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Reprint of: HFD spaces in two problems of countably compact like properties [☆,☆☆]

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ABSTRACT

In this article we will construct a consistent space X such that every power smaller than $2^{\mathfrak{c}}$ of its hyperspace $\mathcal{CL}(X)$ is countably compact, but its $2^{\mathfrak{c}}$-power is not countably compact. This provides a consistent negative answer to a question from I. Juhász and J. E. Vaughan [11]. We also give a consistent negative answer to a question from M. Sanchis and A. Tamariz-Mascarúa [13].

Preliminaries

Every space in this article is a Tychonoff space with more than one point. The letters κ and λ represent cardinal numbers and the letters ξ, ζ, θ and similar represent ordinal numbers. With ω we denote the first infinite cardinal, ω_1 is the first non-countable cardinal and $\mathfrak{c}$ represents the continuum. For an infinite

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cardinal number κ we will denote by $Lim(\kappa)$ the set of all limit ordinals less than κ. Given a set X and a cardinal number κ, we will represent the family $\{A \subseteq X : |A| \leq \kappa\}$ by $[X]^{\leq\kappa}$ and the notations $[X]^{<\kappa}$ and $[X]^{\kappa}$ are defined analogously. If X is a topological space and $A \subseteq X$, then we denote by $cl_X(A)$ (or simply $cl(A)$) the closure of A in X. The *Stone-Čech* compactification $\beta\omega$ of the discrete countable space ω will be identified with the set of all ultrafilters on ω and its remainder ω^* will be identified with the set of all free ultrafilters on ω. Let $f : \omega \longrightarrow \omega$ be a function, the *Stone extension* of f to $\beta\omega$ is the unique continuous function $\bar{f} : \beta\omega \to \beta\omega$ such that $\bar{f}|_\omega = f$. Given two ultrafilters $p, q \in \beta\omega$, we say that $p \leq_{RK} q$ if there exists a function $f : \omega \longrightarrow \omega$ such that $\bar{f}(q) = p$. This relation is known as the *Rudin-Keisler* pre-order on $\beta\omega$. We say that two ultrafilters p and q are *RK-equivalent* if $p \leq_{RK} q$ and $q \leq_{RK} p$ (in symbols, $p \approx_{RK} q$[1]), they are *$\leq_{RK}$-comparable* if either $p \leq_{RK} q$ or $q \leq_{RK} p$ and they are *$\leq_{RK}$-incomparable* if they are not $\leq_{RK}$-comparable. Following the paper [9], given a space X, $p \in \omega^*$ and a sequence $(x_n)_{n\in\omega}$ in X, we will say that $x \in X$ is a *p-limit* of $(x_n)_{n\in\mathbb{N}}$, in symbols $x = p - lim\ x_n$, if $\{n \in \omega : x_n \in W\} \in p$ for each neighborhood W of x and a space X is *p-compact* if every sequence of points has p-limit. It is well known that every p-compact space has very nice properties, one of those is that every p-compact space is countably compact (for more basic properties see [1] and [9]).

For a topological space $(X, \mathcal{T})$, $CL(X)$ denotes the sets of nonempty closed subsets of X and $\mathcal{CL}(X)$ denotes the hyperspace of nonempty closed subsets of X with the Vietoris topology. Remember that the Vietoris topology has the sets of the form

$$U^+ = \{A \in CL(X) : A \subseteq U\} \text{ and } U^- = \{A \in CL(X) : A \cap U \neq \emptyset\}$$

as a subbase, where U is an open subset of X. Given open sets $U_1, \ldots, U_n$ of X, we define

$$\begin{aligned}\langle U_1, \ldots, U_n\rangle \ &= \{T \in CL(X) : T \in (\cup_{1\leq k\leq n} U_k)^+ \\ &\text{and } \ T \in U_k^- \text{ for each } 1 \leq k \leq n\}.\end{aligned}$$

So, the collection

$$\{\langle U_1, \ldots, U_n\rangle \ : n \in \mathbb{N}, U_1, \ldots, U_n \in \mathcal{T}\}$$

constitutes a base for $\mathcal{CL}(X)$. Those notions used and not defined in this article have the meaning given to them in [5].

0. Introduction

In [8], J. Ginsburg studied the countable compactness in the hyperspace $\mathcal{CL}(X)$, and he proved that $\mathcal{CL}(X)$ is p-compact iff X is p-compact for every free ultrafilter p of ω (Theorem 2.1). By following this result and Theorem 2.6 from [9] it is evident that if every power of a space is countably compact, then its hyperspace is countably compact too. Nevertheless, one of the main problems proposed by Ginsburg in that article is to characterize the countable compactness of $\mathcal{CL}(X)$ by properties on X. This problem is the main task of this paper. In Remark 3.2 from the same paper Ginsburg asked whether there is any relation between the countable compactness of X^ω and $\mathcal{CL}(X)$. From that remark, J. Cao, T. Nogura and A. Tomita provided some interesting examples in [2], in particular they showed that there is a countably compact Tychonoff space X such that $X^{\mathfrak{t}}$ is countably compact but $\mathcal{CL}(X)$ is not countably compact and, by using MA, they construct a Tychonoff space X such that X^α is countably compact for every $\alpha < 2^{\mathfrak{c}}$ but $\mathcal{CL}(X)$ is not (Examples 2.6 and 2.9 respectively). In the same paper they ask if $2^{\mathfrak{c}}$ is the best possible cardinal for the

[1] The original definition of $\leq_{RK}$-equivalent ask for a bijection in ω but it is well known that it is enough to have a 1 to 1 function $f : \omega \to \omega$ such that $\bar{f}(p) = q$ to guarantee that $p \approx_{RK} q$.

power of a countably compact space X to guarantee the countable compactness of $\mathcal{CL}(X)$. Following the previously mentioned results from [8] and [9] this is equivalent to ask if the countable compactness of $\mathcal{CL}(X)$ is equivalent to X being p-compact for some $p \in \omega^*$. In [11], I. Juhász and J. E. Vaughan conjecture that this last assertion is true and proved some partial results in that direction. We claim that such conjecture is not true in ZFC, of course this last affirmation does not mean that the conjecture from Juhász and Vaughan cannot be also consistent with ZFC. To do this we will use HFD spaces, introduced in [10] by A. Hajnal and I. Juhász, to construct a consistent example of a space X such that both X^λ and $\mathcal{CL}(X)^\lambda$ are countably compact for every $\lambda < 2^{\mathfrak{c}}$ but X is not p-compact for any $p \in \omega^*$, or equivalently, $X^{2^{\mathfrak{c}}}$ is not countably compact. We cite here the equivalent version of HFD space mentioned in that paper.

Definition 0.1. Let $X = \prod_{\xi<\omega_1} X_\xi$ be a product of topological spaces. A subspace $Z \subseteq X$ is an HFD space if for every $A \in [Z]^\omega$ there is $\gamma < \omega_1$ such that $A|_{[\gamma,\omega_1[} = \{x|_{[\gamma,\omega_1[} : x \in A\}$ is dense in $X = \prod_{\gamma\leq\xi<\omega_1} X_\xi$.

