GLOBAL ANALYTIC HYPOELLIPTICITY AND SOLVABILITY OF CERTAIN OPERATORS SUBJECT TO GROUP ACTIONS

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Abstract. On $T \times G$, where T is a compact real-analytic manifold and G is a compact Lie group, we consider differential operators P which are invariant by left translations on G and are elliptic in T. Under a mild technical condition, we prove that global hypoellipticity of P implies its global analytic-hypoellipticity (actually Gevrey of any order $s \geq 1$). We also study the connection between the latter property and the notion of global analytic (resp. Gevrey) solvability, but in a much more general setup.

Introduction

Two notoriously difficult problems in PDE theory are to determine whether a general linear differential operator P defined, say, on a compact manifold M, is globally hypoelliptic or globally solvable. In very general terms, given a reasonable space of functions $\mathscr{F}$ on M, the first problem means to determine if we can infer from the information $Pu \in \mathscr{F}$ that u itself belongs to $\mathscr{F}$, where u is a priori taken in some larger space of (generalized) functions. The second one means to solve the equation $Pu = f$ for "every" $f \in \mathscr{F}$, with a solution u also in $\mathscr{F}$ – but a subtlety soon arises, for this is generally impossible as the global geometry of both M and P impose natural constraints on the right-hand side f; we compromise by asking instead if we can always solve at least for those "admissible" $f \in \mathscr{F}$.

In order to make the situation more manageable, extra geometrical hypotheses may be imposed to the problem. A traditional one is to assume M endowed with a smooth action of a Lie group G which leaves P invariant; the action may be further assumed transitive (see e.g. [22, Chapter 5] and related works in the references therein) or free, in which case the operator induced by P on the orbit space M/G plays a key role.

Here we are interested in the latter possibility, and deal with the simplest such situation: M is a product $T \times G$, where T is a compact manifold and G is a compact Lie group, which acts freely on M by left-multiplication on the second factor alone (where G acts on itself). Invariance of P under this action allows us to put it in a global canonical form with separate variables (2.2), where the operator P_0 induced by P on the orbit space T can be easily read off; this one will be further required to be elliptic in T. Operators P satisfying these properties essentially are said to belong to class $\mathcal{T}$ (Definition 2.2) and are our main focus. When $M = \mathbb{T}^n_t \times \mathbb{T}^m_x$, where

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$\mathbb{T}^N$ is the N-dimensional torus, the invariance of P under the action of $\mathbb{T}^m$ means that $P = P(t, D_t, D_x)$ and there is a vast literature about global hypoellipticity and solvability of such operators in this ambient. Requiring the ellipticity condition on P_0 imposes certain restrictions, but still there are important classes of such operators, for instance sums of squares of certain real vector fields (see e.g. [11, 13, 1]).

As for the remaining ingredient – the space of functions $\mathscr{F}$ on which our questions will be posed – we are mainly interested in the space of real-analytic functions (for which we of course need to require further regularity of all the objects involved): global analytic hypoellipticity and solvability of P are then addressed. Actually, our approach applies to the more general Gevrey classes of functions $\mathscr{G}^s$ of order $s \geq 1$ (see e.g. [21]) of which real-analytic functions are a special case ($s = 1$), and one can even guess that the program could be carried forward for other ultradifferentiable classes of Roumieu type (even quasianalytic ones).

As far as the regularity problem is concerned, we consider the following question: when does global hypoellipticity (i.e. in the smooth setup) implies global analytic hypoellipticity? Positive answers for this kind of question first appeared in [11], where the authors proved, in contrast to the famous example due to Baouendi and Goulaouic, that the Hörmander condition (so local hypoellipticity – which is much stronger than global hypoellipticity), together with additional hypotheses (that for us reads as the ellipticity condition for the class $\mathcal{T}$), ensure global analytic hypoellipticity. Similar results in a more general framework can be found in [10, 8]. Our main result about this subject is Theorem 4.2, which ensures that every operator P in class $\mathcal{T}$ that is globally hypoelliptic is also globally analytic hypoelliptic (actually globally $\mathscr{G}^s$-hypoelliptic for every $s \geq 1$). Theorem 4.2 is more properly related to [14, Theorem 2.2] and [20, Theorem 1.10] (and indeed a kind of generalization of them) which deal with the case when both T and G are tori. Their results were extended for more general classes of functions in [2].

The proof of Theorem 4.2 is obtained by first retrieving global estimates from microlocal information (Section 3) and then combining them with some results (Section 4) related to the notion of Gevrey vectors of the partial Laplace-Beltrami operator Δ_G (associated with a suitable metric on G), a business of interest in PDE on its own.

In Section 5 we tackle the issue of solvability from a far more abstract viewpoint. We forget the existence of symmetries and prove (Theorem 5.6) that for a general operator P on a compact manifold M satisfying a regularity property much weaker than global $\mathscr{G}^s$-hypoellipticity we have that the map $P : \mathscr{G}^s(M) \to \mathscr{G}^s(M)$ has closed range – which, we argue, is the correct notion of global $\mathscr{G}^s$-solvability (an analogous result in the smooth category has been recently proved to be true [5]). This result follows from a theory of regularity for abstract operators acting on certain pairs of topological vector spaces started in [3] whose development we continue here. The reader will notice that, thanks to the generality of our approach, the proofs here apply equally well to many other classes of operators: certain pseudodifferential operators acting on $\mathscr{G}^s(M)$ (not necessarily of finite order), and so on; either scalar or vector-valued.

Finally, we turn back to the situation with symmetries linking our class $\mathcal{T}$ with the general property addressed in the abstract results of Section 5: a converse of Theorem 5.6 is proved for operators in that class (Proposition 5.7).

