}$ where $Y \sim \mathrm{GE}(\lambda, \alpha)$ and $U \sim \mathrm{Exp}\,(2)$ are independent. This is denoted as $X \sim \mathrm{CMSGE}(\lambda, \alpha)$.*

2.2. Density function

Proposition 2.1. *Let $X \sim \mathrm{MSGE}(\lambda, \alpha, q)$. Then, the pdf of X is given by*

$$f_X(x; \lambda, \alpha, q) = \frac{2q\alpha}{\lambda^q} x^{-(q+1)} J_X(x; \lambda, \alpha, q), \qquad x>0 \tag{5}$$

where $\lambda>0$ is a scale parameter, $\alpha>0$ is a shape parameter, $q > 0$ is a kurtosis parameter and

$$J_X(x; \lambda, \alpha, q) = \int_0^1 [-\log(1-u)]^q u^{\alpha-1} \exp\left[-\left(\frac{-\log(1-u)}{\lambda x}\right)^q\right] du \tag{6}$$

Proof. Using the representation Equation (4), and procedures based on the Jacobian method, the pdf of X is given by

$$f_X(x; \lambda, \alpha, q) = \int_0^\infty 2q\lambda\alpha w^q e^{-(\lambda x w + w^q)} (1 - e^{-\lambda x w})^{\alpha-1}\, dw$$

and considering the change of variable $u = (1-e^{-\lambda x w})$ the result is obtained. □

Remark 2. *Let $X \sim \mathrm{CMSGE}(\lambda, \alpha)$, then the pdf of X is given by*

$$f_X(x; \lambda, \alpha) = \frac{2\alpha}{\lambda} x^{-2} J_X(x; \lambda, \alpha, 1), \qquad x>0$$

where $\lambda>0$ is a scale parameter, $\alpha>0$ is a shape parameter and J_X is as in Equation (6).

2.3. Some properties

In this subsection we present some basic properties of the MSGE distribution.

Let $X \sim \mathrm{MSGE}(\lambda, \alpha, q)$, then

1. $\lim_{q\to\infty} f_X(x; \lambda, \alpha, q) = \alpha\lambda(1 - e^{-\lambda x})^{\alpha-1} e^{-\lambda x}$
2. $\lim_{q\to\infty} f_X(x; \lambda, 1, q) = \lambda e^{-\lambda x}$
3. $F_X(x; \lambda, \alpha, q) = \frac{2q\alpha}{\lambda^q} \int_0^x u^{-(q+1)} J_X(u; \lambda, \alpha, q)\; du$

Remark 3. Property 1 reveals that as $q \to \infty$ the MSGE distribution converges to the usual GE distribution. Property 2 reveals that as $q \to \infty$ (when $\alpha = 1$) the MSGE

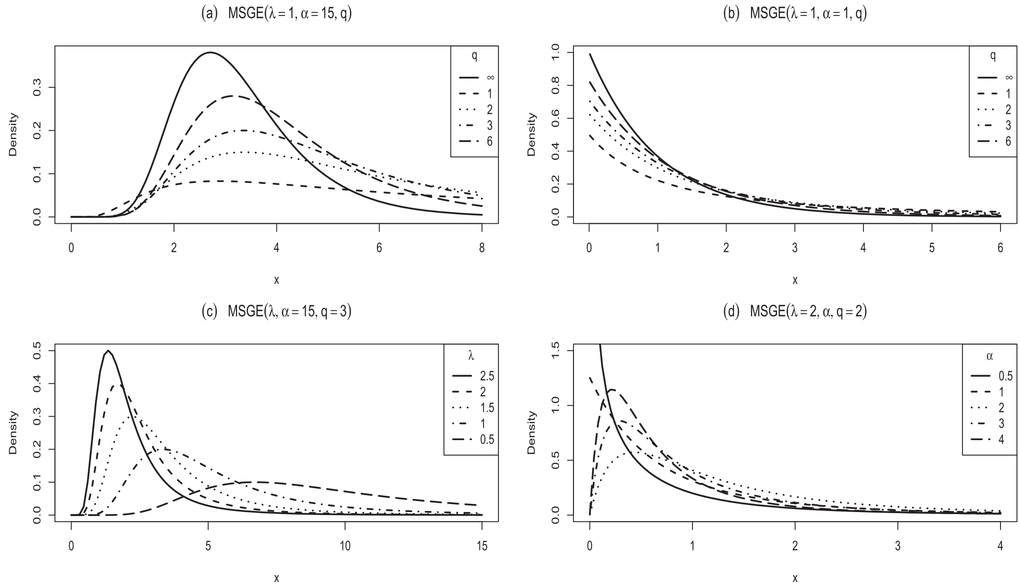

Figure 1. Plot of the MSGE distribution for different values of its parameters.

distribution converges to the classical exponential distribution (Johnson, Kotz, and Balakrishnan 1995). The MSGE density can present the unimodal or non-increasing shapes. Figure 1 shows some shapes of this function for some values of its parameters. It is revealed that parameter λ is a scale type parameter, see part (c) of Figure 1. Parameter α is a shape parameter, according to part (d) of Figure 1. The parameter q is the kurtosis parameter, according to the tails of the distribution given in parts (a) and (b) of Figure 1.

