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ON THE STRUCTURE OF THE INTEGRAL GREEN RING

OF A CYCLIC GROUP OF ORDER p^2

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Introduction

Let p be any prime number, and let R be the ring of p-adic integers. The cyclic group of order p^2 is denoted by $C = <a>$. Its integral representation ring, or Green ring $A = a_{K_p}(RC)$ is the free K-module generated by the $4p+1$ isomorphism classes of the indecomposable RC-lattices, where $K = \mathbb{Z}_p$ is the ring of rational integers localized at p. The multiplication of A is induced by the tensor products of these lattices.

. In [7] V.F. Rudko gave most of the multiplication constants of the K-algebra A. For our study of the ring structure of A [5] we found it necessary to complete his multiplication table in the form stated in Theorem 3.1. The methods of proof for these formulae are also presented here. As an application we show in Corollary 3.2 that the nil radical $N(A)$ of A is a principal ideal.

Using the multiplication table of A given here we determine the Jacobson radical $J(A)$ of A in [5]. If $p = 2$, then A is a local ring. For all odd primes p, the ideal $J(A)$ is generated by 6 elements, and $A/J(A) \simeq F^6$, where $F = GF(p)$ is the residue class field of R and K.

Concerning our terminology and notations we refer to Curtis-Reiner [2].

1. The indecomposable RC-lattices and their reductions.

We begin by listing the $4p+1$ indecomposable RC-lattices.

Let $R_2 = R[\zeta]$ and $R_1 = R[\zeta^p]$, where ζ is a primitive root of 1 of order p^2. Let H be a cyclic group of order p. For every RC-lattice M let $\overline{M} = M/pM$.

The indecomposable RH-lattices are the trivial module R, the group ring RH and the simple module R_1. These are clearly indecomposable RC-lattices. Also R_2 is a simple RC-lattice, and any other indecomposable RC-lattice M is an extension of R_2 by an RH-lattice N. Following [2] we write:

$$M = (R_2, N, \mu) \quad , \quad \mu \in Ext'_{RC}(R_2, N) \simeq \overline{N} .$$

The isomorphism classes of the indecomposable RC-lattices are then represented by the following $4p+1$ lattices (see [2] p. 736).

$$R, R_1, D_1 = RH \ , \ R_2, D = RC \ ,$$
$$E = (R_2, R, 1) \ , \ E_1 = (R_2, R_1, 1) \ ,$$
$$X_i = (R_2, R \oplus R_1, 1+\lambda^i) \ , \ 0 \leq i \leq p-2$$
$$Y_i = (R_2, RH, \lambda^i) \ , \ 1 \leq i \leq p-1$$

If $p \neq 2$,
$$T_i = (R_2, R_1, \lambda^i) \ , \ 1 \leq i \leq p-2 \ , \quad \text{and}$$
$$Z_i = (R_2, R \oplus RH, 1+\lambda^i) \ , \ 1 \leq i \leq p-2 .$$

Our notation is related to that of [7] as follows.

$$R = \Delta_0 \ , \ R_1 = \Delta_1 \ , \ D_1 = \Delta_2 \ , \ R_2 = \Delta_3 \ ,$$
$$E = \Delta_4 \ , \ E_1 = T_0 \ , \ D = \Delta .$$

For the other indecomposables both notations coincide.

We also need the reduction modulo p of these lattices. Let $L(j)$ be the uniserial indecomposable FC-module of rank j. The first seven

lattices of the list are isomorphic to principal ideals of RC, for them $\bar{M} \simeq L(r)$. where $r = \operatorname{rank} M$. The lattices X_i, Y_i, T_i, are isomorphic to ideals generated by two elements of RC, while the Z_i are lattices of rank p^2+1 with two generators, and reduction modulo p in these cases gives (see [1] or [4]):

Lemma 1.1

a) $\bar{X}_{i-1} \simeq \bar{Y}_i \simeq L(p^2-i) \oplus L(i)$ for $1 \leq i \leq p-1$

b) $\bar{T}_i \simeq L(p^2-i-1) \oplus L(i)$ for $1 \leq i \leq p-2$

c) $\bar{Z}_i \simeq L(p^2-i) \oplus L(i+1)$ for $1 \leq i \leq p-2$.

Lemma 1.2   Let H be the subgroup of order p of C, and let M be an indecomposable RC-lattice. Let $0 \leq i,j \leq p$. Then the following assertions hold:

a) If $\bar{M} \simeq L(i)$, then $M_H \simeq iR$

b) If $\bar{M} \simeq L(p^2-j)$, then $M_H \simeq (p-j)D_1 \oplus jR_1$

c) If $\bar{M} \simeq L(p^2-j) \oplus L(i)$, then $M_H \simeq (p-j)D_1 \oplus jR_1 \oplus iR$

For each non-projective (not necessarily R-torsion free) RC-module M, the _Heller module_ of M is denoted by ΩM. From [1] we quote

Lemma 1.3

a) $\Omega R \simeq E_1$,

b) $\Omega R_1 \simeq E$,

c) $\Omega R_2 \simeq D_1$,

d) $\Omega X_i \simeq Y_{i+1}$, for $0 \leq i \leq p-2$,

e) $\Omega T_i \simeq Z_i$, for $1 \leq i \leq p-2$.

Furthermore, $\Omega^2 M \simeq M$ for every indecomposable RC-lattice M.

2. Methods for decomposing tensor products.

In this section the methods for establishing the multiplication table of the Green ring A are presented. The results mentioned are not considered to be new.

<u>Lemma 2.1</u>   For any non-projective RC-lattices M_1 and M_2:

$$M_1 \otimes \Omega M_2 \simeq \Omega(M_1 \otimes M_2) \oplus nD ,$$

where nD denotes the direct sum of n copies of D.

This isomorphism permits to obtain $M_1 \otimes \Omega M_2$ and $\Omega M_1 \otimes M_2$ once $M_1 \otimes M_2$ is known. Observe that the number n of free summands is easily determined by the ranks of M_1 and M_2. For example, for all RC-lattices M we have

$$E_1 \otimes M \simeq \Omega M \oplus nD .$$

We now list J.A. Green's formulas of [3] in the form needed for our computations in the following subsidiary result.