In particular, HFD spaces are hereditarily separable and hereditarily collectionwise normal (Theorems 2 and 4 from [10]). We will show that under these two conditions, the countable compactness of a power of a space is equivalent to the countable compactness of the same power of its hyperspaces. This result is the key which led us to believe that HFD would be useful to find the desired example.

Remark 0.2. In [12] it was studied the analogous problem of pseudocompact products and the hyperspace $\mathcal{CL}(X)$ by using the concept of $(\kappa, \mathcal{D})$-pseudocompactness, which was also introduced by García-Ferreira in [6] and it characterizes pseudocompact powers. In [12] it is proved that if $\omega \subseteq X \subseteq \beta X$ is $(\mathfrak{c}, \omega^*)$-pseudocompact, then $\mathcal{CL}(X)$ is pseudocompact (this in ZFC). Furthermore if $\omega \subseteq X \subseteq \beta X$ and $\mathcal{CL}(X)$ is pseudocompact then X is (κ, ω^*)-pseudocompact for every $\kappa < \mathfrak{h}$ but there exists an example of such an X that is not $(\mathfrak{b}, \omega^*)$-pseudocompact. As we may see these kinds of concepts are intimately related and this fact leads us to believe that we are in the right way.[2] In addition, it is important to say that all the papers mentioned in this introduction have analogs of the same problems, properties and results for pseudocompactness too. We decided to omit them because the advances in this paper are related only with countable compactness and their approaches are different from ours.

A different problem involving p-compactness is to find additional properties to p-pseudcompactness in order to reach the p-compactness. Some interesting advances on this problem can be found in [13]. In that article M. Sanchis and A. Tamariz-Mascarúa define a *ultrapseudocompact*[3] and totally countable compact space (α-pseudocompact, locally compact, sequentially compact space), which is not p-compact for any $p \in \omega^*$ (see Examples 3.3, 3.4 and 3.5). In the same paper the authors note that none of their examples are neither normal nor first countable and they observe that, by following Corollary 6.6 and Theorem 6.8 from [14], every normal finally $\mathfrak{p}$-compact space which is p-pseudocompact for at least a $p \in \omega^*$ is *ultracompact*,[4] concluding that, by assuming $\mathfrak{p} > \omega_1$, every perfectly normal p-pseudocompact space is compact (see last paragraph from [13, pag. 332]). Finally they established the following question:

Question 0.3. [13, Question 3.6] Is it consistent with ZFC that, for every $p \in \omega^*$, each p-pseudocompact space satisfying normality (perfect normality, collectonwise normality, normality + countable paracompactness) must be p-compact?

The second target of this paper is to show that our example is also collectionwise normal and ultrapseudocompact which shows that the negative of the previous question is consistent with ZFC for collectionwise

[2] In fact the present work is previous to [12] and our results motivated us to continue working in the pseudocompact version of the problem. At the end, different circumstances causes that work to be published before its predecessor.

[3] A space is ultrapseudocompact if it is p-pseudocompact for every $p \in \omega^*$.

[4] A space is ultracompact if it is p-compact for every $p \in \omega^*$.

normality. By following the next lemma we know that, if our HFD is dense in a product of compact metric spaces, then our example must be ulstrapseudocompact, so we will also have a consistent example of a collectonwise normal and ultrapseudocompact space which is not p-compact for any $p \in \omega^*$.

Lemma 0.4. *[7, Theorem 2.6] Let $\{X_i : i \in I\}$ be a family of compact metric spaces and let $X = \prod_{i\in I} X_i$. Then, every pseudocompact dense subspace of X is ultrapseudocompact.*

Our main goal is to construct an HFD space, by using forcing, which is dense in $\{0,1\}^{\omega_1}$ and then define a dense subspace Z satisfying that Z^λ is countably compact for each $\lambda < 2^{\mathfrak{c}}$ (also ultrapseudocompact by Lemma 0.4), but $Z^{2^{\mathfrak{c}}}$ is not countably compact and then, by Theorem 2.6 from [9], such subspace cannot be p-compact for any $p \in \omega^*$.

In Section 1 we will define the partial order and some useful technical properties. All the required dense sets of the partial order will be defined in Section 2. Section 3 is devoted to prove that the required HFD space exists and finally, we will construct the promised example in Section 4.

1. The partial order

We will define a partial order to obtain the consistent example. First we need some previous lemmas:

Lemma 1.1. *[4, Lemma 9.1] Let α be a cardinal and let $f : \alpha \to \alpha$ be such that $f(\xi) \neq \xi$ for $\xi < \alpha$. Then there are subsets A_0, A_1 and A_2 of α such that:*

(1) $\alpha = A_0 \cup A_1 \cup A_2$,
(2) $A_i \cap A_j = \emptyset$ *for* $i, j \leq 2$ *and* $i \neq j$, *and*
(3) $A_i \cap f[A_i] = \emptyset$ *for* $i \leq 2$.

Lemma 1.2. *Let X be a set, let p_0 and p_1 two free ultrafilters of ω and let h_0 and h_1 be both 1 to 1 sequences in X. Suppose that either one of the following conditions holds:*

(1) $p_0 = p_1$ *and* $h_0 \not\equiv_{p_0} h_1$, *or*
(2) p_0 *and* p_1 *are non-RK-equivalent ultrafilters.*

Then there exists $A_i \in p_i$ for $i < 2$ such that $\{h_i(n) : n \in A_i$ and $i < 2\}$ are pairwise distinct.

Proof. Without loss of generality we assume that X is countable. First let $p \in \omega^*$ and assume that $h_0 \not\equiv_p h_1$, this means that $\{n \in \omega : h_0(n) \neq h_1(n)\} \in p$. Since each h_i is injective, it is standard to prove that there are bijective functions h'_i such that $h'_i \equiv_p h_i$ for each $i < 2$ and $h'_0(n) \neq h'_1(n)$ for every $n \in \omega$. Let $f = (h'_0)^{-1} \circ h'_1$ and observe that $f : \omega \to \omega$ and $f(n) \neq n$ for every $n \in \omega$. By Lemma 1.1 there is a partition B_j with $j < 3$ of ω, such that $B_j \cap f[B_j] = \emptyset$ for each $j < 3$.

Since p is an ultrafilter, there is an unique $k < 3$ such that $B_k \in p$.

Furthermore $h'_i s$ are bijective so $\emptyset = h'_0[B_k] \cap h'_0[f[B_k]] = h'_0[B_k] \cap h'_1[B_k]$. Now let $C_i \in p$ such that $h'_i|_{C_i} = h_i|_{C_i}$ for each $i < 2$ and define $A_0 = C_0 \cap B_k$ and $A_1 = C_1 \cap B_k$. Then $A_i \in p$ for $i < 2$ and $h_0[A_0] \cap h_1[A_1] = \emptyset$.