2.4. Reliability analysis

The reliability function $R_T(t)$, which is the probability of an item not failing prior to some time t, is defined by $R_T(t) = 1 - F_T(t)$. The reliability function of a MSGE distribution is given by

$$R_X(t; \lambda, \alpha, q) = 1 - \frac{2q\alpha}{\lambda^q} \int_0^t \frac{J_X(u; \lambda, \alpha, q)}{u^{q+1}} \, du \tag{7}$$

where J_X is given in Equation (6). An interesting characteristic of a random variable is its hazard rate function defined by $h_T(t) = \frac{f_T(t)}{1-F_T(t)}$ which is an important quantity characterizing the life-time of a certain phenomenon. It can be loosely interpreted as the conditional probability of failure at time t, given it has survived to time t. The hazard rate function for a MSGE random variable is given by

$$h_X(t; \lambda, \alpha, q) = \frac{J_X(t; \lambda, \alpha, q)}{t^{q+1}} \left(\frac{\lambda^q}{2q\alpha} - \int_0^t \frac{J_X(u; \lambda, \alpha, q)}{u^{q+1}} \, du \right)^{-1} \tag{8}$$

where J_X is given in Equation (6). Figure 2 displays some plots of the hazard rate function of a MSGE (λ, α, q) distribution for different values of its parameters.

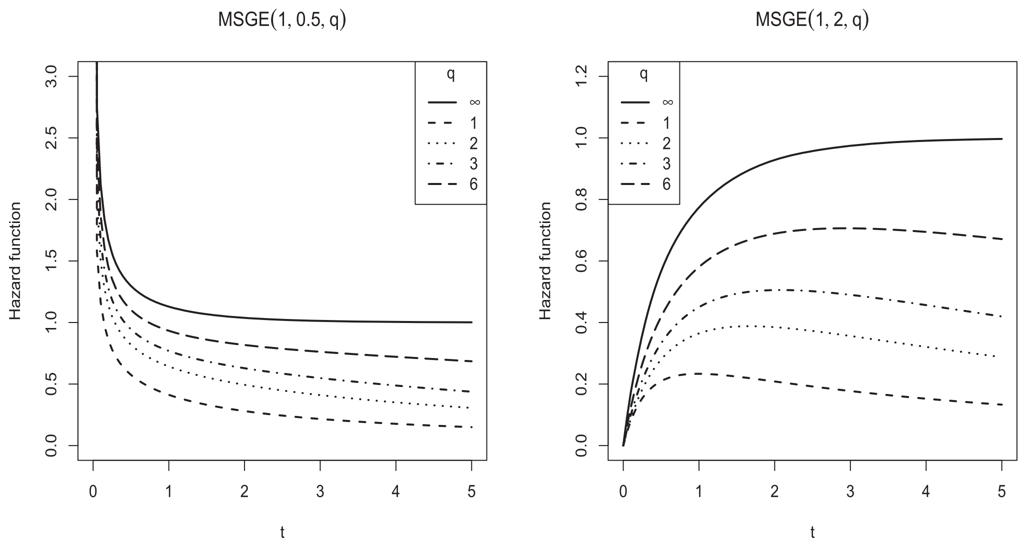

Figure 2. Plots of hazard rate function of a MSGE(λ, α, q) distribution.

2.5. Moments

Proposition 2.2. *Let $X \sim \text{MSGE}(\lambda, \alpha, q)$. Then, for $r = 1, 2, \ldots,$ and $q > r$ the r-th moment of X is given by*

$$\mu_r = E(X^r) = 2^{\frac{r}{q}} \Gamma\left(\frac{q-r}{q}\right) \frac{\alpha \Gamma(r+1)}{\lambda^r} \sum_{i=0}^{\infty} (-1)^i \frac{c(\alpha-1, i)}{(i+1)^{r+1}} \tag{9}$$

where $c(\alpha-1, i) = [(\alpha-1) \times \cdots \times (\alpha - i)]/i!$.

Proof. Using the stochastic representation for the distribution given in Equation (4), we have that

$$\mu_r = E(X^r) = E\left(\left(\frac{Y}{U^{1/q}}\right)^r\right) = E\left(Y^r U^{-r/q}\right) = E(Y^r) E\left(U^{-r/q}\right)$$

where $E(U^{-r/q}) = 2^{\frac{r}{q}} \Gamma(\frac{q-r}{q})$, $q>r$ and $E(Y^r) = \frac{\alpha \Gamma(r+1)}{\lambda^r} \sum_{i=0}^{\infty} (-1)^i \frac{c(\alpha-1,i)}{(i+1)^{r+1}}$ are the moments of the GE(λ, α) distribution. □

Corollary 2.1 . *Let $X \sim \text{MSGE}(\lambda, \alpha, q)$. Then, it follows that*

