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Jordan algebras versus associative algebras

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JORDAN ALGEBRAS VERSUS ASSOCIATIVE ALGEBRAS

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1. Introduction

1.1. The problem. Our paper is devoted to the problem of classification of Jordan bimodules over finite dimensional Jordan algebras. One of our goals is to draw the attention of experts on Jordan algebras to this problem and the attention of experts in representation theory to a very natural and important class of finite dimensional algebras, namely the universal multiplication and special universal envelopes of finite dimensional Jordan algebras. The authors hope that the results of this paper help to fill the gap between these beautiful theories.

We assume the base field $\mathbb{k}$ to be algebraically closed and of characteristics $\neq 2,3$. Recall ([7]), that for a Jordan algebra $\mathcal{J}$ the category $\mathcal{J}-$bimod of $\mathbb{k}$-finite dimensional $\mathcal{J}$-bimodules is equivalent to the category $\mathrm{U}-$mod of (left) finitely dimensional modules over an associative algebra $\mathrm{U}=\mathrm{U}(\mathcal{J})$, which is called the universal multiplication envelope of $\mathcal{J}$. The algebra U is finite dimensional, provided that $\mathcal{J}$ is finite dimensional. It allows us to apply to the category $\mathcal{J}-$bimod all the machinery developed in the representation theory of finite dimensional algebras. In particular, in accordance with the representation type of the algebra U ([4]) one can define Jordan algebras of the finite, tame and wild representation types. As in the case of associative algebra the distinction of the objects of finite and tame representation type is an interesting problem, especially because in these cases we can obtain a complete classification of finite dimensional bimodules over $\mathcal{J}$.

Recall, that the algebra $\mathrm{U}=\mathrm{U}(\mathcal{J})$ decomposes into the product of subalgebras

$$\mathrm{U}=\mathrm{U}_0\times\mathrm{U}_{\frac{1}{2}}\times\mathrm{U}_1,$$

where U_0 is isomorphic to $\mathbb{k}$, $\mathrm{U}_0\oplus\mathrm{U}_{\frac{1}{2}}$ is isomorphic to the so called special universal envelope of the Jordan algebra $\mathrm{S}\,(=\mathrm{S}(\mathcal{J}))$ and U_1 is the so called universal unital multiplication envelope of $\mathcal{J}$. It induces for every $M\in\mathrm{U}-$mod the canonical decomposition $M\simeq M_0\oplus M_{\frac{1}{2}}\oplus M_1$. Moreover $\mathrm{U}-$mod is equivalent to U_0-mod $\oplus\mathrm{U}_{\frac{1}{2}}-$mod $\oplus\mathrm{U}_1-$mod. This splits the problem of defining the representation type of U into the same problem for the algebras S and U_1. Both these problems are of independent interest, but in this paper we investigate the representation type of the algebras S.

There is a classical result([6]), that if $\mathcal{J}$ is a semisimple Jordan algebra, then the algebra $\mathrm{U}(\mathcal{J})$ is a semisimple associative algebra, hence it is of finite representation type. The class of Jordan algebras $\mathcal{J}$ such that $\mathrm{Rad}^2\mathcal{J}=0$, where $\mathrm{Rad}\,\mathcal{J}$ is the Jacobson radical of $\mathcal{J}$, is in some sense closest to the semisimple algebras. For associative algebras analogous to the class of algebras that were considered in the papers [8], [11] (algebras of finite type) and in [3], [12] (algebras of tame type) it was the starting point for a solution the problem of classification of associative algebras of finite representation type. It is interesting that the problem of classification of the indecomposable modules over algebras with radical square zero and so called path algebras is reduced to the classification of the so called quiver representation ([8], [11]). As we show below, this fact has an astonishing analogue for matrix Jordan algebras.

Among other things, in this paper we describe the finite dimensional Jordan algebras $\mathfrak{J} = \mathfrak{L} \oplus \operatorname{Rad}\mathfrak{J}$ such that

(1) $\mathfrak{J}/\operatorname{Rad}\mathfrak{J}$ ($\simeq \mathfrak{L}$) is a semisimple matrix Jordan algebra.
(2) $\operatorname{Rad}^2\mathfrak{J} = 0$.
(3) The associative algebra $\mathrm{S}(\mathfrak{J})$ is of finite or tame representation type.

The main technical difficulty in applying classification results from the theory of associative algebras to algebras of the form $\mathrm{S}(\mathfrak{J})$ is the following: Almost all criterions of finiteness or tameness for associative algebras are formulated in terms of the basic (i.e. Morita reduced) algebras, represented as a quiver with relations. But the algebras $\mathrm{S}(\mathfrak{J})$ are not usually Morita reduced. Hence, to be able to apply a classification result from the representation theory of associative algebras to $\mathrm{S}(\mathfrak{J})$ one should find an idempotent $e \in \mathrm{S}(\mathfrak{J})$ such that the algebra $e\mathrm{S}(\mathfrak{J})e$ is basic and Morita equivalent to $\mathrm{S}(\mathfrak{J})$.

1.2. **Overview of the contents.** In Section 2 we recall briefly the definition of Jordan bimodules and some necessary properties of special universal and universal multiplication envelopes of a Jordan algebra.

In Section 3 we collect the necessary general facts from the theory of representations of finite dimensional associative algebras. In particular, we introduce here the representation of an associative algebra as a quiver with relations and describe an explicit construction of the Morita reduced algebra for an algebra, given by generators and relations (Lemma 3.3), which we apply later. We also give some conditions, under which an involution, defined on a basic algebra $\bar{A}$ can be lifted to an Morita equivalent algebra A (Corollary 3.1). In particular, if $\bar{A}$ is a basic algebra Morita equivalent to $A = \mathrm{S}(\mathfrak{J})$, then it gives a technically important presentation A as subalgebra in a matrix algebra over $\bar{A}$.

In section 4 we introduce the notion of the diagram of a Jordan bimodule and Jordan algebra. These notions are the direct analogue of that of the quiver of an associative algebra. These notions have turned out to be very useful and effective (see subsection 3.4). We also describe the simple modules over the semisimple matrix algebras in a convenient form.

In Sections 5 we investigate the category of Jordan algebras over a fixed Jordan algebra $\mathfrak{L}$. The source of our inspiration is the analogy with associative algebra and the categorical meaning of corresponding constructions.

We would like to draw attention to the notion of the *tensor algebra* $\mathcal{A}(V)$ of a bimodule (in the sense of MacLane [6]) B over algebra $\mathcal{A}$ over $\mathbb{k}$. One may define it as a free object in a convenient category, or construct it as a factor algebra of the free $\mathbb{k}$-linear algebra $F = F[\mathcal{A} \oplus B]$ of the corresponding class, generated by $\mathcal{A} \oplus B$ modulo the ideal, generated by the relations

$$\begin{gathered}[a_1] \star [a_2] - [a_1 \circ a_2], a_1, a_2 \in \mathcal{A}, \\ [b]\star[a] - [b \cdot a], [a] \star [b] - [a \cdot b], a \in \mathcal{A}, b \in B,\end{gathered} \tag{1}$$

where $[x]$ means a generator of F, corresponding to $x \in \mathcal{A}\oplus B$, $\star$ means the product in F, $\circ$ the product in $\mathcal{A}$ and $\cdot$ the action of $\mathcal{A}$ on B.[1] We hope, that for Jordan algebras our paper showed the value of this notion, for both comprehension and as a technical tool.

We investigate the interaction of the notions introduced above with the operations over Jordan algebras and some functors, such as special universal and universal multiplication envelopes. An interesting, although perhaps expected result is some commuting of the functor of the tensor algebra with the functor of the special universal envelope (Theorem 5.1). Namely, if $\mathfrak{L}$ is a Jordan algebra, V a bimodule over $\mathfrak{L}$, then $\mathrm{S}(\mathfrak{L}[V]) \simeq \mathrm{S}(\mathfrak{L})\langle\widetilde{V}\rangle$ for some $\mathrm{S}(\mathfrak{L})$-bimodule $\widetilde{V}$. The Jordan nature of the bimodule $\widetilde{V}$ is explained in 6.

As in the case of associative algebras every finite dimensional Jordan algebra $\mathfrak{J}$ with a Levi subalgebra $\mathfrak{L}$ can be covered by a unique tensor algebra $\pi : \mathfrak{L}[V]\longrightarrow\mathfrak{J}$, such that $\pi|_{\mathfrak{L}} = \mathrm{id}$ and $\pi|_V$ is a monomorphism.

Let $\mathfrak{J}$ be a Jordan algebra over its Levi subalgebra. For the goals of representation theory it seems to be useful represent $\mathfrak{J}$ by a set of generators of $\operatorname{Ker}\pi$, where $\pi : \mathfrak{L}[V]\longrightarrow\mathfrak{J}$, $V = \operatorname{Rad}\mathfrak{J}/(\operatorname{Rad}^2\mathfrak{J} + \mathfrak{L}\operatorname{Rad}^2\mathfrak{J})$. In this case the simple Lemmas 5.9 and 5.10 (together with Lemma 3.2, (2) give an algorithm of construction of an associative algebra A, represented as a quiver with relations, such that A is Morita equivalent to $\mathrm{S}(\mathfrak{J})$.

Section 6 contains a description of the matrix Jordan algebras as the algebras of symmetric elements in their special universal envelopes and generalized classical results for semi-simple algebras. This theorem in the case of a Jordan algebra with a simple Levi subalgebra goes back to Jacobson ([7, Sect. III.5]). We come back to this topic in the section 8. As a corollary (Corollary 6.1) we obtain the reflexivity of tensor algebra $\mathfrak{L}[V]$, where $\mathfrak{L}$ is a matrix semisimple Jordan algebra and V is a unital $\mathfrak{L}$-bimodule.

Section 7 is devoted to the description of the mapping Qui, which transforms a Jordan diagram Γ of a matrix Jordan algebra $\mathfrak{J}$ into the quiver Q of the associative algebra with involution $\mathrm{S}(\mathfrak{J})$. One can consider Qui as an algorithmic realization of the functor $V \mapsto \tilde{V}$. Note, that the Jordan algebras (even finite dimensional), which diagrams belongs to the domain of Qui form more wide class, as matrix Jordan algebras. The arising class of Jordan algebras we will call almost matrix Jordan algebras. Nevertheless, we can apply methods we develop in this new situation.

In the last Section 8 we apply the developed methods and results for investigation of special representation type of almost matrix Jordan algebras. An immediate corollary of the developed techniques is Theorem 8.3, which for a Jordan algebra $\mathfrak{J}$ construct "in principle" a basic algebra A, presented as a quiver with relations (Q, R), such that A Morita equivalent to $\mathrm{S}(\mathfrak{J})$. Further we describes quivers with relations (Q, R), which can be obtained in such way (Theorem 8.2). In subsection 8.3 in terms of the Jordan diagrams and transformation Qui we proof a criterions of special representation finiteness and tameness of almost matrix Jordan tensor algebras and algebras with radical square equals 0 (Theorem 8.4). As in the case of associative algebras the answer is formulated in terms of celebrated Dynkin diagrams. Later we discuss the notion of special Morita equivalence of Jordan algebras and propose an algorithm of construction of Jordan algebras with prescribed associative envelope and calculate some examples.

[1]The trivial split extension $\mathcal{A} \oplus B$ is an obvious factor of $\mathcal{A}(V)$.

1.3. Some notations. We work over the algebraically closed field $\mathbf{k}$ of characteristic $\neq 2, 3$. All associative algebras we will consider are finite dimensional with unity. Unless otherwise stated, the word "module" means "left module". The word "algebra" without the adjective "Jordan" means "associative algebra". We use M_n to denote the associative algebra $n \times n$-matrices over the field $\mathbf{k}$.

The notation "$x \circ y$" denotes a product in a Jordan algebra, "$x \cdot y$" denotes the action of an element of an algebra on an element of a (bi)module. We use the same notation Rad for the Jacobson radical for associative and Jordan algebras.

2. Special universal envelopes

2.1. Jordan bimodules. Recall, that a *Jordan algebra* over the field $\mathbf{k}$ is an algebra $\mathfrak{J}$ with a unique binary operation " $\circ$ ", satisfying the following relations

$$a \circ b = b \circ a \tag{2}$$

$$((a \circ a) \circ b) \circ a = (a \circ a) \circ (b \circ a). \tag{3}$$

for any $a, b \in \mathfrak{J}$.

Let $\mathfrak{J}$ be a Jordan algebra over $\mathbf{k}$, M be a vector space over $\mathbf{k}$ and suppose we have a pair of linear mappings $l : \mathfrak{J} \otimes_{\mathbf{k}} M \longrightarrow M$, $(a \otimes m) \mapsto a \cdot m$, $r : M \otimes_{\mathbf{k}} \mathfrak{J} \longrightarrow M$, $(m, a) \mapsto m \cdot a$, $a \in \mathfrak{J}$, $m \in M$. Define on the vector space $\Omega = \mathfrak{J} \oplus M$ a $\mathbf{k}-$ bilinear product $* : \Omega \times \Omega \longrightarrow \Omega$ by

$$(a_1 + m_1) * (a_2 + m_2) = a_1 \circ a_2 + a_1 \cdot m_2 + m_1 \cdot a_2.$$

for $a_1, a_2 \in \mathfrak{J}$, $m_1, m_2 \in M$, which turns Ω into an algebra, where $\mathfrak{J}$ is a subalgebra and M is an ideal such that $M^2 = 0$. Then we will say that M endowed with two bilinear compositions r, l is a *Jordan bimodule* over $\mathfrak{J}$ if $\Omega = \mathfrak{J} \oplus M$ is a Jordan algebra with respect to "$*$". In this case Ω is called the *null extension* of $\mathfrak{J}$ by the bimodule M.

Since $a \cdot m = m \cdot a$, a Jordan bimodule can be considered as (a Jordan) right or left module. The Jordan bimodules over $\mathfrak{J}$ form an abelian category $\mathfrak{J} - \mathrm{Mod}$, where a morphism of $\mathfrak{J}$-bimodules $f : M \longrightarrow N$ is a $\mathbf{k}$-linear mapping such that $f(a \cdot m) = a \cdot f(m)$, $a \in \mathfrak{J}$, $m \in M$. Since the left and right modules' structures coincide we will use the words "Jordan module" and "Jordan bimodule" synonymously, preferring the term "bimodule" in order to emphasize the existence of both structures.

By $\mathfrak{J} - \mathrm{mod}$ we denote the category of finite dimensional $\mathfrak{J}$-modules.

2.2. Universal multiplication envelope. Following [7], [6] the action of a Jordan algebra $\mathfrak{J}$ on a module can be rewritten as an action of an associative algebra $\mathrm{U}(\mathfrak{J})$ called the *universal multiplication envelope* for the representations (modules). Let $F\langle\mathfrak{J}\rangle$ be the free associative $\mathbf{k}$-algebra generated by the vector space $\mathfrak{J}$ and let $I \subset F\langle\mathfrak{J}\rangle$ be the ideal, generated by elements:

$$2aba + b \circ (a \circ a) - 2a(b \circ a) - b(a \circ a),\ a(a \circ a) - (a \circ a)a,\ a, b \in \mathfrak{J}. \tag{4}$$

Set $\mathrm{U}(\mathfrak{J}) = F\langle\mathfrak{J}\rangle / I$. The mapping $i : \mathfrak{J} \longrightarrow \mathrm{U}(\mathfrak{J}), a \mapsto a + I$ is an injection of vector spaces, hence one can consider $\mathfrak{J}$ as a subspace in $\mathrm{U}(\mathfrak{J})$. We endow every $\mathrm{U}(\mathfrak{J})$-module M with the canonical structure of an $\mathfrak{J}$-module through $a \cdot m \stackrel{def}{=\!=} (a + I)m$, $m \in M$, $a \in \mathfrak{J}$. This defines the isomorphism of categories $\mathfrak{J} - \mathrm{Mod}$ and $\mathrm{U}(\mathfrak{J}) - \mathrm{Mod}$.

We note also that $\mathrm{U}(\mathfrak{J})$ has an involution $^*:\mathrm{U}(\mathfrak{J})\longrightarrow\mathrm{U}(\mathfrak{J}), x\mapsto x^*$, i.e. * is $\mathbf{k}$-linear, $1^*=1$, $(xy)^*=y^*x^*, x,y\in\mathrm{U}(\mathfrak{J})$. This involution is called the *fundamental involution* on $U(\mathfrak{J})$ and is characterized by the property that for $a\in\mathfrak{J}$, $a=a^*$ holds.

If $\dim_{\mathbf{k}}\mathfrak{J}<\infty$, then $\dim_{\mathbf{k}}\mathrm{U}(\mathfrak{J})<\infty$. The construction $\mathrm{U}(\mathfrak{J})$ splits the problem of classification of representations of $\mathfrak{J}$ into two parts:

(1) defining the structure of $\mathrm{U}=\mathrm{U}(\mathfrak{J})$;
(2) investigating the representation for the *associative algebra* U.

2.3. **Special universal and unital universal multiplication envelopes.** Let $\mathfrak{J}$ be a Jordan algebra and let M be a vector space, endowed with a composition $\mathfrak{J}\otimes_{\mathbf{k}}M\to M, a\otimes m\mapsto a\cdot m, a\in\mathfrak{J}, m\in M$ such that for any $a_1,a_2\in\mathfrak{J}, m\in M$ holds

$$2(a_1\circ a_2)\cdot m=a_1\cdot(a_2\cdot m)+a_2\cdot(a_1\cdot m).$$

If we set $m\cdot a=a\cdot m$, then the mappings $(a,m)\to\frac{1}{2}m\cdot a$, $(a,m)\to\frac{1}{2}a\cdot m$ endow M with the structure of a Jordan module for $\mathfrak{J}$. A module of this type will be called *special*. The category of special bimodules will be denoted $\mathfrak{J}-\mathrm{sMod}$.

The full subcategory $\mathfrak{J}-\mathrm{sMod}\subset\mathfrak{J}-\mathrm{Mod}$ can also be described as a category of modules over an associative algebra, namely the so called *special universal envelope*. This is defined to be the algebra $\mathrm{S}(\mathfrak{J})=F\langle\mathfrak{J}\rangle/R_S$, where R_S is the ideal of $F\langle\mathfrak{J}\rangle$ generated by the elements of the form

$$a\otimes b+b\otimes a-2a\circ b \quad a,b\in\mathfrak{J}. \tag{5}$$

We denote the coset $a+R_S$ of $a\in\mathrm{S}(\mathfrak{J})$ by a_S. The isomorphism of the categories $\mathfrak{J}-\mathrm{sMod}$ and $\mathrm{S}(\mathfrak{J})-\mathrm{Mod}$ is settled by the following correspondence: if $a\mapsto S_a$ is a special representation of $\mathfrak{J}$, then $a_S\mapsto 2S_a$ defines a representation of the associative algebra S and vice versa.

Now suppose that $\mathfrak{J}$ is a Jordan algebra with an identity element e. A module M for $\mathfrak{J}$ will be called *unital* if $e\cdot m=m$ for all $m\in M$. The corresponding associative algebra will be called the *unital universal multiplication envelope*. This is the algebra $\mathrm{U}_1(\mathfrak{J})$, that is the factor of $\mathrm{U}(\mathfrak{J})$ by the ideal generated by the elements $ae+ea-2a$, $a\in\mathfrak{J}$. Analogously we can introduce the *special unital universal envelope* $\mathrm{S}^1(\mathfrak{J})$.

The following theorem shows the role of the algebras $\mathrm{S}(\mathfrak{J})$ and $U_1(\mathfrak{J})$.

Theorem 2.1. *([6],II.11.15) Let $\mathfrak{J}$ be a Jordan algebra with identity element e and let $\mathrm{U}=\mathrm{U}(\mathfrak{J})$ be the universal multiplication envelope of $\mathfrak{J}$. Put $E_0=(e-1)(2e-1)$, $E_1=e(2e-1)$, $E_{1/2}=-4e(e-1)$. Then*

(1) E_i are central orthogonal idempotent in U, $E_0+E_1+E_{1/2}=1$ hence $\mathrm{U}=\mathrm{U}_0\times\mathrm{U}_{1/2}\times\mathrm{U}_1$ where $\mathrm{U}_i=\mathrm{U}E_i$ is an ideal.
(2) Moreover, if x_i denotes the component of $x\in\mathrm{U}$ in U_i then U_1 and $a\to a_1$ is a universal unital multiplication envelope $\mathrm{U}_1(\mathfrak{J})$, and $\mathrm{U}_0\oplus\mathrm{U}_{1/2}$ and $a\to 2(a_0+a_{1/2})$ is a special universal envelope for $\mathfrak{J}$.