Now assume that p_0 and p_1 are non-RK-equivalent. Since each h_i is injective, take bijective functions h'_i such that $h'_i \equiv_{p_i} h_i$ for each $i < 2$. Let $f = (h'_0)^{-1} \circ h'_1$ and let $q_1 = \{f[A] : A \in p_1\}$. Observe that q_1 and p_1 are RK-equivalent so, since p_0 and q_1 are not RK-equivalent, there exists $B_0 \in p_0$ and $B_1 \in p_1$ such that $B_0 \cap f[B_1] = \emptyset$. Therefore, $h'_0[B_0] \cap h'_1[B_1] = \emptyset$. Take $C_i \in p_i$ such that $h'_i|_{C_i} = h_i|_{C_i}$ for $i < 2$. Then $A_i = B_i \cap C_i$ for $i < 2$ is as required. □

Assuming the existence of selective ultrafilters it is possible to improve the last lemma.

Lemma 1.3. *Let X be a set, let $\{p_n : n \in \omega\}$ be a family of selective ultrafilters and let $\{h_n : n \in \omega\}$ be a family of sequences in X such that:*

(i) $|h_n^{-1}(x)|$ is finite for every $x \in X$ and $n \in \omega$;

(ii) $\forall\ n, m \in \omega$, either $p_n = p_m$ or p_n and p_m are RK-incomparable;

(iii) $\forall\ n, m \in \omega$, if $n \neq m$ and $p_n = p_m$, then $h_n \not\equiv_{p_n} h_m$.

Then for each $n \in \omega$, there exists $A_n \in p_n$ such that $h_n|_{A_n}$ is 1 to 1 for every $n \in \omega$ and the sets of the family $\big\{\{h_n(k) : k \in A_n\} : n \in \omega\big\}$ are pairwise disjoint.

Proof. First observe that all the p_n's are selective ultrafilters so, for every $n \in \omega$ there is $A_n^* \in p_n$ such that $h_n|_{A_n^*}$ is injective. Now fix, for every pair of different natural numbers n and m, sets $A_n^m \in p_n$ and $A_m^n \in p_m$ such that $A_n^m \subseteq A_n^*$, $A_m^n \subseteq A_m^*$ and

$$\{h_n(k) : k \in A_n^m\} \cap \{h_m(k) : k \in A_m^n\} = \emptyset.$$

This is possible because of Lemma 1.2 and the fact that two RK-incomparable ultrafilters are non-RK-equivalent. Since each p_n is selective, we have that for every $n \in \omega$ there is $A'_n \in p_n$ such that $A'_n \subseteq A_n^*$, $|A'_n \setminus A_n^m| < \omega$ for every $m \neq n$. Note that

$$\{h_n(k) : k \in A'_n\} \cap \{h_m(k) : k \in A'_m\} \text{ is finite }.$$

Set F_n^m for every pair with $m < n$ such that

$$\{h_n(k) : k \in A'_n\} \cap \{h_m(k) : k \in A'_m\} \subseteq \{h_n(k) : k \in F_n^m\}.$$

Define $A_n = A'_n \setminus \big(\bigcup_{i<n} F_n^i\big)$. Of course $A_n \in p_n$ and $\{h_n(k) : k \in A_n\} \cap \{h_m(k) : k \in A_m\} = \emptyset$ for every pair of distinct $n, m \in \omega$. □

Remark 1.4. The previous lemma looks short in its promise to be an improvement of Lemma 1.2 (because we ask for a set of RK-incomparable ultrafilters and this is a stronger request than being non-RK-equivalent). In fact every selective ultrafilter is $\leq_{RK}$-minimal so, two selective ultrafilters which are RK-comparable must be RK-equivalent. Therefore Lemma 1.3 keeps the promise intact.

Now we will establish the initial conditions, notation and the partial order. Let $\kappa = 2^{\omega_1}$ and assume CH, $\beta^\omega < \kappa$ and $2^\beta \leq \kappa$ for every cardinal $\beta < \kappa$. Let $\{L\} \cup \{L_\alpha : \alpha < \kappa\}$ be a partition of the set $Lim(\kappa)$ such that each set has cardinality κ. Let $\{g_\xi : \xi \in Lim(\kappa)\}$ be an enumeration of functions from ω into κ satisfying the following:

- $g_\xi(n) < \xi$ for each $\xi \in Lim(\kappa)$ and $n \in \omega$, and
- $\{g_\xi : \xi \in L_\alpha\}$ is an enumeration of all 1 to 1 functions from ω into κ, for each $\alpha < \kappa$.

Fix a family of κ-many RK-incomparable selective ultrafilters which exist by CH (this is a direct consequence of Lemma 5.1 and Theorem 6.1 from [3]). Split and enumerate such family as follows $\{p_\alpha : \alpha < \kappa\} \cup \{q_n : n < \omega\}$. Finally, for each $\alpha < \kappa$, let $K_\alpha \subseteq L_\alpha$ be a set such that for each 1 to 1 function $g : \omega \longrightarrow \alpha$ there exists a unique $\xi \in K_\alpha$ such that $g \equiv_{p_\alpha} g_\xi$. Note that for every $\alpha < \kappa$, $|\alpha|^\omega < \kappa$. Since κ is regular, K_α is bounded in κ.

Consider the set:

$$\mathbb{U} = \big\{\langle \gamma, I, s, T\rangle : \gamma < \omega_1, I \in [\kappa]^\omega, s : I \longrightarrow 2^\gamma \text{ and } T \in \big[[I]^\omega\big]^\omega\big\}.$$

Observe that $\mathbb{U}$ has cardinality κ. To be short we will use the letters u, v and similar to refer the elements of $\mathbb{U}$ and for a given $u \in \mathbb{U}$ we will write γ_u, I_u, s_u and T_u to denote the coordinates of u except when we specify something different. For each $u \in \mathbb{U}$, $A \in T_u$ and each basic open set U of 2^{γ_u}, let $B_{A,U} = \{\beta \in A : s_u(\beta) \in U\}$. Finally denote by $\mathcal{T}_u$ the family of all such $B_{A,U}$'s, where $A \in T_u$ and $U \subseteq 2^{\gamma_u}$, that are infinite. Note that the set $B_{A,U}$ depends on u but we decided to omit it to simplify the notation since it becomes clear by establishing $\mathcal{T}_u$ or in the choice of A and U. We are ready to define our partial order.

Definition 1.5. Let $\mathbb{P}$ be the set of all $u \in \mathbb{U}$ satisfying the following conditions:

(1) $g_\beta(n) \in I_u$ for each $\beta \in I_u \cap Lim(\kappa)$ and every $n \in \omega$,
(2) p_α-lim $s_u\big(g_\xi(n)\big) = s_u(\xi)$ for each $\alpha < \kappa$ such that $K_\alpha \cap I_u \neq \emptyset$ and every $\xi \in K_\alpha \cap I_u$

Then for two given u, $v \in \mathbb{P}$, we will say that $v \leq u$ if:
(*i*) $\gamma_v \geq \gamma_u$, $I_v \supseteq I_u$ and $T_v \supseteq T_u$,
(*ii*) $s_v(\xi)|_{\gamma_u} = s_u(\xi)$ for each $\xi \in I_u$, and
(*iii*) the set $\{s_v(\beta)|_{[\gamma_u,\gamma_v[} : \beta \in B_{A,U} \setminus F\}$ is dense in $2^{[\gamma_u,\gamma_v[}$ for each $B_{A,U} \in \mathcal{T}_u$ and each finite $F \subseteq B_{A,U}$.