<u>Lemma 2.2</u>

I. $L(i) \otimes L(j) \simeq \coprod\limits_{k=1}^{i} L(j-i+2k-1)$, if $i \leq j$, $i + j \leq p$,

II. $L(i) \otimes L(j) \simeq (j+i-p)L(p) \oplus \coprod\limits_{k=1}^{p-j} L(j-i+2k-1)$, if $i \leq j$, $i+j \geq p$.

III. $L(p^2-i) \otimes L(p^2-j) \simeq (p^2-i-j) L(p^2) \oplus \coprod\limits_{k=1}^{i} L(j-i+2k-1)$,

if $i \leq j$, $i + j \leq p$,

IV. $L(p^2-i) \otimes L(p^2-j) \simeq (i+j-p) L(p) \oplus (p^2-i-j) L(p^2) \oplus \coprod\limits_{k=1}^{p-j} L(j-i+2k-1)$

if $i \leq j$, $i + j \geq p$.

V. $L(i) \otimes L(p^2-j) \simeq \coprod\limits_{k=1}^{i} L(p^2-i-j+2k-1)$, if $i \leq j$, $i + j \leq p$,

VI.　$L(i) \otimes L(p^2-j) \simeq (i-j) L(p^2) \oplus \coprod_{k=1}^{j} L(p^2-i-j+2k-1)$,

　　　　if $i \geq j$, $i + j \leq p$,

VII.　$L(i) \otimes L(p^2-j) \simeq (i+j-p) L(p^2-p) \oplus \coprod_{k=1}^{p-j} L(p^2-2p+i+j+2k-1)$,

　　　　if $i \leq j$, $i + j \geq p$,

VIII.　$L(i) \otimes L(p^2-j) \simeq (i+j-p) L(p^2-p) \oplus (i-j) L(p^2) \oplus \coprod_{k=1}^{p-i} L(p^2-2p+i+j+2k-1)$,

　　　　if $i \geq j$, $i + j \geq p$.

The following result of [6] is best stated in terms of the Heller operator applied to FC-modules, considered as RC-modules.

Lemma 2.3　For every FC-module V and every RC-lattice M:

$$M \underset{R}{\otimes} \Omega V \simeq \Omega(\overline{M} \underset{F}{\otimes} V) \oplus nD$$

Lemma 2.4　a)　$\Omega L(i) \simeq Y_i$ if $1 \leq i \leq p-1$,

　　　　　　b)　$\Omega L(i) \simeq D_1 \oplus R_2$ if $p \leq i \leq p^2-p$

　　　　　　c)　$\Omega L(p^2-i) \simeq X_{i-1}$ if $1 \leq i \leq p-1$,

　　　　　　d)　$\Omega L(p^2) \simeq D$.

Proof.　Let ϕ denote the cyclotomic polynomial of order p^2 . It is easily shown that if $1 \leq i \leq p-1$, and $C = \langle a \rangle$, then

$$D/(RC\phi(a) + RC(a-1)^i) \simeq L(i) .$$

Hence $\Omega L(i) \simeq RC\phi(a) + RC(a-1)^i \simeq$

　　$RC\phi(a) + (R + \ldots + Ra^{p^2-p-1}) (a-1)^i \simeq (R_2, D_1, \lambda^i) \simeq Y_i .$

For $p \leq i \leq p^2-p$ we have $\Omega L(i) \simeq RCp + RC(a-1)^i .$

Therefore it suffices to show that the idempotent $\frac{1}{p} \cdot (a)$ defines an endomorphism of $RC_p + RC(a-1)^i$, and that the given one is the only possible splitting.

The third isomorphism follows from the first one and from the following fact which is easily verified. If $\bar{\Omega}$ indicates Heller's operator on FC-modules, then

$$\Omega(\bar{\Omega}L(i)) \simeq \Omega(\Omega L(i)) .$$

3. The multiplication table of A.

We list now the products of the basis elements of A omitting those products which can be obtained from the given ones by using the Heller operator. In our notation we do not distinguish between a lattice and its isomorphism class. Up to minor corrections the following theorem is due to Rudko [7].

Theorem 3.1   With the notation of section 1 we have:

1) $\quad R_1 \otimes R_1 = R + (p-2)D_1$

2) $\quad D_1 \otimes D_1 = pD_1$

3) $\quad D_1 \otimes R_1 = (p-1)D_1$

4) $\quad D_1 \otimes T_i = iD_1 \oplus (i+1)R_2 \oplus (p-i-1)D$

5) $\quad D_1 \otimes Y_i = i(D_1+R_2) \oplus (p-i)D$

6) $\quad R_1 \otimes T_i = (i-1)D_1 \oplus iR_2 \oplus Z_{p-i-1} \oplus (p-i-2)D$

7) $\quad R_1 \otimes Y_i = (i-1)(D_1+R_2) \oplus Y_{p-i} \oplus (p-i-1)D$

8) a) $\quad T_i \otimes T_j = \overset{i}{\underset{k=1}{\text{⊥}}} T_{j-i+2k-1} \oplus \overset{i}{\underset{k=0}{\text{⊥}}} Z_{j-i+2k} \oplus (p^2-2i-3)D$,

$\quad$ if $\ i < j$, $i+j \leq p-2$,

b) $\quad T_i \otimes T_j = \overset{i}{\underset{k=1}{\text{⊥}}} T_{p-2k} \oplus \overset{i}{\underset{k=1}{\text{⊥}}} Z_{p-2k-1} \oplus D_1 \oplus E \oplus (p^2-2i-3)D$,

$\quad$ if $\ i < j$, $i+j = p-1$,

c) $T_i \otimes T_j = \coprod\limits_{k=1}^{p-j-1} T_{j-i+2k-1} \oplus \coprod\limits_{k=0}^{p-j-1} Z_{j-i+2k} \oplus 2(j+i-p+1)(D_1 \oplus R_2)$

$\oplus (p^2-2i-3)D$, if $i < j$, $i+j \geq p$.

d) $T_i \otimes T_i = \coprod\limits_{k=1}^{i} T_{2k-1} \oplus \coprod\limits_{k=1}^{i} Z_{2k} \oplus R \oplus (p^2-2i-2)D$, if $i < \frac{p-1}{2}$,

e) $T_i \otimes T_i = \coprod\limits_{k=1}^{i} T_{2k-1} \oplus \coprod\limits_{k=1}^{i-1} Z_{2k} \oplus R \oplus D_1 \oplus E \oplus (p^2-p-1)D$,

if $i = \frac{p-1}{2}$,

f) $T_i \otimes T_i = \coprod\limits_{k=1}^{p-i-1} T_{2k-1} \oplus \coprod\limits_{k=1}^{p-i-1} Z_{2k} \oplus R \oplus 2(2i+1-p)(D_1 \oplus R_2)$

$\oplus (p^2-2i-2)D$, if $i > \frac{p-1}{2}$.