This theorem evidently splits the category $\mathfrak{J}-\mathrm{mod}$ in the direct sum of the full subcategories $\mathfrak{J}-\mathrm{mod}_0$, $\mathfrak{J}-\mathrm{mod}_1$, $\mathfrak{J}-\mathrm{mod}_{\frac{1}{2}}$.

3. Preliminaries about finite dimensional algebras

3.1. Representation type of algebra. We refer the reader to [4], [5] or [13] for the details.

Unless otherwise stated, in this section all algebras are finite dimensional algebras with 1 over $\mathbf{k}$ and modules are left. $A-\mathrm{mod}$ denotes the category of left finite dimensional A-modules over the algebra A. Assume Λ is a localization of the polynomial algebra $\mathbf{k}[x]$ by $0 \neq f \in \mathbf{k}[x]$ and M is an $A-\Lambda$ bimodule free as a right Λ-module. Then the *one-parameter family* of A-modules $F = F(\Lambda, M)$ consists of modules of the form $M \otimes_\Lambda U$, where U is a one-dimensional Λ-module. Then A has

(1) a *finite (type)* if there are finitely many isomorphism classes of indecomposable A-modules.
(2) a *tame (type)* if for each dimension d there exists finitely many finitely many one-parameter families $F_1, \dots, F_N, (N = N(d))$ every indecomposable module of dimension d is isomorphic to the module from some F_i.
(3) a *wild (type)* if there exists an $A-\mathbf{k}\langle x,y\rangle$-bimodule M, finitely generated free as a $\mathbf{k}\langle x,y\rangle$-module such that the functor $M \otimes_{\mathbf{k}\langle x,y\rangle} \_$ keeps indecomposability and isomorphism classes.

Due to this definition the algebra of finite type is a tame algebra. The dividing line between these notions are given by the following theorem.

Theorem 3.1. *(1) ([4]) A finite dimensional algebra A is either tame or wild (but not time and wild simultaneously).*
(2) ([1]) Let A be a finite dimensional algebra of tame type. Then either A is of finite type or there exists a dimension, containing infinitely many isoclasses of indecomposable modules.

Let $\mathfrak{J}$ be a finite dimensional Jordan algebra over $\mathbf{k}$. Then all its universal envelopes $\mathrm{U}(\mathfrak{J})$, $\mathrm{U}_1(\mathfrak{J})$, $\mathrm{S}(\mathfrak{J})$ and $\mathrm{S}^1(\mathfrak{J})$ are finite-dimensional algebras. Moreover the category of $\mathfrak{J}$-modules is isomorphic to the direct sum $\mathrm{U}(\mathfrak{J})-\mathrm{mod}$, $\mathrm{U}_1(\mathfrak{J})-\mathrm{mod}$ and $\mathrm{S}(\mathfrak{J})-\mathrm{mod}$. It motivates the following definitions.

Definition 3.1. *A finite dimensional Jordan algebra $\mathfrak{J}$ is of (has) finite, tame or wild type (for special representations, for a unital representation) provided that it is true for $\mathrm{U}(\mathfrak{J})$ (for $\mathrm{S}(\mathfrak{J})$ and for $\mathrm{U}_1(\mathfrak{J})$ correspondingly).*

The representation type for all representations of $\mathfrak{J}$ we define as for the universal algebra $\mathrm{U}(\mathfrak{J})$.

3.2. Morita equivalence. The algebras A and A' are called *Morita equivalent*, if the categories $A-\mathrm{mod}$ and $A'-\mathrm{mod}$ are equivalent. An algebra A is called *basic or Morita reduced*, provided that $A/\operatorname{Rad} A \simeq \mathbf{k}^n, n \geq 1$. In every class of Morita equivalence there exists a unique basic algebra up to isomorphism. The problem of classification of the indecomposable representations of an algebra A can be simplified by passing to a basic algebra B Morita equivalent to A.

A direct summand of A, as a left A-module has a form $A\mathbf{e}$, where $\mathbf{e} \in A$ is an idempotent. If $A\mathbf{e}$ is indecomposable then it is called a *principal indecomposable* A-module and $\mathbf{e}$ is called *primitive*.

Let P be a left ideal of A such that $P = P_1 \oplus \cdots \oplus P_n$, where $\{P_1, \dots, P_n\}$ is a set representative of all isomorphic classes of principal indecomposable modules.

Such a P is unique up to isomorphism and the algebra $B = \mathrm{End}_A(P)$ will be basic and by [13] Proposition 9.6 A and B are Morita equivalent. Moreover (see [13] Proposition 6.4.a and Corollary 6.4.a), there exists an idempotent $\mathrm{e} \in A$ such that $P = A\mathrm{e}$ and $B = \mathrm{End}_A(\mathrm{e}A) \simeq \mathrm{e}A\mathrm{e}$. More precisely

Lemma 3.1. *Let A be an algebra and $1 = \sum\limits_{i=1}^{n}\sum\limits_{j=1}^{k_i} \mathrm{e}_{ij}$ is the decomposition of unity into a sum of orthogonal primitive idempotents such that $A\mathrm{e}_{ij} \simeq A\mathrm{e}_{i'j'}$ if and only if $i = i'$. For any integers $l_1, \dots, l_n$, such that $1 \leqslant l_i \leqslant k_i$, $i = 1, \dots, n$ the algebra $B = eAe$, where $\mathrm{e} = \mathrm{e}_{1l_1} + \cdots + \mathrm{e}_{nl_n}$, is basic and Morita equivalent to A.*

Let A be a (not necessarily finite dimensional) algebra with a decomposition of 1 into a sum of orthogonal idempotents $1 = e_1 + \cdots + e_n$ and $\vec{k} = (k_1, \dots, k_n)$ be an integral vector with non-negative integral components. Then by $A_{\vec{k}}$ we denote the subalgebra in $\mathrm{M}_k(k)$, $k = k_1 + \cdots + k_n$ formed by the block matrices

$$\begin{pmatrix} \mathrm{M}_{k_1\times k_1}(e_1Ae_1) & \mathrm{M}_{k_1\times k_2}(e_1Ae_2) & \dots & \mathrm{M}_{k_1\times k_n}(e_1Ae_n) \\ \mathrm{M}_{k_2\times k_1}(e_2Ae_1) & \mathrm{M}_{k_2\times k_2}(e_2Ae_2) & \dots & \mathrm{M}_{k_2\times k_n}(e_2Ae_n) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{M}_{k_n\times k_1}(e_nAe_1) & \mathrm{M}_{k_n\times k_2}(e_nAe_2) & \dots & \mathrm{M}_{k_n\times k_n}(e_nAe_n) \end{pmatrix} \tag{6}$$

with the natural multiplication. Then $A_{\vec{k}}$ is Morita equivalent to A and if A is basic, then in this way we obtain all (up to isomorphism) algebras Morita equivalent to A. The entries in such matrices are naturally indexed by the pair $((s_1, i_1), (s_2, i_2))$, $1 \leqslant s_1, s_2$, $1 \leqslant i_1 \leqslant k_{s_1}$, $1 \leqslant i_2 \leqslant k_{s_2}$: it corresponds to the entry $(\sum\limits_{j=1}^{s_1-1} k_i + i_1, \sum\limits_{j=1}^{s_2-1} k_i + i_2)$. The corresponding unit matrix is denoted by $E_{(s_1,i_1),(s_2,i_2)}$.

3.3. Quivers and relations. An *oriented graph* or *quiver* Q is defined by its set of *vertices* of *points* Q_0 and set of *arrows* Q_1 together with two maps $s, e : \mathrm{Q}_1 \longrightarrow \mathrm{Q}_0$, which send an arrow to its *start* and *end* vertex correspondingly. We say that the arrow $x \in \mathrm{Q}_0$ *leads* from the vertex $s(x)$ to the arrow $e(x)$. Let us denote by QAs the category of (possibly infinite) quivers, where the morphism F from $Q = (Q_0, Q_1, s, e)$ to $Q' = (Q'_0, Q'_1, s, e)$ is a pair of maps $F_0 : Q_0 \longrightarrow Q'_0$, $F_1 : Q_1 \longrightarrow Q'_1$, such that $sF_1 = F_0 s$, $eF_1 = F_0 e$.

Let $S = S_1 \times \cdots \times S_n$, where S_i is a matrix algebra. We denote by V_i a simple left S_i-module. Any finitely generated S-bimodule V (equivalently any $S^\circ \otimes_{\mathbf{k}} S$-module) is isomorphic to the S-bimodule $\bigoplus\limits_{i,j=1}^{n} (V_j \otimes_{\mathbf{k}} V_i^*)^{k_{ij}}$ for some $k_{ij} \geqslant 0$, where V_i^* is dual to the V_i space with the natural structure of a right S_i-module. The *diagram of bimodule* V is the quiver $\mathrm{Q} = \mathrm{Q}(V)$, $\mathrm{Q}_0 = \{1, \dots, n\}$ and from the vertex i to the vertex j, $1 \leqslant i, j \leqslant n$ lead k_{ij} arrows. Conversely, any (finite) quiver Q defines uniquely up to isomorphism a bimodule $V = V_{\mathrm{Q}}$, such that $\mathrm{Q} = \mathrm{Q}(V)$.

Remark 3.1. *Let V_1, V_2 be S-bimodules. then $\mathrm{Q}(V_1 \oplus V_2)_1 = \mathrm{Q}(V_1) \sqcup \mathrm{Q}(V_2)$ with the same s and e.*

Let A be an algebra and $S = A/\operatorname{Rad} A \simeq S = S_1 \times \cdots \times S_n$ is as above. Then the *quiver* (the *diagram*, *scheme*) $\mathrm{Q}(A)$ of the algebra A is called the diagram of the S-bimodule $\operatorname{Rad} A/\operatorname{Rad}^2 A$. $\mathrm{Q}(A)$ is an invariant of the class of Morita equivalence.

We need some standard facts about algebras ([5], [13]). Let A be an algebra and W be an A-bimodule, then by $A\langle W\rangle$ (see 1.2) we denote the tensor algebra (not necessarily finite dimensional) of the bimodule V over A, endowed with the canonical A-structure, i.e. a homomorphism $\imath = \imath_{A\langle W\rangle} : A \longrightarrow A\langle W\rangle$. It is well known, that in the case of associative algebras

$$A\langle W\rangle = A \oplus \bigoplus_{i=1}^{\infty} W^{\otimes i}, \ \imath(a) = a, a \in A. \tag{7}$$

$A\langle W\rangle$ is a graded algebra: $\deg A = 0, \deg W^{\otimes i} = i, i \geqslant 1$.

Lemma 3.2. *Denote by $S \subset A$ a Levi subalgebra in A (i.e. $A = S \dotplus \operatorname{Rad} A$) and by V the S-bimodule* $\operatorname{Rad} A/\operatorname{Rad}^2 A$.

(1) *Let $p : \operatorname{Rad} A \longrightarrow V$ be the canonical projection, $s : V \longrightarrow \operatorname{Rad} A$ be a S-bimodule splitting of p, $S\langle V\rangle$ be the tensor algebra of V over S. Then the algebra homomorphism $\pi : S\langle V\rangle \longrightarrow A$, which is identical on S and $\pi|_V = s$, is an epimorphism and* $\operatorname{Ker} \pi \subset \sum_{i\geqslant 2} V^{\otimes 2}$.

(2) *Assume for a S-bimodule W that there exists an epimorphism $\pi : S\langle W\rangle \longrightarrow A$, identical on S and such that $\operatorname{Ker}\pi \subset \sum_{i\geqslant 2} W^{\otimes i}$. Then $V \simeq W$.*

If A is basic and $1 = \mathrm{e}_1 + \cdots + \mathrm{e}_n$ is the decomposition of its unit in the sum of orthogonal primitive idempotents, then $k_{ij} = \dim_{\mathbf{k}} \mathrm{e}_j(\operatorname{Rad} A/\operatorname{Rad}^2 A)\mathrm{e}_i$.

With a quiver Q, $|\,\mathrm{Q}_0\,| \leqslant \infty$ is associated with the (not necessarily finite dimensional!) *path algebra* $\mathbf{k}[\mathrm{Q}]$ of the quiver Q. This algebra is isomorphic to the tensor algebra over the semisimple algebra $\mathbf{k}^{\mathrm{Q}_0}$ of the $\mathbf{k}^{\mathrm{Q}_0}$-bimodule V_{Q}. The standard basis of $\mathbf{k}[\mathrm{Q}]$ forms oriented paths in Q, i.e. sequences $x_1 \dots x_k$, $x_i \in \mathrm{Q}_1$, such that $s(x_i) = e(x_{i+1})$, $i = 1, \dots, k-1$, provided that $k \geqslant 1$ and the vertices start from Q_0 if $k = 0$. We set $s(w) = s(x_k), e(w) = e(x_1)$ and k is called the *length* of the path. The product p_1p_2 of paths $p_1 = x_1 \dots x_k$ and $p_2 = y_1 \dots y_l$ equals $x_1 \dots x_k y_1 \dots y_l$, if $s(x_k) = e(y_1)$ and 0 otherwise. The unit element of $\mathbf{k}[\mathrm{Q}]$ is $\sum_{P\in\mathrm{Q}_0} P$.

Let A be a basic algebra, $\mathrm{Q} = \mathrm{Q}(A)$. Following Lemma 3.2 there exists an epimorphism $\pi : \mathbf{k}[\mathrm{Q}] \longrightarrow A$. Fix a set $R \subset \mathbf{k}[\mathrm{Q}]$ of generators of the ideal $\operatorname{Ker}\pi$. We call the pair (Q, R) a *quiver with relations*, which represents the algebra A.

3.4. Construction of a quiver with relations. Usually the universal multiplication and special universal envelopes of a finite dimensional Jordan algebra are not basic. Lemma 3.3 below gives a method to construct Morita reduced subalgebras as quivers with relations.

Let A be an algebra, $S = \mathrm{M}_{k_1} \times \cdots \times \mathrm{M}_{k_n} \subset A$ a Levy subalgebra,

$$1 = \sum_{s=1}^{n} \sum_{i=1}^{k_s} \mathrm{e}_{ii}^s, \text{ where } \mathrm{e}_{ii}^s \text{ are diagonal matrix units in } \mathrm{M}_{k_s} \tag{8}$$

the decomposition of the unit of A in the sum of primitive idempotents, V be a S-bimodule such that $A \simeq S\langle V\rangle/I$, $I \subset \bigoplus_{i\geqslant 2} V^{\otimes i}$, $\mathrm{e} = \sum_{s=1}^{n} \mathrm{e}^s_{11}$, $\bar{A} = \mathrm{e}A\mathrm{e}$. Following Lemma 3.1 $\bar{A}$ is a basic algebra, Morita equivalent to A. Moreover, $A \simeq A_{\vec{k}}$, $\vec{k} = (k_1, \dots, k_n)$.

We need an implicit isomorphism $\Phi_A : A \longrightarrow \bar{A}_{\vec{k}}$. Φ_A which sends $a \in A$ to the matrix $\Phi_A(a)$, such that

$$\Phi_A(a)_{(s_1,i_1),(s_2,i_2)} = \mathrm{e}^{s_1}_{1\,i_1} a \mathrm{e}^{s_2}_{i_2\,1}, \quad 1 \leqslant s_1, s_2 \leqslant n, 1 \leqslant i_1 \leqslant k_{s_1}, 1 \leqslant i_2 \leqslant k_{s_2}. \tag{9}$$

Denote $\bar{S} = \mathrm{e}S\mathrm{e}$, $\bar{V} = \mathrm{e}V\mathrm{e}$. Then the canonical embeddings $\bar{S} \hookrightarrow S$ and $\bar{V} \hookrightarrow V$ induce a homomorphism of algebras[2] $i : \bar{S}\langle\bar{V}\rangle \longrightarrow S\langle V\rangle$. Since $\bar{S}$ and S are semisimple, i is an embedding.

We prove $\operatorname{Im} i = \mathrm{e}S\langle V\rangle\mathrm{e}$. Evidently $\operatorname{Im} i$ is a graded subalgebra in $\mathrm{e}S\langle V\rangle\mathrm{e}$ and in the degrees 0 and 1 they equal $\bar{S}$ and $\bar{V}$ correspondingly. To finish the proof make the step of induction in deg from $n-1$ to n. Let $x \in \mathrm{e}S\langle V\rangle\mathrm{e}$, $\deg x = n$. Then $x = \sum_i y_i z_i$, $\deg y_i < n$, $\deg z_i < n$. But for $y, z \in S\langle V\rangle$, $\deg y < n$, $\deg z < n$

$$\mathrm{e}yz\mathrm{e} = \mathrm{e}y1z\mathrm{e} = \sum_{s=1}^{n}\sum_{i=1}^{k_s} \mathrm{e}x\mathrm{e}^s_{ii}y\mathrm{e} = \sum_{s=1}^{n}\sum_{i=1}^{k_s} (\mathrm{e}x\mathrm{e}^s_{i1})(\mathrm{e}^s_{1i}y\mathrm{e})$$

holds. By induction, all $\mathrm{e}x\mathrm{e}^s_{i1}, \mathrm{e}^s_{1i}y\mathrm{e} \in \operatorname{Im} i$, which completes the proof.

Then we have the following commutative diagram

$$\begin{tikzcd} S\langle V\rangle \arrow[r, "\Phi_{S\langle V\rangle}"] \arrow[d, "\pi"'] & \bar{S}\langle\bar{V}\rangle_{\vec{k}} \arrow[d, "\pi_{\vec{k}}"] \\ A \arrow[r, "\Phi_A"] & A_{\vec{k}} \end{tikzcd}, \tag{10}$$

where the horizontal arrows are isomorphisms, π is a canonical projection and $\bar{\pi} : \bar{S}\langle\bar{V}\rangle \longrightarrow \bar{A}$ is induced by π. Hence we obtain the following lemma.

Lemma 3.3. *Let $A \simeq S\langle V\rangle/I$ and $R \subset \bigoplus_{i\geqslant 2} V^{\otimes i}$ be a family of generators of the ideal I. Then*

(1)

$$\operatorname{Ker} \bar{\pi}_{\vec{k}} = \begin{pmatrix} \mathrm{M}_{k_1\times k_1}(e_1\bar{I}e_1) & \mathrm{M}_{k_1\times k_2}(e_1\bar{I}e_2) & \dots & \mathrm{M}_{k_1\times k_n}(e_1\bar{I}e_n) \\ \mathrm{M}_{k_2\times k_1}(e_2\bar{I}e_1) & \mathrm{M}_{k_2\times k_2}(e_2\bar{I}e_2) & \dots & \mathrm{M}_{k_2\times k_n}(e_2\bar{I}e_n) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{M}_{k_n\times k_1}(e_n\bar{I}e_1) & \mathrm{M}_{k_n\times k_2}(e_n\bar{I}e_2) & \dots & \mathrm{M}_{k_n\times k_n}(e_n\bar{I}e_n) \end{pmatrix},$$

where $\bar{I} = \operatorname{Ker}\bar{\pi}$

(2) The entries of the matrices $\Phi_{S\langle V\rangle}(r)$, $r \in R$ form a family of generators $\Phi_{S\langle V\rangle}(R)$ of the ideal $\bar{I} \subset \bar{S}\langle\bar{V}\rangle$.

Proof. The statement (1) is obvious.

To prove (2) denote $\bar{J} \subset \bar{S}\langle\bar{V}\rangle$ the ideal, generated by $\Phi_{S\langle V\rangle}(R)$. By (1) $\bar{J} \subset \bar{I}$. On the other hand $\Phi_{S\langle V\rangle}$ induces an isomorphism from $\operatorname{Ker}\pi$ to $\operatorname{Ker}\bar{\pi}_{\vec{k}}$, so $\Phi_{S\langle V\rangle}(R)$

[2] Note, that i sends the unit in $\bar{S}$ to an idempotent in S.

generates $\operatorname{Ker}\bar{\pi}_{\bar{k}}$ over $\bar{S}\langle\bar{V}\rangle_{\bar{k}}$. Hence all entries of matrices from $\operatorname{Ker}\bar{\pi}_{\bar{k}}$ belong to $\bar{J}$, that completes the proof. □

3.5. Morita equivalence of associative algebras with involution. Let, in the assumptions of subsection 3.4, the algebra S be endowed with an involution $*$, such that $(S_i)^* = S_{\sigma(i)}$ and for all $i, \sigma(i) = i$ is defined $\varepsilon(i) = \pm 1$, provided that $*$ when restricted to S_i is either a transposition or a symplectic involution.