To end this section let us prove some properties of $(\mathbb{P}, \leq)$.

Lemma 1.6. $(\mathbb{P}, \leq)$ *is transitive.*

Proof. Suppose that $u_2 < u_1$ and $u_1 < u_0$. Clearly $\gamma_{u_2} \geq \gamma_{u_0}$, $I_{u_2} \supseteq I_{u_0}$, $T_{u_2} \supseteq T_{u_0}$ and $s_{u_2}(\xi)|_{\gamma_{u_0}} = s_{u_0}(\xi)$ for each $\xi \in I_{u_0}$, so items (*i*) and (*ii*) from Definition 1.5 are satisfied for u_2 and u_0. Hence it suffices to show that $\{s_{u_2}(\beta)|_{[\gamma_{u_0},\gamma_{u_2}[} : \beta \in B_{A,U} \setminus F\}$ is dense in $2^{[\gamma_{u_0},\gamma_{u_2}[}$ for each $B_{A,U} \in \mathcal{T}_{u_0}$ and each finite $F \subseteq B_{A,U}$. Let W be a basic open set in $2^{[\gamma_{u_0},\gamma_{u_2}[}$, then there exists W_i basic open set of $2^{[\gamma_{u_i},\gamma_{u_{i+1}}[}$ for $i \in \{0, 1\}$, such that $W = W_0 \times W_1$. Fix $B_{A,U} \in \mathcal{T}_{u_0}$ and $F \in [B_{A,U}]^{<\omega}$, then $\{s_{u_1}(\beta)|_{[\gamma_{u_0},\gamma_{u_1}[} : \beta \in B_{A,U} \setminus F\}$ is dense in $2^{[\gamma_{u_0},\gamma_{u_1}[}$ because $u_1 \leq u_0$. Also $B_{A,U} \subseteq A \in I_{u_0} \subseteq I_{u_1}$, since $s_{u_2}(\beta)|_{[\gamma_{u_0},\gamma_{u_1}[} = s_{u_1}(\beta)|_{[\gamma_{u_0},\gamma_{u_1}[}$ for each $\beta \in B_{A,U}$, $\{s_{u_2}(\beta)|_{[\gamma_{u_0},\gamma_{u_1}[} : \beta \in B_{A,U} \setminus F\}$ is dense in $2^{[\gamma_{u_0},\gamma_{u_1}[}$. Thus $\{s_{u_2}(\beta)|_{[\gamma_{u_0},\gamma_{u_1}[} : \beta \in B_{A,U} \setminus F\} \cap W_0$ is infinite. Even more, $U \subseteq 2^{\gamma_{u_0}}$ and W_0 is a basic open set of $2^{[\gamma_{u_0},\gamma_{u_1}[}$, so $U \times W_0$ is an open set of $2^{\gamma_{u_1}}$, then $B_{A,U\times W_0} \in \mathcal{T}_{u_1}$. By $u_2 \leq u_1$, $\{s_{u_2}(\beta)|_{[\gamma_{u_1},\gamma_{u_2}[} : \beta \in B_{A,U\times W_0} \setminus F\}$ is dense in $2^{[\gamma_{u_1},\gamma_{u_2}[}$. Thus $\{s_{u_2}(\beta)|_{[\gamma_{u_1},\gamma_{u_2}[} : \beta \in B_{A,U\times W_0} \setminus F\} \cap W_1$ is infinite. If $\beta \in B_{A,U\times W_0} \setminus F$ and $s_{u_2}(\beta)|_{[\gamma_{u_1},\gamma_{u_2}[} \in W_1$, then $\beta \in B_{A,U} \setminus F$, $s_{u_2}(\beta)|_{[\gamma_{u_0},\gamma_{u_1}[} \in W_0$ and $s_{u_2}(\beta)|_{[\gamma_{u_1},\gamma_{u_2}[} \in W_1$, so $s_{u_2}(\beta)|_{[\gamma_{u_0},\gamma_{u_2}[} \in W$. Therefore the set $\{s_{u_2}(\beta)|_{[\gamma_{u_0},\gamma_{u_2}[} : \beta \in B_{A,U} \setminus F\} \cap W$ is infinite and we are done. □

Lemma 1.7. $(\mathbb{P}, \leq)$ *is* ω_2*-cc.*

Proof. Let $\{u_\alpha : \alpha < \omega_2\}$ be a family of elements of $\mathbb{P}$. To simplify we will replace u_α for α to enumerate s_{u_α}, I_{u_α} and so on.

There is $\gamma < \omega_1$ and $J \in [\omega_2]^{\omega_2}$ such that $\gamma_\alpha = \gamma$ for all $\alpha \in J$. As $\omega_1^\omega = \omega_1$, by the Δ-system Lemma, there are $R \subseteq \kappa$ and $J' \in [J]^{w_2}$ such that $I_\alpha \cap I_\beta = R$ for each $\alpha, \beta \in J'$ distinct. Since $|\gamma| \leq \omega$ and $|R| \leq \omega$, there are at most $(2^\omega)^\omega = \omega_1^\omega = \omega_1$ functions from R to 2^γ, so there exist $J'' \in [J']^{w_2}$ and $t : R \longrightarrow 2^\gamma$ such that $s_\alpha|R = t$ for every $\alpha \in J''$. Picking any two distinct elements $\alpha, \beta \in J''$, set $I = I_\alpha \cup I_\beta$, $s = s_\alpha \cup s_\beta$ and $T = T_\alpha \cup T_\beta$. It is clear that $\langle \gamma, I, s, T\rangle \in \mathbb{P}$ and since $\gamma = \gamma_\alpha = \gamma_\beta$, it follows that $\langle \gamma, I, s, T\rangle$ is below u_α and u_β. □

Lemma 1.8. $(\mathbb{P}, \leq)$ *is* ω_1*-closed.*

Proof. The proof is standard and uses the fact that the support of a basic open set is finite. □

2. Dense sets

Let us define the dense sets of $\mathbb{P}$ that we will use. The next lemma will be useful for that.

Lemma 2.1. *Let $u \in \mathbb{P}$ and let $A \in [\kappa]^{\leq\omega}$. Then there is $v \in \mathbb{P}$ such that $v \leq u$ and $A \subseteq I_v$.*

Proof. Take $u \in \mathbb{P}$, $\gamma = \gamma_u$ and $T = T_u$. Choose a countable subset I of κ such that:
(i) $I_u \cup A \subseteq I$ and
(ii) $\{g_\xi(n) : n \in \omega\} \subseteq I$ for all $\xi \in I \cap Lim(\kappa)$.

This is possible because A is countable.