1. $\mu_1 = E(X) = 2^{\frac{1}{q}} \Gamma(\frac{q-1}{q}) \frac{d_1}{\lambda}, \ q>1$
2. $\mu_2 = E(X^2) = 2^{\frac{2}{q}} \Gamma(\frac{q-2}{q}) \frac{(d_2 + d_1^2)}{\lambda^2}, \ q>2$
3. $\mu_3 = E(X^3) = 2^{\frac{3}{q}} \Gamma(\frac{q-3}{q}) \frac{(d_3 + 3d_1 d_2 + d_1^3)}{\lambda^3}, \ q>3$
4. $\mu_4 = E(X^4) = 2^{\frac{4}{q}} \Gamma(\frac{q-4}{q}) \frac{(d_4 + 3d_2^2 + 4d_1 d_3 + 6d_1^2 d_2 + d_1^4)}{\lambda^4}, \ q>4$
5. $Var(X) = \frac{2^{\frac{2}{q}}}{\lambda^2} \left[\Gamma(\frac{q-2}{q})(d_2 + d_1^2) - \Gamma^2(\frac{q-1}{q}) d_1^2\right], \ q>2$

where

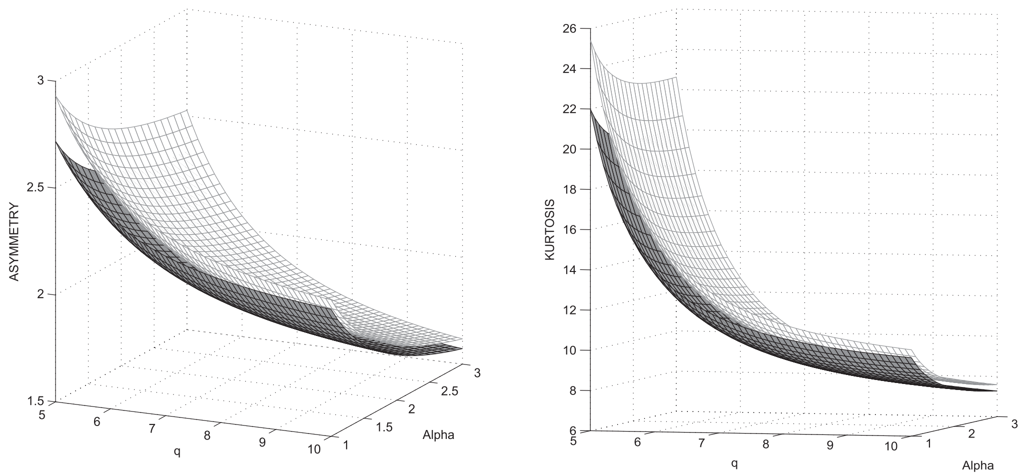

Figure 3. Plots of asymmetry and kurtosis coefficients for the SGE (lower grid) and MSGE (upper grid) models.

$d_1 = \psi(\alpha+1) - \psi(1)$, $d_2 = \psi'(1) - \psi'(\alpha+1)$, $d_3 = \psi''(\alpha+1) - \psi''(1)$, $d_4 = \psi'''(1) - \psi'''(\alpha+1)$ and $\psi^m(x)$, is the polygamma function of order m.

Corollary 2.2. *Let $X \sim \text{MSGE}(\lambda, \alpha, q)$. Then, the asymmetry and kurtosis coefficients, denoted by β_1 and β_2, are given as below*

$$\beta_1 = \frac{\Gamma\left(\frac{q-3}{q}\right)\left(d_3 + 3d_1d_2 + d_1^3\right) - 3\Gamma\left(\frac{q-1}{q}\right)\Gamma\left(\frac{q-2}{q}\right)\left(d_1d_2 + d_1^3\right) + 2\Gamma^2\left(\frac{q-1}{q}\right)d_1^3}{\left[\Gamma\left(\frac{q-2}{q}\right)\left(d_2 + d_1^2\right) - \Gamma^2\left(\frac{q-1}{q}\right)d_1^2\right]^{\frac{3}{2}}}, \quad q>3$$

and

$$\beta_2 = \frac{K_1}{\left[\Gamma\left(\frac{q-2}{q}\right)\left(d_2 + d_1^2\right) - \Gamma^2\left(\frac{q-1}{q}\right)d_1^2\right]^2}, \quad q>4$$

where

$$\begin{aligned} K_1 = &\Gamma\left(\frac{q-4}{q}\right)\left(d_4 + 3d_2^2 + 4d_1d_3 + 6d_1^2d_2 + d_1^4\right) - 4\Gamma\left(\frac{q-1}{q}\right)\Gamma\left(\frac{q-3}{q}\right)\left(d_1d_3 + 3d_1^2d_2 + d_1^4\right) \\ &+ 6\Gamma^2\left(\frac{q-1}{q}\right)\Gamma\left(\frac{q-2}{q}\right)\left(d_1^2d_2 + d_1^4\right) - 3\Gamma^4\left(\frac{q-1}{q}\right)d_1^4 \end{aligned}$$