Corollary 3.1. *Let, in assumption above, V be a bimodule with involution, $*$, which induces a permutation on the set*

$$\{e^s_{ii} \mid s = 1, \dots, n; i = 1, \dots, k_s\},$$

$\mathbf{e}^* = \mathbf{e}$ and $I^* = I$. Then

(1) $*$ induces an involution on A and $\bar{A}^* = \bar{A}$.
(2) Let $1 = e_1 + \dots + e_n$ be the decomposition of unity into the sum of minimal orthogonal idempotents, $*$ be an involution on $\bar{A}$ and the functions σ, ε define uniquely an involution on $A_{\bar{k}}$, which turns Φ_A into an isomorphism of algebras with involution.

Proof. The statement (1) is obvious. Since Φ_A is an isomorphism, it endows $A_{\bar{k}}$ by an involution, such that Φ_A is an isomorphism of algebras with involution. We need to prove, that this structure is uniquely defined by $*|_{\bar{A}}$, σ and ε.

By definition $\Phi_A(e^s_{ij})$ is the unit matrix $E_{(s,i),(s,j)} \in \bar{A}_{\bar{k}}$. By definition of the involution $*$, $(e^k_{ij})^* = e^{s'}_{i'j'}$ holds and $\Phi_A(e^{s'}_{i'j'}) = E_{(s',i'),(s',j')}$. Note that i', j', s' are defined by σ and ε.

Let $a \in A$ be such that $e^{s_1}_{i_1 i_1} a e^{s_2}_{i_2 i_2} = a$. Let $a^* = e^{t_2}_{j_2 j_2} a^* e^{t_1}_{j_1 j_1}$. Then

$$e^{t_1}_{j_1 j_1} = (e^{s_1}_{i_1 i_1})^* = (e^{s_1}_{i_1 1}\mathbf{e}\, e^{s_1}_{1 i_1})^* = (e^{s_1}_{1 i_1})^* \mathbf{e}\, (e^{s_1}_{i_1 1})^*,$$

hence $(e^{s_1}_{1 i_1})^* = e^{t_1}_{j_1 1}$ and $(e^{s_1}_{i_1 1})^* = e^{t_1}_{1 j_1}$. Analogously $(e^{s_2}_{i_2 1})^* = e^{s_2}_{1 i_2}$. This implies that the diagram

$$\begin{array}{ccc} a = e^{s_1}_{i_1 i_1} a e^{s_2}_{i_2 i_2} & \xrightarrow{\quad \Phi_A \quad} & (e^{s_1}_{1 i_1} a e^{s_2}_{i_2 1}) E_{(s_1,i_1),(s_2,i_2)} \\ {\scriptstyle *}\Big\downarrow & & \Big\downarrow{\scriptstyle *|_{\bar{A}}} \\ e^{t_2}_{j_2 j_2} a^* e^{t_1}_{j_1 j_1} & \xrightarrow{\quad \Phi_A \quad} & (e^{t_2}_{1 j_2} a^* e^{t_1}_{j_1 1}) E_{(t_2,j_2),(t_1,j_1)} \end{array} \tag{11}$$

commutes. Hence the involution of $\bar{A}_{\bar{k}}$ is defined by $*|_{\bar{A}}$. □

4. Bimodules over semisimple matrix Jordan algebras and their diagrams

4.1. Diagrams of Jordan bimodules and algebras. One can observe, in representation theory of finite dimensional algebras the important role played by some geometrical objects - both as technical tools and as new sources of intuition. We mention only the notions of the quiver (quiver with relations) of an algebra, the two dimensional complex associated with the Auslander-Reiten quiver of an algebra, covering techniques etc. ([9], [10])

This inspires us to introduce the notion of the *diagram* of a finite dimensional Jordan algebra. It is analogue of the notion of the quiver of an algebra. We try to show its importance in applications to representation theory of Jordan algebras.

Let $\mathfrak{L}_1, \dots, \mathfrak{L}_n$ be simple Jordan algebras, $\mathfrak{L} = \mathfrak{L}_1 \oplus \cdots \oplus \mathfrak{L}_n$. Then $\mathrm{U}(\mathfrak{L})$ canonically decomposes into a product of algebras ([6], II.2, Theorem 5, II.11, Theorem 16).

$$\begin{aligned}\mathrm{U}(\mathfrak{L}) \simeq \underbrace{\Big(\bigotimes_{i=1}^{n} \mathrm{U}_0(\mathfrak{L}_i)\Big)}_{\mathrm{U}_0(\mathfrak{L})} \times \underbrace{\Big(\prod_{i=1}^{n} \mathrm{U}_{\frac{1}{2}}(\mathfrak{L}_i)\Big)}_{\mathrm{U}_{\frac{1}{2}}(\mathfrak{L})} \times \\ \underbrace{\Big(\big(\prod_{1\leqslant i<j\leqslant n} \mathrm{U}_{\frac{1}{2}}(\mathfrak{L}_i) \otimes \mathrm{U}_{\frac{1}{2}}(\mathfrak{L}_j)\big) \times \big(\prod_{i=1}^{n} \mathrm{U}_1(\mathfrak{L}_i)\big)\Big)}_{\mathrm{U}_1(\mathfrak{L})}.\end{aligned} \tag{12}$$

Hence the category $\mathrm{U}(\mathfrak{L})-\mathrm{mod}$ is the direct sum of the categories $\mathrm{U}_0(\mathfrak{L})-\mathrm{mod}$, $\mathrm{U}_{\frac{1}{2}}(\mathfrak{L}_i)-\mathrm{mod}$, $\mathrm{U}_{\frac{1}{2}}(\mathfrak{L}_i)\otimes\mathrm{U}_{\frac{1}{2}}(\mathfrak{L}_j)-\mathrm{mod}$ and $\mathrm{U}_1(\mathfrak{L}_i)-\mathrm{mod}$, where $1 \leqslant i < j \leqslant n$. Then for an indecomposable $\mathrm{U}(\mathfrak{L})$-module M we say its *type* $t(M)$ equals 0, $(\frac{1}{2})_{\mathfrak{L}_i}$, $1_{\mathfrak{L}_i\mathfrak{L}_j}$, $1_{\mathfrak{L}_i}$ provided it belongs to one of the categories above. All these categories are semisimple. The algebra U_0 is isomorphic to $\mathbb{k}$ and the category $\mathrm{U}_0(\mathfrak{J})-\mathrm{mod}$ consists of modules with zero action of $\mathfrak{J}$.

Denote by $\mathsf{C}_{\mathfrak{L}}$ the set of isoclasses of simple $\mathfrak{L}$-modules. Due to the type of module, $\mathsf{C}_{\mathfrak{L}}$ decomposes in the disjoint union of the subsets $\mathsf{C}_{\mathfrak{L}}(0)$, $\mathsf{C}_{\mathfrak{L}}((\frac{1}{2})_i)$, $\mathsf{C}_{\mathfrak{L}}(1_{ij})$, $\mathsf{C}_{\mathfrak{L}}(1_i)$. We associate with every $\mathfrak{L}$-bimodule M some geometrical object, which allows us to identify the isoclass of M.

A (non-oriented)) *graph* Γ is defined by the set of its *vertices* Γ_0 and the set of its *edges* Γ_1 with an *incidence function* $v : \Gamma_1 \to (\Gamma_0 \times \Gamma_0)/\sim$, where $\sim$ means equivalence on $\Gamma_0 \times \Gamma_0$ such that $(i,j) \sim (j,i)$. We say v sends an edge to the pair of its *border vertices*. Usually we will write $v(x) = (i,j)$ instead of $v(x) = \{(i,j),(j,i)\}$.

Let C_{JA} denote the set of isoclasses of simple finite dimensional Jordan algebras. The *diagram* $\Gamma = \Gamma(\mathfrak{L})$ of the algebra $\mathfrak{L} = \mathfrak{L}_1 \oplus \ldots \oplus \mathfrak{L}_n$ is a graph with the empty set of arrows, besides $\Gamma_0 = \{1,\dots,n\}$ is endowed with a mapping $\mathsf{c}_a : \Gamma_0 \longrightarrow \mathsf{C}_{JA}$, $\mathsf{c}_a L(i) = [\mathfrak{L}_i]$, where $[\mathfrak{L}_i]$ means the isoclass.

Let M be a $\mathrm{U}(\mathfrak{L})$-module, $M \simeq \bigoplus_{i=1}^{t} M_i$ is a decomposition of M in a direct sum of simple modules [3]. By a *diagram* of M we mean the graph $\Gamma = \Gamma(M)$, with the set of vertices $\Gamma(M)_0 \overset{def}{=\!=} \Gamma(\mathfrak{L})_0 \sqcup \{0\}$ and the set of edges $\Gamma(M)_1 = \{a_1,\dots,a_t\}$ endowed with the mapping $\mathsf{c}_m : \Gamma(M)_1 \longrightarrow \mathsf{C}_{\mathfrak{L}}$, which sends every M_i to its isoclass. The incidence function v is defined on an edge $x \in \Gamma_1(M)$, $k = 1,\dots,t$ as follows.

$$v(x) = \begin{cases} (0,0), & \text{if type of } \mathsf{c}_m(x) \text{ is } 0, \\ (i,i), & \text{if type of } \mathsf{c}_m(x) \text{ is } 1_{\mathfrak{L}_i}, \\ (i,i), & \text{if type of } \mathsf{c}_m(x) \text{ is } (\frac{1}{2})_{\mathfrak{L}_i}, \\ (i,j), & \text{if type of } \mathsf{c}_m(x) \text{ is } 1_{\mathfrak{L}_i,\mathfrak{L}_j}, 1 \leqslant i < j \leqslant n. \end{cases} \tag{13}$$

The diagram $\Gamma(M)$ defines the semisimple Jordan algebra $\mathfrak{L}$ and the bimodule M up to isomorphism. Moreover, there is a bijection between the set of finite diagrams

[3]Sometimes we will understan $\mathsf{c}_a(i)$ and $\mathsf{c}_m(x)$ as a representative of the corresponding isoclass.

$\Gamma = (\Gamma_0, \Gamma_1, v, \mathsf{c}_a, \mathsf{c}_m)$ and the set of isoclasses of bimodules over semisimple Jordan algebras.

A category of Jordan diagrams JorD is defined analogously, where a morphism F from $\Gamma = (\Gamma_0, \Gamma_1, v, \mathsf{c}_a, \mathsf{c}_m)$ to $\Gamma' = (\Gamma'_0, \Gamma'_1, v, \mathsf{c}_a, \mathsf{c}_m)$ is the pair $F_0 : \Gamma_0 \longrightarrow \Gamma'_0$, $F_1 : \Gamma_1 \longrightarrow \Gamma'_1$, such that $F_0 \mathsf{c}_a = \mathsf{c}_a F_0$, $F_1 \mathsf{c}_m = \mathsf{c}_m F_1$ and $((F_0 \times F_0)/\sim)v = vF_1$.

Remark 4.1. *Let V_1, V_2 be two bimodules over a semisimple Jordan algebra $\mathfrak{J}$. Then*

$$\Gamma(V_1 \oplus V_2)_1 = \Gamma(V_1) \sqcup \Gamma(V_2)$$

with the same c_m.

The *diagram* $\Gamma(\mathfrak{J})$ of a Jordan algebra $\mathfrak{J}$ with semisimple part $\mathfrak{L}$ we call the diagram of the $\mathfrak{L}$-bimodule $\operatorname{Rad}\mathfrak{J}/(\operatorname{Rad}^2\mathfrak{J} + \mathfrak{L}\operatorname{Rad}^2\mathfrak{J})$. In particular, if $\operatorname{Rad}^2\mathfrak{J} = 0$, then its diagram defines the algebra $\mathfrak{J}$ uniquely up to isomorphism (in spite of the quiver of an algebra, see also 3.3).

Note that $\mathfrak{J}$ is a Jordan algebra with unity if and only if Γ does not contain edges of the type $(\frac{1}{2})_{\mathfrak{L}_i}$ for some i. If Γ contains such an arrow, then we will add a unit to $\mathfrak{J}_\Gamma$. We will denote the algebra so obtained by $\tilde{\mathfrak{J}}$.

Lemma 4.1. *Let Γ be a diagram of a Jordan algebra $\mathfrak{J}$, $\tilde{\mathfrak{J}}$ be the algebra obtained from $\mathfrak{J}$ by adding the unit, i.e. $\tilde{\mathfrak{J}} = \mathbf{k}e \oplus \mathfrak{J}$, the multiplication on $\mathfrak{J} \subset \tilde{\mathfrak{J}}$ coincide with the multiplication in $\mathfrak{J}$ and $e \circ a = a \circ e = a$ for any $a \in \tilde{\mathfrak{J}}$, S the set of $x \in \Gamma_1$ of type $(\frac{1}{2})\}$, $\tilde{\Gamma} = \Gamma(\tilde{\mathfrak{J}})$. Then*

(1) $\tilde{\Gamma}_0 = \Gamma_0 \sqcup \{E\}$, where $\mathsf{c}_{\tilde{\mathfrak{J}}}(E) = \mathbf{k}$ and $\mathsf{c}_{\tilde{\mathfrak{J}}}|_{\mathfrak{J}} = \mathsf{c}_{\mathfrak{J}}$.

(2) $\tilde{\Gamma}_1 = (\Gamma_1 \setminus S) \sqcup \{y_x | x \in S\}$, where v, c in $\tilde{\Gamma}_1$ restricted to $(\Gamma_1 \setminus S)$ coincides with those for Γ and if for $x \in S$ $v(x) = (i,i)$ holds, then $v(y_x) = (E,i)$, $\mathsf{c}_m(y_x)$ coincides with $\mathsf{c}_m(x) \otimes_{\mathbf{k}} \mathbf{k}$.

Proof. It's obvious. □

As in the case of associative algebras the following obvious lemma holds.

Lemma 4.2. $\Gamma(\mathfrak{J}_1 \times \mathfrak{J}_2) = \Gamma(\mathfrak{J}_1) \sqcup \Gamma(\mathfrak{J}_2)$, *i.e. a Jordan algebra $\mathfrak{J}$ is indecomposable into a direct product if and only if $\Gamma(\mathfrak{J})$ is connected.*

4.2. **Associative algebras with involution and Jordan matrix algebras.** Let $A = (A, *)$ be a unital associative algebra with an involution $*$ and $\mathfrak{J} = H_n(A)$ be the Jordan algebra of $n \times n$ Hermitian matrices over A. Recall the following classical result ([6], Corollary V.6.2).

Theorem 4.1. *Let $\mathfrak{J}$ be a finite-dimensional simple Jordan algebra over an algebraically closed field $\mathbf{k}$. Then we have the following possibilities for $\mathfrak{J}$: (1) $\mathfrak{J} = \mathbf{k}$ is the basic field, (2) $\mathfrak{J} = \mathbf{k}1 \oplus V$ is the Jordan algebra of a nondegenerate symmetric bilinear form f in a finite-dimensional vector space V with $\dim V > 1$, (3) $\mathfrak{J} = \mathrm{H}_n(D, J)$, $n \geq 3$, where $(A, *)$ is a composition algebra of dimension $1, 2$ or 4 if $n \geq 4$ and of dimension $1, 2, 4, 8$ if $n = 3$.*

Following [7], we define a functor $\mathcal{H}_n$ from the category $(A, *) - \text{Bimod}$ of unital associative bimodules with involution over A into the category $\mathfrak{J} - \text{Mod}_1$ of unital Jordan bimodules over $\mathfrak{J}$.

Let $(W, *) \in (A, *) - \text{Bimod}$ and $E = A \oplus W$ be the split null extension of $(A, *)$ by $(W, *)$. Then we let $*$ be the linear mapping of E which extends the given linear mappings $*$ on A and W. Then $(E,*)$ is an associative algebra with involution and identity element 1, the identity of A. We can form the Jordan matrix algebra $K = H_n(E)$ which contains $\mathfrak{J}$ as a subalgebra. Also K contains the ideal $V = W_n \cap K = H_n(W)$ which is just the set of matrices of K whose entries are in the ideal $(W,*)$ of $(E,*)$. Then V is a unital Jordan bimodule for $\mathfrak{J}$ relative to the multiplication defined in K. We shall call V the $\mathfrak{J}$-bimodule *associated* with the given bimodule with involution $(W,*)$ of $(A,*)$ and denote

$$V = \mathcal{H}_n(W).$$

Since $E = A \oplus W$ we have $K = \mathfrak{J} \oplus V$. Also $W^2 = 0$ in E implies $V^2 = 0$ in K so K is the split null extension of $\mathfrak{J}$ by its bimodule V.

It is proved in [7, Sect. III.5] that $\mathcal{H}_n$ for $n \geq 4$ is a functor which establishes an isomorphism from the category $(A,*)-$ Bimod into the category $(H_n(A))-$ Mod.

In the case that $n = 3$, Jordan matrix algebras may have as coordinating algebras not only associative algebras but also alternative ones. Recall that an algebra A is called *alternative* if it satisfies the identities

$$(xx)y = x(xy), \ (xy)y = x(yy).$$

The best known example of an alternative non associative algebra is provided by the 8-dimensional octonion algebra O. An involution $*$ of a (non associative) algebra A is called *nuclear* if the $*$-symmetric elements lie in the nucleus (= associative center) of A. Now, if $(A,*)$ is an alternative algebra with nuclear involution then the algebra $H_3(A)$ of 3×3 $*$-Hermitian matrices over A is Jordan, and the category $H_3(A)-$ Mod is isomorphic to the category $(A,*)-\mathrm{Bimod}_{Alt}$ of unital alternative bimodules with nuclear involution over A [7].

4.3. Simple Jordan bimodules over simple Jordan matrix algebras. Let now J be a special Jordan matrix algebra, that is, $\mathfrak{J} = H(D_n)$ be an algebra of $n\times n$ hermitian matrices over an associative composition algebra $(D,*)$, where $n \geq 3$. Due to the previous section, every unital simple bimodule V for $\mathfrak{J}$ has a form $V = \mathcal{H}_n(W)$, where W is a unital simple associative D-bimodule with involution (alternative bimodule with nuclear involution for $n = 3$). Therefore, it suffices to give the list of such bimodules.

If W is a D-bimodule with involution then it is easy to see that W with the mapping $v \mapsto -v^*$ is also a D-bimodule with involution, which we will denote by $-W$.

(1) $D = \mathbf{k}$, $* = \mathrm{id}_{\mathbf{k}}$. In this case we have two non-isomorphic bimodules

$$W = \mathrm{Reg}\ \mathbf{k}, \ W = -\mathrm{Reg}\ \mathbf{k}.$$

(2) $D = \mathbf{k}\oplus\mathbf{k} = \mathbf{k}e_1 \oplus \mathbf{k}e_2$, $(a,b)^* = (b,a)$. Here we have five non-isomorphic bimodules

$$W = \mathrm{Reg}\ D;\ W = \mathrm{Cay}_i = \mathbf{k}v_i,\ v_i = v_i^* = e_i v_i e_{1-i},\ i = 1,2;\ W = -\mathrm{Cay}_i,\ i = 1,2.$$

(3) $D = \mathrm{M}_2(\mathbf{k})$ with symplectic involution. In this case for $n > 3$ there are two non-isomorphic bimodules

$$W = \mathrm{Reg}\ D,\ W = -\mathrm{Reg}\ D.$$

For $n = 3$ this algebra has also a non-associative alternative simple bimodule with nuclear involution. Let W be a simple left D-module. We give a bimodule structure on it by setting $x \cdot a = ax$, $a \cdot x = \bar{a}x$, where $a \mapsto \bar{a}$ is the involution in D. The obtained bimodule with the involution $v^* = -v$ we denote by Cay D, it is called a *Cayley bimodule* over D.

To describe the non-unital bimodules over $\mathfrak{L} = H(D_n)$ we need a modification of construction of $\mathcal{H}_n$. Every non-unital V is just a unital bimodule over $\mathbf{k} \times \mathfrak{L}$. It is an algebra of the form $H(A, *)$ for some algebra A with involution.

On other hand, on the set of irreducible unital $\mathfrak{L}$-bimodules with involution exists the involution $W \longleftrightarrow W^o$ of taking an opposite bimodule together with an D-bimodule anti-isomorphism $o : W \longrightarrow W^o$

$$W^o = \{w^o \mid w \in W\},\ (w^o)^* = (w^*)^o.$$

On the isoclasses o exchanges $\pm\,\mathrm{Cay}_i \overset{o}{\longleftrightarrow} \pm\,\mathrm{Cay}_{2-i}$, $v_i \overset{o}{\longleftrightarrow} v_{2-i}$, $i = 1, 2$ and is trivial otherwise.