9) a) $Y_i \otimes Y_j = \coprod\limits_{k=0}^{i-1} X_{j-i+2k} \oplus \coprod\limits_{k=1}^{i} Y_{j-i+2k-1} \oplus (p^2-2i)D$,

if $i \leq j$, $i+j \leq p$,

b) $Y_i \otimes Y_j = \coprod\limits_{k=0}^{p-j-1} X_{j-i+2k} \oplus \coprod\limits_{k=1}^{p-j} Y_{j-i+2k-1} \oplus 2(i+j-p)(D_1 \oplus R_2)$

$\oplus (p^2-2i)D$, if $i \leq j$, $i+j \geq p$.

10) a) $T_i \otimes Y_j = \coprod\limits_{k=0}^{i} X_{j-i+2k-1} \oplus \coprod\limits_{k=1}^{i} Y_{j-i+2k-1} \oplus (p^2-2i-2)D$,

if $i < j$, $i+j \leq p-1$,

b) $T_i \otimes Y_j = \coprod\limits_{k=1}^{j} X_{i-j+2k-1} \oplus \coprod\limits_{k=1}^{j} Y_{i-j+2k-1} \oplus (p^2-2j-1)D$,

if $i \geq j$, $i+j \leq p-1$,

c) $T_i \otimes Y_j = \coprod\limits_{k=0}^{p-j-1} X_{j-i+2k-1} \oplus \coprod\limits_{k=1}^{p-j} Y_{j-i+2k-1} \oplus (2(i+j-p)+1)(D_1 \oplus R_2)$

$\oplus (p^2-2i-2)D$ if $i < j$, $i+j > p-1$,

d) $T_i \otimes Y_j = \coprod\limits_{k=1}^{p-i-1} X_{i-j+2k-1} \oplus \coprod\limits_{k=1}^{p-i} Y_{i-j+2k-1} \oplus (2(i+j-p)+1)(D_1 \oplus R_2)$

$\oplus (p^2-2j-1)D$, if $i \geq j$, $i+j > p-1$.

Proof. The fact that tensoring with lattices preserves exactness, and the use of Schanuel's lemma suffice to determine the products of the RH-lattices.

Since $D_1 = R^C$ and $R_2 = R_1{}^C$, the products of D_1 and R_2 with any other lattice M are obtained by using Lemma 1.2 and the general isomorphism $N^C \otimes M \simeq (N \otimes M_H)^C$.

The products of lattices M_1 and M_2 not isomorphic to X_i or Y_i can be determined by means of Lemma 2.2, because the indecomposable direct summands of $\overline{M}_1 \otimes \overline{M}_2$ can be lifted uniquely to indecomposable RC-lattices, except when $M_1 \simeq M_2 \simeq T_{\frac{p-1}{2}}$.

For example, applying VIII and II we obtain

$\overline{R}_1 \otimes \overline{T}_i \simeq L(p-1) \otimes (L(p^2-i-1) \oplus L(i)) \simeq$

$iL(p^2-p) \oplus (p-i-2) L(p^2) \oplus L(p^2-p+i+1) \oplus (i-1) L(p) \oplus L(p-i) \simeq$

$i\overline{R}_2 \oplus (p-i-2)\overline{D} \oplus (i-1)\overline{D}_1 \oplus \overline{Z}_{p-i-1}$.

Computing $T_i \otimes T_i$ for $i = \frac{p-1}{2}$ by this method, it is found that there are three possible liftings of $\overline{T}_i \otimes \overline{T}_i$. But $(T_1-R)T_i{}^2$ can be written as a linear combination of basis elements in A if $p \neq 3$, and the three liftings of $\overline{T}_i \otimes \overline{T}_i$ give three different products when multiplied by T_1-R, so this enables us to determine the correct lifting. The case $p = 3$ can be verified separately.

The products involving X_i or Y_i are found by application of Lemmas 2.2, 2.3 and 2.4.

For example, if $i < j$ and $i+j \leq p-1$, using I and VI of Lemma 2.2, we get:

$$T_i \otimes Y_j \simeq T_i \otimes \Omega L(j) \simeq \Omega(\overline{T}_i \otimes L(j)) \oplus nD$$

$$\simeq \Omega[(j-i-1)L(p^2) \oplus \coprod_{k=1}^{i+1} L(p^2-i-j+2k-2) \oplus \coprod_{k=1}^{i} L(j-i+2k-1)] \oplus nD$$

$$\simeq \coprod_{k=0}^{i} X_{j-i+2k-1} \oplus \coprod_{k=1}^{i} Y_{j-i+2k-1} \oplus mD .$$

This completes the proof.

Assertion c) of the following result is due to Rudko [7] and Reiner [6] .

Corollary 3.2.   The integral Green ring A of the cyclic group C of order p^2 has the following ideals.

a) $I(A) = KD_1 + KR_2 + KD = R_2A + D_1A$

b) $N(A) = \sum_{i=0}^{p-2} K(X_i - Y_{i+1}) = (X_0 - Y_1)A$

c) $N(A)$ is the nil radical of A, and $N(A)^2 = 0$.

Proof.   a) Notice that $I(A) = \sum_N KN^C$, where N runs through all the indecomposable RH-lattices. This implies that $I(A)$ is an ideal of A. Since $E_1D_1 = R_2 + (p-1)D$, $I(A)$ can be generated by R_2 and D_1 as an ideal of A.

b) Now $N(A) = \sum K(\Omega V - \Omega W)$, where the sum extends over all pairs of indecomposable FC-modules V,W such that $\overline{\Omega V} \simeq \overline{\Omega W}$. By Lemma 2.3, for every RC-lattice M we have

$M(\Omega V - \Omega W) = \Omega(\overline{M} \otimes V) - \Omega(\overline{M} \otimes W)$, $\overline{\Omega(\overline{M} \otimes V)} \simeq \overline{\Omega(\overline{M} \otimes W)}$, as $\overline{D}$ cannot be a direct summand of any of these two FC-modules, because otherwise D would be a direct summand of a Heller module. Hence $N(A)$ is an ideal of A.