1. Preliminaries

Let M be a real-analytic manifold, assumed throughout to be compact, connected and oriented. The space $\mathscr{G}^s(M)$ of globally defined Gevrey functions [21] can be characterized by means of the powers of a suitable real-analytic elliptic operator [18, 7]: here, we endow M with a real-analytic Riemannian metric (which is always possible [12]) and denote by Δ_M the underlying Laplace-Beltrami operator acting on functions, in which case a smooth f belongs to $\mathscr{G}^s(M)$ if and only if there exist $C, h > 0$ such that

$$\|(I + \Delta_M)^k f\|_{L^2(M)} \leq C h^k k!^{2s}, \quad \forall k \in \mathbb{Z}_+.$$

The case $s = 1$ describes the space of real-analytic functions. As in [3], we furnish topologies to these spaces as follows: for each $h > 0$ we let

$$\mathscr{G}^{s,h}(M) \doteq \left\{ f \in \mathscr{C}^\infty(M) \ ; \ \sup_{k \in \mathbb{Z}_+} h^{-k} k!^{-2s} \|(I + \Delta_M)^k f\|_{L^2(M)} < \infty \right\}$$

which is a Banach space; as the parameter h increases, these are contained into one another in a continuous and compact fashion, meaning that their union $\mathscr{G}^s(M)$, now endowed with the injective limit topology, is what one calls a DFS space.

We denote by $\sigma(\Delta_M)$ the spectrum of Δ_M. By ellipticity of Δ_M, its λ-eigenspace E^M_λ is a finite dimensional subspace of $\mathscr{G}^1(M)$; connectedness of M ensures that $E^M_0 = \mathbb{C}$. Denoting by $\mathcal{F}^M_\lambda : L^2(M) \to E^M_\lambda$ the orthogonal projection we have

$$f = \sum_{\lambda \in \sigma(\Delta_M)} \mathcal{F}^M_\lambda(f) \tag{1.1}$$

with convergence in $L^2(M)$; this is an abstract analog of Fourier series. As such, we extend it to Schwartz distributions: given $f \in \mathscr{D}'(M)$ – which we identify with a continuous linear functional on $\mathscr{C}^\infty(M)$ using the underlying volume form – we let $\mathcal{F}^M_\lambda(f)$ be the unique element in E^M_λ such that

$$\langle \mathcal{F}^M_\lambda(f), \phi \rangle_{L^2(M)} = \langle f, \bar{\phi} \rangle, \quad \forall \phi \in E^M_\lambda.$$

In this situation (1.1) still holds, but with convergence in $\mathscr{D}'(M)$; when f is smooth this convergence takes place in $\mathscr{C}^\infty(M)$, and so on. Thanks to Weyl's asymptotic estimates [9, p. 155], a distribution f belongs to $\mathscr{G}^s(M)$ if and only if there exist $C, h > 0$ such that

$$\|\mathcal{F}^M_\lambda(f)\|_{L^2(M)} \leq C e^{-h(1+\lambda)^{\frac{1}{2s}}}, \quad \forall \lambda \in \sigma(\Delta_M).$$

This allows us to consider alternatively the adapted norms

$$\|f\|_{\tilde{\mathscr{G}}^{s,h}(M)} \doteq \Big(\sum_{\lambda \in \sigma(\Delta_M)} e^{2h(1+\lambda)^{\frac{1}{2s}}} \|\mathcal{F}^M_\lambda(f)\|^2_{L^2(M)} \Big)^{\frac{1}{2}} \tag{1.2}$$

and the spaces

$$\tilde{\mathscr{G}}^{s,h}(M) \doteq \{ f \in \mathscr{C}^\infty(M) \ ; \ \|f\|_{\tilde{\mathscr{G}}^{s,h}(M)} < \infty \} \tag{1.3}$$

which are better suited to some applications (e.g. in the proof of Lemma 3.3); as $h > 0$ increases, this produces new Banach spaces compactly contained in one another and forming a directed system equivalent to the previous one, hence with the same injective limit $\mathscr{G}^s(M)$ – both set-theoretically and topologically. On time, it is also convenient to introduce the following adapted Sobolev norms:

$$\|f\|_{\mathscr{H}^t(M)} \doteq \Big(\sum_{\lambda \in \sigma(\Delta_M)} (1+\lambda)^{2t} \|\mathcal{F}^M_\lambda(f)\|^2_{L^2(M)} \Big)^{\frac{1}{2}}, \quad t > 0. \tag{1.4}$$

When M is a product $T \times G$ of two such manifolds and carrying the product metric, an abstract theory of *partial* Fourier series can also be developed (for details see e.g. [4, 5]). In that case one proves that $\Delta_M = \Delta_T + \Delta_G$ as differential operators on M, that any $\alpha \in \sigma(\Delta_M)$ is of the form $\alpha = \mu + \lambda$ for some $\mu \in \sigma(\Delta_T)$ and $\lambda \in \sigma(\Delta_G)$ (and vice versa), and

$$E^M_\alpha = \bigoplus_{\substack{\mu \in \sigma(\Delta_T) \\ \lambda \in \sigma(\Delta_G) \\ \mu + \lambda = \alpha}} E^T_\mu \otimes E^G_\lambda.$$

Moreover, given $\mu \in \sigma(\Delta_T)$ and $\lambda \in \sigma(\Delta_G)$ we will fix, whenever necessary, bases $\psi^\mu_1, \ldots, \psi^\mu_{d^T_\mu}$ for E^T_μ and $\phi^\lambda_1, \ldots, \phi^\lambda_{d^G_\lambda}$ for E^G_λ, both orthonormal w.r.t. the inner products inherited from $L^2(T), L^2(G)$, respectively, in which case

$$\mathcal{S} \doteq \{\psi^\mu_i \otimes \phi^\lambda_j \ ; \ 1 \leq i \leq d^T_\mu, \ 1 \leq j \leq d^G_\lambda, \ \mu \in \sigma(\Delta_T), \ \lambda \in \sigma(\Delta_G)\}$$

is a Hilbert basis for $L^2(T \times G)$. Now given $f \in \mathscr{D}'(T \times G)$ and $\lambda \in \sigma(\Delta_G)$ one defines an object $\mathcal{F}^G_\lambda(f) \in \mathscr{D}'(T; E^G_\lambda) \cong \mathscr{D}'(T) \otimes E^G_\lambda$ that in terms of our choice of basis can be concretely written as

$$\mathcal{F}^G_\lambda(f) = \sum_{j=1}^{d^G_\lambda} \mathcal{F}^G_\lambda(f)_j \otimes \phi^\lambda_j,$$

where $\mathcal{F}^G_\lambda(f)_j \in \mathscr{D}'(T)$ is defined by $\langle \mathcal{F}^G_\lambda(f)_j, \psi \rangle \doteq \langle f, \psi \otimes \overline{\phi^\lambda_j} \rangle$ for $\psi \in \mathscr{C}^\infty(T)$. The "total" Fourier projection of f can then be recovered from the partial ones as

$$\mathcal{F}^M_\alpha(f) = \sum_{\mu+\lambda=\alpha} \mathcal{F}^T_\mu \mathcal{F}^G_\lambda(f), \quad \alpha \in \sigma(\Delta_M).$$