Consider the null extension of associative algebra $A = (\mathbf{k} \times D) \oplus (W^o \oplus W)$, where D is the algebra from the list above and W is a simple unital D-bimodule with involution and $\mathbf{k}$ acts just by the multiplications. Endow A with the involution, which coincides with the canonical on $\mathbf{k} \times D$ and $(w_1^o, w_2)^* = (w_2^o, w_1)$. Note, that if $-A$ be the algebra, constructed by the bimodule $-W$, then A isomorphic to A as an algebra with involution. The isomorphism $\varphi : A \longrightarrow -A$ is an identity on $\mathbf{k} \times D$ and $\varphi(w_1^o, w_2) = (w_1^o, -w_2)$. Then φ commutes with the involution on $\mathbf{k} \times D$ and

$$(w_1^o, w_2) \overset{\varphi}{\longmapsto} (w_1^o, -w_2) \overset{-*}{\longmapsto} (w_2^o, -w_1), \quad (w_1^o, w_2) \overset{*}{\longmapsto} (w_2^o, w_1) \overset{\varphi}{\longmapsto} (w_2^o, -w_1).$$

One can present A as the algebra with involution of matrices

$$\begin{pmatrix} \mathbf{k} & W \\ W^o & D \end{pmatrix} \text{ with involution } \begin{pmatrix} \lambda & w_1 \\ w_2^o & d \end{pmatrix}^* = \begin{pmatrix} \lambda^* & w_2^* \\ (w_1^*)^o & d^* \end{pmatrix}.$$

Consider the algebra $A_{1,n} = M_{1,n}(A))$ (see 3.2) of the matrices

$$\begin{pmatrix} \mathbf{k} & \mathrm{M}_{1\times n}(W) \\ \mathrm{M}_{n\times 1}(W^o) & \mathrm{M}_n(D) \end{pmatrix}$$

with the involution induced from A. Then $A_{1,n}$ has the radical $W_{1,n} \simeq \mathrm{M}_{1\times n}(W) \oplus \mathrm{M}_{n\times 1}(W^o)$ and we denote $\mathcal{H}_{1,n}(W) = \mathcal{H}(W_{1,n})$. It has an obvious structure of a $\mathbf{k} \times \mathrm{M}_n(\mathfrak{L})$-bimodule, hence is a non-unital bimodule over $\mathrm{M}_n(\mathfrak{J})$.

It turns out, that using construction $\mathcal{H}_{1,n}$ we can describe all irreducible non-unital bimodules for the simple matrix algebras. Below we give the list of irreducible unital D-bimodules W, where D is a composition algebra from the list above and such that $\mathcal{H}_{1,n}(W)$ gives an non-unital irreducible module over $\mathcal{H}(\mathrm{M}_n(D), *)$. In all examples the first multiplier $\mathbf{k}$ acts by multiplication on $\frac{1}{2}$.

(1) $D = \mathbf{k}$, $W = \pm\,\mathrm{Reg}\,D$.
(2) $D = \mathbf{k} \oplus \mathbf{k} = \mathbf{k}e_1 \oplus \mathbf{k}e_2$, $W = \mathrm{Reg}\,D$. Then $\mathcal{H}_{1,1}(W)$ splits in direct sum of two irreducible bimodules $W_1 \oplus W_2$, which differs by the transposition. A basic vector in W and W' one can choose as

$$\begin{pmatrix} 0 & e_2 \\ e_1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & e_1 \\ e_2 & 0 \end{pmatrix} \tag{14}$$

which are isomorphic to $\mathcal{H}_{1,1}(\pm\,\mathrm{Cay}_1)$ and $\mathcal{H}_{1,1}(\pm\,\mathrm{Cay}_2)$ correspondingly.

(3) $D = \mathrm{M}_2(\mathbf{k})$ with symplectic involution. In this case for $n > 3$ there are two non-isomorphic bimodules

$$W = \pm \operatorname{Reg} D.$$

4.4. Simple bimodules over semisimple matrix Jordan algebras. We need explicit basics in simple bimodules over matrix Jordan algebras.

A faithful simple unital module M over a semisimple Jordan algebra $\mathfrak{L}$ exist if either $\mathfrak{L}$ is simple or if $\mathfrak{L}$ is a product of two simple algebras $\mathfrak{L}_1 \times \mathfrak{L}_2$. In the last case M is isomorphic to the tensor product $M_1 \otimes M_2$, where M_i is a $\frac{1}{2}$ module over $\mathfrak{J}_i$, $i = 1, 2$. The interested reader we direct to [6], in particular VII.3, Theorem 7, [7]. The lists of bimodules contain two parts, relating unital and non-unital modules.

If V is a simple unital module over a simple matrix algebra, then we do it using the functor $\mathcal{H}_n$ for convenient n. If $\mathfrak{L}$ is a simple algebra, then a unital module over it we will denote by capital letters M, M', N, N', R and their basics by the corresponding small letters with indices (m_{ij}, m'_{ij}, etc). The range of indices i, j and the action are defined uniquely by the functor $\mathcal{H}_n$. R means always the regular module.

If $\mathfrak{L}$ is simple, then we present here diagrams of its simple bimodules. The structure of the diagram of a finite dimensional bimodule over any semisimple $\mathfrak{L}$ we present in subsection 4.5. In a graphical representation a vertex corresponding to the algebra $\mathrm{HM}_i\, i \geqslant 3$ (the involution (1)) is depicted as a square $\boxed{i}$, a vertex, corresponding to the algebra $\mathrm{M}_i^+, i \geqslant 3$ (the involution (2)) we will depict as a circle $ⓘ$, and a vertex corresponding to the symplectic symmetric algebra HM_{iS} (the involution (3)) is depicted as a $\langle i \rangle$.

Besides matrix semisimple Jordan algebras we need the basis in modules over the simple Jordan algebra $\mathfrak{J} = \mathbf{k}$ (see Lemma 4.1). In the graphical representation we will denote it by $\boxed{1}$, since in some aspects of theory of special representation it behaves similar to $\boxed{n}$, $n \geq 3$.

4.4.1. $1_{\mathfrak{L}}$-*bimodules for* $\mathfrak{L} = \mathrm{HM}_n(\mathbf{k}), n \geqslant 3$. $D = \mathbf{k}$, $* = \mathrm{id}_{\mathbf{k}}$.

$$R = \begin{tikzcd} \boxed{n} \arrow[loop above, "R"] \end{tikzcd}, W = \operatorname{Reg} D\,,\ N = \begin{tikzcd} \boxed{n} \arrow[loop above, "N"] \end{tikzcd}, W = -\operatorname{Reg} D.$$

4.4.2. $1_{\mathfrak{L}}$-*bimodules for* $\mathfrak{L} = \mathrm{M}_n^+, n \geqslant 3$. $D = \mathbf{k} \oplus \mathbf{k}$, $(a, b)^* = (b, a)$.

$$R = \begin{tikzcd} ⓝ \arrow[loop above, "R"] \end{tikzcd}, W = \operatorname{Reg} D.\ M = \begin{tikzcd} ⓝ \arrow[loop above, "M"] \end{tikzcd}, W = \mathrm{Cay}_1\,.\ M' = \begin{tikzcd} ⓝ \arrow[loop above, "M'"] \end{tikzcd}, W = \mathrm{Cay}_2\,.$$

$$N = \begin{tikzcd} ⓝ \arrow[loop above, "N"] \end{tikzcd}, W = -\mathrm{Cay}_1\,.\ N' =, \begin{tikzcd} ⓝ \arrow[loop above, "N'"] \end{tikzcd}, W = -\mathrm{Cay}_2\,.$$

4.4.3. $1_{\mathfrak{L}}$-*bimodules for* $\mathfrak{L} = \mathrm{HM}_{nS}(\mathbf{k}), n \geqslant 3$. $D = \mathrm{M}_2(\mathbf{k})$, X^* is the adjacent matrix, $X \in \mathrm{M}_2(\mathbf{k})$.

$$R = \begin{tikzcd} \langle n \rangle \arrow[loop above, "R"] \end{tikzcd}, W = \operatorname{Reg} D\,,\ N = \begin{tikzcd} \langle n \rangle \arrow[loop above, "N"] \end{tikzcd}, W = -\operatorname{Reg} D.$$

4.4.4. $\frac{1}{2}$*-bimodules for* $\mathcal{L} = \mathrm{HM}_n(\mathbf{k})), n \geqslant 3$. $D = \mathbf{k}$, $* = \mathrm{id}_{\mathbf{k}}$.

$$P = \boxed{n}\ , V = \mathcal{H}_{1,1}(\mathrm{Reg}\, D).$$

4.4.5. $\frac{1}{2}$*-bimodules for* $\mathcal{L} = \mathrm{M}_n^+, n \geqslant 3$. $D = \mathbf{k} \oplus \mathbf{k}$, $(a,b)^* = (b,a)$.

$$P = (n)\ , V = \mathcal{H}_{1,1}(\mathrm{Cay}_1),$$

$$P' = (n)\ , V = \mathcal{H}_{1,1}(\mathrm{Cay}_2).$$

4.4.6. $\frac{1}{2}$*-bimodules for* $\mathcal{L} = \mathrm{HM}_{nS}(\mathbf{k}), n \geqslant 3$. $D = \mathrm{M}_2(\mathbf{k})$, X^* is the adjacent matrix, $X \in \mathrm{M}_2(\mathbf{k})$.

$$P = \boxed{n}\ , V = \mathcal{H}_{1,1}(\mathrm{Reg}\, D).$$

4.5. **Diagrams of Jordan algebras of matrix type.** We apply here the graphical convention for depicting the diagram $\Gamma = \Gamma(\mathfrak{J})$, where the Jordan algebra $\mathfrak{J}$ is such that $\mathcal{L} = \mathfrak{J}/\mathrm{Rad}\,\mathfrak{J} \simeq \mathcal{L}_1 \times \cdots \times \mathcal{L}_n$ and all $\mathcal{L}_i$ are simple matrix Jordan algebras. Then $\Gamma_0 = \{1, \dots, n\}$ and the point $i \in \Gamma_0$ is depicted by one of the symbols $\bigcirc$, $\square$, $\diamond$ in correspondence with 4.4.

Let $V = \mathrm{Rad}\,\mathfrak{J}/(\mathrm{Rad}\,\mathfrak{J}^2 + \mathfrak{J}\,\mathrm{Rad}\,\mathfrak{J}^2)$. Then we have enough to define a convention for the depiction of a simple direct summand of M.

Assume first M $t(M) = 1_{\mathcal{L}_i, \mathcal{L}_j}, 1 \leqslant i < j \leqslant n$. Then M is isomorphic to the tensor product $M_i \otimes_{\mathbf{k}} M_j$ where M_i (M_j) is a simple bimodule over $\mathcal{L}_i$ $(\mathcal{L}_j)$ and $t(M_i) = (\frac{1}{2})_{\mathcal{L}_i}$ $(t(M_j) = (\frac{1}{2})_{\mathcal{L}_j})$. As said in 4.4 both of the bimodules M_i are isomorphic to the bimodules P or P'. Following 4.1 a bimodule of such a form will be depicted as an edge x_M between vertices i and j and $c_m(M)$ is the isoclass of M. If $M_i \simeq P'$, then we put an arrow on x_M in i. Analogously, if $M_j \simeq P'$, then we put arrow on x_M in j. Thus we obtain 4 types of arrows x_M. Note, that the algebras $\mathrm{HM}_i, i \geqslant 3$ do not have modules P', hence in our diagrams there are no edges with an arrow in the vertices $\boxed{i}$, $i \geqslant 3$.

If $t(M) = 1_{\mathcal{L}_i}$, then it gives in Γ_1 the loop x_M in the vertex i, $c_M(x_M)$ is the isoclass of $\mathcal{L}_i$. This loop will be depicted as in 4.4.

If $t(M) = \dfrac{1}{2}_{\mathcal{L}_i}$ we depict it as in the case $t(M) = 1_{\mathcal{L}_i}$. But, since in this case the algebra $\mathfrak{J}$ does not contain unity, *we will change* $\mathfrak{J}$ *to* $\tilde{\mathfrak{J}}$ *as in Lemma 4.1 and consider instead of diagram* Γ *the diagram* $\tilde{\Gamma}$. The diagram $\tilde{\Gamma}$ contains a unique vertex, corresponding to $\mathbf{k}$. There are no loops in this new vertex. It will be depicted as $\boxed{1}$, since $\mathbf{k}$ does not have the simple module of the type P'. Γ is reconstructed by $\tilde{\Gamma}$ in the obvious way.

For convenience in the case of the unital bimodule V *we set* $\tilde{\mathfrak{J}} = \mathfrak{J}$, $\tilde{\Gamma} = \Gamma$.

5. Tensor Jordan algebras

5.1. Relative Jordan algebras. The category of Jordan algebras over $\mathbf{k}$ we will denote Jord. By $\mathrm{F} : \mathbf{k}-\mathrm{mod} \to \mathrm{Jord}$ we denote the functor of the free Jordan algebra over $\mathbf{k}$ as left adjoint to the forgetful functor. For a $\mathbf{k}$-space M and $x \in M$ let $[x]$ denote the corresponding generator of $\mathrm{F}(M)$.

Let us fix $\mathcal{L} \in \mathrm{Ob\,Jord}$ and define the category $\mathrm{Jord}_{\mathcal{L}}$ of *relative Jordan algebras* or *Jordan algebras* over $\mathcal{L}$. An object of the category $\mathrm{Jord}_{\mathcal{L}}$ is a Jordan algebra $\mathcal{A}$ endowed with a homomorphism of Jordan algebras $\imath : \mathcal{L} \longrightarrow \mathcal{A}$. The morphism from $\imath_1 : \mathcal{L} \longrightarrow \mathcal{A}_1$ to $\imath_2 : \mathcal{L} \longrightarrow \mathcal{A}_2$ is a homomorphism $f : \mathcal{A}_1 \to \mathcal{A}_2$, such that $\imath_2 = f\imath_1$.

Usually in the notations of a Jordan algebra over $\mathcal{L}$ $\imath : \mathcal{L} \to \mathcal{A}$ we will skip the structure homomorphism $\imath$ and write simply $\mathcal{A}$ and (if necessary) denote corresponding homomorphism $\imath_{\mathcal{A}}$.

5.2. Sums and products, free objects. Consider two Jordan S-algebras $\imath_1 : \mathcal{L} \longrightarrow \mathcal{A}_1$ and $\imath_1 : \mathcal{L} \longrightarrow \mathcal{A}_2$, $\mathrm{F}([\mathcal{A}_1] \oplus [\mathcal{A}_2])$, its multiplication we denote by $\star$. Then the product $\mathcal{A}_1 * \mathcal{A}_2$ ($\mathcal{A}_1 *_{\mathcal{L}} \mathcal{A}_2$) is the factor of $\mathrm{F}([\mathcal{A}_1] \oplus [\mathcal{A}_2])$ by the ideal, generated by the relations $[x_1] \star [y_1] - [x_1 \circ y_1]$, $x_1, y_1 \in \mathcal{A}_1$, $[x_2] \star [y_2] - [x_2 \circ y_2]$, $x_2, y_2 \in \mathcal{A}_2$, $[\imath_1(s)] - [\imath_2(s)]$, $s \in \mathcal{L}$, where by $\circ$ is denoted the multiplication in $\mathcal{A}_1$ and $\mathcal{A}_2$ correspondingly.

Lemma 5.1. *$\mathcal{A}_1 * \mathcal{A}_2$ is the sum of $\mathcal{A}_1$ and $\mathcal{A}_2$ in the category* $\mathrm{Jord}_{\mathcal{L}}$.

Proof. We will denote the elements of $\mathcal{A}_1 * \mathcal{A}_2$ by their representative in the free algebra. There exists a canonical homomorphism $\imath : \mathcal{L} \longrightarrow \mathcal{A}_1 * \mathcal{A}_2$, defined as $\imath(s) = [\imath_1(s)]$ $(= [\imath_2(s)])$, which makes $\mathcal{A}_1 * \mathcal{A}_2$ an algebra over S. The structure morphisms $\sigma_1 : \mathcal{A}_1 \to \mathcal{A}_1 * \mathcal{A}_2$ and $\sigma_2 : \mathcal{L}_2 \to \mathcal{A}_1 * \mathcal{A}_2$ are defined analogously. The universal property of the sum for $\mathcal{A}_1 * \mathcal{A}_2$, i.e. the isomorphism of the functors $\mathrm{Jord}_{\mathcal{L}} \longrightarrow \mathrm{Sets}$

$$\mathrm{Hom}_{\mathrm{Jord}_{\mathcal{L}}}(\mathcal{A}_1 * \mathcal{A}_2, \mathcal{A}) \simeq \mathrm{Hom}_{\mathrm{Jord}_{\mathcal{L}}}(\mathcal{A}_1, \mathcal{A}) \times \mathrm{Hom}_{\mathrm{Jord}_{\mathcal{L}}}(\mathcal{A}_2, \mathcal{A}),\ f \longmapsto (f\sigma_1, f\sigma_2)$$

follows immediately from the definitions. □

Remark 5.1. *The product "$*$" endows the category of Jordan algebras over $\mathcal{L}$ with a structure of symmetrical monoidal category with the unit $\mathcal{L}$.*

5.3. Relatively free (tensor) Jordan algebras. Let $\varepsilon : \mathcal{A} \longrightarrow \mathcal{J}$ be a morphism of Jordan algebras. Due to the standard definitions it induces the functor of the multiplication envelopes $\mathrm{U}(\varepsilon) : \mathrm{U}(\mathcal{A}) \to \mathrm{U}(\mathcal{J})$ and $\mathrm{U}(\varepsilon)$ induces the canonical functor $F^*_\varepsilon : \mathrm{U}(\mathcal{J}) - \mathrm{mod} \to \mathrm{U}(\mathcal{A}) - \mathrm{mod}$, or, equivalently, $F^*_\varepsilon : \mathcal{J} - \mathrm{mod} \to \mathcal{A} - \mathrm{mod}$.

On other hand, for every $\mathcal{L} \in \mathrm{Jord}_{\mathcal{L}}$ the structure morphism $\imath_{\mathcal{A}} : \mathcal{L} \to \mathcal{A}$ endows $\mathcal{A}$ with the structure of an $\mathcal{L}$-bimodule, which defines a restriction functor $R : \mathrm{Jord}_{\mathcal{L}} \to \mathcal{L} - \mathrm{mod}$. This functor allows left adjoint $\mathcal{L}[] : \mathcal{L} - \mathrm{mod} \to \mathrm{Jord}_{\mathcal{L}}$, $B \longmapsto \mathcal{L}[B]$, i.e

$$\tau : \mathrm{Hom}_{\mathcal{L}}(B, F(\mathcal{A})) \simeq \mathrm{Hom}_{\mathrm{Jord}_{\mathcal{L}}}(\mathcal{L}[B], \mathcal{A}), \quad B \in \mathcal{L} - \mathrm{mod}, \mathcal{A} \in \mathrm{Jord}_{\mathcal{L}}\,. \tag{15}$$

The Jordan algebra $\mathcal{L}[B]$ is defined as a factor of the free Jordan algebra $F([\mathcal{L}] \oplus [B])$ by the ideal, generated by the relations $[l_1] \star [l_2] - [l_1 \circ l_2]$, $l_1, l_2 \in \mathcal{L}$, $[b] \star [l] - [b \cdot l]$, $[l] \star [b] - [l \cdot b]$, $s \in \mathcal{L}$, $b \in B$. The algebra $\mathcal{L}[B]$ is canonically graded by $\deg x = 0$, $x \in \mathcal{L}$, $\deg b = 1$, $b \in B$.

The following lemma connects two of the notions just introduced.

Lemma 5.2. $\mathfrak{L}[B_1] * \mathfrak{L}[B_2] \simeq \mathfrak{L}[B_1 \oplus B_2]$.