Remark 4. The asymmetry and kurtosis coefficients were obtained using

$$\beta_1 = \frac{\mu_3 - 3\mu_1\mu_2 + 2\mu_1^3}{\left(\mu_2 - \mu_1^2\right)^{\frac{3}{2}}} \qquad \textit{and} \qquad \beta_2 = \frac{\mu_4 - 4\mu_1\mu_3 + 6\mu_2\mu_1^2 - 3\mu_1^4}{\left(\mu_2 - \mu_1^2\right)^2}$$

Figure 3 depicts plots for the asymmetry and kurtosis coefficients of the SGE and MSGE distributions, respectively. Notice that asymmetry and kurtosis coefficients are larger in MSGE model than in SGE model. Also, these coefficients do not depend on λ

because it is a scale parameter. Since α is a shape parameter, it is clear that it has an effect on both the asymmetry and kurtosis of the model. This is observed in Corollary 2.2. On the other hand, the kurtosis of the MSGE distribution is controlled mainly by the parameter q. This fact is clearly seen in Figure 3.

3. Inference

In this section, we discuss moments and maximum likelihood estimation for parameters of the MSGE (λ, α, q) distribution. First, we present the moment approach.

3.1. Moment estimators

Proposition 3.1. *For a random sample $X_1, ..., X_n$ from the distribution* $\mathrm{MSGE}(\lambda, \alpha, q)$, *the moment estimators for (α, q) are given by the roots of the following system of equations*

$$\overline{X^2}\Gamma^2\left(\frac{\hat{q}_M-1}{\hat{q}_M}\right)d_1^2 - \bar{X}^2\Gamma\left(\frac{\hat{q}_M-2}{\hat{q}_M}\right)\left(d_2+d_1^2\right) = 0 \tag{10}$$

$$\overline{X^3}\Gamma^3\left(\frac{\hat{q}_M-1}{\hat{q}_M}\right)d_1^3 - \bar{X}^3\Gamma\left(\frac{\hat{q}_M-3}{\hat{q}_M}\right)\left(d_3+3d_1d_2+d_1^3\right) = 0 \tag{11}$$

and the moment estimator for λ is given by

$$\hat{\lambda}_M = \frac{2^{1/q}}{\bar{X}}\Gamma\left(\frac{\hat{q}_M-1}{\hat{q}_M}\right)d_1 \tag{12}$$

where

$$d_1 = \psi(\alpha+1) - \psi(1)$$
$$d_2 = \psi'(1) - \psi'(\alpha+1)$$
$$d_3 = \psi''(\alpha+1) - \psi''(1)$$

where $\psi^m(x)$ is the poligamma function of order m.

Proof. From Corollary 2.1, and considering the first three equations in the moments approach, it follows that

$$\bar{X} = 2^{1/\hat{q}_M}\Gamma\left(\frac{\hat{q}_M-1}{\hat{q}_M}\right)\frac{d_1}{\hat{\lambda}_M}, \quad \overline{X^2} = 2^{2/\hat{q}_M}\Gamma\left(\frac{\hat{q}_M-2}{\hat{q}_M}\right)\frac{d_2+d_1^2}{\hat{\lambda}_M^2}$$

$$\overline{X^3} = 2^{3/\hat{q}_M}\Gamma\left(\frac{\hat{q}_M-3}{\hat{q}_M}\right)\frac{d_3+3d_1d_2+d_1^3}{\hat{\lambda}_M^3}$$

Now, considering the above equations:

a. by solving the first equation for λ we obtain $\hat{\lambda}_M$ given in Equation (12).
b. by solving the first and second equations for λ, and equating the resulting expressions, it is obtained the first equation in Equation (10).

c. by solving the first and third equations for λ, and equating the resulting expressions, we obtain the second equation in Equation (11). □

The system of equation given in Equations (10) and (11) can be solved using numerical procedures such as the Newton-Raphson procedure. An alternative is to use the NumDeriv routine with the R software (R Core Team 2014).

3.2. Maximum likelihood estimators

For a random sample $X_1, \ldots, X_n$ from the distribution MSGE(λ, α, q), the log-likelihood function can be written as

$$l(\lambda, \alpha, q) = c(\lambda, \alpha, q) - (q+1) \sum_{i=1}^{n} \log(x_i) + \sum_{i=1}^{n} \log J(x_i) \tag{13}$$

where $c(\lambda, \alpha, q) = n \log(2) + n \log(q) + n \log(\alpha) - nq \log(\lambda)$ and $J_X(x) = J_X(x; \lambda, \alpha, q)$ is defined in Equation (6). The maximum likelihood (ML) estimators are obtained by maximizing the log-likelihood function given in Equation (13). Deriving the log-likelihood function with respect to each parameter, the following estimating equations are obtained:

$$-\frac{nq}{\lambda} + \sum_{i=1}^{n} \frac{J_1(x_i)}{J(x_i)} = 0 \tag{14}$$

$$\frac{n}{\alpha} + \sum_{i=1}^{n} \frac{J_2(x_i)}{J(x_i)} = 0 \tag{15}$$

$$\frac{n}{q} - n \log(\lambda) - \sum_{i=1}^{n} \log(x_i) + \sum_{i=1}^{n} \frac{J_3(x_i)}{J(x_i)} = 0 \tag{16}$$

where $J(x_i) = J_X(x_i; \lambda, \alpha, q)$, $J_1(x_i) = \frac{\partial}{\partial \lambda} J(x_i)$, $J_2(x_i) = \frac{\partial}{\partial \alpha} J(x_i)$ and $J_3(x_i) = \frac{\partial}{\partial q} J(x_i)$. Solutions for Equations (14–16) can be obtained by using numerical procedures such as the Newton-Raphson procedure. An alternative is to use the optim routine with the R software (R Core Team 2014).

3.3. The observed information matrix

Let $X \sim$MSGE(λ, α, q), then the observed information matrix is given by

$$I_n(\lambda, \alpha, q) = \begin{pmatrix} a_{\lambda\lambda} & a_{\lambda\alpha} & a_{\lambda q} \\ & a_{\alpha\alpha} & a_{\alpha q} \\ & & a_{qq} \end{pmatrix}$$

such that

$$a_{\lambda\lambda} = \frac{nq}{\lambda^2} + \sum_{i=1}^{n} \frac{J_{11}(x_i)}{J(x_i)} - \sum_{i=1}^{n} \left(\frac{J_1(x_i)}{J(x_i)} \right)^2$$

$$a_{\lambda\alpha} = \sum_{i=1}^{n} \frac{J_{12}(x_i)}{J(x_i)} - \sum_{i=1}^{n} \frac{J_1(x_i)J_2(x_i)}{(J(x_i))^2}$$

$$a_{\lambda q} = -\frac{n}{\lambda} + \sum_{i=1}^{n} \frac{J_{13}(x_i)}{J(x_i)} - \sum_{i=1}^{n} \frac{J_1(x_i)J_3(x_i)}{(J(x_i))^2}$$

$$a_{\alpha\alpha} = -\frac{n}{\alpha^2} + \sum_{i=1}^{n} \frac{J_{22}(x_i)}{J(x_i)} - \sum_{i=1}^{n} \left(\frac{J_2(x_i)}{J(x_i)} \right)^2$$

$$a_{\alpha q} = \sum_{i=1}^{n} \frac{J_{23}(x_i)}{J(x_i)} - \sum_{i=1}^{n} \frac{J_2(x_i)J_3(x_i)}{(J(x_i))^2}$$

$$a_{qq} = -\frac{n}{q} + \sum_{i=1}^{n} \frac{J_3(x_i)}{J(x_i)} - \sum_{i=1}^{n} \left(\frac{J_3(x_i)}{J(x_i)} \right)^2$$

where $J_k(x_i)$, with $k = 1, 2, 3$, are given in subsection 3.2 and

$$J_{11}(x) = \frac{\partial J_1(x_i)}{\partial \lambda}, \quad J_{12}(x) = \frac{\partial J_1(x_i)}{\partial \alpha}, \quad J_{13}(x) = \frac{\partial J_1(x_i)}{\partial q}$$
$$J_{22}(x) = \frac{\partial J_2(x_i)}{\partial \alpha}, \quad J_{23}(x) = \frac{\partial J_2(x_i)}{\partial q}, \quad J_{33}(x) = \frac{\partial J_3(x_i)}{\partial q}$$

3.4. Simulation study

In this section a simulation study is performed to illustrate the behavior of the ML estimates for parameters λ, α and q. We generate 1000 random samples of sizes $n = 50$, 150 and 300 from the MSGE distribution for fixed values of the parameters. Random numbers $X \sim \text{MSGE}(\lambda, \alpha, q)$ can be generated as follows:

Step 1. Generate $Y \sim \text{GE}(\lambda, \alpha)$.
Step 2. Generate $U \sim \text{Exp}\,(2)$.
Step 3. Compute $X = YU^{-1/q}$.

The estimates can be obtained using the optimization method based on Nelder-Mead, quasi-Newton and conjugate-gradient algorithms and implemented in the statistical package R. Measures and empirical standard deviations are presented in Table 1.

The main conclusion is that as sample size increases, estimates become closer to the true parameter values. Further, results indicate that estimated standard errors become smaller as sample size increases.

4. Real data illustration

In this section, we present two illustrations using real data sets aiming at comparing in terms of model fitting the MSGE, SGE and GE models. Comparisons are made using

Table 1. ML estimates computed using generated samples of sizes 50, 150 and 300 for different values of λ, α and q.