Using Theorem 3.1 we get

$T_1(Y_1 - X_0) = T_1 Y_1 - \Omega(T_1 Y_1) = Y_1 - X_0 - (Y_2 - X_1)$, and for $i > 1$:

$$T_1(Y_i - X_{i-1}) = TY_i - \Omega(T_1 Y_i)$$

$$= (X_{i-2} - Y_{i-1}) + (X_i - Y_{i+1}) + (Y_i - X_{i-1}) .$$

Therefore we get by induction that $N(A) = (Y_1 - X_0)A$. Clearly $(Y_1 - X_0)^2 = 0$, and so $N(A)^2 = 0$.

c) It suffices to show that $\widetilde{A} = A/N(A)$ does not have nilpotent elements. Let

$$a = r_0 R + r_1 R_1 + r_2 E + r_3 E_1 + r_4 D_1 + r_5 R_2 + r_6 D + \sum_{i=1}^{p-1} y_i Y_i + \sum_{i=1}^{p-2} t_i T_i +$$

$$+ \sum_{i=1}^{p-2} z_i Z_i \text{ of } \widetilde{A} \text{ satisfy } a^2 = 0 , \text{ where } r_i, y_i, t_i, z_i \in K .$$

Let M, N be indecomposable RC-lattices, then by Theorem 3.1 R is a direct summand of $M \otimes N$ if and only if $M \simeq N$ and p does not divide the rank of M. Thus $a^2 = 0$ implies

$$r_0^2 + r_1^2 + r_2^2 + r_3^2 + \sum_{i=1}^{p-2} t_i^2 + \sum_{i=1}^{p-2} z_i^2 = 0 .$$

Thus $r_j = 0$, $t_i = 0$, $z_i = 0$ for all the above indices, and

$a \in I(A) + \sum_{i=1}^{p-1} KY_i$. Let $\bar{a} = a + I(A) \in A/I(A)$. Then

$\bar{a} = \sum_{i=1}^{p-1} y_i Y_i$, and $\bar{a}^2 = (0)$. Then, since by Theorem 3.1 Y_1 is

not a direct summand of $Y_i \otimes Y_j$ for $i \neq j$, it follows that the coefficient of Y_1 in $\bar{a}^2$ is:

$$0 = y_1^2 + y_2^2 + \ldots + y_{p-1}^2 , \text{ and so } y_i = 0$$

for $1 \leq i \leq p-1$. Thus $a \in I(A)$, and so $a = r_4 D_1 + r_5 R_2 + r_6 D$. Applying Theorem 3.1 again we get

$$0 = a^2 = (r_4^2 + r_5^2)pD_1 + 2r_4 r_5 pR_2 + nD$$

for some integer n . Hence $r_4 = r_5 = 0$. As D is not nilpotent, $r_6 = 0$. Thus $a = 0$. This completes the proof.

Acknowledgements

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THE JACOBSON RADICAL OF THE INTEGRAL GREEN RING OF A

CYCLIC GROUP OF ORDER p^2

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Introduction

For any prime number p, let R be the ring of p-adic integers, and let $K = \mathbb{Z}_p$ be the ring of rational integers localized at p. The cyclic group of order p^2 is denoted by $C = \langle a \rangle$. The integral representation ring, or integral Green ring of C, is the free K-module $A = a_K(RC)$, generated by the $4p+1$ isomorphism classes of indecomposable RC-lattices. The multiplication of A is induced by the tensor products of these lattices.

In [3] Rudko determined the nil radical $N(A)$ of A. It is the purpose of this article to show that for all odd primes p the Jacobson radical $J(A)$ of A is generated by 6 elements, and that $A/J(A) \simeq F^6$, where $F = GF(p)$ is the residue class field of R and of K (Theorem 11). If $p = 2$, then A is a local ring.

Concerning our terminology and notations we refer to Curtis-Reiner [1] and our previous paper [2].

1. The multiplication table of A.

Let $R_2 = R[\rho]$ and $R_1 = R[\rho^p]$, where ρ is a primitive root of 1 of order p^2. Let H be a cyclic group of order p. The isomorphism classes of the indecomposable RC-lattices are then represented by the following $4p+1$ lattices in the notation of Curtis-Reiner [1], p. 736:

$$R \quad, \quad R_1 \quad, \quad D_1 = RH \quad, \quad R_2 \quad, \quad D = RC \quad, $$

$$E = (R_2, R, 1) \quad, \quad E_1 = (R_2, R_1 1) \quad, $$

$$X_i = (R_2, R \oplus R_1, 1+\lambda^i) \quad, \quad 0 \le i \le p-2$$

$$Y_i = (R_2, RH, \lambda^i) \quad, \quad 1 \le i \le p-1$$

If $p \ne 2$, then we also have the lattices,

$$T_i = (R_2, R_1, \lambda^i) \quad, \quad 1 \le i \le p-2$$

$$Z_i = (R_2, R \oplus RH, 1+\lambda^i) \quad, \quad 1 \le i \le p-2 .$$

We now state Rudko's [3] multiplication table of A in its complete version as it is given in our paper [2]. The missing products can be obtained from the given ones by means of the use of the Heller operator Ω, see [2].

Theorem 1.   With the above notation we have:

$$p^2(p-1)^2 = p^2(p^2+1-2p)$$

1) $R_1 R_1 = R + (p-2)D_1$

2) $D_1 D_1 = pD_1$ $R_2^2 = pRH + (p-2)RC$ $RH.R_2 =$

3) $D_1 R_1 = (p-1)D_1$

4) $D_1 T_i = iD_1 + (i+1)R_2 + (p-i-1)D$

5) $D_1 Y_i = i(D_1 + R_2) + (p-i)D$

6) $R_1 T_i = (i-1)D_1 + iR_2 + Z_{p-i-1} + (p-i-2)D$

7) $R_1 Y_i = (i-1)(D_1+R_2) + Y_{p-i} + (p-i-1)D$

8) a) $T_i T_j = \sum_{k=1}^{i} T_{j-i+2k-1} + \sum_{k=0}^{i} Z_{j-i+2k} + (p^2-2i-3)D$,

   if $i < j$, $i+j \leq p-2$,

   b) $T_i T_j = \sum_{k=1}^{i} T_{p-2k} + \sum_{k=1}^{i} Z_{p-2k-1} + D_1 + E + (p^2-2i-3)D$,

   if $i < j$, $i+j = p-1$,

   c) $T_i T_j = \sum_{k=1}^{p-j-1} T_{j-i+2k-1} + \sum_{k=0}^{p-j-1} Z_{j-i+2k} + 2(j+i-p+1)(D_1+R_2)$

