# The generalized curvature ellipses and minimal surfaces

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# THE GENERALIZED CURVATURE ELLIPSES AND MINIMAL SURFACES

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## §0. INTRODUCTION

The well known curvature ellipse is a very useful tool for the study of surfaces with higher codimensions, especially,for those with codimension 2. However, the classical curvature ellipses always lie on the first normal spaces of the surface. Thus,only the first order osculating spaces are taken into considerations, that, in the higher condimensional cases, makes the curvature ellipses less effective. In this paper, motivated by a joint work with Góes [3], in each normal space we introduce an ellipse in the orthogonal complement to the previous normal space, of surfaces with parallel mean curvature vector in a constantly curved Riemannian manifold. These ellipses at each point are clearly mutually orthogonal, and, we use their areas to generalize the normal curvature. Finally, we apply this new notion to study minimal surfaces in euclidean spaces, and we show that, in fact, the generalized curvature ellipses turn out to be more effective.

## §1. DEFINITIONS

For a general setting, we consider a Riemannian submanifold

$$(1.1) \qquad x: M \hookrightarrow \overline{M}$$

To each $p \in M$, let $N_k(p)$ denote the k-th normal space of M at p. The (k+1)st fundamental form is the multi-linear tensor

(1.2)
$$B^k : T_p M \times \overset{(k+1)}{\cdots} \times T_p M \longrightarrow N_k(p)$$

defined by

(1.3)
$$B^k(X_0, \ldots, X_k) = (\overline{\nabla}_{\tilde{X}_k} \cdots \overline{\nabla}_{\tilde{X}_1} \tilde{X}_0)^{\pi_k}$$

where $\overline{\nabla}$ is the riemannian connection of $\overline{M}$, $\tilde{X}_j$ local field which extends $X_j$, and $\pi_k$ denotes the projection onto $[T_p M \oplus N_{k-1}(p)]^\perp$. It's known that when $\overline{M}$ has constant sectional curvature, $B^k$ is symmetric. Details can be found in Spivak [9, p.240].

## PROPOSITION 1.1

Let $x: M^2 \hookrightarrow \overline{M}(c)$ be a surface with parallel mean curvature vector (i.e. $\nabla^\perp H \equiv 0$) in a constantly curved riemannian manifold. Then $\forall p \in M$, $\forall k \geq 1$

(1.4)
$$\varepsilon_k(p) = \{E^k(X) = B^k(X, \ldots, X) \mid X \in T_p M, \ |X| = 1\}$$

is an ellipse.

Proof. (i) For $k=1$, $\varepsilon_1(p)$ is the classical curvature ellipse centered at $H(p)$.

(ii) For $k>1$, choose $e_1, e_2$ an orthonormal base for $T_p M$. Since $\nabla^\perp H = 0$,

$$B^k(e_2, e_2, *, \ldots, *) = -B^k(e_1, e_1, *, \ldots, *)$$

Thus, if $X = \cos\theta\, e_1 + \sin\theta\, e_2$, from the identity

$$(\cos\theta + i\, \sin\theta)^{k+1} = (e^{i\theta})^{k+1} = \cos(k+1)\theta + i\, \sin(k+1)\theta,$$

we get immediately

(1.5) $E^k(X) = \cos(k+1)\theta\, B^k(e_1, \ldots, e_1) + \sin(k+1)\theta\, B^k(e_1, \ldots, e_1, e_2)$

This shows that $\varepsilon_k(p)$ goes around k+1 times on an ellipse in $N_k(p) \cap N_{k-1}(p)^\perp$, centered at p.

<div align="right">Q.E.D.</div>

Therefore we call $\varepsilon_k(p)$ the <u>k-th order curvature ellipse</u> at p, and we define the <u>k-th order normal curvature</u> at p to be

$$(1.6) \qquad K_{N_k}(p) = \frac{2}{\pi} \text{ area } \varepsilon_k(p)$$

<u>REMARK</u>. For k=1, our definitions coincide with the classical ones.