Gevrey functions can also be described by the "partial" Fourier projections:

Proposition 1.1. *If $f \in \mathscr{G}^s(T \times G)$ then $\mathcal{F}^G_\lambda(f) \in \mathscr{G}^s(T; E^G_\lambda)$ for every λ and*

$$f = \sum_{\lambda \in \sigma(\Delta_G)} \mathcal{F}^G_\lambda(f)$$

with convergence in $\mathscr{G}^s(T \times G)$.

2. A class of invariant operators

Let T be a real-analytic Riemannian manifold, assumed compact, connected and oriented, and let G be a compact, connected Lie group, carrying a Riemannian metric which is ad-invariant [16, Proposition 4.24]. We denote by $\mathfrak{g}$ the Lie algebra associated with G. Let P be a LPDO on $T \times G$ that is invariant by the left action of G, i.e., if $L_g : T \times G \longrightarrow T \times G$ is defined by $L_g(t,x) \doteq (t, gx)$ then for every $u \in \mathscr{C}^\infty(T \times G)$ we have

$$(L_g)^*(Pu) = P[(L_g)^*u], \quad \forall g \in G. \tag{2.1}$$

We call such operators G*-invariant*. By choosing a basis $\mathrm{X}_1, \dots, \mathrm{X}_m \in \mathfrak{g}$ – which we regard as a global frame for the tangent space of G – we may write P as follows:

$$P = \sum_{|\alpha| \le r} P_\alpha \mathrm{X}^\alpha \tag{2.2}$$

where P_α is a LPDO on T and $\mathrm{X}^\alpha = \mathrm{X}_1^{\alpha_1} \dots \mathrm{X}_m^{\alpha_m}$ for each $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{Z}_+^m$.

It is then clear that $[P, \Delta_G] = 0$ – thanks to the general remark that left-invariant vector fields on G always commute with the Laplace-Beltrami operator associated to an ad-invariant metric. As such, P acts as an endomorphism of both $\mathscr{C}^\infty(T; E_\lambda^G)$ and $\mathscr{D}'(T; E_\lambda^G)$ for each $\lambda \in \sigma(\Delta_G)$, which induces by restriction a differential operator $\widehat{P}_\lambda$ on $T \times E_\lambda^G$, the latter regarded as a trivial vector bundle over T. These operators are expressed as follows: given $\psi \in \mathscr{C}^\infty(T; E_\lambda^G)$, written as

$$\psi = \sum_{i=1}^{d_\lambda^G} \psi_i \otimes \phi_i^\lambda, \quad \psi_i \in \mathscr{C}^\infty(T),$$

we have

$$\widehat{P}_\lambda \psi = \sum_{i=1}^{d_\lambda^G} (P_0 \psi_i) \otimes \phi_i^\lambda + \sum_{i=1}^{d_\lambda^G} \sum_{0 < |\alpha| \le r} (P_\alpha \psi_i) \otimes (\mathrm{X}^\alpha \phi_i^\lambda)$$

which we put on canonical form recalling that

$$\mathrm{X}^\alpha \phi_i^\lambda = \sum_{j=1}^{d_\lambda^G} \gamma_{ij}^{\lambda\alpha} \phi_j^\lambda, \quad \gamma_{ij}^{\lambda\alpha} \in \mathbb{C},$$

hence

$$\widehat{P}_\lambda \psi = \sum_{j=1}^{d_\lambda^G} \left(P_0 \psi_j + \sum_{i=1}^{d_\lambda^G} \sum_{0 < |\alpha| \le r} \gamma_{ij}^{\lambda\alpha} P_\alpha \psi_i \right) \otimes \phi_j^\lambda. \tag{2.3}$$

From here on we will assume that P and P_0 have the same order r. Thus the order of P_α is at most $r - |\alpha|$, and expression (2.3) reveals that $\widehat{P}_\lambda$, as a differential operator on the vector bundle $T \times E_\lambda^G$ over T, also has order r: its principal part is essentially the same as that of P_0. Actually, under such circumstances the following identity between their principal symbols holds:

$$\mathrm{Symb}_{(t_0,\tau_0)}(\widehat{P}_\lambda) = \mathrm{Symb}_{(t_0,\tau_0)}(P_0) \cdot \mathrm{id}_{E_\lambda^G}, \quad \forall (t_0, \tau_0) \in T^*T \setminus 0. \tag{2.4}$$

In particular:

Proposition 2.1. *Suppose that P as in (2.2) has the property that P and P_0 have the same order. Then the following are equivalent:*

(1) *P_0 is elliptic in T.*
(2) *$\widehat{P}_\lambda$ is elliptic in $T \times E_\lambda^G$ for some $\lambda \in \sigma(\Delta_G)$.*
(3) *$\widehat{P}_\lambda$ is elliptic in $T \times E_\lambda^G$ for every $\lambda \in \sigma(\Delta_G)$.*

This motivates us to introduce the following class of operators on $T \times G$.

Definition 2.2. We say that a LPDO P on $T \times G$ belongs to *class* $\mathcal{T}$ if

(1) P is G-invariant,
(2) P and P_0 in (2.2) have the same order and
(3) P_0 is elliptic in T.