   $+ (p^2-2i-3)D$, if $i < j$, $i+j \geq p$.

   d) $T_i T_i = \sum_{k=1}^{i} T_{2k-1} + \sum_{k=1}^{i} Z_{2k} + R + (p^2-2i-2)D$, if $i < \frac{p-1}{2}$,

   e) $T_i T_i = \sum_{k=1}^{i} T_{2k-1} + \sum_{k=1}^{i-1} Z_{2k} + R + D_1 + E + (p^2-p-1)D$, if $i = \frac{p-1}{2}$

   f) $T_i T_i = \sum_{k=1}^{p-i-1} T_{2k-1} + \sum_{k=1}^{p-i-1} Z_{2k} + R + 2(2i+1-p)(D_1+R_2)$

   $+ (p^2-2i-2)D$, if $i > \frac{p-1}{2}$.

9) a) $Y_i Y_j = \sum_{k=0}^{i-1} X_{j-i+2k} + \sum_{k=1}^{i} Y_{j-i+2k-1} + (p^2-2i)D$,

   if $i \leq j$, $i+j \leq p$,

   b) $Y_i Y_j = \sum_{k=0}^{p-j-1} X_{j-i+2k} + \sum_{k=1}^{p-j} Y_{j-i+2k-1} + 2(i+j-p)(D_1+R_2)$

   $+ (p^2-2i)D$, if $i \leq j$, $i+j \geq p$.

10) a) $T_i Y_j = \sum_{k=0}^{i} X_{j-i+2k-1} + \sum_{k=1}^{i} Y_{j-i+2k-1} + (p^2-2i-2)D$,

    if $i < j$, $i+j \leq p-1$,

    b) $T_i Y_j = \sum_{k=1}^{j} X_{i-j+2k-1} + \sum_{k=1}^{j} Y_{i-j+2k-1} + (p^2-2j-1)D$,

    if $i \geq j$, $i+j \leq p-1$,

c) $T_i Y_j = \sum\limits_{k=0}^{p-j-1} X_{j-i+2k-1} + \sum\limits_{k=1}^{p-j} Y_{j-i+2k-1} + (2(i+j-p)+1)(D_1+R_2)$

$+ (p^2-2i-2)D$, if $i < j$, $i+j > p-1$,

d) $T_i Y_j = \sum\limits_{k=1}^{p-i-1} X_{i-j+2k-1} + \sum\limits_{k=1}^{p-i} Y_{i-j+2k-1} + (2(i+j-p)+1)(D_1+R_2)$

$+ (p^2-2j-1)D$, if $i \geq j$, $i+j > p-1$.

2. The structure of A.

For the determination of the Jacobson radical $J(A)$ of the integral Green ring A we need several ideals of A defined in the following subsidiary result. The notations given there will be kept in the following without further mention.

Lemma 2. a) The following K-submodules of A are ideals:

$I(A) = KD_1 + KR_2 + KD = DA + R_2A$.

$N(A) = \sum\limits_{i=0}^{p-2} K(X_i - Y_{i+1}) = (X_0 - Y_1)A$.

$L = I(A) + N(A) + pA$.

$B = I(A) + \sum\limits_{i=0}^{p-2} KX_i + \sum\limits_{i=1}^{p-1} KY_i$.

b) $N(A)$ is the nil radical of A, and $N(A)^2 = 0$.

c) $L^2 \leq pA$, and $L \leq J(A)$.

Proof. By Corollary 3.2 of [2], b) and the first two assertions of a) hold.

Clearly L is an ideal. As $I(A) \cap N(A) = 0$, it follows that $L^2 \leq I^2 + pA \leq pA$. Hence $L \leq J(A)$.

By Lemma 1.2 of [2], $B = I(A) + \sum_V R\Omega V$, where V runs through the indecomposable FC-modules and Ω denotes the Heller operator. Hence by Lemma 2.1 of [2], B is an ideal of A.

<u>Theorem 3.</u> If $p = 2$, then A is a local ring with Jacobson radical

$$J(A) = 2A + B + K(R_1-R) + K(E-R) + K(E_1-R) \quad ,$$

and $A/J(A) \simeq F$.

<u>Proof.</u> As $p = 2$, Theorem 1 implies

$$Y_1^2 = X_0^2 \in K(X_0+Y_1) + I(A) \leq N(A) + 2A + I(A) = L .$$

Therefore $B^2 \subseteq L$. Hence $2A + B \leq J(A)$. Now

$$A/(2A+B) \simeq FR + FR_1 + FE + FE_1 .$$

Since modulo $I(A)$ we have $R_1^2 = E^2 = E_1^2 = R$, and $R_1 E = E_1$, it follows that $A/(2A+B)$ is isomorphic to the group algebra FV_4 of the Klein four group V_4. This implies that $J(A)$ has the given form, and $A/J(A) \simeq F$.

From now on we assume that p is an odd prime, and we keep the following

<u>Notation.</u> $\hat{A} = A/L$, $\hat{B} = (B+pA)/L = \sum_{i=1}^{p-1} FY_i$,

$$\hat{C} = \hat{A}/\hat{B} = FR + FE + FR_1 + FE_1 + \sum_{i=1}^{p-2} FT_i + \sum_{i=1}^{p-2} FZ_i .$$

Observe that $\hat{A} = \hat{B} + \hat{C}$, and $\hat{B} \cap \hat{C} = 0$.

<u>Lemma 4.</u> Let $e_1 = \frac{1}{4}(Y_1 + Y_{p-1})$ and $e_2 = \frac{1}{4}(Y_1 - Y_{p-1})$.

Then e_1 and e_2 are orthogonal idempotents of $\hat{B}$ such that
$\hat{B} = e_1\hat{B} \oplus e_2\hat{B}$.

<u>Proof.</u> From Theorem 1 it follows that in $\hat{B}$, we have:

$$e_1 = \frac{1}{4} Y_1(R+R_1) \ , \ e_2 = \frac{1}{4} Y_1(R-R_1) \ , \ Y_1^2 = 2Y_1 \ , \ (R-R_1)^2 = 2(R-R_1)$$

$$(R+R_1)^2 = 2(R+R_1) \ , \ R^2-R_1^2 = 0 \ .$$

Hence $e_1^2 = e_1$, $e_2^2 = e_2$ and $e_1e_2 = 0$.