Now we will use induction to define a function $s : I \to \{0,1\}^\gamma$. Let $\xi \in I$ the least ordinal for which $s(\xi)$ has not been defined. If $\xi \in I_u$, then define $s(\xi) = s_u(\xi)$. Suppose that $\xi \in (I \setminus I_u) \cap K_\alpha$ for some $\alpha < \kappa$. Since $s(\eta)$ is already defined for every $\eta \in I \cap [0,\xi)$ we have that $s(g_\xi(n))$ is defined for each $n \in \omega$ because of condition (ii) and the fact that $g_\xi(n) < \xi$ for all $n \in \omega$. As $\{0,1\}^\gamma$ is compact, there exists the p_α-limit of the set $\{s_u(g_\xi(n)) : n \in \omega\}$. Then define $s(\xi) = p_\alpha - lim\ s(g_\xi(n))$. Otherwise choose $s(\xi) \in 2^\gamma$ arbitrarily. Thus, function s satisfies that $s|_{I_u} = s_u$. Finally let $v = \langle \gamma, I, s, T\rangle$. Clearly $v \in \mathbb{P}$ and $v \leq u$ since $\gamma_u = \gamma_v$. □

Lemma 2.2. *The set $\mathcal{D}_\beta = \{u \in \mathbb{P} : \beta \in I_u\}$ is dense in $\mathbb{P}$ for each $\beta < \kappa$.*

Proof. This lemma follows directly from Lemma 2.1 □

Lemma 2.3. *The set $\mathcal{E}_A = \{u \in \mathbb{P} : A \in T_u\}$ is dense for each $A \in [\kappa]^\omega$.*

Proof. Fix $v \in \mathbb{P}$. By Lemma 2.1, there is $w \in \mathbb{P}$ such that $w \leq v$ and $A \subseteq I_w$. Let $\gamma = \gamma_w$, $I = I_w$, $s = s_w$ and $T = T_w \cup \{A\}$. Therefore $\langle \gamma, I, s, T\rangle \in \mathbb{P}$ and $\langle \gamma, I, s, T\rangle \leq v$. □

Notation 2.4. For the following lemmas we will use the next notation. If $\xi \in Lim(\kappa)$, then denote by $\alpha(\xi)$ the ordinal α such that $\xi \in L_\alpha$.

Lemma 2.5. *Let $F \in [\kappa]^{<\omega}$, let I be a countable subset of κ which satisfies condition (1) from Definition 1.5 and let $J = I \cap (\bigcup_{\beta<\kappa} K_\beta)$. Then for every family $\{A_k : k \in \omega\}$ of infinite subsets of I and every function $f : F \longrightarrow 2$ there exist a family of sets $\{B_\xi \in p_{\alpha(\xi)} : \xi \in J\}$ and a function $h : F \cup I \longrightarrow 2$ such that $f \subseteq h$, $h^{-1}[\{i\}] \cap A_k$ is infinite for every $i < 2$ and $k \in \omega$; and $\{n \in \omega : h\big(g_\xi(n)\big) = h(\xi)\}$ contains a cofinite subset of B_ξ for each $\xi \in J$.*

Proof. Split each A_k in two infinite sets A_k^i with $i < 2$ and fix a 1 to 1 enumeration σ_k^i of A_k^i for each $i < 2$ and $k \in \omega$. Relabel the previously fixed set of ultrafilters $\{q_n : n < \omega\}$ as $\{q_k^i : k \in \omega \text{ and } i < 2\}$ and recall that the ultrafilters of the set

$$\mathcal{Z} = \{p_\alpha : K_\alpha \cap I \neq \emptyset\} \cup \{q_k^i : k \in \omega \text{ and } i < 2\}$$

were chosen to be selective and pairwise RK-incomparable since the beginning. Since $|\mathcal{Z}| = \omega$ we can find, by Lemma 1.3, sets $B_\xi \in p_{\alpha(\xi)}$ for each $\xi \in J$ and sets $C_k^i \in q_k^i$ for each $k \in \omega$ and $i < 2$ such that the elements of

$$\{\{g_\xi(n) : n \in B_\xi\} : \xi \in J\} \cup \{\{\sigma_k^i(n) : n \in C_k^i\} : k \in \omega,\ i < 2\} \cup \{F\}$$

are pairwise disjoint. We will define the function $h : F \cup I \to 2$ in four disjoint parts of the domain.
- Let $h(\xi) = f(\xi)$ for every $\xi \in F$,
- let $h(\xi) = i$ for every $\xi \in \{\sigma^i_k(n) : k \in \omega,\ i < 2 \text{ and } n \in C^i_k\}$,
- enumerate the set $\{g_\xi(n) : \xi \in J \text{ and } n \in B_\xi\}$ by $\{a_m : m \in \omega\}$, using induction define $h(a_m) = h(\xi)$ when $n \in B_\xi$, where $a_m = g_\xi(n)$ and $\xi \in F \cup \{\sigma^i_k(n) : k \in \omega,\ i < 2 \text{ and } n \in C^i_k\} \cup \{a_j : j < m\}$, and define $h(a_n) = 0$ otherwise; and
- define $h(\xi) = 0$ otherwise.

It is clear that the function h is well defined and satisfies $f \subseteq h$. Also, by the definition of h, the set $\{n \in \omega : h\big(g_\xi(n)\big) = h(\xi)\}$ contains a cofinite subset of B_ξ for each $\xi \in J$. Finally, since σ^i_k is 1 to 1, $C^i_k \in q^i_k$ and $\{\sigma^i_k(n) : n \in C^i_k\} \subseteq h^{-1}[\{i\}] \cap A_k$ then $h^{-1}[\{i\}] \cap A_k$ is infinite for each $k \in \omega$ and $i < 2$. □

Lemma 2.6. *For every $u \in \mathbb{P}$ and every pair $\eta, \zeta \in I_u$, there exists $w \leq u$ such that $\gamma_w = \gamma_u + 1$ and $s_w(\eta)(\gamma_u) \neq s_w(\zeta)(\gamma_u)$.*

Proof. Let $\gamma = \gamma_u + 1$, $I = I_u$, $J = I \cap \big(\bigcup_{\beta<\kappa} K_\beta\big)$, $T = T_u$ and let $f : \{\eta, \zeta\} \to 2$ be a function such that $f(\eta) \neq f(\zeta)$. Applying the previous lemma to $\{\eta, \zeta\}$, I, $\mathcal{T}_u$ and f, there is a family of sets $\{B_\xi \in p_{\alpha(\xi)} : \xi \in J\}$ and a function $h : I \to 2$ such that $f \subseteq h$, $h^{-1}[\{i\}] \cap B_{A,U}$ is infinite for each $B_{A,U} \in \mathcal{T}_u$ and $\{n \in \omega : h\big(g_\xi(n)\big) = h(\xi)\}$ contains a cofinite subset of B_ξ for each $\xi \in J$. Set $s(\xi) = s_u(\xi)^\frown(h(\xi))$.