2.1. **Examples.**

2.1.1. *Vector fields.* Consider Y a vector field on $T \times G$ that can be written as

$$\mathrm{Y} = \mathrm{W} + \sum_{j=1}^{m} a_j(t)\mathrm{X}_j \tag{2.5}$$

where $a_1, \dots, a_m \in \mathscr{C}^\infty(T)$ and $\mathrm{W} \in \mathfrak{X}(T)$ is a complex vector field. Notice that W has the same order of Y unless it vanishes identically. Therefore, Y belongs to $\mathcal{T}$ if and only if W is elliptic on T.

When W is a real vector field the latter condition forces T to be one-dimensional i.e. $T = S^1$, in which case W is a non-vanishing multiple of ∂_t, and Y is equivalent to a vector field of the form

$$\partial_t + \sum_{j=1}^{m} a_j(t)\mathrm{X}_j \quad \text{on } S^1 \times G.$$

The case $\dim T = 2$ is also possible but in that case W must be complex with $\mathsf{Re}\mathrm{W}$ and $\mathsf{Im}\mathrm{W}$ linearly independent everywhere; if we additionally assume that these real vector fields commute then one can endow T with the structure of a Riemann surface such that W is essentially the Cauchy-Riemann operator i.e. Y is of the form

$$\partial_{\bar{z}} + \sum_{j=1}^{m} a_j(t)\mathrm{X}_j \quad \text{on } T \times G.$$

When $\dim T \geq 3$ no vector field Y on $T \times G$ will belong to $\mathcal{T}$ as in that case W can never be elliptic.

2.1.2. *Certain second-order operators.* Let Q be a second-order operator on T and

$$\mathrm{Y}_\ell \doteq \mathrm{W}_\ell + \sum_{j=1}^{m} a_{\ell j}(t)\mathrm{X}_j, \quad \ell \in \{1, \dots, N\},$$

be vector fields of the form (2.5). Then

$$P \doteq Q - \sum_{\ell=1}^{N} \mathrm{Y}_\ell^2$$

is G-invariant, and moreover belongs to $\mathcal{T}$ if and only if

$$Q - \sum_{\ell=1}^{N} \mathrm{W}_\ell^2$$

is elliptic of order 2 on T. Real operators in this class were investigated e.g. when $Q = \Delta_T$ [4, 5]; or when $Q = 0$ and $\mathrm{W}_1, \dots, \mathrm{W}_N$ span the tangent bundle of T everywhere [8] (further references therein).

3. From microlocal analysis to global Gevrey estimates

Lemma 3.1. *Let P be a real-analytic LPDO on $T \times G$ belonging to class $\mathcal{T}$. If $u \in \mathscr{D}'(T \times G)$ is such that $Pu \in \mathscr{G}^s(T \times G)$ then for every $\phi \in \mathscr{G}^s(G)$ we have that $\tilde{u}(\phi) \doteq \langle u, \cdot \otimes \phi \rangle \in \mathscr{G}^s(T)$.*

Proof. Fix $t \in T$ and notice that $\mathrm{WF}_s(u)$ does intercept the conormal bundle of $\{t\} \times G$. Indeed, a covector $(t, \tau, x, \xi) \in T^*(T \times G)$ annihilates $T_{(t,x)}(\{t\} \times G)$ if and only if $\xi = 0$, and none of these belongs to the characteristic set of P: as one can easily compute,

$$\mathrm{Symb}_{(t,\tau,x,0)}(P) = \mathrm{Symb}_{(t,\tau)}(P_0)$$

and P_0 is elliptic in T by assumption. Our claim follows since $\mathrm{WF}_s(u) \subset \mathrm{Char}(P)$ [15, Theorem 5.1]. We are then allowed to restrict u to $\{t\} \times G$, i.e. to pull it back via the map $x \in G \mapsto (t, x) \in T \times G$, yielding in this way a distribution $u_t \in \mathscr{D}'(G)$, and by [15, Theorem 4.1] the function $t \in T \mapsto \langle u_t, \phi \rangle \in \mathbb{C}$ belongs to $\mathscr{G}^s(T)$ whatever $\phi \in \mathscr{G}^s(G)$. That function is, however, none other than the distribution $\psi \in \mathscr{C}^\infty(T) \mapsto \langle u, \psi \otimes \phi \rangle \in \mathbb{C}$. □

Corollary 3.2. *Under the hypotheses of Lemma 3.1 we have $\mathcal{F}^G_\lambda(u) \in \mathscr{G}^s(T; E^G_\lambda)$ for every $\lambda \in \sigma(\Delta_G)$.*

Proof. Apply Lemma 3.1 to a basis of E^G_λ. □

For the next result, given $\theta \in (0, 1)$ we define the set

$$\Gamma_\theta \doteq \{(\mu, \lambda) \in \sigma(\Delta_T) \times \sigma(\Delta_G) \ ; \ \lambda \leq \theta\mu\}.$$

Lemma 3.3. *Suppose that $u \in \mathscr{D}'(T \times G)$ is such that $\tilde{u}(\phi) \in \mathscr{G}^s(T)$ for every $\phi \in \mathscr{G}^s(G)$. Then there exist $C, h > 0$ and $\theta \in (0, 1)$ such that*

$$\|\mathcal{F}^T_\mu \mathcal{F}^G_\lambda(u)\|_{L^2(T\times G)} \leq C e^{-h(1+\mu+\lambda)^{\frac{1}{2s}}}, \quad \forall (\mu, \lambda) \in \Gamma_\theta. \tag{3.1}$$

Proof. We employ the DFS space characterization of the Gevrey spaces as in (1.2)-(1.3). By hypothesis we have $\tilde{u}(\tilde{\mathscr{G}}^{s,1}(G)) \subset \mathscr{G}^s(T)$ hence the induced linear map $\tilde{u} : \tilde{\mathscr{G}}^{s,1}(G) \to \mathscr{G}^s(T)$ is continuous by De Wilde's Closed Graph Theorem [19, p. 57] (whose applicability is granted by the fact that Banach spaces are ultrabornological and DFS spaces are webbed): indeed, notice that its graph is closed thanks to the continuity of $\tilde{u} : \mathscr{C}^\infty(G) \to \mathscr{D}'(T)$. As such, $\tilde{u}$ maps bounded sets in $\tilde{\mathscr{G}}^{s,1}(G)$ to bounded sets in $\mathscr{G}^s(T)$ so by [17, Lemma 3] we may assert the existence of an $h > 0$ such that

$$\tilde{u}(\{\phi \in \tilde{\mathscr{G}}^{s,1}(G) \ ; \ \|\phi\|_{\tilde{\mathscr{G}}^{s,1}(G)} \leq 1\}) \subset \tilde{\mathscr{G}}^{s,h}(T).$$