$n = 50$					
λ	α	q	$\hat{\lambda}$ (SD)	$\hat{\alpha}$ (SD)	$\hat{q}$ (SD)
1.0	0.5	2	1.1513 (0.4012)	0.5113 (0.0812)	2.2192 (0.8812)
		3	1.1877 (0.4213)	0.5214 (0.0776)	2.9539 (1.0301)
		4	1.1491 (0.3902)	0.5288 (0.0715)	3.9071 (1.3486)
3.0	2.0	2	3.3714 (1.0921)	2.1508 (0.6012)	2.2261 (0.8054)
		3	3.1561 (0.9517)	2.1612 (0.5101)	3.4811 (1.5703)
		4	3.1326 (0.9115)	2.1719 (0.5066)	4.3004 (1.8381)
5.0	3.0	2	5.2816 (1.4811)	3.3006 (1.5384)	2.1391 (0.5219)
		3	5.1762 (1.4452)	3.2791 (1.1561)	3.4842 (1.2873)
		4	5.1508 (1.3972)	3.2549 (1.0817)	4.3672 (1.5715)
$n = 150$					
λ	α	q	$\hat{\lambda}$ (SD)	$\hat{\alpha}$ (SD)	$\hat{q}$ (SD)
1.0	0.5	2	1.0419 (0.3745)	0.5067 (0.0606)	2.1947 (0.7921)
		3	1.0306 (0.3395)	0.5138 (0.0564)	2.9719 (0.9400)
		4	1.0807 (0.3416)	0.5131 (0.0569)	3.9155 (1.1217)
3.0	2.0	2	3.1112 (0.8515)	2.1331 (0.5696)	2.1497 (0.7051)
		3	3.1007 (0.8187)	2.1188 (0.4835)	3.4407 (1.5446)
		4	3.0895 (0.7831)	2.1020 (0.4338)	4.2881 (1.6819)
5.0	3.0	2	5.2251 (1.4263)	3.2561 (1.1213)	2.0778 (0.4745)
		3	5.1046 (1.3154)	3.2286 (1.0607)	3.4278 (1.1547)
		4	5.1354 (1.1718)	3.2077 (0.8593)	4.3154 (1.4912)
$n = 300$					
λ	α	q	$\hat{\lambda}$ (SD)	$\hat{\alpha}$ (SD)	$\hat{q}$ (SD)
1.0	0.5	2	1.0285 (0.2917)	0.5011 (0.0278)	2.1191 (0.4590)
		3	1.0161 (0.2755)	0.5071 (0.0327)	3.0171 (0.4571)
		4	1.0421 (0.1896)	0.5098 (0.0441)	3.9867 (0.8971)
3.0	2.0	2	3.0427 (0.5565)	2.0527 (0.3263)	2.0615 (0.3030)
		3	3.0452 (0.4971)	2.0549 (0.3129)	3.1035 (0.5537)
		4	3.0276 (0.3826)	2.4760 (0.3517)	4.1529 (0.8255)
5.0	3.0	2	5.1658 (1.0382)	3.1212 (0.8639)	2.0527 (0.3826)
		3	5.0741 (0.8714)	3.1590 (0.7746)	3.2573 (0.8146)
		4	5.0504 (0.9057)	3.1008 (0.6931)	4.1797 (0.9367)

Table 2. Descriptive statistics for the failure data.

n	$\bar{x}$	s^2	b_1	b_2
188	92.074	107.916	2.139	8.023

the Kolmogorov-Smirnov Goodness-of-fit test and the likelihood approach, namely the AIC and BIC type criteria.

4.1. Number of failures of a conditioning system

The first data set represents the number of successive failures for the air conditioning system for each member in a fleet of 13 Boeing 720 jet airplanes (Proschan 1963). Table 2 presents summary statistics for the data set where b_1 and b_2 correspond to the sample asymmetry and kurtosis coefficients, respectively.

Computing initially the moment estimators under the MSGE model we have the following estimates: $\hat{\lambda}_M = 0.024, \hat{\alpha}_M = 1.174$ and $\hat{q}_M = 2.277$. Using the moment estimators as initial values, the ML estimates are computed and presented in Table 3 with the standard errors in parentheses. For each model we report the log-likelihood estimated values.

Table 3. ML estimates with its respective standard error (SE) for fitting models on the failure data.

Parameters	GE (SE)	SGE (SE)	MSGE (SE)
$\hat{\lambda}$	0.010 (0.001)	0.016 (0.008)	0.024 (0.006)
$\hat{\alpha}$	0.910 (0.086)	1.014 (0.194)	1.175 (0.175)
$\hat{q}$	–	2.995 (2.903)	2.277 (0.539)
Log-likelihood	−1037.750	−1036.385	**−1035.319**

Table 4. AIC, BIC, KSS, and *p*-values for each fitted model.