Proof. It follows from the following chain of isomorphisms of the functors from $\mathrm{Jord}_{\mathfrak{L}}$ to the Sets:

$$\begin{aligned}&\mathrm{Hom}_{\mathbf{Jord}_{\mathfrak{L}}}(\mathfrak{L}[B_1] * \mathfrak{L}[B_2], \_) \simeq\\ &\mathrm{Hom}_{\mathbf{Jord}_{\mathfrak{L}}}(\mathfrak{L}[B_1], \_) \times \mathrm{Hom}_{\mathbf{Jord}_{\mathfrak{L}}}(\mathfrak{L}[B_2], \_) \simeq\\ &\mathrm{Hom}_{\mathfrak{L}-\mathbf{mod}}(B_1, \_) \times \mathrm{Hom}_{\mathfrak{L}-\mathbf{mod}}(B_2, \_) \simeq\\ &\mathrm{Hom}_{\mathfrak{L}-\mathbf{mod}}(B_1 \oplus B_2, \_) \simeq \mathrm{Hom}_{\mathbf{Jord}_{\mathfrak{L}}}(\mathfrak{L}[B_1 \oplus B_2], \_).\end{aligned}$$

□

Lemma 5.3. *Let $\mathfrak{J} = \mathfrak{L} \oplus \mathrm{Rad}\,\mathfrak{J}$ be a finite dimensional Jordan algebra and* $\mathrm{U}(\mathfrak{J})$ *be its multiplicative envelope. Then* $\mathrm{Rad}\,\mathrm{U}(\mathfrak{J})$ *is generated by* $\mathrm{Rad}\,\mathfrak{J}$ *and* $\mathrm{U}(\mathfrak{J}) \simeq \mathrm{U}(\mathfrak{L}) \oplus \mathrm{Rad}\,\mathrm{U}(\mathfrak{J})$.

Proof. Following [6], Chapter VI, section 2, Theorem 2 $\mathrm{Rad}\,\mathfrak{J}$ generates a nilpotent ideal I in $\mathrm{U}(\mathfrak{J})$, hence $I \subset \mathrm{Rad}\,\mathrm{U}(\mathfrak{J})$. On the other hand $\mathrm{U}(\mathfrak{J}) = \mathrm{U}(\mathfrak{L}) + I$, where $\mathrm{U}(\mathfrak{L})$ is a semisimple algebra, which completes the proof. □

Lemma 5.4. *Let $\mathfrak{J}$ be a finite dimensional Jordan algebra, $\mathfrak{J}_n \subset \mathfrak{J}$ an ideal, generated by the non-associative words in $\mathfrak{J}$, where at least n letters belongs to* $\mathrm{Rad}\,A$. *Then* $\mathrm{Rad}\,\mathfrak{J}$ *is strongly nilpotent, i.e. there exists $N \geq 1$, such that $\mathfrak{J}_k = 0$ for any $k \geqslant N$.*

Proof. Let n be the degree of nilpotency of $\mathrm{Rad}\,\mathrm{U}(\mathfrak{J})$. Then in $\mathrm{U}(\mathfrak{J})$ every associative word, containing at least n letters from $\mathrm{Rad}\,\mathrm{U}(\mathfrak{J})$, in particular from $\mathrm{Rad}\,\mathfrak{J}$, equals 0. Then the equality in $\mathrm{U}(\mathfrak{J})$

$$\begin{aligned}a_1 \cdot (a_3 \cdot a_2) = -a_1a_2a_3 - a_3a_2a_1 +&\\ a_1(a_2 \cdot a_3) + a_2(a_1 \cdot a_3) + a_3(a_1 \cdot a_2),&\quad a_1, a_2, a_3 \in \mathfrak{J}.\end{aligned}$$

shows, that every non-associative word in $\mathfrak{L}$, containing at least $N = 2^n$ elements from $\mathrm{Rad}\,\mathfrak{J}$ is zero in $\mathrm{U}(\mathfrak{J})$, hence zero in $\mathfrak{J}$. □

Proposition 5.1. *Let $\mathfrak{J}$ be a finite dimensional Jordan algebra, $\mathfrak{L}$ a Levi subalgebra,* $\pi : \mathrm{Rad}\,\mathfrak{J} \to V = \mathrm{Rad}\,\mathfrak{J}/(\mathrm{Rad}^2\,\mathfrak{J} + \mathfrak{J}\,\mathrm{Rad}^2\,\mathfrak{J})$ *be the canonical projection, and* $M \subset \mathrm{Rad}\,\mathfrak{J}$ *be a subspace, such that $\pi(M) = V$. Then the subalgebra in $\mathfrak{J}$, generated by $\mathfrak{L}$ and M, coincides with $\mathfrak{J}$.*

In particular, if $s : V \longrightarrow \mathrm{Rad}\,\mathfrak{J}$ is a $\mathfrak{L}$-module homomorphism, such that $\pi s = \mathrm{id}_V$, then the homomorphism $f : \mathfrak{L}[V] \longrightarrow \mathfrak{J}$ of Jordan algebras over $\mathfrak{L}$ induced by s is surjective.

Proof. Obviously, $\mathfrak{L} \subset \mathfrak{J}$, hence we should prove $\mathrm{Rad}\,A \subset \mathfrak{J}$. Then for every $a \in \mathfrak{I}_n$ there exists $x \in <\mathfrak{L}, M>$, such that $a - x \in \mathfrak{I}_{n+1}$. But there exists $N \geq 0$, such that $\mathfrak{J}_N = 0$, that completes the proof of the first statement. The second statement is a corollary of the first. □

5.4. Jordan algebras with Levi decomposition and completion. Let $\mathfrak{L}$ be a semisimple finite dimensional algebra. Denote $\mathrm{Jords}_{\mathfrak{L}} \subset \mathrm{Jord}_{\mathfrak{L}}$ the full subcategory of $\imath : \mathfrak{L} \to A$, such that $\imath$ is injective and A is a direct sum of $\mathrm{Im}\,\imath$ and $\mathrm{Rad}\,A$, where $\mathrm{Rad}\,A$ is the Jacobson radical of A.

Lemma 5.5. *Any two objects of* $\mathrm{Jords}_{\mathfrak{L}}$ $\imath_1 : \mathfrak{L} \to A$ *and* $\imath_2 : \mathfrak{L} \to A$ *are isomorphic.*

Proof. It is the Levi-Maltsev theorem for Jordan algebras. □

In this category there exists a product $\mathcal{J}_1 \times_{\mathcal{L}} \mathcal{J}_2 = \mathcal{L} \oplus \operatorname{Rad}\mathcal{J}_1 \oplus \mathcal{J}_2$ with an obvious composition. If as a sum we define $\mathcal{J}_1 *_{\mathcal{L}} \mathcal{J}_2$, then the natural complement to $\mathcal{L}$ will not necessarily be a radical of $\mathcal{J}_1 *_{\mathcal{L}} \mathcal{J}_2$.

Assume $\mathcal{A} \in \operatorname{Ob}\operatorname{Jord}_{\mathcal{L}}$ is endowed by a descent separable filtration of Jordan ideals $\mathcal{A} = \mathfrak{I}_0 \supset \mathfrak{I}_1 \supset \dots$ such that $\mathfrak{I}_i \circ \mathfrak{I}_j \subset \mathfrak{I}_{i+j}$. The filtration defines a degree $\deg (= \deg_{\mathcal{A}})$ on $\mathcal{A}$: $\deg x = k$, provided that $x \in \mathfrak{I}_k \setminus \mathfrak{I}_{k+1}$. The filtration $\deg_{\mathcal{A}}$ endows the algebra $\mathrm{U}(\mathcal{A})$ with a filtration, defined $\deg_{\mathrm{U}(\mathcal{A})} x = \deg_{\mathcal{A}} x, x \in \mathcal{A}$.

We denote by $\hat{\mathcal{J}} = \varprojlim_k \mathcal{A}/\mathfrak{I}_k$ the *completion* of $\mathcal{A}$ and by $\hat{\mathfrak{I}}_k \subset \hat{\mathcal{A}}$ denote the complete ideal, generated by the image of $\mathfrak{I}_k$. The Jordan algebra $\hat{\mathcal{A}}$ is endowed with the topology of an inverse limit.

We call a (not necessarily finite dimensional) Jordan algebra a *complete* Jordan algebra provided that the following holds:

(1) $\mathcal{J} = \mathcal{L} \oplus \operatorname{Rad}\mathcal{J}$, where $\mathcal{L}$ is a finite dimensional semisimple Jordan algebra.

(2) $\mathcal{J}$ is complete and separable in the topology, defined by the chain of ideals

$$\{\operatorname{Rad}^2\mathcal{J} + \mathcal{J}\operatorname{Rad}^2\mathcal{J} \supset \operatorname{Rad}^3\mathcal{J} \supset \dots \supset \operatorname{Rad}^n\mathcal{J} \supset \dots\}.$$

If the $\mathcal{L}$-module $\mathfrak{Q}(\mathcal{J}) = \operatorname{Rad}\mathcal{J}/(\operatorname{Rad}^2\mathcal{J} + \mathcal{J}\operatorname{Rad}^2\mathcal{J})$ is finite dimensional, then we call $\mathcal{J}$ *complete finitely generated.*

The corresponding category we denote Jordc, where the morphisms in Jordc will be the continuous homomorphisms of Jordan algebras. Obviously, the category Jordc contains Jord as a full subcategory. The category $\mathcal{J}$ – mod denotes the category of finite dimensional modules. The action of $\mathcal{J}$ on such a module M is continuous, provided that M is endowed with discrete topology. In this category a product is defined as a natural completion of $\mathcal{J}_1 *_{\mathcal{L}} \mathcal{J}_2$.

5.5. Complete tensor Jordan algebras. We apply this construction for $\mathcal{A} = \mathcal{L}[V]$. $\mathcal{L}[V]$ is a graded Jordan algebra and the completion of $\mathcal{A}$ in the associated filtration $\{\mathfrak{I}_i | i \geq 0\}$ we will call a *completed tensor Jordan algebra* and will denote it by $\widehat{\mathcal{L}[V]}$. Since $\mathcal{L}[V]$ is graded, the filtration $\mathfrak{I}_n$ is separable, i.e. $\cap_{i\geq 0}\mathfrak{I}_n = 0$.

Assume, $\mathcal{L}$ is a semisimple Jordan algebra. Then the complete ideal $\hat{I}_V \subset \mathrm{U}(\widehat{\mathcal{L}[V]})$, generated by V is the (topological) radical in $\mathrm{U}(\widehat{\mathcal{L}[V]})$. The following lemmas are standard.

Lemma 5.6. *(1) Let $\mathcal{J}$ be a Jordan algebra over $\mathcal{L}$, $\mathcal{J} = \mathcal{L} \oplus \operatorname{Rad}\mathcal{J}$ and $\mathcal{J}$ be complete in* $\operatorname{Rad}\mathcal{J}$*-topology. Then there exists a functorial isomorphism*

$$\hat{\tau} : \operatorname{Hom}_{\mathcal{L}}(V, \mathcal{J}) \sim \operatorname{Hom}_{\mathrm{Jord}}(\widehat{\mathcal{L}[V]}, \mathcal{J})_c, \tag{16}$$

where $\operatorname{Hom}_{\mathrm{Jord}}(\widehat{\mathcal{L}[V]}, \mathcal{J})_c$ *consists of continuous homomorphisms.*

(2) A continuous homomorphism of $\mathcal{L}$-algebras $\varphi : \widehat{\mathcal{L}[V]} \longrightarrow \widehat{\mathcal{L}[V]}$ is an isomorphism if and only if $\mathcal{L}$- module induced homomorphism $V \longrightarrow V$ is an isomorphism.

Lemma 5.7. *Let $\mathcal{J}$ be a Jordan algebra over $\mathcal{L}$, complete and separable in the topology defined by an ideal $\mathfrak{I} \subset \mathcal{J}$, $\mathcal{J} = \mathcal{L} \oplus \mathfrak{I}$, $\pi : \mathfrak{I} \to \mathfrak{I}/(\mathfrak{I}^2 + \mathcal{L}\mathfrak{I}^2) = V$ the canonical projection and $M \subset \mathfrak{I}$ a subspace such that $\pi(M) = V$. Then the subalgebra in $\mathcal{J}$, generated by $\mathcal{L}$ and V is dense in $\mathcal{J}$, i.e. M generates $\mathcal{J}$ in a topological sense.*

In particular, for a $\mathfrak{L}$-module map $s : V \longrightarrow \mathfrak{J}$, $\pi s = \mathrm{id}_V$, the continuous homomorphism of algebras over $\mathfrak{L}$ $f : \widehat{\mathfrak{L}[\bar{\mathfrak{J}}]} \longrightarrow \mathfrak{J}$ induced by s is an epimorphism.

Proof. The proof is analogous to the proof of Proposition 5.1. ☐

5.6. Minimal tensor cover of a Jordan algebra. The foll owing proposition shows the minimality of the construction in Proposition 5.1 and in Lemma 5.7. The proofs of these sta temets are standard.

Let $\mathfrak{J}$ be a Jordan algebra over the semisimple Jordan algebra $\mathfrak{L}$, R be its (topological) r adical and $\mathfrak{J} = \mathfrak{L} \oplus R$ hold, $V = R/(R^2 + R^2\mathfrak{J})$ be finite dimensional and $\pi : R \to V$ the canonical projection. Since $\mathfrak{L}$ is semisimple, there exists a $\mathfrak{L}$-bilinear morphism $\tau : V \to R$, such that $\pi\sigma = \mathrm{id}_V$.

Proposition 5.2. *In the situation above the following holds:*

1. *If $\mathfrak{J}$ is complete in R-adic topology (in pa rticular, if $\mathfrak{J}$ is finite dimensional), then $\widehat{\mathfrak{L}[t]} : \hat{T} \longrightarrow \mathfrak{J}$ is an epimorphism.*
2. *If $\mathfrak{J}$ is finite dimensional, then $\mathfrak{L}[t] : \mathfrak{L}[B] \longrightarrow \mathfrak{J}$ is an epimorphism.*
3. *Let $\mathfrak{J}$ be complete in R-adic topology, B a $\mathfrak{L}$-bimodule, $p : \widehat{\mathfrak{L}[B]} \longrightarrow \mathfrak{J}$ a continuous epimorphism over $\mathfrak{L}$. Then there exists a continuous homomorphism over $\mathfrak{L}$ $\varphi : \widehat{\mathfrak{L}[B]} \longrightarrow \widehat{\mathfrak{L}[V]}$ such that $p = \pi\varphi$. Besides, φ induces an epimorphism of $\mathfrak{L}$-modules $B \to V$.*

Corollary 5.1. *Let $\mathfrak{L}$ be a semisimple Jordan algebra, V a finite dimensional bimodule, $\pi : \mathfrak{L}[V] \longrightarrow \mathfrak{J}$ be a $\mathfrak{L}$-epimorphism, such that $\mathrm{Ker}\,\pi \in V \circ V + \mathfrak{L}(V \circ V)$. If there exists another $\mathfrak{L}$-epimorphism $\varphi : \mathfrak{L}[W] \longrightarrow \mathfrak{J}$, such that $\mathrm{Ker}\,\varphi \in W \circ W + \mathfrak{L}(W \circ W)$, then $W \simeq V$ as a $\mathfrak{L}$-module.*

Proof. Let $\mathfrak{I}_n = \mathfrak{I}_n(V)$ be the chain of ideals in $\mathfrak{J}$, defined in Lemma 5.4. The statement is obvious in the case of a finite dimensional Jordan algebra. In the general situation there exists $n \geq 1$, such that $\mathfrak{I}_n \cap \pi(V) = \mathfrak{I}_n \cap \varphi(W) = 0$ and we reduce the problem to the case of the finite dimensional Jordan algebra $\mathfrak{J}/\mathfrak{I}_n$. ☐

5.7. Jordan bimodules over relatively free Jordan algebras. Let $\mathfrak{L}$ be a Jordan algebra, V be an $\mathfrak{L}$-module. Consider the $\mathrm{S}(\mathfrak{L})$-bimodule

$$\tilde{V} = \mathrm{S}(\mathfrak{L}) \otimes_{\mathbf{k}} V \otimes_{\mathbf{k}} \mathrm{S}(\mathfrak{L})/I, \text{ where } I \text{ is generated by} \tag{17}$$
$$c(s, v) = s \otimes v \otimes 1 + 1 \otimes v \otimes s - 2(1 \otimes s \cdot v \otimes 1),\ s \in \mathfrak{L}, v \in V.$$

$\widetilde{V}$ is a bimodule with involution $*$ over the algebra with involution $\mathrm{S}(\mathfrak{L})$

$$(s_1 \otimes v \otimes s_2 + I)^* = s_2^* \otimes v \otimes s_1^* + I,\ s_1, s_2 \in \mathrm{S}(\mathfrak{L}), v \in V.$$

Remark 5.2.

$\widetilde{V_1 \oplus V_2} \simeq \tilde{V}_1 \oplus \tilde{V}_2$.

Then $\tilde{}$ defines a functor $\tilde{} : \mathrm{S}(\mathfrak{L}) - \mathrm{mod} \longrightarrow \mathrm{S}(\mathfrak{L}) - \mathrm{bimod}$. This functor has a universal property. If S is an algebra, then by As_S we denote the category of algebras over S. Assume A an algebra and A^+ endowed with the structure of a Jordan $\mathfrak{L}$-algebra $\imath : \mathfrak{L} \longrightarrow A^+$. Then A is obviously an $\mathrm{S}(\mathfrak{L})$-algebra, in particular it has the $\imath$-induced structure of an $\mathrm{S}(\mathfrak{L})$-bimodule.

Lemma 5.8. *There exists a functorial isomorphism*

$$\beta : \mathrm{Hom}_{\mathfrak{L}}(V, A^+) \simeq \mathrm{Hom}_{\mathrm{S}(\mathfrak{L})-\mathrm{bimod}}(\widetilde{V}, A) \tag{18}$$

Proof. Let $f : V \longrightarrow A^+$ be any $\mathbb{k}$-linear map. Then it defines uniquely an S-bimodule morphism $F : \mathrm{S} \otimes_{\mathbb{k}} V \otimes_{\mathbb{k}} \mathrm{S} \longrightarrow A$, such that $F(1 \otimes v \otimes 1) = f(v)$. But if f is an $\mathcal{L}$-module homomorphism, then

$$F(c(s,v)) = \imath(s)f(v) + f(v)\imath(s) - 2f(s \cdot v) = 2(s \cdot f(v) - f(s \cdot v)) = 0,$$

which defines $\beta(f)$ on $\tilde{V}$.

On the other hand, if $F : \tilde{V} \longrightarrow A$ is an $\mathrm{S}(\mathcal{L})$-bimodule homomorphism and $j : V \longrightarrow \tilde{V}$, $j(v) = 1 \otimes v \otimes 1 + I$, then the composition $G = Fj : V \longrightarrow A^+$ is an $\mathcal{L}$-module homomorphism

$$G(s \cdot v) = F(1 \otimes s \cdot v \otimes 1) = \frac{1}{2}F(s \otimes v \otimes 1 + 1 \otimes v \otimes s) = \frac{1}{2}\big(\imath(s)F(v) + F(v)\imath(s)\big) = s \cdot G(v). \tag{19}$$

Obviously, β and β^{-1} are functorial and mutually inverse. $\square$

Let $\mathrm{As}_{\mathrm{S}(\mathcal{L})}$ be the category of algebras over $\mathrm{S}(\mathcal{L})$. Then there exists a standard functor $+ : \mathrm{As}_{\mathrm{S}(\mathcal{L})} \to \mathrm{Jord}_{\mathcal{L}}$ and the functor of the special universal envelope $\mathrm{S} : \mathrm{Jord}_{\mathcal{L}} \to \mathrm{As}_{\mathrm{S}(\mathcal{L})}$. These functors form an adjoint pair, i.e

$$\alpha : \mathrm{Hom}_{\mathrm{Jord}_{\mathcal{L}}}(\mathfrak{J}, A^+) \simeq \mathrm{Hom}_{\mathrm{As}_{\mathcal{L}}}(\mathrm{S}(\mathfrak{J}), A). \tag{20}$$

Let $(A, *)$ be an associative algebra with involution and $\mathfrak{J} \subset A$ a Jordan subalgebra in A^+. We call $\mathfrak{J} \subset A$ an *involutive pair*, provided that $\mathfrak{J}$ coincides with the subalgebra of symmetric elements $\mathrm{H}(A, *)$ in A and the canonical inclusion $\imath : \mathfrak{J} \hookrightarrow A^+$ induces an isomorphism $\mathrm{S}(\imath) : \mathrm{S}(\mathfrak{J}) \longrightarrow A$, i.e. $(A, *)$ is reflexive.

If $f : \mathfrak{J} \longrightarrow A^+$, then the diagram below shows, that this is the same as defining a homomorphism $\mathrm{S}(f) : \mathrm{S}(\mathfrak{J}) \longrightarrow A$ of associative algebras over $\mathrm{S}(\mathcal{L})$.