Of course, we may assume that $h > 2$. By linearity, $\tilde{u}(\tilde{\mathscr{G}}^{s,1}(G)) \subset \tilde{\mathscr{G}}^{s,h}(T)$ and again $\tilde{u} : \tilde{\mathscr{G}}^{s,1}(G) \to \tilde{\mathscr{G}}^{s,h}(T)$ is continuous by the Closed Graph Theorem (the classical one). Therefore, there exists a constant $C > 0$ such that

$$\|\tilde{u}(\phi)\|_{\tilde{\mathscr{G}}^{s,h}(T)} \leq C\|\phi\|_{\tilde{\mathscr{G}}^{s,1}(G)}, \quad \forall \phi \in \tilde{\mathscr{G}}^{s,1}(G).$$

When we take $\phi = \overline{\phi^\lambda_j}$ – an element of our orthonormal basis of E^G_λ – we obtain

$$\|\tilde{u}(\overline{\phi^\lambda_j})\|^2_{\tilde{\mathscr{G}}^{s,h}(T)} = \sum_{\mu \in \sigma(\Delta_T)} e^{2h(1+\mu)^{\frac{1}{2s}}} \sum_{i=1}^{d^T_\mu} |\langle u, \overline{\psi^\mu_i \otimes \phi^\lambda_j} \rangle|^2$$

hence

$$\sum_{j=1}^{d^G_\lambda} \|\tilde{u}(\overline{\phi^\lambda_j})\|^2_{\tilde{\mathscr{G}}^{s,h}(T)} = \sum_{\mu \in \sigma(\Delta_T)} e^{2h(1+\mu)^{\frac{1}{2s}}} \|\mathcal{F}^T_\mu \mathcal{F}^G_\lambda(u)\|^2_{L^2(T\times G)}$$

from which we conclude that

$$e^{2h(1+\mu)^{\frac{1}{2s}}} \|\mathcal{F}_\mu^T \mathcal{F}_\lambda^G(u)\|^2_{L^2(T\times G)} \leq \sum_{j=1}^{d_\lambda^G} \|\tilde{u}(\overline{\phi_j^\lambda})\|^2_{\tilde{\mathscr{G}}^{s,h}(T)} \leq d_\lambda^G C^2 e^{2(1+\lambda)^{\frac{1}{2s}}}$$

where we used that $\|\overline{\phi_j^\lambda}\|_{\tilde{\mathscr{G}}^{s,1}(G)} = e^{(1+\lambda)^{\frac{1}{2s}}}$.

By Weyl's asymptotic formula $d_\lambda^G = O(\lambda^{m/2})$ we have, enlarging C if necessary,

$$e^{2h(1+\mu)^{\frac{1}{2s}}} \|\mathcal{F}_\mu^T \mathcal{F}_\lambda^G(u)\|^2_{L^2(T\times G)} \leq C^2 e^{4(1+\lambda)^{\frac{1}{2s}}}, \quad \forall (\mu,\lambda) \in \sigma(\Delta_T)\times\sigma(\Delta_G).$$

For $(\mu,\lambda)\in\Gamma_\theta$ we then have $1+\lambda \leq 1+\mu$ so

$$\|\mathcal{F}_\mu^T \mathcal{F}_\lambda^G(u)\|_{L^2(T\times G)} \leq C e^{2(1+\lambda)^{\frac{1}{2s}} - h(1+\mu)^{\frac{1}{2s}}} \leq C e^{(2-h)(1+\mu)^{\frac{1}{2s}}}$$

but also $1+\mu+\lambda \leq 2(1+\mu)$ hence

$$\|\mathcal{F}_\mu^T \mathcal{F}_\lambda^G(u)\|_{L^2(T\times G)} \leq C e^{-h'(1+\mu+\lambda)^{\frac{1}{2s}}}$$

where $h' \doteq (h-2)/(2^{\frac{1}{2s}})$. ☐

Proposition 3.4. *If $u \in \mathscr{D}'(T\times G)$ is such that*

1. *there exist $C, h > 0$ and $\theta \in (0,1)$ such that (3.1) holds and*
2. *there exist $C', h' > 0$ such that*

$$\|\mathcal{F}_\lambda^G(u)\|_{L^2(T\times G)} \leq C' e^{-h'(1+\lambda)^{\frac{1}{2s}}}, \quad \forall \lambda \in \sigma(\Delta_G) \tag{3.2}$$

then $u \in \mathscr{G}^s(T\times G)$.

Proof. For $(\mu,\lambda)\in\sigma(\Delta_T)\times\sigma(\Delta_G)$ not in Γ_θ we have, since $1/\theta > 1$, that

$$(1+\lambda)^{\frac{1}{2s}} \geq (\theta/2)^{\frac{1}{2s}} (1+\mu+\lambda)^{\frac{1}{2s}}.$$

We conclude that

$$\|\mathcal{F}_\mu^T \mathcal{F}_\lambda^G(u)\|_{L^2(T\times G)} \leq \begin{cases} C' e^{-h'(\theta/2)^{\frac{1}{2s}}(1+\mu+\lambda)^{\frac{1}{2s}}}, & \text{in } \Gamma_\theta^c \\ C e^{-h(1+\mu+\lambda)^{\frac{1}{2s}}}, & \text{in } \Gamma_\theta \end{cases}$$

and therefore $u \in \mathscr{G}^s(T\times G)$. ☐

4. Gevrey vectors for the partial Laplacian

Proposition 4.1. *Let P be a real-analytic LPDO on $T\times G$ that is globally hypoelliptic and commutes with Δ_G. If $u\in\mathscr{C}^\infty(T\times G)$ is such that $Pu\in\mathscr{G}^s(T\times G)$ then u is a s-*Gevrey vector *for Δ_G, that is, there exist $C,h>0$ such that*

$$\|\Delta_G^k u\|_{L^2(T\times G)} \leq C h^k k!^{2s} \tag{4.1}$$

for every $k\in\mathbb{Z}_+$.