Criterion	GE	SGE	MSGE
AIC	2079.500	2078.770	**2076.638**
BIC	2082.297	2082.966	**2080.834**
KSS	0.080	0.075	**0.059**
P-values	0.588	0.674	**0.904**

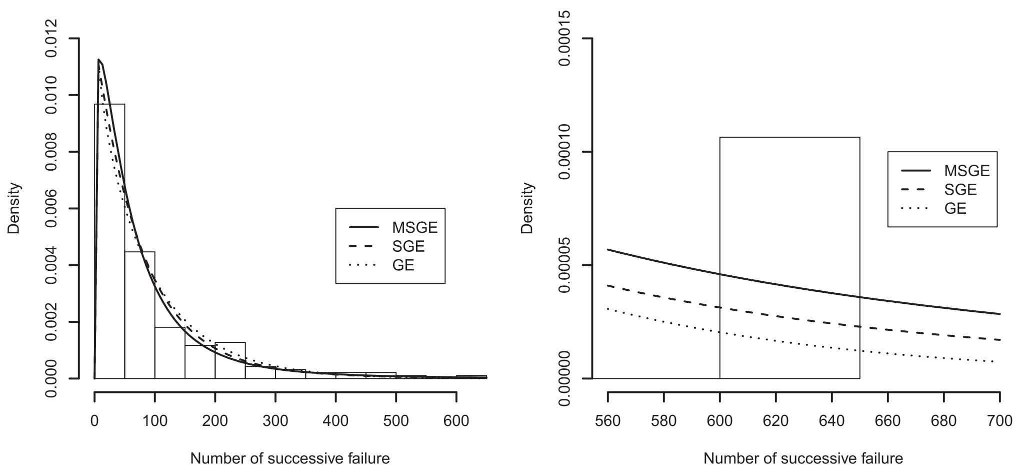

Figure 4. Left panel: models fitted by the maximum likelihood approach for failure data set. Right panel: Plots of the tails for models.

In order to compare the distributions, we computed the usual Akaike information criterion (AIC) (Akaike 1974) and Bayesian information criterion (BIC) (Schwarz 1978). It is known that AIC $= 2k - 2\log lik$ and BIC $= k[\log(n) - \log(2\pi)] - 2\log lik$ where k is the number of parameters in the model, n is the sample size and $\log lik$ is the maximized value of the log-likelihood function. In addition, we applied the goodness-of-fit test the Kolmogorov-Smirnov statistic (KSS). Table 4 shows the corresponding AIC, BIC, KSS, and p-values for each model. For these data, based on the AIC, BIC, KSS, and p-values, the MSGE model presents better fitting values than the ones for the other models. Figure 4 depicts plots of the fitted GE, SGE and MSGE models using the ML estimates. Notice that the fitted MSGE model present heavier tails than the SGE and GE models. The QQ-plots for the SGE and MSGE models are presented in Figure 5.

4.2. Time failures related to pascal programing

The data considered in this illustration were collected from a single-user workstation at the Center for Software Reliability (CSR) and represents the time failures related to

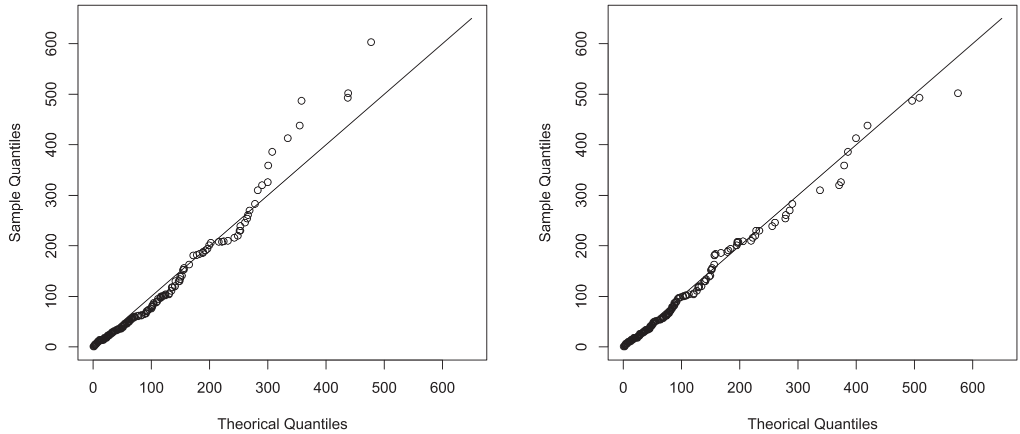

Figure 5. Left panel: QQ-plot for SGE model. Right panel: QQ-plot for MSGE model.

Table 5. Descriptive statistics for the failure data.

n	$\bar{x}$	s^2	b_1	b_2
104	147.783	60071.770	3.000	14.579

Table 6. ML estimates with its respective standard error (SE) for fitting models on the data set.

Parameters	GE (SE)	SGE (SE)	MSGE (SE)
$\hat{\lambda}$	0.004 (0.001)	0.013 (0.046)	0.036 (0.021)
$\hat{\alpha}$	0.512 (0.059)	0.637 (0.584)	0.906 (0.251)
$\hat{q}$	–	1.450 (3.976)	1.071 (0.258)
Log-likelihood	−604.166	−602.0543	**−599.863**

Table 7. AIC, BIC, KSS, and p-values for each fitted model.

Criterion	GE	SGE	MSGE
AIC	1212.333	1210.109	**1205.727**
BIC	1217.621	1218.042	**1213.660**
KSS	0.154	0.125	**0.067**
P-values	0.171	0.391	**0.973**

Pascal programing. This data can be found in Lyu (1996). Table 5 presents summary statistics for the data set.