Claim. *$\langle \gamma, I, s, T\rangle \in \mathbb{P}$ and $\langle \gamma, I, s, T\rangle < u$.*

Proof of Claim. Items (1), (3), (i), (ii) and (iii) from Definition 1.5 follows directly from our selection of $\langle \gamma, I, s, T\rangle$. By item (2) observe that for each $\xi \in J$,

$$p_{\alpha(\xi)} - lim\ s\big(g_\xi(n)\big) = p_{\alpha(\xi)} - lim\ \Big(s_u\big(g_\xi(n)\big)^\frown\big(h\big(g_\xi(n)\big)\big)\Big) =$$
$$p_{\alpha(\xi)} - lim\ s_u\big(g_\xi(n)\big)^\frown\big(p_{\alpha(\xi)} - lim\ h\big(g_\xi(n)\big)\big) = s_u(\xi)^\frown(h(\xi)) = s(\xi),$$

where the equality between the lines holds because $\{n \in \omega : h\big(g_\xi(n)\big) = h(\xi)\} \in p_{\alpha(\xi)}$ and the fact that for every free ultrafilter $p \in \omega^*$ and every sequence in a topological product, the p-limit of the sequence is exactly the point whose coordinates are the p-limits of the projection of the sequence in each coordinate. □

Lemma 2.7. *The set $\mathcal{F}_{\delta,\theta_0,\theta_1} =$*

$$\{u \in \mathbb{P} : \gamma_u > \delta,\ \theta_i \in I_u,\ s_u(\theta_0)(\alpha) \neq s_u(\theta_1)(\alpha) \text{ for some } \delta < \alpha < \gamma_u\}$$

is dense for every pair $\{\theta_0, \theta_1\} \in [\kappa]^2$ and every $\delta < \omega_1$.

Proof. Pick v and note that by Lemma 2.1, we may assume without loss of generality, that $\theta_0, \theta_1 \in I_v$. Since $\delta < \omega_1$ and $\mathbb{P}$ is ω_1-closed we have by Lemma 2.6 that there is $u \in \mathcal{F}_{\delta,\theta_0,\theta_1}$ with $u < v$ and such that u satisfies all the required conditions. □

Let $u \in \mathbb{P}$. From now to the end we will denote the set $\bigcup_{\alpha<\kappa}(K_\alpha \cap I_u)$ by K_u.

Lemma 2.8. *For each $\beta, \zeta \in L$ let $\mathcal{G}_{\beta,\zeta}$ be the set of elements $u \in \mathbb{P}$ satisfying:*
(i) $[\beta, \beta + \omega[\cup [\zeta, \zeta + \omega[\subseteq I_u$, and
(ii) there is $\alpha < \gamma_u$ such that $s_u(\beta + n)(\alpha) = 1 - s_u(\zeta + n)(\alpha)$ for every $n \in \omega$. Then $\mathcal{G}_{\beta,\zeta}$ is dense for every pair $\beta, \zeta \in L$.

Proof. Let $u \in \mathbb{P}$, by Lemma 2.1 there is $v < w$ such that $[\beta, \beta+\omega[\cup[\zeta, \zeta+\omega[\subseteq I_v$. Let $\{A_k : k \in \omega\}$ be an enumeration of $\mathcal{T}_v$, split each A_k in two sets A_k^i and let $\sigma_k^i : \omega \to A_k^i$ be a bijection for each $i < 2$. Define, for each $\xi \in K_v$, the function $\tilde{g}_\xi : \omega \to \kappa$ by:

$$\tilde{g}_\xi(n) = \begin{cases} g_\xi(n) & \text{when } g_\xi(n) \notin [\beta, \beta+\omega[, \\ \zeta + m & \text{when } g_\xi(n) = \beta + m \end{cases}$$

and similarly, define the functions $\tilde{\sigma}_k^i : \omega \to A_k^i$ for every $k \in \omega$ and $i \in \{0, 1\}$ as:

$$\tilde{\sigma}_k^i(n) = \begin{cases} \sigma_k^i(n) & \text{when } \sigma_k^i(n) \notin [\beta, \beta+\omega[, \\ \zeta + m & \text{when } \sigma_k^i(n) = \beta + m. \end{cases}$$

Note that g_ξ's and σ_k^i's are 1-1, therefore, $\tilde{g}_\xi$'s and $\tilde{\sigma}_k^i$'s are finite to one. By Lemma 1.3, there are sets $B_\xi \in p_{\alpha(\xi)}$ for each $\xi \in K_v$ and sets $C_k^i \in q_k^i$ for each $k \in \omega$ and $i \in \{0, 1\}$ such that $\tilde{g}_\xi|_{B_\xi}$ is 1-1 for each $\xi \in K_w$ and $\tilde{\sigma}_k^i|_{C_k^i}$ is 1-1 for each $k \in \omega$ and $i < 2$ and

$$\{\{\tilde{g}_\xi(n) : n \in B_\xi\} : \xi \in K_v\} \cup \{\{\tilde{\sigma}_k^i(n) : n \in C_k^i\} : k \in \omega, i < 2\}$$

are pairwise disjoint. Then

$$\tilde{O} = \{\tilde{g}_\xi(n) : \xi \in K_v \text{ and } n \in B_\xi\} \cup \{\tilde{\sigma}_k^i(n) : k \in \omega, i < 2 \text{ and } n \in C_k^i\}$$

are pairwise distinct.

Let

$$O = \{g_\xi(n) : \xi \in K_v \text{ and } n \in B_\xi\} \cup \{\sigma_k^i(n) : k \in \omega, i < 2 \text{ and } n \in C_k^i\}.$$

We claim that $|\{\beta + m, \zeta + m\} \cap O| \leq 1$ for every $m < \omega$.
In fact, if that was not the case, we would have $|\{\beta + m, \zeta + m\} \cap O| = 2$ for some $m < \omega$ then there would be two different elements in O that correspond to the same element in O^* and this contradicts the fact that the indexed elements of O^* are pairwise distinct. Fix a function $j_0 : \omega \to [\beta, \beta+\omega[\cup[\zeta, \zeta+\omega[$ such that $j_0(m) \in \{\beta + m, \zeta + m\} \setminus O$ for each $m \in \omega$. We note that here is important that the elements in $[\beta, \beta+\omega[\cup[\zeta, \zeta+\omega[$ are not assigned to be p_α-limits and thus they can be dealt at the end.

We will extend v to u so that $\gamma_u = \gamma_v + 1$, $I_u = I_v$ and $T_u = T_v$. We will define a new coordinate.

Enumerate $I_v \setminus ran\, j_0$ as $\{a_m : m \in \omega\}$ faithfully. Define inductively

- let $h(a_m) = i$ if $a_m \in \{\sigma_k^i(n) : k \in \omega, i < 2 \text{ and } n \in C_k^i\}$,
- let $h(a_m) = h(\xi)$ if $a_m = g_\xi(n)$ for some $\xi \in K_v$, $n \in B_\xi$ and there exists $l < m$ with $a_l = \xi$, and
- let $h(a_m) = 0$ otherwise.