Proof. We follow an approach similar to that of [8], for which we start with some preliminary remarks. Let $u\in\mathscr{C}^\infty(T\times G)$ be such that $Pu\in\mathscr{G}^s(T\times G)$. Since P is globally hypoelliptic, by standard functional analytic arguments (see e.g. [6, Lemma 3.1]) there exists $t\in\mathbb{R}$ and $C_1>0$ such that

$$\|\Delta_G^k u\|_{L^2(T\times G)} \leq C_1(\|P\Delta_G^k u\|_{\mathscr{H}^t(T\times G)} + \|\Delta_G^k u\|_{\mathscr{H}^{-1}(T\times G)}), \quad \forall k\in\mathbb{Z}_+.$$

We make use of the Sobolev norms (1.4). Then the last term in the inequality above can be estimated as follows:

$$\|\Delta_G^{k+1}u\|^2_{\mathscr{H}^{-1}(T\times G)} = \sum_{\substack{\mu\in\sigma(\Delta_T)\\ \lambda\in\sigma(\Delta_G)}} \frac{\lambda^2}{(1+\mu+\lambda)^2}\|\mathcal{F}^T_\mu\mathcal{F}^G_\lambda(\Delta_G^k u)\|^2_{L^2(T\times G)} \leq \|\Delta_G^k u\|^2_{L^2(T\times G)},$$

for every $k\in\mathbb{Z}_+$. Moreover, as P commutes with Δ_G we have

$$\|P\Delta_G^k u\|^2_{\mathscr{H}^t(T\times G)} = \|\Delta_G^k Pu\|^2_{\mathscr{H}^t(T\times G)} = \sum_{\substack{\mu\in\sigma(\Delta_T)\\ \lambda\in\sigma(\Delta_G)}} (1+\mu+\lambda)^{2t}\lambda^{2k}\|\mathcal{F}^T_\mu\mathcal{F}^G_\lambda(Pu)\|^2_{L^2(T\times G)}$$

and since $Pu\in\mathscr{G}^s(T\times G)$ there exists $A>0$ such that

$$\|\mathcal{F}^T_\mu\mathcal{F}^G_\lambda(Pu)\|_{L^2(T\times G)} \leq A^{\ell+1}\ell!^{2s}(1+\mu+\lambda)^{-\ell}, \quad \forall\ell\in\mathbb{Z}_+,\ \mu\in\sigma(\Delta_T),\ \lambda\in\sigma(\Delta_G)$$

hence

$$\|P\Delta_G^k u\|^2_{\mathscr{H}^t(T\times G)} \leq (A^{\ell+1}\ell!^{2s})^2 \sum_{\substack{\mu\in\sigma(\Delta_T)\\ \lambda\in\sigma(\Delta_G)}} (1+\mu+\lambda)^{2t-2\ell}\lambda^{2k}. \tag{4.2}$$

We choose $\ell_0\in\mathbb{Z}_+$ such that

$$B \doteq \Bigg(\sum_{\substack{\mu\in\sigma(\Delta_T)\\ \lambda\in\sigma(\Delta_G)}} (1+\mu+\lambda)^{2t-2\ell_0}\Bigg)^{1/2} < \infty$$

which always exists thanks to Weyl's asymptotic formula, so plugging $\ell=\ell_0+k$ into (4.2) yields:

$$\begin{aligned}\|P\Delta_G^k u\|^2_{\mathscr{H}^t(T\times G)} &\leq (A^{\ell_0+k+1}(\ell_0+k)!^{2s})^2 \sum_{\substack{\mu\in\sigma(\Delta_T)\\ \lambda\in\sigma(\Delta_G)}} (1+\mu+\lambda)^{2t-2\ell_0}\frac{\lambda^{2k}}{(1+\mu+\lambda)^{2k}}\\ &\leq B^2(A^{\ell_0+k+1}(\ell_0+k)!^{2s})^2\end{aligned}$$

and in particular we have that, for every $k\in\mathbb{Z}_+$,

$$\begin{aligned}\|\Delta_G^{k+1}u\|_{L^2(T\times G)} &\leq C_1(BA^{\ell_0+k+2}(\ell_0+k+1)!^{2s} + \|\Delta_G^k u\|_{L^2(T\times G)})\\ &\leq C_1(BA^{\ell_0+k+2}4^{s(\ell_0+k+1)}\ell_0!^{2s}(k+1)!^{2s} + \|\Delta_G^k u\|_{L^2(T\times G)}).\end{aligned} \tag{4.3}$$

We are in position to assume by induction that there exist $C,h>0$ such that (4.1) holds up to a certain $k_0\in\mathbb{Z}_+$: we will check that it also holds for $k=k_0+1$. We may assume w.l.o.g. that

$$C > 2C_1BA^{\ell_0+1}4^{s\ell_0}\ell_0!^{2s}, \quad h>\max\{2C_1, A4^s\}. \tag{4.4}$$