## PROPOSITION 1.2.

$\varepsilon_k(p)$ is a circle if and only if for any orthogonal base {X,Y} in $T_pM$ (i.e $X \perp Y$ and $|X|=|Y|>0$)

$$(1.7) \quad B^k(X,\ldots,X) \perp B^k(X,\ldots,X,Y) \text{ and } |B^k(X,\ldots,X)|=|B^k(X,\ldots,X,Y)|$$

which is equivalent to

$$(1.8) \quad 2|B^k(X,\ldots,X) \wedge B^k(X,\ldots,X,Y)| = |B^k(X,\ldots,X)|^2 + |B^k(X,\ldots,X,Y)|^2$$

Proof. We need only to observe that

$$(1.9) \quad 2|B^k(X,\ldots,X) \wedge B^k(X,\ldots,X,Y)| \leq 2|B^k(X,\ldots,X)| \, |B^k(X,\ldots,X,Y)|$$
$$\leq |B^k(X.,,,X)|^2 + |B^k(X,\ldots,X,Y)|^2$$

and the equalities hold if and only if (1.7) holds.

<div align="right">Q.E.D.</div>

Finally, we set

$$(1.10) \qquad C_k = \{p \in M \,|\, \varepsilon_k(p) \text{ is a circle}\}$$

## §2. APPLICATIONS

In this section we make some applications to minimal sur

faces in euclidean spaces. A classical theorem of Eisenhart [4] calls naturally to our first attention. The theorem says that a minimal surface $x:M^2 \hookrightarrow R^4$ is a holomorphic curve if and only if $C_1=M$. The theorem fails immediately if we raise the codimension. For example, take any simply connected non-flat minimal surface in $R^3$ and consider its generalized Gauss map $\phi=(\phi_1,\phi_2,\phi_3)$. Set

$$g_j = \int \phi_j(\zeta)d\zeta, \quad j=1,2,3.$$

Then

$$x = \mathrm{Re} \int (\tfrac{1}{2}(1-\textstyle\sum g_j^2), \; \tfrac{i}{2}(1+\textstyle\sum g_j^2), \; g_1,g_2,g_3)d\zeta$$

gives a minimal surface in $R^5$ with $C_1=M$, but $x(M)$ cannot be viewed as a holomorphic curve in some affine subspace of $R^5$. In the subsequent discussions we will explain why is that so.

Let $x:M^2 \hookrightarrow R^n$ now be an orientable minimal surface, and let $\zeta=\xi+i\eta$ be the complex parameter formed by isothermal parameters. The generalized Gauss map is defined to be $[\phi]$ where

(2.1)
$$\phi(\zeta) = x_\xi - ix_\eta$$

which is holomorphic and satisfies

(2.2)
$$(\phi,\phi) \equiv \textstyle\sum \phi_j^2 = 0$$

We will use the generalized Gauss map to interpret the behavior of the generalized curvature ellipses. First we observe that

## LEMMA 2.1.

Let $(\cdot,\cdot)$ denote the symmetric product between complex vectors. Then $(u,v)=0$ if and only if $<\mathrm{Re}u,\mathrm{Re}v>=<\mathrm{Im}u,\mathrm{Im}v>$ and $<\mathrm{Re}u,\mathrm{Im}v>=-<\mathrm{Im}u,\mathrm{Re}v>$.

PROPOSITION 2.2.

Let $x:M^2 \hookrightarrow R^n$ be a minimal surface. Then

(i) $p \epsilon C_1 \iff (\phi'(p),\phi'(p))=0$

(ii) if $C_1=\ldots=C_{m-1}=M$ for some $m>1$, then

$p \epsilon C_m \iff (\phi^m(p),\phi^m(p))=0$

Proof. (i) Since $(\phi,\phi)=0$, $(\phi,\phi')=0$. Observe that $\phi'(\zeta)=x_{\xi\xi}-ix_{\xi\eta}$ and use Lemma 2.1 to get

$$x_{\xi\xi}=ax_\xi+bx_\eta+B^1(x_\xi,x_\xi)$$

(2.3)

$$x_{\xi\eta}=-bx_\xi+ax_\eta+B^1(x_\xi,x_\eta)$$

for some functions $a,b$. Then, from Prop.1.2, we get immediately $p \epsilon C_1$ iff $(\phi'(p), \phi'(p))=0$

(ii) By induction, since $(\phi^r,\phi^r)\equiv 0 \; \forall r<m$ implies

$$(\phi,\phi^m)=(\phi',\phi^m)=\ldots=(\phi^{m-1},\phi^m)=0,$$

as observed in (i), applying Lemma 2.1 and Prop.1.2, a straightforward calculation shows that $p \epsilon C_m \iff (\phi^m(p),\phi^m(p))=0$

Q.E.D.