Define $j_1 : \omega \to [\beta, \beta+\omega[\cup[\zeta, \zeta+\omega[$ such that $j_1(m)$ is the unique element of $\{\beta + m, \zeta + m\} \setminus \{j_0(m)\}$. Note that $h(j_1(m))$ is already defined for each $m \in \omega$. Extend h to I_v so that $h(j_0(m)) = 1 - h(j_1(m))$. Let $s_u : I_u \to 2^{\gamma_v+1}$ so that $s_u(\mu)(\alpha) = s_v(\mu)(\alpha)$ for each $\mu \in I_v$ and $\alpha < \gamma_v$ and $s_u(\mu)(\gamma_v) = h(\mu)$ for each $\mu \in I_v$.

Finally let $u = \langle \gamma_v + 1, I_v, s_u, T_v \rangle$. It is evident from the definition of h that $s_u(\beta + m)(\gamma_v) = 1 - s_u(\zeta + m)(\gamma_v)$ for each $m < \omega$ with $\gamma_v < \gamma_u$. To show that $u \in \mathbb{P}$ and $u \leq v$ just follow the same proof of Lemma 2.6, so $u \in \mathcal{G}_{\beta,\zeta}$. Therefore $\mathcal{G}_{\beta,\zeta}$ is dense. □

Lemma 2.9. *The set*

$$\mathcal{H}_{\theta,\alpha,x} = \{u \in \mathbb{P} : \beta > \theta \text{ for some } \beta \in I_u,\ \alpha \leq \gamma_u \text{ and } s_u(\beta)|_\alpha = x\},$$

is dense for every $\theta < \kappa$, $\alpha < \omega_1$ and $x \in 2^\alpha$.

Proof. Pick $w \in \mathbb{P}$, by Lemmas 1.8 and 2.6 there is $v \leq w$ such that $\gamma_v > \alpha$. Choose $\beta \in \kappa \setminus I_v$ a successor ordinal such that $\beta > \theta$ and choose $y \in 2^{\gamma_v}$ such that $x \subseteq y$. Set $\gamma_u = \gamma_v$, $I_u = I_v \cup \{\beta\}$, $T_u = T_v$ and let $s_u : I_u \to 2^{\gamma_u}$ such that $s_u(\xi) = s_v(\xi)$ for $\xi \in I_v$ and $s_u(\beta) = y$. Since $\beta \notin L$, we have $K_u = K_v$ and as $\gamma_u = \gamma_v$, it is not difficult to check that $u \in \mathbb{P}$ and $u \leq v$. So, $u \in \mathcal{H}_{\theta,\alpha,x}$ which is dense. $\square$

3. The HFD space

In this section we will show that, in the generic extension of $\mathbb{P}$, there exists a nice HFD space as we require.

Theorem 3.1. *Assume CH, $2^{\omega_1} = \kappa$ be regular, and $\alpha^\omega < \kappa$ and $2^\alpha \leq \kappa$ for every cardinal $\alpha < \kappa$. Then in the generic extension of $\mathbb{P}$, there exists an HFD space $X = \{x_\mu : \mu < \kappa\}$ such that:*

(1) *X is dense in $\{0,1\}^{\omega_1}$;*
(2) *for every $\lambda, \zeta < \kappa$, every $Y \subseteq X$ and every sequence in Y^λ there is $\zeta < \eta < \kappa$ such that the sequence has p_α-limit in $(Y \cup \{x_\xi : \xi \in K_\alpha\})^\lambda$ for every $\alpha \geq \eta$; and*
(3) *for each ultrafilter p and each $\theta < \eta < \kappa$ there exists $\beta \in L$ such that $\beta \geq \eta$ and the p-limit of the sequence $\{x_{\beta+n} : n \in \omega\}$ does not exist or is not an element of $\{x_\mu : \mu < \theta\}$.*

Furthermore, in the extension, $\alpha^\omega < \kappa$ and $2^\alpha \leq \kappa$ for every cardinal $\alpha < \kappa$.

Proof. Let G be a $\mathbb{P}$-generic filter, then G intersects all the dense sets that we previously defined: $\mathcal{D}_\beta$ with $\beta < \kappa$, $\mathcal{E}_A$ with $A \in [\kappa]^\omega$, $\mathcal{F}_{\delta,\theta_0,\theta_1}$ with $\{\theta_0, \theta_1\} \in [\kappa]^2$ and $\delta < \omega_1$, $\mathcal{G}_{\beta,\zeta}$ with $\beta, \zeta \in L$, and $\mathcal{H}_{\theta,\alpha,x}$ with $\theta < \kappa$, $\alpha < \omega_1$ and $x \in 2^\alpha$. In the extension, CH holds, $2^{\omega_1} = \kappa$ and the ground model ultrafilters (selective ultrafilters) are still ultrafilters (selective ultrafilters). In the extension, there are no new countable sets, therefore, α^ω is the same cardinal as before smaller than κ, hence it is smaller than κ in the extension, since cardinals are preserved. Since $\mathbb{P}$ is ω_2-cc, the number of nice names for a subset of a cardinal $\lambda < \kappa$ is $|(\kappa^{\omega_1})^\lambda| \leq \kappa$. Therefore, $2^\lambda \leq \kappa$ in the extension.

Denote as x_μ the function $\bigcup_{u \in G, \mu \in I_u} s_u(\mu)$. Observe that every $\beta < \kappa$ is in some I_u with $u \in G$ due to Lemma 2.2. By the dense sets from Lemma 2.7, one can see that $x_\mu \in \{0,1\}^{\omega_1}$ and all the elements in the set $X = \{x_\mu : \mu < \kappa\}$ are pairwise distinct.

To see that X is HFD take a countable infinite subset A of κ. The forcing does not add new countable sets, thus A is in the ground model. By Lemma 2.3, there exists $u \in G$ such that $A \in T_u$. Let U be a basic open set in $\{0,1\}^{[\gamma_u, \omega_1[}$ let $\delta < \omega_1$ be greater than the support of U. By density and genericity of G, there exists $w \in G$ such that $w < u$ and $\gamma_w > \delta$. Observe that $A = B_{A,\{0,1\}^{\gamma_u}} \in \mathcal{T}_u$ and then, by item (iii) from Definition 1.5 we obtain that $\{s_w(\mu)|_{[\gamma_u,\gamma_w[} : \mu \in A\}$ is dense in $\{0,1\}^{[\gamma_u,\gamma_w[}$. So

$$\{\mu \in A : x_\mu|_{[\gamma_u,\omega_1[} \in U\} = \{\mu \in A : s_w(\mu)|_{[\gamma_u,\gamma_w[} \in \pi_{[\gamma_u,\gamma_w[}[U]\}$$

which is infinite. Then X has the HFD property.

By Lemma 2.9 we have that for each $x \in \{0,1\}^\alpha$ for some $\alpha < \omega_1$, there exists β such that $x_\beta \supseteq x$. In particular, it follows that X is dense in $\{0,1\}^{\omega_1}$ and (1) holds.