By (4.3) we have

$$\|\Delta_G^{k_0+1}u\|_{L^2(T\times G)} \leq C_1(BA^{\ell_0+k_0+2}4^{s(\ell_0+k_0+1)}\ell_0!^{2s}(k_0+1)!^{2s} + Ch^{k_0}k_0!^{2s})$$

hence

$$\frac{\|\Delta_G^{k_0+1}u\|_{L^2(T\times G)}}{Ch^{k_0+1}(k_0+1)!^{2s}} \leq \frac{C_1BA^{\ell_0+1}4^{s\ell_0}\ell_0!^{2s}}{C}\left(\frac{A4^s}{h}\right)^{k_0+1} + \frac{C_1}{h(k_0+1)^{2s}}$$

is less than 1 thanks to (4.4). This concludes our proof. □

Notice that if $u \in \mathscr{C}^\infty(T \times G)$ is a s-Gevrey vector for Δ_G then

$$\lambda^k \|\mathcal{F}^G_\lambda(u)\|_{L^2(T\times G)} = \|\mathcal{F}^G_\lambda(\Delta^k_G u)\|_{L^2(T\times G)} \leq \|\Delta^k_G u\|_{L^2(T\times G)} \leq Ch^k k!^{2s}$$

for some $C, h > 0$ independent of both $\lambda \in \sigma(\Delta_G)$ and $k \in \mathbb{Z}_+$. Assuming that $h > 1$ we obtain

$$(1+\lambda)^k \|\mathcal{F}^G_\lambda(u)\|_{L^2(T\times G)} \leq C(2h)^k k!^{2s},$$

which, by standard computations, is equivalent to (3.2) for some constants $C', h' > 0$. Summing up with the results from Section 3 we immediately get:

Theorem 4.2. *Let P be a real-analytic LPDO on $T \times G$ belonging to class $\mathcal{T}$. If P is globally hypoelliptic in $T \times G$ then for each $s \geq 1$ it is also globally $\mathscr{G}^s$-hypoelliptic:*

$$\forall u \in \mathscr{D}'(T \times G),\ Pu \in \mathscr{G}^s(T \times G) \Longrightarrow u \in \mathscr{G}^s(T \times G).$$

5. A general result and an application on global Gevrey solvability

Our goal in this section is to introduce an abstract notion of hypoellipticity of an operator and show that this notion implies closedness of range of the same operator. As a consequence of this relationship we prove that a very weak notion of Gevrey hypoellipticity is sufficient to global solvability in the corresponding setup.

We take a look at the *category of pairs* of topological vector spaces: its objects are 2-tuples of topological vector spaces $(E^\sharp, E)$, where E is a linear subspace of $E^\sharp$ carrying a topology finer than that inherited from $E^\sharp$, while its morphisms are *maps of pairs* $T : (E^\sharp, E) \to (F^\sharp, F)$, meaning that $T : E^\sharp \to F^\sharp$ is a continuous linear map such that $T(E) \subset F$ and the induced map $T : E \to F$ is continuous. In this context:

Definition 5.1. We shall say that T satisfies:

- *property* $(\mathcal{H})$ if for every $u \in E^\sharp$ such that $Tu \in F$ we have that $u \in E$;
- *property* $(\mathcal{H}')$ if for every $u \in E^\sharp$ such that $Tu \in F$ there exists $v \in E$ such that $Tv = Tu$.

Clearly $(\mathcal{H})$ holds if and only if T satisfies both $(\mathcal{H}')$ and $\ker T \subset E$. Moreover, one has that $(E^\sharp/\ker T, E/(E \cap \ker T))$ is also a pair of topological vector spaces and T descends to the quotient as a map of pairs

$$T' : (E^\sharp/\ker T, E/(E\cap \ker T)) \longrightarrow (F^\sharp, F) \tag{5.1}$$

and a simple argument shows that T satisfies $(\mathcal{H}')$ if and only if T' satisfies $(\mathcal{H})$.

Our goal is to investigate closedness of the range of $T : E \to F$ as a continuous linear map.

Lemma 5.2. *Let $T : (E^\sharp, E) \to (F^\sharp, F)$ be a map of pairs of topological vector spaces with $F^\sharp$ Hausdorff. If T satisfies $(\mathcal{H})$ then the graph of $T : E \to F$ is closed in $E^\sharp \times F$.*

Proof. We prove that the range of the continuous map

$$\gamma_T : E \ni u \longmapsto (u, Tu) \in E^\sharp \times F$$

is closed. Take $\{u_\alpha\}$ a net in E such that $(u_\alpha, Tu_\alpha) \to (u, f)$ in $E^\sharp \times F$ for some $(u, f) \in E^\sharp \times F$. Then:

- $u_\alpha \to u$ in $E^\sharp$, and as $T : E^\sharp \to F^\sharp$ is continuous we have $Tu_\alpha \to Tu$ in $F^\sharp$;
- $Tu_\alpha \to f$ in F, hence also in $F^\sharp$.

Since $F^\sharp$ is Hausdorff we have that $Tu = f \in F$: therefore $u \in E$ thanks to property $(\mathcal{H})$, hence $(u, f) = (u, Tu) \in \operatorname{ran} \gamma_T$. ☐

From here on we focus on the particular situation in which $E, E^\sharp, F$ are DFS spaces, and we fix $\{E_j\}_{j\in\mathbb{Z}_+}$, $\{E^\sharp_k\}_{k\in\mathbb{Z}_+}$, $\{F_k\}_{k\in\mathbb{Z}_+}$ injective sequences of Banach spaces with compact inclusion maps whose injective limits are $E, E^\sharp, F$ respectively. We also assume the following condition (stronger than $E \hookrightarrow E^\sharp$):

$$E \hookrightarrow E^\sharp_0 \text{ continuously.} \tag{5.2}$$

Theorem 5.3. *Let $E, E^\sharp, F$ be DFS as above and $F^\sharp$ be a Hausdorff space. If T satisfies $(\mathcal{H})$ then $T : E \to F$ has closed range.*

Proof. By Lemma 5.2 the graph map γ_T has closed range. Since both E and $E^\sharp \times F$ are DFS spaces, and moreover γ_T is injective, the following criterion for closedness of its range applies [3, Lemma 2.3]: for every $k \in \mathbb{Z}_+$ there exists $j \in \mathbb{Z}_+$ such that

$$\forall u \in E,\ \gamma_T(u) \in E^\sharp_k \times F_k \Longrightarrow u \in E_j$$

which, thanks to (5.2) and the definition of γ_T, is equivalent to

$$\forall u \in E,\ Tu \in F_k \Longrightarrow u \in E_j$$

in turn implying that $T : E \to F$ has closed range [3, Theorem 2.5]. ☐

Since the class of DFS spaces is closed under taking closed subspaces and quotients [17], the whole picture above is preserved if we replace T by T' (5.1). For instance, $H \doteq E/(E \cap \ker T)$ is a DFS space; actually, $H_j \doteq E_j/(E_j \cap \ker T)$ is a Banach space, the inclusion map $E_j \hookrightarrow E_{j'}$ $(j < j')$ descends to a compact injection $H_j \hookrightarrow H_{j'}$, and we have $H \cong \varinjlim H_j$ where we regard H_j as a subspace of H by means of the identification

$$u + (E_j \cap \ker T) \in H_j \longmapsto u + (E \cap \ker T) \in H$$

(which the reader may check at once to be well-defined, continuous and injective). The same goes for $E^\sharp/\ker T$, and (5.2) descends to a continuous injection $E/(E \cap \ker T) \hookrightarrow E^\sharp_0/(E^\sharp_0 \cap \ker T)$ – the first step in the sequence that naturally defines the injective limit topology on $E^\sharp/\ker T$.