Let $q = \frac{p-1}{2}$. Since $Y_1Y_i = 2Y_i$, we get

$$e_1\hat{B} = (R+R_1) \sum_{i=1}^{p-1} FY_i = \sum_{i=1}^{p-1} F(Y_i+Y_{p-i}) = \sum_{i=1}^{q} F(Y_i+Y_{p-i}) \ .$$

Similarly $e_2\hat{B} = \sum_{i=1}^{q} F(Y_i-Y_{p-i})$. Thus $e_1\hat{B} \oplus e_2\hat{B} = \hat{B}$.

<u>Remark 5.</u> For $1 \leq i \leq q = \frac{p-1}{2}$ let $y_i = \frac{1}{2}(Y_i+Y_{p-i})$ and
$y_i' = \frac{1}{2}(Y_i-Y_{p-i})$. Then in $\hat{B}$ we have $y_i = e_1Y_i$ and $y_i' = e_2Y_i$,
hence $y_iy_j = e_1Y_iY_j$ and $y_i'y_j' = e_2Y_iY_j$.

Thus from Theorem 1 it follows that for all $1 \leq i \leq j \leq q$:

$$y_iy_j = 2 \sum_{k=1}^{i} y_{j-i+2k-1} \ , \ y_i'y_j' = 2 \sum_{k=1}^{i} y_{j-i+2k-1}' \ .$$

For the proofs of the following results we require some subsidiary results on certain arithmetical triangles.

For $k \geq 0$ and $n \geq 1$ let $C(k,n) = \sum_{j=0}^{k} \binom{n}{j}\binom{j}{k-j}$, where
$\binom{j}{i} = 0$, if $j < i$. As is well known,
$C(k,n) = C(k,n-1) + C(k-1,n-1) + C(k-2,n-1)$, see Vilenkin [4] .

For $k \geq 0, -1$ and $n \geq 2$ let the sequences of integers $a_n^{(k)}$ and $b_n^{(k)}$ be inductively defined by:

(1) $\quad a_2^{(1)} = a_n^{(0)} = 1$, $\quad a_n^{(k)} = a_{n-1}^{(k-1)} + a_{n-1}^{(k)}$ for $2 \leq 2k < n$,

and $a_n^{(k)} = a_{n-1}^{(k-1)}$, if $2k = n$,

(2) $\quad b_2^{(2)} = b_2^{(1)} = b_n^{(0)} = 1$, $\quad b_n^{(-1)} = 0$,

$b_n^{(k)} = b_{n-1}^{(k)} + b_{n-1}^{(k-1)} + b_{n-1}^{(k-2)}$ for $1 \leq k < n$,

and $b_n^{(n)} = b_{n-1}^{(n-2)}$.

Lemma 6. Let p be any odd prime, $n \geq 1$ and $k \geq -1$. With the above notations we have:

(a) $\quad a_n^{(k)} = \binom{n}{k-1} \frac{n-2k+1}{k}$ for $2 \leq 2k \leq n$,

(b) $\quad b_n^{(k)} = C(k, n-1) - C(k-3, n-1)$ for $3 \leq k \leq n-1$,

(c) $\quad a_p^{(k)} \equiv 0 \pmod{p}$ for $2 \leq 2k \leq n$,

(d) $\quad b_p^{(k)} \equiv 0 \pmod{p}$ for $2 \leq k \leq p-1$,

(e) $\quad b_p^{(p)} \equiv 1 \bmod p$ and $b_p^{(1)} \equiv -2 \bmod p$.

Proof. Assertions a) and b) follow easily by induction, and a) implies c).

Clearly $b_n^{(1)} = n-1$, and using induction we get that $b_n^{(2)} = \binom{n}{2}$ for all n. Thus $b_p^{(2)} \equiv 0 \bmod p$. Clearly

$$C(k, p-1) = \sum_{j=0}^{k} \binom{p-1}{j} \binom{j}{k-j} \equiv c(k, p-1) = \sum_{j=0}^{k} (-1)^j \binom{j}{k-j} \bmod p.$$

Let $k = 2s$ or $k = 2s + 1$. Then

$$c(k,p-1) = (-1)^k \binom{k}{0} + (-1)^{k-1}\binom{k-1}{1} + \ldots + (-1)^{k-s}\binom{k-s}{s} , \quad \text{and so}$$

$$c(k,p-1) = \begin{cases} 0 & \text{if } k \equiv 2 \bmod 3 \\ 1 & \text{if } k \equiv 0 \bmod 3 \\ -1 & \text{if } k \equiv 1 \bmod 3 \end{cases}$$

As for $k \geq 3$ we have $b_p^{(k)} = C(k,p-1) - C(k-3,p-1)$, assertion d) follows.

Since $b_p^{(1)} = p-2$ it remains to show that $b_p^{(p)} = b_{p-1}^{(p-2)} \equiv 1$ mod p. Applying d) we see that

$$\sum_{j=0}^{p-2} b_{p-1}^{(j)} = b_{p-1}^{(0)} + b_p^{(p-2)} + b_p^{(p-5)} + \ldots + b_p^{(3)} \equiv b_{p-1}^{(0)} \bmod p$$

$$\sum_{j=0}^{p-2} b_{p-1}^{(j)} = b_{p-1}^{(p-2)} + b_p^{(p-3)} + b_p^{(p-6)} + \ldots + b_p^{(2)} \equiv b_{p-1}^{(p-2)} \bmod p$$

Thus it follows that $b_p^{(p)} = b_{p-1}^{(p-2)} \equiv b_{p-1}^{(0)} \equiv 1 \bmod p$, and the proof is complete.

With the notation of Remark 5 we now can state

Lemma 7. For all i with $1 \leq i \leq q = \frac{p-1}{2}$, we have $y_i^p = (-1)^{i+1} i\, y_1$ in $\hat{B}$.

Proof. Certainly $y_1^p = y_1$. We now show that $y_2^p = -2y_1$.