Now we prove a generalization of the Eisenhart's theorem:

THEOREM 1

Let $x:M^2 \hookrightarrow R^n$ be a minimal surface. Then $x(M)$ is a holomorphic curve with respect to an orthogonal complex structure on $R^{2m} \subset R^n$ if and only if $C_k=M \forall k \le [\frac{n-1}{2}]$.

Proof. We first invoke a theorem of Lawson [7,p.173] which states that $x(M)$ is a holomorphic curve if and only if $(\phi^r,\phi^s)=0 \quad \forall r,s \ge 0$ $(\phi^0=\phi)$ which is equivalent to $(\phi^r,\phi^r)=0 \; \forall r \ge 0$, and, the generalized

Weierstrass representation formula developed recently by Hoffman-Ossermam [5,p.43]

$$(2.4) \qquad \phi = \frac{f}{2}(1 - \textstyle\sum g_j^2, i(1 + \textstyle\sum g_j^2), 2g_1, \ldots, 2g_{n-2})$$

From the Eisenhart's theorem (an alternative proof can be found in [3]) and Prop.2.2, we see that for $n \leq 4$,

$$(2.5) \qquad (\phi, \phi) = (\phi', \phi') = 0 \implies (\phi^r, \phi^r) = 0 \;\; \forall r \geq 0$$

Now we rewrite (2.4) into

$$(2.6) \qquad \phi = \frac{f}{2}(1 - \textstyle\sum g_j^2, i(1 + \textstyle\sum g_j^2), \mu)$$

with $\mu = (2g_1, \ldots, 2g_{n-2})$, and observe that

$$(2.7) \qquad (\phi, \phi) = \ldots = (\phi^m, \phi^m) = 0 \iff (\mu', \mu') = \ldots = (\mu^m, \mu^m) = 0$$

for $m > 0$.

Therefore, by induction on n, using (2.5) and Prop.2.2, we prove the theorem.

$$\text{Q.E.D.}$$

COROLLARY. x(M) is a holomorphic curve if and only if all the gene ralized curvature ellipses are circles.

REMARKS

1. Using (2.4) and (2.7), we can easily construct examples of minimal surfaces whose curvature ellipses stop from being circles from certain order on, as the one given in the beginning of this section.

2. In the proof of Th.1, we obtain an interesting application of the generalized Weierstrass representation formula.

Next, we study the generalized normal curvatures. For mi nimal surfacez in $R^4$, it's well known [8] that $K_{N_1}(p) \leq -K(p)$ where K

is the Gaussian curvature, and the equality holds iff $p \epsilon C_1$. We extend this result in a general sense. For a general setting, we start with a minimal surface $x:M^2 \hookrightarrow R^n$. Since we are only concerned with local properties, we may assume, without loss of generality, that M is simply connected. Since x is harmonic, let y be its harmonic conjugate. Then

$$(2.8) \qquad \Psi = \frac{\sqrt{2}}{2}(x+iy)$$

represents a holomorphic curve in $C^n = R^n \oplus iR^n = R^{2n}$, which is isometric to x.

In [6], Lawson introduces to holomorphic curves a sequence of intrinsic curvature functions, $K_k^\Psi$, based on the work of Calabi [1,2]. In our setting, we have

$$K_{N_1}^\Psi = \frac{2}{\pi} \text{ area } \epsilon_1^\Psi = 2|B_\Psi^1(e_1,e_1)|^2 = 2K_1^\Psi$$

$(2.9)$

$$\vdots$$

$$K_{N_k}^\Psi = \frac{2}{\pi} \text{ area } \epsilon_k^\Psi = 2|B_\Psi^k(e_1,\ldots,e_1)|^2 = 2K_1^\Psi \ldots K_k^\Psi$$

Since $K_{N_k}^\Psi$'s are intrinsic, we call

$$(2.10) \qquad K_k = -K_{N_k}^\Psi$$

the k-th order generalized Gaussian curvature associated to x. Observe that we don't need M be simply connected to define $K_k$, and $K_1$ is, in fact, the classical Gaussian curvature of $x(M)$. Comparing the normal curvatures and Gaussian curvatures we get

THEOREM 2

Let $x:M^2 \hookrightarrow R^n$ be a minimal surface. Then

(1) $K_{N_1}(p) \leq -K_1(p)$ and equality holds iff $p \epsilon C_1$.