Corollary 5.4. *If T satisfies $(\mathcal{H}')$ then $T : E \to F$ has closed range.*

Proof. In that case, T' (5.1) satisfies $(\mathcal{H})$: by our previous digression, we are entitled to apply Theorem 5.3 to it, yielding closedness of the range of $T' : E/(E \cap \ker T) \to F$, which equals that of $T : E \to F$. ☐

5.1. **Global Gevrey solvability.** Let M be a compact real-analytic manifold as in Section 1. Given $s \geq 1$, we denote by $\mathscr{D}'_s(M)$ the topological dual of $\mathscr{G}^s(M)$, the so-called space of Gevrey ultradistributions of order s (when $s = 1$ this is the space of hyperfunctions on M).

Given P a Gevrey LPDO on M, we are interested in solving the equation $Pu = f$ for $f \in \mathscr{G}^s(M)$. If there exists a solution $u \in \mathscr{G}^s(M)$ to this problem then for any $v \in \mathscr{D}'_s(M)$ we have

$$\langle v, f\rangle = \langle v, Pu\rangle = \langle {}^{\mathrm{t}}Pv, u\rangle$$

where ${}^{\mathrm{t}}P$ denotes the transpose of P. A necessary condition on f to the solvability of $Pu = f$ is then that

$$\langle v, f\rangle = 0 \quad \text{for every } v \in \mathscr{D}'_s(M) \text{ such that } {}^{\mathrm{t}}Pv = 0. \tag{5.3}$$

This leads us to the following:

Definition 5.5. We say that P is *globally $\mathscr{G}^s$-solvable* if for every $f \in \mathscr{G}^s(M)$ satisfying (5.3) there exists $u \in \mathscr{G}^s(M)$ such that $Pu = f$.

It turns out [3, Lemma 2.2] that P is globally $\mathscr{G}^s$-solvable if and only if the map between DFS spaces $P : \mathscr{G}^s(M) \to \mathscr{G}^s(M)$ has closed range. In [5, Theorem 3.5] we proved that the following property, weaker that global hypoellipticity, implies global solvability of P in the smooth setting (i.e. closedness of the range of $P : \mathscr{C}^\infty(M) \to \mathscr{C}^\infty(M)$):

$$\forall u \in \mathscr{D}'(M),\ Pu \in \mathscr{C}^\infty(M) \Longrightarrow \exists v \in \mathscr{C}^\infty(M) \text{ such that } Pv = Pu.$$

Corollary 5.4 gives, in the Gevrey framework, that a weak notion of global Gevrey hypoellipticity implies global solvability.

Theorem 5.6. *Let $s \geq 1$ and suppose that for some $s_+ > s$ we have*

$$\forall u \in \mathscr{G}^{s_+}(M),\ Pu \in \mathscr{G}^s(M) \Longrightarrow \exists v \in \mathscr{G}^s(M) \text{ such that } Pv = Pu. \tag{5.4}$$

Then P is globally $\mathscr{G}^s$-solvable.

Proof. Let

$$E \doteq \mathscr{G}^s(M), \quad E^\sharp \doteq \mathscr{G}^{s_+}(M), \quad F \doteq \mathscr{G}^s(M), \quad F^\sharp \doteq \mathscr{D}'_s(M).$$

Since with these choices (5.4) is per se property $(\mathcal{H}')$ for $T \doteq P$, all the hypotheses of Corollary 5.4 are automatically fulfilled. □

The converse of Theorem 5.6 holds for operators in the class $\mathcal{T}$:

Proposition 5.7. *Let P be a real-analytic LPDO on $T \times G$ in class $\mathcal{T}$. If P is globally $\mathscr{G}^s$-solvable then*

$$\forall u \in \mathscr{D}'(T \times G),\ Pu \in \mathscr{G}^s(T \times G) \Longrightarrow \exists v \in \mathscr{G}^s(T \times G) \text{ such that } Pv = Pu.$$

Proof. Let $u \in \mathscr{D}'(T \times G)$ be such that $f \doteq Pu \in \mathscr{G}^s(T \times G)$. Thanks to Proposition 1.1 we have

$$f = \sum_{\lambda \in \sigma(\Delta_G)} \mathcal{F}^G_\lambda(f) = \lim_{\nu \to \infty} \sum_{|\lambda| \leq \nu} \widehat{P}_\lambda \mathcal{F}^G_\lambda(u) = \lim_{\nu \to \infty} P \sum_{|\lambda| \leq \nu} \mathcal{F}^G_\lambda(u) \tag{5.5}$$

with convergence in $\mathscr{G}^s(T \times G)$. Since each $\widehat{P}_\lambda$ is elliptic we have that $\widehat{P}_\lambda \mathcal{F}^G_\lambda(u) = \mathcal{F}^G_\lambda(f) \in \mathscr{G}^s(T; E^G_\lambda)$ implies that $\mathcal{F}^G_\lambda(u) \in \mathscr{G}^s(T; E^G_\lambda)$, hence (5.5) ensures that f belongs to the closure of the range of $P : \mathscr{G}^s(T \times G) \to \mathscr{G}^s(T \times G)$. As the latter is closed in $\mathscr{G}^s(T \times G)$ by assumption, there exists $v \in \mathscr{G}^s(T \times G)$ such that $Pv = f = Pu$. □

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