From $y_2 y_1 = 2y_2$ and $y_2 y_j = 2(y_{j-1}+y_{j+1})$ for $j > 1$, we obtain successively:

$$y_2^2 = 2(y_1+y_3) \ , \ y_2^3 = 2^2(2y_2+y_4) \ , \ \ldots$$

$$y_2^{p-2} = 2^{p-3}(a_{p-2}^{(q-1)} \, y_2 + a_{p-2}^{(q-2)} \, y_4 + \ldots + a_{p-2}^{(1)} \, y_{p-3} + y_{p-1}) \ ,$$

where the integers $a_n^{(k)}$ are defined as in the preceding lemma. Since $y_2 y_{p-1} = y_2 y_1 = 2y_{p-2}$, we get

$$y_2^{p-1} = 2^{p-2}(a_{p-1}^{(q)} \, y_1 + a_{p-1}^{(q-1)} \, y_3 + \ldots + a_{p-1}^{(1)} \, y_{p-2}) \ .$$

And from $y_2 y_{p-2} = y_2^2 = 2(y_1+y_{p-3})$ we have

$$y_2^p = 2^{p-1}(a_{p-1}^{(1)} \, y_1 + a_p^{(q)} \, y_2 + \ldots + a_p^{(2)} \, y_{p-3}) \ .$$

But $a_{p-1}^{(1)} = p-2$ and $a_p^{(k)} = 0 \pmod{p}$ if $1 < k \leq q$, so $y_2^p = -2y_1$.

Since $y_2^2 = 2(y_1+y_3)$, we have $y_2^{2p} = 2y_1^p + 2y_3^p = 4y_1^2 = 8y_1$. Hence $y_3^p = 3y_1$.

Suppose now that $y_i^p = (-1)^{i+1} i \, y_1$ for all i with $3 \leq i \leq k$. Then from $y_2 y_k = 2(y_{k-1}+y_{k+1})$ we get

$$y_2^p y_k^p = 2y_{k-1}^p + 2y_{k+1}^p = (-1)^k 2(k-1)y_1 + 2y_{k+1}^p = (-1)^{k+2} 4ky_1 \ .$$

Therefore $y_{k+1}^p = (-1)^{k+2}(k+1)y_1$.

<u>Proposition 8.</u> a) The orthogonal idempotents $e_1 = \frac{1}{4}(Y_1+Y_{p-1})$ and $e_2 = \frac{1}{4}(Y_1-Y_{p-1})$ of $\hat{B}$ are primitive.

b) $\hat{B} = e_1\hat{B} \oplus e_2\hat{B}$

c) Each ideal $e_i\hat{B}$ of $\hat{A}$ is a local uniserial ring of composition length $q = \frac{1}{2}(p-1)$ with $e_i\hat{B}/J(e_i\hat{B}) \simeq F$.

d) $J(\hat{B}) = (2Y_1+Y_2)\hat{A}$.

Proof. From the proof of Lemma 4 we know that $\{Y_1, Y_2, \ldots, Y_q\}$ is an F-vector space basis of $e_1\hat{B}$. By Lemma 7 $[(k+1)y_k + ky_{k+1}]^p = 0$ for $1 \le k \le q-1$. This implies that $\{2y_1+y_2, \ldots, qy_{q-1}+(q-1)y_q\}$ is a basis of $J(e_1\hat{B})$ over F . Thus $\dim_F(Je_1\hat{B}) = q-1$. Hence $e_1\hat{B}$ is a local ring, and e_1 is primitive.

Let $n = y_1 + \frac{p+1}{2} y_2$. Then $n \in J(e_1\hat{B})$. For every integer $1 \le k \le q-1$ the element y_2^k is a linear combination of $y_1, y_2, \ldots, y_{k+1}$, where y_{k+1} has coefficient $2^{k-1} \in F$. Hence $y_1, y_2, y_2^2, \ldots, y_2^{k-1}, y_2^k$ are linearly independent over F for $1 \le k \le q-1$. Now

$$n^{q-1} = \sum_{i=0}^{q-1} \binom{q-1}{i} y_1^{q-1-i} (\frac{p+1}{2})^i y_2^i$$

$$= 2^{q-2}y_1 + \sum_{i=1}^{q-1} \binom{q-1}{i} 2^{q-2-2i} y_1 y_2^i$$

$$= 2^{q-2}y_1 + \sum_{i=1}^{q-1} \binom{q-1}{i} 2^{q-1-2i} y_2^i .$$

Thus $n^{q-1} \ne 0$, and $J(e_1\hat{B}) = n\hat{B}$ has Loewy series

$$J(e_1\hat{B}) = n\hat{B} > n^2\hat{B} > \ldots > n^{q-1}\hat{B} > n^q\hat{B} = 0 ,$$

because $\dim_F J(e_1\hat{B}) = q-1$. Therefore $e_1\hat{B}$ is a uniserial local ring with composition length $l(e_1\hat{B}) = q$. This completes the proof.

With the integers $b_n^{(k)}$ defined in Lemma 6 we now state

Lemma 9. a) For every odd $n < p$ we have mod $I(A)$:

$$T_1^n = b_n^{(n)} R + b_n^{(n-1)} T_1 + b_n^{(n-2)} Z_2 + \ldots + b_n^{(1)} Z_{n-1} + b_n^{(0)} T_n$$

b) For every even $n < p$ we have mod $I(A)$:

$$T_1^n = b_n^{(n)} R + b_n^{(n-1)} T_1 + b_n^{(n-2)} Z_2 + \ldots + b_n^{(1)} T_{n-1} + b_n^{(o)} Z_n$$

c) $\quad T_1^p = b_p^{(p)} R + b_p^{(p-1)} T_1 + b_p^{(p-2)} Z_2 + \ldots + b_p^{(1)} Z_{p-1} + b_{p-1}^{(1)} E$

d) In the ring $\hat{C}$ the following relations hold:

$$(T_{2k-1})^p = (2k-1)R - 2kE \qquad \text{for} \quad 1 \leq k \leq \tfrac{1}{2}(p-1)$$

$$(T_{2k})^p = -2kR_1 + (2k+1)E_1 \qquad \text{for} \quad 1 \leq k \leq \tfrac{1}{2}(p-3)$$

$$(Z_{2k-1})^p = -2kR_1 + (2k-1)E_1 \qquad \text{for} \quad 1 \leq k \leq \tfrac{1}{2}(p-1)$$

$$(Z_{2k})^p = (2k+1)R - 2kE \qquad \text{for} \quad 1 \leq k \leq \tfrac{1}{2}(p-3)$$