(21)
$$\begin{tikzcd} & \mathcal{L} \arrow[dl, "\imath_{\mathfrak{J}}"'] \arrow[dr, "\imath_{A^+}"] & \\ \mathfrak{J} \arrow[rr, "f"] & & A^+ \end{tikzcd} \quad \Longleftrightarrow \quad \begin{tikzcd} & \mathrm{S}(\mathcal{L}) \arrow[dl, "\alpha(\imath_{\mathfrak{J}})"'] \arrow[dr, "\mathrm{S}(\imath_{A^+})"] & \\ \mathrm{S}(\mathfrak{J}) \arrow[rr, "\mathrm{S}(f)"] & & A \end{tikzcd}$$

If $F : \mathrm{As}_A \longrightarrow A - \text{bimod}$ is a forgetful functor, then

$$\gamma : \mathrm{Hom}_{A-A}(V, F(V)) \simeq \mathrm{Hom}_{\mathrm{As}_A}(A\langle V\rangle, V). \tag{22}$$

Theorem 5.1. $\mathrm{S}(\mathcal{L}[V]) \simeq \mathrm{S}(\mathcal{L})\langle\tilde{V}\rangle$.

Proof. Let V be an $\mathcal{L}$-module. Then there exists the following chain of functorial isomorphisms.

$$\begin{aligned} &\mathrm{Hom}_{\mathrm{As}_{\mathrm{S}(\mathcal{L})}}(\mathrm{S}(\mathcal{L}[V]), A) \simeq \text{ (due to the ajointness (20))} \\ &\mathrm{Hom}_{\mathrm{Jord}_{\mathcal{L}}}(\mathcal{L}[V], A^+) \simeq \text{ (universal property (15))} \\ &\mathrm{Hom}_{\mathcal{L}}(V, A^+) \simeq \text{ (Lemma 5.8)} \\ &\mathrm{Hom}_{\mathrm{S}(\mathcal{L})-\mathrm{S}(\mathcal{L})}(\tilde{V}, A) \simeq \text{ (universal property of tensor algebra (22))} \\ &\mathrm{Hom}_{\mathrm{As}_{\mathrm{S}(\mathcal{L})}}(\mathrm{S}(\mathcal{L})\langle\tilde{V}\rangle, A). \end{aligned} \tag{23}$$

□

We say, that a Jordan algebra $\mathcal{J}$ with a Levi subalgebra $\mathcal{L}$ is *finitely represented* over $\mathcal{L}$, if $V = \operatorname{Rad}\mathcal{J}/(\operatorname{Rad}^2\mathcal{J}+\mathcal{J}\operatorname{Rad}^2\mathcal{J})$ is finitely generated over $\mathcal{L}$ and the kernel of a projection $\pi : \mathcal{L}[V] \twoheadrightarrow \mathcal{J}$ is finitely generated over $\mathcal{L}[V]$.

Lemma 5.9. *A finite dimensional Jordan algebra $\mathcal{J}$ is finitely represented over its Levy subalgebra $\mathcal{L}$.*

Proof. Let n be the degree of strong nilpotency of $\operatorname{Rad}\mathcal{J}$. Then the ideal $\mathfrak{I}_n$ in $\mathcal{L}[v]$ consisting of words with at least n letters in V is finitely generated, the Jordan algebra $\mathcal{J}/\mathfrak{I}_n$ is finite dimensional and this algebra covers $\mathcal{J}$, which completes the proof. □

Lemma 5.10. *Let $\imath : \mathcal{L}[V]\longrightarrow \mathrm{S}(\mathcal{L}[V])^+$ be a canonical homomorphism of Jordan algebras and $\pi : \mathcal{L}[V]\longrightarrow\mathcal{J}$ an epimorphism of algebras over $\mathcal{L}$, $I = \operatorname{Ker}\pi$ and $(I) \subset \mathrm{S}(\mathcal{L}[V])$ be the ideal, generated by I. Then $\mathrm{S}(\mathcal{J}) \simeq \mathrm{S}(\mathcal{L}[V])/(I)$.*

Proof. Obviously, the canonical epimorphism $\mathcal{L}(\pi)$: $\mathrm{S}[\mathcal{L}[V]]\longrightarrow\mathrm{S}(\mathcal{J})$ factorizes through $p : \mathrm{S}(\mathcal{L}[V])\longrightarrow\mathrm{S}(\mathcal{L}[V])/(I)$. But for any associative algebra A

$$\begin{aligned}
&\mathrm{Hom}_{\mathrm{Jord}_{\mathcal{L}}}(\mathcal{J}, A^+) \simeq\\
&\{f \in \mathrm{Hom}_{\mathrm{Jord}_{\mathcal{L}}}(\mathcal{L}[V], A^+) \mid f(I) = 0\} \simeq\\
&\{F \in \mathrm{Hom}_{\mathrm{Ass}(\mathcal{L})}(\mathrm{S}(\mathcal{L}[V]), A) | F(I) = 0\} \simeq\\
&\mathrm{Hom}_{\mathrm{Ass}(\mathcal{L})}(\mathrm{S}(\mathcal{L}[V])/(I), A)
\end{aligned}\tag{24}$$

holds. □

Following Theorem 5.1, the $\imath$ induced canonical homomorphism Ψ : $\mathcal{L}[V]\longrightarrow\mathrm{S}(\mathcal{L})\langle\tilde{V}\rangle$, sends any non-associative word $w = x_1 \dots x_N$ with letters from $\mathcal{L}$ and V in the corresponding Jordan element into the algebra $\mathrm{S}(\mathcal{L})\langle\tilde{V}\rangle$, where for the letter $s \in \mathcal{L}$, $\Psi(s) = s$ holds and for $v \in V$ $\Psi(v)$ is the class of $1 \otimes v \otimes 1$ in $\tilde{V}$.

Corollary 5.2. *If $R_{\mathcal{J}}$ is a system of generators of the Jordan ideal I, then $\Psi(R_{\mathcal{J}})$ is a system of generators of the associative (I).*

Proof. Let $R_{\mathcal{J}} = \{r_k\}$, where k runs some set of indices. We should prove, that $\Psi(I)$ belongs to ideal, generated by all $\Psi(r_k)$. If $x \in I$, then $x = x_1 \dots x_n$ for some non-associative word $x_1 \dots x_n$, where some $x_i = r_k$. If $n = 1$ then all proved, otherwise $x = (x_1 \dots x_j) \circ (x_{j+1} \dots x_n)$ for some j, $\Psi(x) = \Psi(x_1 \dots x_j)\Psi(x_{j+1} \dots x_n) + \Psi(x_{j+1} \dots x_n)\Psi(x_1 \dots x_j)$ and induction in n completes the proof. □

The corollary above together with Lemma 3.3 gives us possibility to write down a basic algebra, which is Morita equivalent to $\mathrm{S}(J)$, where $\mathcal{J}$ is an algebra over $\mathcal{L}$, presented by generator and relations.

6. Structure of special universal envelope of matrix Jordan algebra

Theorem 6.1. *Let $\mathcal{J} = \mathcal{L} + \operatorname{Rad}\mathcal{J}$, $\mathcal{L} = \bigoplus_{i=1}^{k} \mathrm{H}_{n_i}(D_i)$, $n_i \geqslant 3$, $(D_i, *)$ associative composition algebras, $1 = e_1 + \dots + e_n$ the corresponding decomposition of unit of $\mathcal{J}$ in the sum of orthogonal idempotents, $N = \operatorname{Rad}\mathcal{J}$, $N = \bigoplus_{1\leqslant i\leqslant j\leqslant n} N_{ij}$, $N_{ij} = e_i, N, e_j$, where x, y, z for $x, y, z \in \mathcal{J}$ means the Jordan triple product. Then*

(1)

$$\mathrm{S}(\mathcal{J}) = \Big(\bigoplus_{i=1}^{k} \mathrm{M}_{n_i}(\tilde{D}_i)\Big) \bigoplus \Big(\bigoplus_{1 \leqslant i \leqslant j \leqslant n} (\tilde{N}_{ij} + \tilde{N}_{ji})\Big), \tag{25}$$

where $(\tilde{D}_i, *)$ is an algebra with involution, which contains an ideal $(R_i, *)$, such that $D_i = \tilde{D}_i / R_i$, $N_{ii} = \mathrm{H}_{n_i}(R_i)$ and $\tilde{N}_{ij} = e_i N_{ij} e_j$.

(2) $\mathcal{J} = \mathrm{H}(\mathrm{S}(\mathcal{J}), *)$, where $*$ is the principal involution in $\mathrm{S}(\mathcal{J})$ ($a^* = a$ for any $a \in \mathcal{J}$). In other words, $(\mathrm{S}(\mathcal{J}), *)$ is reflexive.

(3) The map

$$n \longmapsto e_i n e_j$$

gives $1-1$-correspondence between N_{ij} and $\tilde{N}_{ij}$. In particular

$$\dim N_{ij} = \dim \tilde{N}_{ij} = \dim \tilde{N}_{ji}$$

for any $i \neq j$.

(4) If $\tilde{N} = \mathrm{Rad}\,\mathrm{S}(\mathcal{J})$, then

$$\mathrm{S}(\mathcal{J}/(N^2 + \mathcal{J}N^2) \simeq \mathrm{S}(\mathcal{J})/\tilde{N}^2.$$

In particular, if $N^2 = 0$, then $\tilde{N}^2 = 0$.

Proof. We have

$$\begin{aligned} &\mathrm{S}(\mathcal{J}) = \mathrm{S}(\mathcal{L}) + \tilde{N}, \tilde{N} = \mathrm{Rad}\,\mathrm{S}(\mathcal{J}), \\ &\mathrm{S}(\mathcal{L}) = \mathrm{S}\Big(\bigoplus_{i=1}^{k} \mathrm{H}_{n_i}(D_i)\Big) = \bigoplus_{i=1}^{k} \mathrm{S}(\mathrm{H}_{n_i}(D_i)) = \bigoplus_{i=1}^{k} \mathrm{M}_{n_i}(D_i), \\ &\tilde{N} = \bigoplus_{i,j=1}^{k} e_i \tilde{N} e_j = \bigoplus_{i,j=1}^{k} \tilde{N}_{ij}. \end{aligned} \tag{26}$$

Thus

$$\mathrm{S}(\mathcal{J}) = \bigoplus_{i=1}^{k} \mathrm{M}_{n_i}(D_i) + \bigoplus_{i,j=1}^{k} \tilde{N}_{ij}. \tag{27}$$

Consider the "tetrad-eating" ideal $Z_{48}(\mathcal{J})$ in $\mathcal{J}$ (see [?], [14]). Evidently, for any $\mathrm{H}_{n_i}(D_i)$ we have $Z_{48}(\mathrm{H}_{n_i}(D_i)) \neq 0$ (since $n_i \geqslant 3$), hence all $e_i \in Z_{48}(\mathcal{J})$ and so $1 \in Z_{48}(\mathcal{J})$, hence $Z_{48}(\mathcal{J}) = \mathcal{J}$. Thus, by [14], $\mathrm{S}(\mathcal{J})$ is reflexive, that is

$$\mathcal{J} = \mathrm{H}(\mathrm{S}(\mathcal{J}), *),$$

where $*$ is the principal involution in $\mathrm{S}(\mathcal{J})$. Observe, that $*$ in restriction on $\mathrm{M}_{n_i}(D_i)$, coincides with a canonical involution in $\mathrm{M}_{n_i}(D_i)$, given by $(a_{ij})^* = (a^*_{ji})$. Thus we have $\mathrm{H}(\tilde{N}, *) = N$, $\mathrm{H}(\tilde{N}_{ii}, *) = N_{ii}$, $\mathrm{H}(\tilde{N}_{ij} + \tilde{N}_{ji}, *) = N_{ij}$. For any $\tilde{n} \in \tilde{N}$ we have

$$n_{ij} = e_i \tilde{n} e_j + e_j \tilde{n}^* e_i \in \mathrm{H}(\tilde{N}_{ij} + \tilde{N}_{ji}, *) = N_{ij}.$$

Now $e_i n_{ij} e_j = e_i \tilde{n} e_j = \tilde{n}_{ij}$. This proves, that $\tilde{N}_{ij} = e_i N_{ij} e_j$.

Assume, that $e_i n e_j \neq 0$ for some $n \in N_{ij}$. Then $0 = (e_i n e_j)^* = e_j n e_i$, hence $\{e_i n e_j\} = \frac{1}{2} n = 0$.

So, we have proved (2) and (3). Let us prove (1). Consider $A = \mathrm{M}_{n_i} + \tilde{N}_{ii} \subset \mathrm{S}(\mathcal{J})$. Evidently, $A^* = A$, $\tilde{N}_{ii} = \mathrm{Rad}\,A$ and $\tilde{N}^*_{ii} = \tilde{N}_{ii}$.

By associative coordinalization theorem $A = M_{n_i}(\tilde{D}_i)$, where $(\tilde{D}_i, *)$ is an algebra with involution, which has an ideal $(R_i, *)$, such that $\tilde{D}_i/R_i = D_i$, $\tilde{N}_{ii} = \mathrm{M}_{n_i}(R_i, *)$. We have

$$\mathrm{H}(A, *) = \mathrm{H}_{n_i}(\tilde{D}_i) = \mathrm{H}_{n_i}(D_i) + N_{ii}.$$

Since for the matrix units e_{ij} in $\mathrm{M}_{n_i}(D_i)$ we have $e_{ii}^* = e_{ii}, e_{ij}^* = e_{ji}$, it follows easily, that for any $r_{ij} \in R_i$ we have

$$(\sum_{i,j} r_{ij}e_{ij})^* = \sum_{i,j} r_{ij}^* e_{ji},$$

hence $*$ from A in restriction to $M_{n_i}(D_i)$ coincides with the canonical involution. In particular, we have the known formulas

$$(re_{ij} + r^*e_{ji}) \circ (se_{jk} + s^*e_{kj}) = (rs)e_{ik} + (rs)^*e_{ki}, \; r, s \in D_i. \tag{28}$$

This implies

$$\mathrm{H}(\tilde{N}_{ii}^2) = N_{ii} \circ N_{ii}.$$

We start already to prove (4). Let us prove, that the ideal $(N \circ N)$, generated in $\mathrm{S}(\mathfrak{J})$ by $N \circ N$ coincides with $\tilde{N}^2$. This is equivalent to (4). Clearly, we need to prove only, that $\tilde{N}^2 \subset (N \circ N)$. Without loss of generality we may assume, that $N \circ N = 0$. First of all for any $r, s \in R_i$, we have for $j \neq k$ in $\mathrm{M}_{n_i}(R_i)$

$$(rs)e_{jk} = ((rs)e_{jk} + (rs)^*e_{kj})e_{kk} \in (N \circ N)e_{kk} = 0,$$

and $(rs)e_{jj} = (rs)e_{jk} \circ e_{kj} = 0$, which proves, that $\tilde{N_{ii}}^2 = 0$.

Consider $\tilde{N}_{ij}\tilde{N}_{jk}$ for $i \neq j$, $i \neq k$. It suffices to prove that $N_{ij}N_{jk} = 0$. We have

$$2n_{ij}n_{jk} = n_{ij} \circ n_{jk} + [n_{ij}, n_{jk}] = [n_{ij}, n_{jk}] =$$
$$[e_i \circ n_{ij}, n_{jk}] = [e_i, n_{ij} \circ n_{jk}] + [n_{ij}, e_i \circ n_{jk}] = 0.$$

It remains to prove that $N_{ij}N_{ij} = 0$. Let

$$e = e_i, e = \sum_t^r f_t, \; r \geq 3, f_t f_s = \delta_{ts} f_t,$$

then $N_{ij} = \sum_t \{f_t N_{ij} e_j\}$ and it is sufficient to prove that for any $n, m \in N_{ij}$

$$\{f_r n e_j\}\{f_{r'} m e_j\} = 0.$$

If $r \neq r'$, we may argue as before. Assume now that $r = r' = 1$ and denote $n_1 = \{f_1 n e_j\}$, $m_1 = \{f_1 m e_j\}$. It follows from the structure of N_{ij} as $\mathfrak{J}_{ss}$-module, (it is a special module over $\{e_i \mathfrak{J} e_i\} = \mathrm{H}_{n_i}(D_i)$), that there exists $f_{12} \in \mathrm{H}_{n_i}(D_i)$ such that $m_1 = m_2 \cdot f_{12}$, for some $m_2 \in \{f_2 N_{ij} e_j\}$. Now

$$[n_1, m_1] = [n_1, m_2 \cdot f_{12}] =$$
$$[n_1 \cdot m_2, f_{12}] + [n_1 \cdot f_{12}, m_2] = [n_1 \cdot f_{12}, m_2] \in [\{f_2 N_{ij} e_i\}, \{f_2 N_{ij} e_i\}].$$

In particular, we have

$$f_1[n_1, m_1] = [n_1, m_1]f_1 = 0,$$

which proves, that $[n_1, m_1] \in \tilde{N}_{jj}$. Making the similar decomposition for e_j, we prove that $[n_1, m_1] \in \tilde{N}_{ii}$ and finally $[n_1, m_1] = 0$. □

Corollary 6.1. *Let $\mathfrak{L}$ be a semisimple matrix Jordan algebra, V an finite dimensional unital $\mathfrak{L}$-module. Then the tensor algebra $\mathfrak{L}[V]$ is reflexive.*

Proof. Let cI_n be the ideal of non-associative words in $\mathfrak{L}[V]$ containing at least n letters in V, $x \in \mathrm{S}(\mathfrak{L}[V])$ is a symmetrical element. Since $\mathfrak{L}[V]$ is graded and its homogeneous components are $*$-invariant we can assume x homogeneous of degree n. Consider a finite dimensional Jordan algebra $\mathfrak{L}[V]/\mathfrak{I}^{n+1}$.

Consider the commutative diagram

$$\begin{tikzcd} \mathfrak{L}[V] \arrow[r, "\pi"] \arrow[d, "\imath"'] & \mathfrak{L}[V]/\mathfrak{I}^{n+1} \arrow[d, "\imath"] \\ \mathrm{S}(\mathfrak{L}[V]) \arrow[r, "\mathrm{S}(\pi)"] & \mathrm{S}(\mathfrak{L}[V]/\mathfrak{I}^{n+1}) \end{tikzcd},$$

where horizontal arrows are canonical projection and vertical are maps in its associative envelope. Note that both π and $\mathrm{S}(\pi)$ are mono in restriction on n-th graded component of $\mathfrak{L}[V]$ and $\mathrm{S}(\mathfrak{L}[V])$ correspondingly. Hence x is symmetrical, since $\mathrm{S}(\pi)(x)$ is symmetric and by Theorem 6.1 $x \in \imath(\mathfrak{L}[V]/\mathfrak{I}^{n+1})$. □

7. Building blocks for quiver special universal envelope

7.1. **Considered class of Jordan algebras.** In this section we define the mapping $\mathrm{Qui} : \mathrm{JorD} \longrightarrow \mathrm{QAs}$, which sends the diagram of a Jordan algebra $\mathfrak{J}$ of matrix type to the quiver of associative algebra $\mathrm{S}(\mathfrak{J})$. The quiver from $\mathrm{Q} \in \mathrm{Im\,Qui}$ is endowed with some extra structures, which reflect some features of $\mathrm{S}(\mathfrak{J})$. The functor Qui and the extra structures are defined "locally", i.e. on the subdiagrams of Γ containing one or two vertices and one edge.

We emphasize, that we consider both finite dimensional and infinite dimensional Jordan algebras. We say a Jordan algebra $\mathfrak{J}$ over a semisimple finite dimensional Jordan algebra $\mathfrak{J}$ has a diagram Γ, provided that there exists $\mathfrak{L}$-epimorphism $\pi : \mathfrak{L}[V] \longrightarrow \mathfrak{J}$ for a finite dimensional $\mathfrak{L}$-bimodule V, $\mathrm{Ker}\,\pi \subset V \circ V + \mathfrak{L} \circ (V \circ V)$. Without loss of generality we will assume, that the module V is unital, possibly passing from the diagram Γ to $\tilde{\Gamma}$. Recall also, that a bimodule V is uniquely (up to isomorphism) defined by $\mathfrak{J}$ (Corollary 5.1).