Computing initially the moment estimators under the MSGE model we have the following estimates: $\hat{\lambda}_M = 0.008, \hat{\alpha}_M = 0.629$ and $\hat{q}_M = 3.004$. Using the moment estimators as initial values, the ML estimates are computed and presented in Table 6 with the standard errors in parentheses. For each model we also report the log-likelihood estimated values. Table 7 reports the corresponding AIC, BIC, KSS, and p-values for each model. For these data, based on the AIC, BIC, KSS, and p-values, the MSGE model presents better fitting values than the ones for the other models. Figure 6 depicts plots of the fitted GE, SGE and MSGE models using the ML estimates. Notice that the fitted MSGE model present heavier tails than the SGE and GE models. The QQ-plots for the SGE and MSGE models are presented in Figure 7.

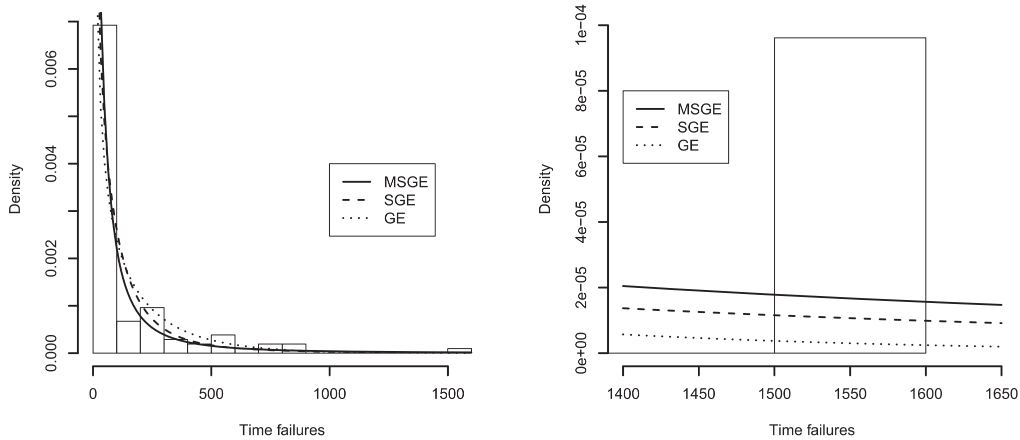

Figure 6. Left panel: models fitted by the maximum likelihood approach for failure data set. Right panel: Plots of the tails for models.

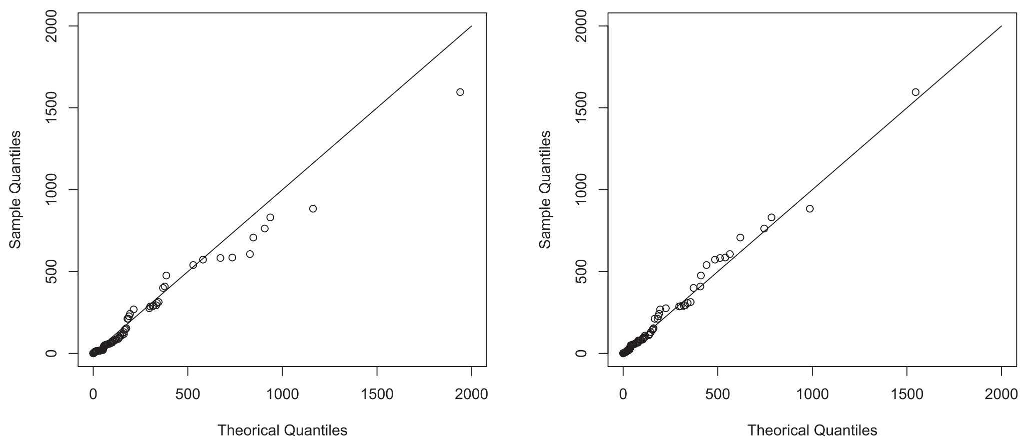

Figure 7. Left panel: QQ-plot for SGE model. Right panel: QQ-plot for MSGE model.

5. Concluding remarks

In this paper, we introduced a new extension of the GE distribution in order to get a flexible family of distributions for positive data modeling. This family presents a wide range of kurtosis indexes, which enable capturing outlying observations in non-negative data. This proposed distribution can be represented as the ratio of two independent random variables, one being the GE distribution and the other a power of the exponential distribution. We call this distribution as the MSGE distribution. Parameter estimation was conducted by using moments and maximum likelihood approaches, where the moment estimates were used for initializing maximum likelihood estimation through the Newton-Raphson procedure. By computing the asymmetry and kurtosis coefficients we illustrated the fact that the MSGE model is able to accommodate data with higher kurtosis. Finally, we fit MSGE models to two real data sets to demonstrate the potentiality of the proposed model. These applications show that the MSGE distribution presents a better fit than other competing models such as the GE and SGE models.

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