<u>Proof.</u> The assertions a), b) and c) follow from Theorem 1.
By c) and Lemma 6 , $\quad T_1^p = R-2E$. Let

$$t_{2k-1} = T_{2k-1} - (2k-1)R + 2kE \ ,$$

$$t_{2k} = T_{2k} + 2kR_1 - (2k+1)E_1 \ ,$$

$$z_{2k-1} = Z_{2k-1} - (2k-1)E_1 + 2kR_1 \ ,$$

$$z_{2k} = Z_{2k} + 2kE - (2k+1)R \ .$$

Then $t_1^p = 0$. As $t_{p-2} = Et_1$ we thus get $t_{p-2}^p = 0$.
Using the multiplication table of A and $\hat{C}$ again we get
$t_1^2 = -t_1 + z_2 + 4t_{p-2}$. Hence $z_2^p = 0$, and so
$t_2^p = 0$. For $2 \leq j \leq \tfrac{1}{2}(p-1)$ we have:

$$t_1 t_j = -jt_1 + z_{j-1} + z_{j+1} + 2t_{p-j-1} + (j+1)t_{p-2} , \quad \text{if} \quad j \text{ is odd,}$$

$$t_1 t_j = -(j+1)z_1 + z_{j-1} + z_{j+1} + 2t_{p-j-1} + jz_{p-2} , \quad \text{if} \quad j \text{ is even.}$$

Suppose that for $1 \leq i \leq j$ we know that $t_i^p = 0$, then
$z_{j+1}^p = 0$. Therefore $t_{j+1}^p = 0$. Hence assertion d) holds

for all k by induction. This completes the proof.

Proposition 10. Let $p \neq 2$. Then $\hat{C} = FV_4 + \sum_{i=1}^{p-2} FT_i + \sum_{i=1}^{p-2} FZ_i$

has four primitive orthogonal idempotents

$e_3 = \frac{1}{4}(R + R_1 + E + E_1)$

$e_4 = \frac{1}{4}(R + R_1 - E - E_1)$

$e_5 = \frac{1}{4}(R - R_1 - E + E_1)$

$e_6 = \frac{1}{4}(R - R_1 + E - E_1)$, and

$$\hat{C} = e_3\hat{C} \oplus e_4\hat{C} \oplus e_5\hat{C} \oplus e_6\hat{C}$$

where each $e_i\hat{C}$ is a uniserial local ring of composition length

$$l(e_3\hat{C}) = l(e_5\hat{C}) = \frac{1}{2}(p-1) \quad ,$$

$$l(e_4\hat{C}) = l(e_6\hat{C}) = \frac{1}{2}(p-3) \quad .$$

Furthermore, $J(\hat{C}) = (T_1 - R + 2E)\hat{C}$.

Proof. As $p \neq 2$, $FV_4 = FR + FR_1 + FE + FE_1$ is semi-simple, and $\{e_i \mid 3 \leq i \leq 6\}$ is the complete set of all orthogonal primitive idempotents of FV_4 .

Let x be a primitive idempotent of $\hat{C}$ and
$x = f_0R + f_1R_1 + f_2E + f_3E_1 + \sum_{i=1}^{p-2} v_iT_i + \sum_{i=1}^{p-2} w_iZ_i$,

for some $f_i, v_j, w_k \in F$. As $x = x^2$, we get

$x = x^p = f_0R + f_1R_1 + f_2E + f_3E_1 + \sum_{i=1}^{p-2} v_iT_i^p + \sum_{i=1}^{p-2} w_iZ_i^p$.

By Lemma 9d) each T_i^p , Z_i^p belongs to FV_4 .
Thus $x \in \{e_i \mid 3 \leq i \leq 6\}$.

Let $u = e_3 + e_4$. Then $u\hat{C} = \sum_{i=1}^{p-2} FuT_i + FuR + FuE$.

By the multiplication table of A and Lemma 9 the F-vector space $u\hat{C}$ is generated by the powers of uT_1 . Hence $u\hat{C}$ is a finite-dimensional epimorphic image of the polynomial ring $F[X]$. Therefore $u\hat{C}$ is a direct sum of uniserial local rings. Thus the local rings $e_3\hat{C}$ and $e_4\hat{C}$ are uniserial.

Since $e_3\hat{C} = \sum_{i=1}^{\frac{1}{2}(p-1)} Fe_3T_i + Fe_3$, $e_4\hat{C} = \sum_{i=1}^{\frac{1}{2}(p-3)} Fe_4T_i + Fe_4$,

it follows that the Loewy lengths are $l(e_3\hat{C}) = \frac{1}{2}(p-1)$ and $l(e_4\hat{C}) = \frac{1}{2}(p-3)$.

By Lemma 9d) $J(\hat{C})$ contains $T_1 - R + 2E$. As $e_3\hat{C}$ is generated as an F-vector space by the powers of e_3T_1 , it follows from

$$e_3(T_1 - R + 2E) = e_3T_1 + e_3$$

that the non-zero powers of $e_3(T_1 - R + 2E)$ are linearly independent, and therefore they are a vector space basis of $J(e_3\hat{C})$. Thus

$$J(e_3\hat{C}) = e_3(T_1 - R + 2E)\hat{C} .$$

Using the equation $e_4(T_1 - R + 2E) = e_4 - 3e_4$ it follows similarly that $J(e_4\hat{C}) = e_4(T_1 - R + 2E)\hat{C}.$

The corresponding results for $e_5\hat{C}$ and $e_6\hat{C}$ are proved analogously.

Since $J(\hat{C}) = \sum_{i=3}^{6} \oplus J(e_i\hat{C})$, we get $J(\hat{C}) = (T_1 - R + 2E)\hat{C}$,

which completes the proof.

Combining the above results we obtain the following

Theorem 11. Let $p \neq 2$. Let A be the integral Green ring
of the cyclic group C of order p^2. Then:

a) $J(A) = pA + DA + R_2A + (Y_1 - X_0)A + (2Y_1 + Y_2)A + (T_1 - R + 2E)A$.

b) $A/J(A) \simeq GF(p)^6$.

c) The ideal $L = pA + I(A) + N(A) \leq J(A)$, and $\hat{A} = A/L$ is
isomorphic to a direct sum of 6 local uniserial rings, 4 of which
have Loewy length $\frac{1}{2}(p-1)$ and 2 of which have Loewy length $\frac{1}{2}(p-3)$.

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