Definition 7.1. *We call a Jordan algebra $\mathfrak{J}$ almost matrix Jordan algebra if in its diagram $\tilde{\Gamma}$ the set Γ_0 consists of matrix simple Jordan algebras and fields $\mathbf{k}$ and there are no edge $a \in \tilde{\Gamma}_1$, such that both ends of a are fields.*

Let Q be a quiver. The opposite quiver Q^o is defined by $\mathrm{Q}_0^o = \mathrm{Q}_0$, $\mathrm{Q}_1^o = \{x^o \mid x \in \mathrm{Q}_1\}$, $s(x^o) = e(x), e(x^o) = s(x)$, $x \in \mathrm{Q}_1$. Obviously $(\mathrm{Q}^o)^o \simeq \mathrm{Q}$. An *involution on the quiver* Q consists of two involutive bijections $* : \mathrm{Q}_0 \longrightarrow \mathrm{Q}_0$ and $* : \mathrm{Q}_1 \longrightarrow \mathrm{Q}_1$, such that $s(x^*) = (e(x))^*$, $e(x^*) = (s(x))^*$.

Let Q be a quiver of $A = \mathrm{S}(\mathfrak{J})$, $\mathfrak{J} = \mathfrak{L} \oplus \mathrm{Rad}\,\mathfrak{J}$, $\mathfrak{L} = \mathfrak{L}_1 \times \cdots \times \mathfrak{L}_n$, where all $\mathfrak{L}_i$ are simples, so for $\Gamma = \Gamma(\mathfrak{J})$, $\Gamma_0 = \{1, \ldots, n\}$, where i corresponds to $\mathfrak{L}_i$. Analogously, if $\mathrm{Rad}\,\mathfrak{J}/(\mathrm{Rad}^2\,\mathfrak{J} + \mathfrak{J} \cdot \mathrm{Rad}^2\,\mathfrak{J}) = V_1 \oplus \cdots \oplus V_N \subset \mathfrak{J}$ is a sum of simple modules, so we will identify Γ_1 with a family $\{V_1, \ldots, V_N\}$.

Then Q will be endowed with the following structures.

(1) There exists an involution $* : \mathrm{Q} \longrightarrow \mathrm{Q}$, induced by the involution on $\mathrm{S}(\mathfrak{J})$.

(2) There are given maps

$$alg : \mathrm{Q}_0 \longrightarrow \left\{ \boxed{1},\ n_1,\ \boxed{n_2},\ n_3 \right\},\ n_1, n_2 \geq 3, n_3 \geq 4 \tag{29}$$

$$mod : \mathrm{Q}_1 \longrightarrow \{\mathbf{k}, M, M', N, N', R\}. \tag{30}$$

If $x \in \mathrm{Q}_0$ and $e_x \in A$ is the idempotent corresponding to x, then there exits a unique $\mathcal{L}_i$, such that $e_x \in \mathrm{S}(\mathcal{L}_i) \subset A$ and we set $alg(x)$ to be the graphical presentation of $\mathcal{L}_i$. On other hand, every edge a of Γ by construction (see) can be identified with an element of some V_i, $i = 1, \dots, N$ and $mod(a)$ equals graphical presentation of V_i. Usually we will skip n_1, n_2, n_3 (see 8.4)

7.2. **Vertices.** For any matrix Jordan algebra $\mathcal{J}$ we have $\mathcal{L}(\mathrm{S}^1)(\mathcal{J}) = \mathrm{S}^1(\mathcal{L}(\mathcal{J}))$. Further $\mathrm{S}^1(\mathcal{J}_1 \oplus \mathcal{J}_2) = \mathrm{S}^1(\mathcal{J}_1) \oplus \mathrm{S}^1(\mathcal{J}_2)$, therefore it is enough to construct $Q_0(\mathcal{J})$ for $\mathcal{J} = \mathbf{k}$, M_n^+, HM_n, $\mathrm{HM}_{2n}(J_s)$.

(31) $\boxed{n}$ $\qquad \mathcal{J}_n$, for $\mathcal{J} \in \{\mathbf{k}, \mathrm{H}, \mathrm{H}(J_s)\}$,

$e_{11}\bullet$ $\qquad \mathrm{S}^1(\mathcal{J}) = \mathrm{M}_n$.

(32) n $\qquad \mathcal{J} = \mathrm{M}_n^+$,

$E_1\bullet$ $\qquad \mathrm{S}^1(\mathcal{J}) = \mathrm{M}_n \oplus \mathrm{M}_n$.

$E_2\bullet$

We write the explicit basis for $\mathrm{S}^1(\mathcal{J})$, for $\mathcal{J}$ from (1)-(3). Here by E_{ij} we denote the elementary matrix with a 1 in the (i,j) entry and 0's elsewhere.
1. $\mathcal{J} = \mathrm{M}_n^+ = \langle e_{ij} = E_{ij},\ i,j = 1, \cdots, n \rangle$ then

$$\mathrm{S}^1(\mathrm{M}_n^+) = \begin{pmatrix} e_{11}e_{12}e_{21} & e_{11}e_{12} & \dots & e_{11}e_{1n} \\ e_{22}e_{21} & e_{22}e_{23}e_{32} & \dots & e_{22}e_{2n} \\ \dots & \dots & \dots & \dots \\ e_{nn}e_{n1} & e_{nn}e_{n2} & \dots & e_{nn}e_{n1}e_{1n} \end{pmatrix} \oplus$$
$$\begin{pmatrix} e_{11}e_{21}e_{12} & e_{11}e_{21} & \dots & e_{11}e_{n1} \\ e_{22}e_{12} & e_{22}e_{32}e_{23} & \dots & e_{22}e_{n2} \\ \dots & \dots & \dots & \dots \\ e_{nn}e_{1n} & e_{nn}e_{n2} & \dots & e_{nn}e_{1n}e_{n1} \end{pmatrix}. \tag{33}$$

2. $\mathcal{J} = \mathrm{HM}_n(\mathbf{k}, \tau) = \langle e_{ii} = E_{ii},\ \tilde{e}_{ij} = E_{ij} + E_{ji},\ 1 \leq i,j \leq n,\ i < j \rangle$,

$$\mathrm{S}^1(\mathrm{HM}_n(\mathbf{k}, \tau)) = \begin{pmatrix} e_{11} & e_{11}\tilde{e}_{12} & \dots & e_{11}\tilde{e}_{1n} \\ e_{22}\tilde{e}_{12} & e_{22} & \dots & e_{22}\tilde{e}_{2n} \\ \dots & \dots & \dots & \dots \\ e_{nn}\tilde{e}_{1n} & e_{nn}\tilde{e}_{2n} & \dots & e_{nn} \end{pmatrix}. \tag{34}$$

3. $\mathcal{J} = \mathrm{HM}_{2n}(\mathbf{k}, J_s)$. Then $A \in \mathrm{HM}_{2n}(\mathbf{k}, J_s)$ if $A \in \mathrm{M}_{2n}$ and $A = S^{-1}A^{\tau}S$. We obtain that A has the following form:

$$\begin{pmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ \bar{A}_{12} & A_{22} & \dots & A_{2n} \\ \dots & \dots & \dots & \dots \\ \bar{A}_{1n} & \bar{A}_{2n} & \dots & A_{nn} \end{pmatrix},$$

where A_{ii}, $i = 1, \dots, n$ is a diagonal matrix 2×2, $A_{ij} = \left(\begin{smallmatrix} a_{ij} & b_{ij} \\ c_{ij} & d_{ij} \end{smallmatrix}\right)$ is 2×2 matrix with $\bar{A} = \left(\begin{smallmatrix} d_{ij} & -b_{ij} \\ -c_{ij} & a_{ij} \end{smallmatrix}\right)$. Therefore $\mathrm{HM}_{2n}(\mathbf{k}, J_s) = \langle \varepsilon_i = E_{2i-1,2i-1} + E_{2i,2i},\ a_{ij} = E_{2i-1,2j-1} + E_{2j,2i},\ b_{ij} = E_{2i-1,2j} - E_{2j-1,2i},\ c_{ij} = E_{2i,2j-1} - E_{2j,2i-1},\ d_{ij} = E_{2i,2j} + E_{2j-1,2i-1},\ 1 \le i, j \le n,\ i < j\rangle$ and

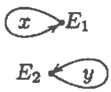
(35)

$$\mathrm{S}^1(\mathrm{HM}_{2n}(\mathbf{k}, J_s)) = \begin{pmatrix} \varepsilon_1 a_{12} d_{12} & b_{12} a_{12} & \varepsilon_1 a_{12} & \varepsilon_1 b_{12} & \dots & \varepsilon_1 a_{1n} & \varepsilon_1 b_{1n} \\ c_{12} d_{12} & \varepsilon_1 d_{12} a_{12} & \varepsilon_1 c_{12} & \varepsilon_1 d_{12} & \dots & \varepsilon_1 c_{1n} & \varepsilon_1 d_{1n} \\ \varepsilon_2 d_{12} & -\varepsilon_2 b_{12} & \varepsilon_2 a_{23} d_{23} & b_{23} a_{23} & \dots & \varepsilon_2 a_{2n} & \varepsilon_2 b_{2n} \\ -\varepsilon_2 c_{12} & \varepsilon_2 a_{12} & c_{23} d_{23} & \varepsilon_2 d_{12} a_{12} & \dots & \varepsilon_2 c_{2n} & \varepsilon_2 d_{2n} \\ \dots & \dots & \dots & \dots & & & \\ \varepsilon_n d_{1n} & -\varepsilon_n b_{1n} & \varepsilon_n d_{2n} & -\varepsilon_n b_{2n} & \dots & \varepsilon_n d_{1n} a_{1n} & b_{n-1n} d_{n-1n} \\ -\varepsilon_n c_{1n} & \varepsilon_n a_{1n} & -\varepsilon_n c_{2n} & \varepsilon_n a_{2n} & \dots & c_{n-1n} d_{n-1n} & \varepsilon_n a_{1n} d_{1n} \end{pmatrix}.$$

7.3. **One-point building blocks.** Let us calculate $\mathrm{Q}(\mathrm{S}(\mathcal{L} + R))$, where $\mathcal{L} = \mathrm{H}(D_n, *)$ is a Jordan matrix algebra and R is a simple unital $\mathcal{L}$-bimodule and $\mathcal{L} + R$ is the trivial split extension. By Subsection 4.2, we have $R = \mathrm{H}(W)$, where W is one of simple unital D-bimodules with involution, and the algebra $\mathcal{L} + R$ is isomorphic to the Jordan matrix algebra $\mathrm{H}_n(D + W)$. Let us first exclude the case $D = \mathrm{M}_2$, $n = 3$, $W = \mathrm{Cay}$. Then by the Martindale theorem [7], we have the isomorphism $\mathrm{S}(\mathrm{H}_n(D + W)) \cong \mathrm{M}_n(D + W)$. Finally, the algebra $M_n(D + W)$ is Morita equivalent to $D + W$. Thus we have $\mathrm{Q}(\mathrm{S}(\mathcal{L} + R)) = \mathrm{Q}(D + W)$. In the last case, in order to determine the representation type of the algebra $\mathcal{L} + R$, by the isomorphism of the categories $\mathcal{L} - \mathrm{mod}$ and $(D, *)_{Alt} - \mathrm{bimod}$ we have to study the structure of unital indecomposable alternative bimodules with nuclear involution over the alternative algebra with involution $\mathrm{M}_2 + \mathrm{Cay}$.

(36) 

M_n^2 for $n \ge 3$,



$+$ $\Phi(r_{kl}) = E_{kl}x + E_{l+n,k+n}y$,

$x = r_{11}e_{12}e_{21},\ y = r_{11}e_{21}e_{12},\ x^* = y.$

$+$

(37) 

M_n^2 for $n \ge 3$,

$$\begin{tikzcd} E_1 \bullet \\ E_2 \bullet \arrow[u, "x"] \end{tikzcd}$$

$\Phi(m_{kl}) = (E_{k,l+n} + E_{l,k+n})x$, for $k \ne l$,

$\Phi(m_{kk}) = E_{k,n+k}x,\ x = m_{11},\ x^* = x.$

(38)
$$\begin{tikzcd} \textcircled{n} \arrow[loop above, "M'"] \end{tikzcd} \qquad \mathrm{M}_n^2 \quad \text{for} \quad n \geq 3,$$

$$\begin{tikzcd} E_1 \bullet \arrow[d, "x"] \\ E_2 \bullet \end{tikzcd} \qquad \begin{array}{l} \Phi(m'_{kl}) = (E_{k+n,l} + E_{l+n,k})x, \quad \text{for} \quad k \neq l, \\ \Phi(m'_{kk}) = E_{k+n,k}x, \ x = m'_{11}, \ x^* = x. \end{array}$$

(39)
$$\begin{tikzcd} \textcircled{n} \arrow[loop above, "N"] \end{tikzcd} \qquad \mathrm{M}_n^2 \quad \text{for} \quad n \geq 3,$$

$$\begin{tikzcd} E_1 \bullet \\ E_2 \bullet \arrow[u, "x"] \end{tikzcd} \qquad \begin{array}{l} \Phi(n_{kl}) = (E_{k,l+n} - E_{l,k+n})x, \\ x = n_{12}e_{12}, \ x^* = -x. \end{array}$$

(40)
$$\begin{tikzcd} \textcircled{n} \arrow[loop above, "N'"] \end{tikzcd} \qquad \mathrm{M}_n^2 \quad \text{for} \quad n \geq 3,$$

$$\begin{tikzcd} E_1 \bullet \arrow[d, "x"] \\ E_2 \bullet \end{tikzcd} \qquad \begin{array}{l} \Phi(n'_{kl}) = (E_{k+n,l} - E_{l+n,k})x, \quad \text{for} \quad k \neq l, \\ x = n_{12}e_{21}, \ x^* = -x. \end{array}$$

(41)
$$\begin{tikzcd} \boxed{n} \arrow[loop above, "N"] \end{tikzcd} \qquad \mathrm{M}_n, \quad \Phi(\tilde{n}_{kl}) = (E_{kl} - E_{lk})x,$$

$$\begin{tikzcd} e_{11} \bullet \arrow[loop above, "x"] \end{tikzcd} \qquad x = \tilde{n}_{12}\tilde{e}_{12}, \quad x^* = -x.$$

(42)
$$\begin{tikzcd} \boxed{n} \arrow[loop above, "R"] \end{tikzcd} \qquad \mathrm{M}_n, \quad \Phi(\tilde{r}_{kl}) = (E_{kl} + E_{lk})x, \quad l \neq k,$$

$$\begin{tikzcd} e_{11} \bullet \arrow[loop above, "x"] \end{tikzcd} \qquad \Phi(\tilde{r}_{kk}) = E_{kk}x, \quad x = \tilde{r}_{12}\tilde{e}_{12}, \quad x^* = x.$$

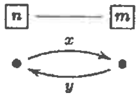
(43)
$$\begin{tikzcd} \Diamond n \arrow[loop above, "N"] \\ e_{11} \bullet \arrow[loop above, "x"] \end{tikzcd} \qquad \begin{array}{c} \Phi_A(n\varepsilon_k) = (E_{2k-1,2k-1} - E_{2k,2k})x, \\ \Phi_A(na_{kl}) = (E_{2k-1,2l-1} - E_{2l,2k})x, \\ \Phi_A(nb_{kl}) = (E_{2k-1,2l} + E_{2l-1,2k})x, \\ \Phi_A(nc_{kl}) = (E_{2k,2l-1} + E_{2l,2k-1})x, \\ \Phi_A(nd_{kl}) = (E_{2k,2l} - E_{2l-1,2k-1})x, \\ \Phi_A(g_{kk}) = E_{2k-1,2k}x, \quad \Phi_A(g'_{kk}) = E_{2k,2k-1}x, \\ M_n, \quad x = x^*, \quad x = n\varepsilon_1 a_{12} d_{12}. \end{array}$$

(44)

$$\begin{gathered}\Phi_A(r\varepsilon_k) = (E_{2k-1,2k-1} + E_{2k,2k})x,\\ \Phi_A(ra_{kl}) = (E_{2k-1,2l-1} + E_{2l,2k})x,\\ \Phi_A(rb_{kl}) = (E_{2k-1,2l} - E_{2l-1,2k})x,\\ \Phi_A(rc_{kl}) = (E_{2k,2l-1} - E_{2l,2k-1})x,\\ \Phi_A(rd_{kl}) = (E_{2k,2l} + E_{2l-1,2k-1})x,\\ M_n, \quad x = x^*, \quad x = n\varepsilon_1 a_{12} d_{12}.\end{gathered}$$

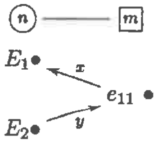
7.4. **Two-point building blocks.** All the bimodules here are tensor products of two non-unital modules over the left and the right Jordan algebra correspondingly (4.5) The basic vectors for the module over the left (the right) algebra are denoted by p_i (by q_j).

(45)



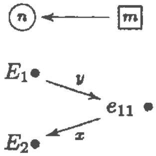
$$\begin{gathered}\mathrm{S}_n \times \mathrm{S}_m, \quad \text{with } \mathrm{S} \in \{\mathbf{k}, \mathrm{H}, \mathrm{H}(J_s)\},\\ \Phi(p_k \otimes q_l) = E_{k,l+n}x + E_{l,k+n}y,\\ x = e^1_{11}p_1 \otimes q_1, \ y = p_1 \otimes q_1 e^1_{11}, \ * = \mathrm{id}.\end{gathered}$$

(46)



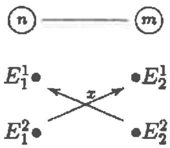
$$\begin{gathered}\mathrm{M}^2_n \times \mathrm{S}_m \quad \text{for } n \geq 3, \ \mathrm{S} \in \{\mathbf{k}, \mathrm{H}, \mathrm{H}(J_s)\},\\ \Phi(p_k \otimes q_l) = E_{k,2n+l}y + E_{2n+l,k+n}x,\\ x = e_{11}p_1 \otimes q_1, \ y = p_1 \otimes q_1 e_{11}, \ x^* = y,\\ \mathrm{e} = E_{11} + e_{11}, \ \text{for } m = 1 \quad e_{11} = \mathrm{e}.\end{gathered}$$

(47)



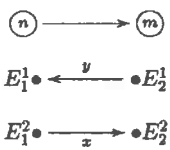
$$\begin{gathered}\mathrm{M}^2_n \times \mathrm{S}_m \quad \text{for } n \geq 3, \ \mathrm{S} \in \{\mathbf{k}, \mathrm{H}, \mathrm{H}(J_s)\},\\ \Phi(p_k \otimes q_l) = E_{k+n,2n+l}y + E_{2n+l,k}x,\\ x = e_{11}p_1 \otimes q_1, \ y = p_1 \otimes q_1 e_{11}, \ x^* = y,\\ \mathrm{e} = E_{11} + e_{11}, \ \text{for } m = 1 \quad e_{11} = \mathrm{e}.\end{gathered}$$

(48)



$$\begin{gathered}\mathrm{M}^2_n \times \mathrm{M}^2_m, \ \text{for } m, n \geq 3,\\ \Phi(p_k \otimes q_l) = E_{n+k,2n+l}x + E_{2n+m+l,k}y,\\ x = E^2_2 p_1 \otimes q_1, \ y = E^2_1 p_1 \otimes q_1, \ x^* = y.\end{gathered}$$

(49)



$$\begin{gathered}\mathrm{M}^2_n \times \mathrm{M}^2_m, \ \text{for } m, n \geq 3,\\ \Phi(p_k \otimes q_l) = E_{n+k,2n+m+l}x + E_{2n+l,k}y,\\ x = E^2_1 p_1 \otimes q_1, \ y = E^1_2 p_1 \otimes q_1, \ x^* = y.\end{gathered}$$

$$
(50) \qquad \begin{array}{ll}
\begin{tikzcd} (n) \arrow[r, leftrightarrow] & (m) \end{tikzcd} & \mathrm{M}_n^2 \times \mathrm{M}_m^2, \text{ for } m, n \geq 3, \\
\begin{tikzcd} E_1^1 \bullet \arrow[dr, "y"] & \bullet E_2^1 \arrow[dl] \\ E_1^2 \bullet & \bullet E_2^2 \end{tikzcd} & \begin{array}{l} \Phi(p_k \otimes q_l) = E_{k,2n+m+l}x + E_{2n+l,k+n}y, \\ \\ x = E_2^1 p_1 \otimes q_1, \ y = E_1^1 p_1 \otimes q_1, \ x^* = y. \end{array}
\end{array}
$$

7.5. Associative quiver of associative envelope of matrix Jordan algebra.

Theorem 7.1. *Let $\mathfrak{J}$ be a Jordan algebra of matrix type over $\mathfrak{L}$. Then*

$$\beta : \mathrm{Q}(\mathrm{S}(\mathfrak{J})) \simeq \mathrm{Qui}(\Gamma(\mathfrak{J})).$$

Proof. This statement is enough to prove for the tensor Jordan algebra $\mathfrak{L}[V]$. Note, that on the vertices β is checked immediately. The proof of coincidence on arrows gives the following. Let $\Gamma = \Gamma(V)$.

$$
\begin{aligned}
&\mathrm{Qui}(\Gamma)_1 \simeq \ \text{(by Remark 4.1)} \\
&\bigsqcup_{x \in \Gamma_1} \mathrm{Qui}(\mathrm{c}_m(x))_1 = \ \text{(by construction above)} \\
&\bigsqcup_{x \in \Gamma_1} D(\widetilde{\mathrm{c}_m(x)})_1 = \ \text{by remark 3.1} \\
&D(\widetilde{\bigoplus_{x \in \Gamma_1} \mathrm{c}_m(x)})_1 = \ \text{by Remark 5.2} \\
&D(\widetilde{\bigoplus_{x \in \Gamma_1} \mathrm{c}_m(x)})_1 = \ \text{by Theorem 5.1} \\
&\mathrm{Q}(\mathrm{S}(\mathfrak{L}[V]))_1.
\end{aligned}
$$

□

Corollary 7.1. *Let in assumption above $A = \mathrm{S}(\mathfrak{L}[V])$, $\mathrm{e} \in A$ be an idempotent, such $\mathrm{e}^* = \mathrm{e}$ and $\bar{A} = \mathrm{e}A\mathrm{e}$ is basic Morita equivalent to A. The mapping* Qui *on Γ_1 coincides with the restriction of the homomorphism $\Phi_A : A \longrightarrow \bar{A}_{\vec{k}}$ for convenient $\vec{k}$.*

Proof. See Corollary 3.1. □

Corollary 7.2. *Let* Q *be a quiver with involution $*$. Then there exists a Jordan diagram Γ, such that $\mathrm{Qui}(\Gamma) = \mathrm{Q}$ if and only if for any edge a, connecting X and X^* holds $a = a^*$.*

8. Special representations of matrix algebras

8.1. Criterion of finiteness (tameness) for special representation. The following theorem reduces the problem of classification of indecomposable quiver with relations to the classification of representation of a quiver with relations.