(ii) if $K_{N_j} = -K_j$ on $M \forall j < m$, then at each $p \in M$,

$$K_{N_m}(p) \leq -K_m(p) \text{ and equality holds iff } p \in C_m.$$

Proof. (i) From (2.8), we have

(2.11)
$$|\Psi_\xi| = |\Psi_\eta| = |x_\xi| = |x_\eta| = \lambda$$

and

(2.12)
$$|\Psi_{\xi\xi}|^2 + |\Psi_{\xi\eta}|^2 = |x_{\xi\xi}|^2 + |x_{\xi\eta}|^2$$

(2.13)
$$\langle \Psi_{\xi\xi}, \Psi_\xi \rangle = \langle x_{\xi\xi}, x_\xi \rangle$$

$$\langle \Psi_{\xi\xi}, \Psi_\eta \rangle = \langle x_{\xi\xi}, x_\eta \rangle$$

Comparing (2.3) and from (2.11), (2.12), (2.13) and Lawson [6], we have

$$-K_1(p) = 2K_1^\Psi(p) = |B^1(\frac{\Psi_\xi}{\lambda}, \frac{\Psi_\xi}{\lambda})_p|^2 + |B^1(\frac{\Psi_\xi}{\lambda}, \frac{\Psi_\eta}{\lambda})_p|^2$$

$$= |B^1(\frac{x_\xi}{\lambda}, \frac{x_\xi}{\lambda})_p|^2 + |B^1(\frac{x_\xi}{\lambda}, \frac{x_\eta}{\lambda})_p|^2 \geq K_{N_1}(p)$$

And, from Prop.1.2, the equality holds iff $p \in C_1$. This proves (i).

(ii) Following the same reasoning, we have

(2.14) $$|\Psi_{\xi\cdots\xi}|^2 + |\Psi_{\xi\cdots\xi\eta}|^2 = |x_{\xi\cdots\xi}|^2 + |x_{\xi\cdots\xi\eta}|^2$$

By induction on m, from the hypotheses and Prop.2.2, we have

(2.15) $$(\phi, \phi) = \ldots = (\phi^{m-1}, \phi^{m-1}) = 0$$

Since

$$(\phi_\Psi^r, \phi_\Psi^r) = 0 \quad \forall r \geq 0,$$

we have

$$|B^j(\Psi_\xi, \ldots, \Psi_\xi)| = |B^j(\Psi_\xi, \ldots, \Psi_\xi, \Psi_\eta)| =$$

$$= |B^j(x_\xi, \ldots, x_\xi)| = |B^j(x_\xi, \ldots, x_\xi, x_\eta)|$$

$\forall j < m$. Then a straightforward computation shows that

$$-K_m(p) = |B^m(\frac{\Psi_\xi}{\lambda}, \ldots, \frac{\Psi_\xi}{\lambda})_p|^2 + |B^m(\frac{\Psi_\xi}{\lambda}, \ldots, \frac{\Psi_\xi}{\lambda}, \frac{\Psi_\eta}{\lambda})_p|^2$$

$$= |B^m(\frac{x_\xi}{\lambda}, \ldots, \frac{x_\xi}{\lambda})_p|^2 + |B^m(\frac{x_\xi}{\lambda}, \ldots, \frac{x_\xi}{\lambda}, \frac{x_\eta}{\lambda})_p|^2 \geq K_{N_m}(p)$$

And, again from Prop.2.2, the equality holds iff $p \in C_m$

<div align="right">Q.E.D.</div>

REMARK. This theorems shows a curious fact that, in certain sense, the generalized normal curvatures are bounded by the intrinsic generalized Gaussian curvatures. And the limits are all achieved if and only if the surface is, indeed, a holomorphic curve. Probably, because the sum of the squares of the norms of the semi-axes of each ellipse is an intrinsic invariant, which is $-K_*(p)$.

## §3. SOME COMMENTS

1. Since osculating spaces of all order are taken into considerations, the generalized curvature ellipses should be useful for the study of surfaces with parallel mean curvature vector in constantly curved ambient spaces.

2. In particular, combining with the generalized Gauss map, the generalized curvature ellipses should improve the understanding of minimal surfaces in euclidean spaces.

3. There are lots of questions to be answered about the generalized curvature ellipses. Some studies are being undertaken by several people and the results will be announced soon.

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