Theorem 8.1. *Let $\mathfrak{J}$ be an almost matrix Jordan algebra, given in the form $\mathfrak{L}[V]/\mathfrak{I}$, where $\mathfrak{L}$ is a semisimple Jordan algebra of matrix type, $\mathfrak{I}$ is an ideal, generated by finitely many non-associative words $w_1, \ldots, w_r$ in alphabet $\mathfrak{L} \sqcup V$, every of which* *contains at least 2 letters from V, $\mathrm{Q} = \mathrm{Qui}(\Gamma(\mathfrak{J}))$ and $I \subset \mathrm{S}(\mathfrak{L}[V])$, generated by $\Phi_{\mathrm{S}(\mathfrak{L})\langle\hat{V}\rangle}\Psi(w_i)$, $i = 1, \dots, r$. Then*

$$\mathrm{S}(\mathfrak{J}) \simeq \mathbf{k}[\mathrm{Q}]/I.$$

Proof. It follows immediately from Lemma 3.3, (2) and Corollary 5.2. □

8.2. Reconstruction of Jordan algebras. The statements, proved in this section are variations of Theorem 6.1. The main idea of this subsection is to work with the algebra with involution $\Phi_{\mathrm{S}(\mathfrak{L}[V])}$ instead of $\mathrm{S}(\mathfrak{L}[V]$.

Proposition 8.1. *Let $\mathfrak{L}$ be a semisimple Jordan algebra, V be finite dimensional $\mathfrak{L}$-module, such that $\mathfrak{L}[V]$ is almost matrix, $\Gamma = \Gamma(\mathfrak{L}[V])$ and $\mathrm{Q} = \mathrm{Qui}(\Gamma)$, $A = \mathbf{k}[\mathrm{Q}]$ be the path algebra, $w = a_1a_2\dots a_n$, $n \geq 1$, $a_i \in \mathrm{Q}_1$, $i = 1, \dots, n$ an oriented path, leading from X to Y, $X, Y \in \mathrm{Q}_0$. Then for every admissible pair of indices i, j, $i \neq j$ in $\mathfrak{L}[V]$ exists a non-associative word $x = x_1 \circ \dots \circ x_n$, where all x_i-th are either matrix units from $\mathfrak{L}$ or the standard basic vectors of V, $\Phi_{\mathrm{S}(\mathfrak{L}[V])}(x) = E_{ij}w + w^*E^*_{ij}$.[4] The notion of admissible pair depends on the type of Jordan algebra.*

Proof. We use induction in n. If $n = 1$, the diagrams from 7 proves the lemma. Assume for $n-1$ the lemma holds. In particular,. there exists $y = x_1 \dots x_{k-1}$ such that for $v = a_1 \dots a_{n-1}$ holds $\Phi_{\mathrm{S}(\mathfrak{L}[V])}(y) = E_{ik}v + v^*E^*_{ik}$.

Assume first, that the path w contains an inner vertex $Z \neq \mathbf{0}$. Then $w = w_1w_2$ for some pathes $w_1 : Z \longrightarrow Y$, $w_2 : X \longrightarrow Z$. If both X, Y do not coincide with $\mathbf{0}$, then by induction assumption we can assume for some Jordan words

$$\Phi_{\mathrm{S}(\mathfrak{L}[V])}(y_1) = E_{ik}w_1 + w_2^*E^*_{ik}, \Phi_{\mathrm{S}(\mathfrak{L}[V])}(w_2) = E_{kj}y_2 + y_2^*E^*_{kj} \tag{51}$$

and $i \neq j$, $k \neq i$, $k \neq j$, since all matrix algebras there are of dimension ≥ 3.

Then $E_{kj}E_{ik} = 0$, since $i \neq j$, $0 = (E_{kj}E_{ik})^* = E^*_{ik}E^*_{kj}$. Note also, that for any considered involution holds the following: if $p \neq q$ and $E^*_{pq} = E_{p'q'}$, then $p \neq p'$. It gives us $E_{ik}E^*_{kj} = 0$, $E^*_{ik}E_{kj} = 0$ and, applying involution, $E_{kj}E^*_{ik} = 0$, $E^*_{kj}E_{ik} = 0$.

Due the calculation above

$$\begin{aligned} &\Phi_{\mathrm{S}(\mathfrak{L}[V])}(y_1 \circ y_2) = \Phi_{\mathrm{S}(\mathfrak{L}[V])}(y_1) \circ \Phi_{\mathrm{S}(\mathfrak{L}[V])}(y_2) = \\ &(E_{ik}w_1 + w_1^*E^*_{ik}) \circ (E_{kj}w_2 + w_2^*E^*_{kj}) = \\ &E_{kj}E_{jk} = E^*_{jk}E^*_{kj} = E_{ik}E^*_{kj} = E^*_{ik}E_{kj} = E_{kj}E^*_{ik} = E^*_{kj}E_{ik} = 0) \\ &E_{ik}w_1E_{kj}w_2 + w_2^*E^*_{kj}w_1^*E^*_{ik} = E_{ij}w + w^*E^*_{ij}. \end{aligned} \tag{52}$$

Assume then, that $X = Y = \mathbf{0}$, that means in (51) holds $i = j = 0$ and, since $Z \neq \mathbf{0}$, $k \neq 0$. Then as in (52) we obtain

$$\Phi_{\mathrm{S}(\mathfrak{L}[V])}(y_1 \circ y_2) = E_{00}w + w^*E^*_{00} + E_{kk}w_2w_1 + (w_2w_1)^*E^*_{kk}. \tag{53}$$

Then if $e_0 \in \mathfrak{J}$, corresponding $\boxed{1}$, then $x = \Phi_{\mathrm{S}(\mathfrak{L}[V])}(e_0 \circ (y_1 \circ y_2))$.

So it remains consider the case, when all inner vertices of w coincide with $\mathbf{0}$. Since there are no loops in $\mathbf{0}$ holds

$$w = x_1x_2,\ x_1 : \mathbf{0} \longrightarrow Y,\ x_2 : X \longrightarrow \mathbf{0},\ X \neq \mathbf{0},\ Y \neq \mathbf{0}.$$

[4]The notion of admissible pair depends on the type of Jordan algebra - especially in the case of full matrix algebra

Since $X, Y \neq \mathbf{0}$, in the case $X \neq Y$ we can conclude as in (51) and in the case $X = Y$ as in the case (53). □

Corollary 8.1. *Let in assumption above $L \subset \mathbf{k}[\mathrm{Q}]$ be a semisimple subalgebra, generates by Q_0, $\mathbf{k}[\mathrm{Q}]^+ \subset \mathbf{k}[\mathrm{Q}]$ the ideal of pathes of length ≥ 1, $J(\mathrm{Q}) \subset \mathbf{k}[\mathrm{Q}]^+$ the space, generated either by pathes w in Q, such that $alg(s(w)) \neq \boxed{1}$ or $alg(e(w)) \neq \boxed{1}$ or $s(w) \neq e(w)$ and by the sums $w + w*$ provided that $s(w) = e(w)$, $alg(s(w)) = alg(e(w)) = \boxed{1}$. Then the space $F \subset \mathbf{k}[Q]^+$, generated by entries of matrices $\Phi_{\mathrm{S}(\mathcal{L}[V])}(x)$, where x runs $\mathcal{L}[V]$, coincide with $L + J(\mathrm{Q})$.*

Proof. Evidently, entries of $\Phi_{\mathrm{S}(\mathcal{L}[V])}(\mathcal{L})$ generate L. Let $a = \sum_{k=1}^{n} \lambda_k w_k$, where $\lambda_k \in \mathbf{k}$ and w_k are different pathes and $X = s(w_1) = \dots = s(w_n)$ and $Y = e(w_1) = \dots = e(w_n)$. Assume, at least one from X, Y is not $\boxed{1}$. Then by the Proposition 8.1 there exists $i \neq j$ and $x_k \in \mathcal{L}[V]$, such that $\Phi_{\mathrm{S}(\mathcal{L}[V])}(x_k) = E_{ij} w_k + w_k^* E_{ij}^*$. Let $x = \sum_{k=1}^{n} \lambda_k x_k \in \mathcal{L}[V]$. Then $\Phi_{\mathrm{S}(\mathcal{L}[V])}(x) = E_{ij} a + a^* E_{ij}^*$, therefore $a \in F$.

In the case, when X and Y are $\boxed{1}$, then Proposition 8.1 gives us $x_k \in \mathcal{L}[V]$, such that $\Phi_{\mathrm{S}(\mathcal{L}[V])}(x_k) = E_{0_X 0_Y} w_k + w_k^* E_{0_Y 0_X}$, since $E_{0_X 0_Y}^* = E_{0_Y 0_X}$. Putting x as above, we obtain, that if $X \neq Y$, then $w \in F$ and for $X = Y$ holds $w + w^* \in F$. To finish the proof note, that if $x \in \mathcal{L}[V]_{00}$, then $\Phi_{\mathrm{S}(\mathcal{L}[V])}(x_k)$ is just 1×1 matrix over $\mathbf{k}[\mathrm{Q}]$. The involution on its entry is induced by the involution from $\mathbf{k}[\mathrm{Q}]$, hence it is symmetrical in $\mathbf{k}[\mathrm{Q}]$. □

In such way we obtain another proof of Corollary 6.1 including the case of non-unital bimodules.

Corollary 8.2. *Let $\mathcal{L}[V]$ be almost matrix Jordan algebra. Then the algebra $\mathrm{S}(\mathcal{L}[V])$ is reflexive.*

Proof. There is enough to prove reflexivity for $\Phi_{\mathrm{S}(\mathcal{L}[V])}(\mathrm{S}(\mathcal{L}[V]))$, which is isomorphic to $\mathrm{S}(\mathcal{L}[V]$ as an algebra with involution. But Proposition 8.1 shows, that the image of $\mathcal{L}[V]$ contains all symmetric elements from $\Phi_{\mathrm{S}(\mathcal{L}[V])}(\mathrm{S}(\mathcal{L}[V]))$. □

If $x_1 \circ x_2 \circ \dots \circ x_n \in \mathfrak{J}$, then call the expression (and its value)

$$x_1 \dots x_n = (\dots((x_1 \circ x_2) \circ x_3) \circ \dots) \circ x_{n-1}) \circ x_n$$

a *normalized word* in $\mathfrak{J}$.

Remark 8.1. *Assume in Proposition 8.1 the bimodule V is unital. Then from the proof follows, that we can choose the word x normalized.*

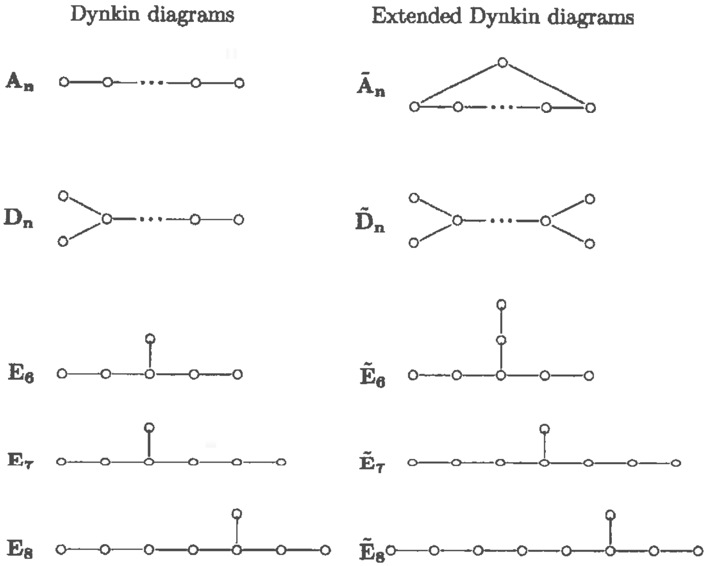
Theorem 8.2. *Let $\mathcal{L}$ be a semisimple matrix Jordan algebra, V be finite dimensional $\mathcal{L}$-module, $\mathcal{L}[V]$ almost matrix Jordan algebra, $\Gamma = \Gamma(\mathcal{L}[V])$ and $\mathrm{Q} = \mathrm{Qui}(\Gamma)$, $A = \mathbf{k}[\mathrm{Q}]$ be the path algebra, $I \subset \sum_{i=2}^{\infty} \mathbf{k}\,\mathrm{Q}^i$ an ideal in A, generated by some $*$-invariant subset $S \subset J(\mathcal{L}[V])$, i.e. $I^* = I$. Then there exists unique (up to isomorphism) Jordan algebra $\mathfrak{J}$ with Levi subalgebra $\mathcal{L}$, such that $\mathrm{eS}(\mathfrak{J})\mathrm{e} = \overline{\mathrm{S}(\mathfrak{J})}$ is isomorphic to A/I as an algebra with involution.*

Proof. Let $R \subset \mathbf{k}[Q]$ be a set of generators of I. Using Corollary 8.1 we lift all elements of R and obtain $R' \subset \mathcal{L}[V]$. Then the set of entries of $\Phi_{\mathrm{S}(\mathcal{L}[V])}(R)$ coincides with $R \cup R^* = R$, which generates I. Set $\mathfrak{J} = \mathcal{L}[V]/(R')$. Uniqueness follows from Corollary 3.1. □

8.3. An example: Jordan algebras with zero radical square and tensor Jordan algebras. Let Q be a quiver. Then the *quiver double* $D(\mathrm{Q})$ of the quiver Q is defined as follows:

(54) $$D(\mathrm{Q})_0 = \{ X^+, X^- \mid X \in \mathrm{Q}_0 \},\ D(\mathrm{Q})_1 = \{ \tilde{a} : s(a)^- \longrightarrow e(a)^+ \mid a \in \mathrm{Q}_1 \}.$$

Recall, that unoriented graphs from the following lists



are called Dynkin digrams and extended Dynkin diagrams without multiply connections. For short in this subsection we will call they just Dynkin diagrams. A Dynkin diagram (extended Dynkin diagram) we can endow with an orientation of edges and call obtained quiver an oriented (extended) Dynkin diagram.

The following results are classical.

Theorem 8.3. *(1) Let $A = \mathbf{k}[Q]$ is the path algebra of a quiver* Q*. Then A if of finite (tame) representation type if and only if* Q *is a disjoint union of oriented Dynkin diagrams (extended Dynkin diagrams).*

(2) Let A be a finite dimensional associative algebra, such that $\mathrm{Rad}^2 A = 0$, Q *its quiver. Then A is of finite (tame) representation type if and only if $D(\mathrm{Q})$ is a disjoint union of oriented Dynkin diagrams (extended Dynkin diagrams).*

We can prove some analogues of these statement for matrix Jordan algebras.

Theorem 8.4. *(1) Let $\mathfrak{L}$ be a semisimple matrix Jordan, Γ be a Jordan diagram over $\mathfrak{L}$, $V = V(\Gamma)$ and $\mathfrak{J} = \mathfrak{L}[V]$. Then $\mathfrak{J}$ is of special representation finite (tame) type, if and only if* $\mathrm{Qui}(\Gamma)$ *is a disjoint union of oriented Dynkin diagrams (extended Dynkin diagrams).*

(2) *Let $\mathfrak{J}$ be a finite dimensional Jordan algebra of matrix type,* $\operatorname{Rad}\mathfrak{J}^2 = 0$ *and* Γ *its Jordan diagram. Then $\mathfrak{J}$ is of special representation finite (tame) type, if and only if* $D(\mathrm{Qui}(\Gamma))$ *is a disjoint union of oriented Dynkin diagrams (extended Dynkin diagrams).*

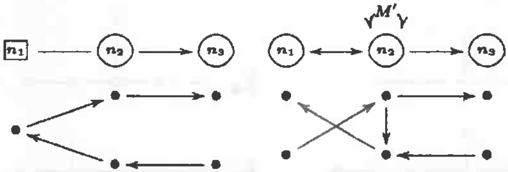
Proof. The statement (1) follows immediately from Theorem 5.1, Theorem 7.1 and Theorem 8.3, (1).

In the same way, the in the case, when the bimodule V is unital, the statement (2), follows from Theorem 6.1, (4) and Theorem 8.3, (2).

Let us extend the statement (2) to the general case. Let $\mathfrak{L} \oplus V$ be the trivial split extension, $\pi : \mathfrak{L}[V] \longrightarrow \mathfrak{L} \oplus V$ the canonical projection. Then $\operatorname{Ker}\pi \subset \mathfrak{L}[V]$ is generated by $V \circ V$, $\mathfrak{L} \circ (V \circ V)$. But following Proposition 8.1 and Corollary 8.1 the ideal in $\mathbf{k}[\mathrm{Qui}(\Gamma)]$, generated by entries of the matrices $\Phi_{\mathfrak{L}[V]}(V \circ V)$ and $\Phi_{\mathfrak{L}[V]}(\mathfrak{L} \circ (V \circ V))$ contains all pathes of length 2 in $\mathrm{Qui}(\Gamma)$. $\square$

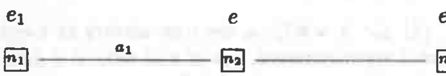
Note, that if in a diagram Γ is a Dynkin diagram and the vertices are matrix algebras, then $\mathrm{Qui}(\Gamma)$ is the disjoint union of two oppositely oriented copies of this diagram.

Example 8.1. *The examples below shows, how to work the map* Qui.



8.4. Morita equivalence for matrix Jordan algebras. The theorem above shows, that the class of Morita equivalence of $\mathrm{S}(\mathfrak{J})$ depends, in fact, on some ideal, generated by Jordan words in the quiver algebra $\mathbf{k}[\mathrm{Qui}(\Gamma(\mathfrak{J}))]$. We show, the $\mathfrak{J}$ and $\mathfrak{J}'$ are special Morita equivalent if they are almost matrix and $\mathrm{Qui}(\mathfrak{J})$ and $\mathrm{Qui}(\mathfrak{J}')$ are isomorphic as quivers with involution, endowed with the maps *alg* and *mod*.

In the class of special Morita equivalence Proposition 8.1 allows a rule to lift of associative words in a Jordan words in $\mathfrak{L}[V]$. Hence if $a_1, \ldots, a_n \in \Gamma_1$, and $a_1 \ldots a_n$ is a non-associative word, we will consider as a lift of some associative word.



$$e_1 \circ (a_1 \circ a_1) - e_3 \circ (a_2 \circ a_2)$$

$$x_1y_1 - y_2x_2$$

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