To show (2) let $\lambda, \zeta < \kappa$, $Y \subseteq X$ and let $\{y_\mu : \mu < \lambda\}$ be a family of sequences in Y. Then there exists $\eta' < \kappa$ such that

$$\{y_\mu(n) : \mu < \lambda \text{ and } n \in \omega\} \subseteq \{x_\xi : \xi < \eta'\}.$$

Consider the first ordinal β_0 such that $K_\beta \cap (\zeta+1) = \emptyset$ for every $\beta \geq \beta_0$ and let $\eta = \max\{\zeta, \eta', \beta_0\}$. Observe that for each $\mu < \lambda$, the family $\{\{n \in \omega : y_\mu(n) = x\} : x \in y_\mu[\omega]\}$ is a partition for ω, thus for every selective ultrafilter p, the sequence y_μ must be p-equivalent to a constant or an injective sequence. In particular, for every $\alpha > \eta$ and every $\mu < \lambda$, y_μ is p_α-equivalent to a constant sequence or an 1 to 1 sequence (because p_α is still selective in the extension). It is clear that in the first case, the sequence y_μ has a p_α-limit in Y. Suppose that y_μ is p_α-equivalent to an 1 to 1 function. Since $y_\mu \subseteq \{x_\xi : \xi < \eta'\}$, we have by $\alpha > max\{\eta', \beta_0\}$ and Lemma 2.2 that there are $\xi \in K_\alpha$ and $w \in G$ such that $\xi \in I_w$ and $\{n \in \omega : y_\mu(n) = x_{g_\xi(n)}\} \in p_\alpha$, where $g_\xi(n) \in I_w$ for every $n \in \omega$ because of (1) from Definition 1.5. Since $\xi > \zeta$ we obtain by the definition of the $x'_\mu s$ and item (2) from Definition 1.5 that x_ξ is the p_α-limit of y_μ.

Finally, for (3) let p be an ultrafilter. If for every $\theta < \kappa$ there exists $\beta \in L$ such that $\beta > \theta$ and the sequence $\{x_{\beta+n} : n \in \omega\}$ has no p-limit in $\{x_\mu : \mu < \kappa\}$, so we are done.

Suppose then that for a given $\theta < \kappa$ the sequence $\{x_{\beta+n} : n \in \omega\}$ has p-limit in $\{x_\mu : \mu < \kappa\}$ for every $\beta \in L$ with $\theta < \beta < \kappa$. In this case we claim that for every $\beta, \zeta \in L$ with $\theta < \beta < \zeta < \kappa$, the sequences $\{x_{\beta+n} : n \in \omega\}$ and $\{x_{\zeta+n} : n \in \omega\}$ have different p-limits. Otherwise suppose that there are $\beta, \zeta \in L$ such that $\theta < \beta < \zeta < \kappa$ and $\{x_{\beta+n} : n \in \omega\}$ and $\{x_{\zeta+n} : n \in \omega\}$ have x_η as p-limit. By Lemma 2.8, it follows that there exists $\alpha < \omega_1$ such that $x_{\beta+n}(\alpha) = 1 - x_{\zeta+n}(\alpha)$ for each $n \in \omega$. Since the sequences have p-limits, it follows that their coordinates have p-limits. Therefore $x_\eta(\alpha) = p$-limit $\{x_{\beta+n}(\alpha) : n \in \omega\} = p$-limit $1 - \{x_{\zeta+n}(\alpha) : n \in \omega\} = 1 - x_\eta(\alpha)$, which is a contradiction. Therefore the set of p-limits of $\{x_{\beta+n} : n \in \omega\}$ for $\beta > \theta$ are pairwise distinct and (3) holds. □

4. The main example

In this last section we will consider an HFD space X as Theorem 3.1 to define our main example. Let begin setting the theorems that will guarantee that such example will be ultrapseudocompact and each of its powers under $2^{\mathfrak{c}}$ is countably compact. First it is necessary to define the $(\kappa, \mathcal{D})$-compactness, a property introduced in [6] by S. García-Ferreira. This property is weaker but close enough to p-compactness (of course stronger than countable compactness) and characterize the spaces whose κ-powers are countably compact for every cardinal κ.

Definition 4.1. [6, Definition 3.1] Let $\emptyset \neq \mathcal{D} \subseteq \omega^*$ and let $1 \leq \kappa$ be a cardinal number. A space X is said to be $(\kappa, \mathcal{D})$-compact if for every set $\{(x^\xi_n)_{n<\omega} : \xi < \gamma\}$ of γ-many sequences, for $\gamma \leq \kappa$, of points of X, there are $p \in \mathcal{D}$ and $x_\xi \in X$, for each $\xi < \gamma$, such that $x_\xi = p - lim\ x^\xi_n$ for each $\xi < \gamma$.

Theorem 4.2. *[6, Theorem 3.2] Let $1 \leq \kappa$ be a cardinal and let X be a space. Then X^κ is countably compact iff there is $\mathcal{D} \subseteq \omega^*$ such that X is $(\kappa, \mathcal{D})$-compact.*

Now we will provide conditions for a space in order to guarantee that every power under $2^{\mathfrak{c}}$ of its hyperspace of closed sets is countably compact.

Theorem 4.3. *Let Z be a normal space such that $hd(Z) \leq \mathfrak{c}$ and suppose that λ is a cardinal for which $\mathfrak{c} \leq \lambda \leq 2^{\mathfrak{c}}$. Then Z^λ is countably compact iff $\mathcal{CL}(Z)^\lambda$ is countably compact.*

Proof. Of course the countable compactness of $\mathcal{CL}(Z)^\lambda$ implies that Z^λ has such property. Suppose that Z^λ is countably compact. By Theorem 4.2, it is enough to prove that for every family of at most λ sequences of $\mathcal{CL}(Z)$, there is a free ultrafilter $p \in \omega^*$ such that every sequence in that family has p-limit. So let $\{(B^\xi_n)_{n\in\omega} : \xi < \lambda\}$ be a family of sequences of $\mathcal{CL}(Z)$. For every $\xi < \lambda$ and $n \in \omega$ fix D^ξ_n a dense subset of B^ξ_n in Z with cardinality less or equal than the continuum. Consider in Z the family $\mathcal{A}_\xi$ of all sequences in $\{(x^\xi_n)_{n\in\omega} : x^\xi_n \in D^\xi_n\}$ for each $\xi < \lambda$. Let $\mathcal{A} = \bigcup_{\xi<\lambda} \mathcal{A}_\xi$ and note that $|\mathcal{A}| \leq \lambda$ because $|\mathcal{A}_\xi| \leq \mathfrak{c}$ for every $\xi < \lambda$. Then by Theorem 4.2, there is $p \in \omega^*$ such that every sequence in $\mathcal{A}$ has p-limit. Let $E_\xi = \{p - lim\ x^\xi_n : (x^\xi_n)_{n\in\omega} \in \mathcal{A}_\xi\}$. We claim that $cl(E_\xi) = p - lim\ B^\xi_n$ in $\mathcal{CL}(Z)$ for every $\xi < \lambda$. In fact, fix

$\xi < \lambda$ and let $U_1, \ldots, U_k$ be open sets of Z such that $cl(E_\xi) \in \mathcal{U} = \langle U_0, \ldots, U_m \rangle$. Since Z is normal, there is an open set W such that

$$cl(E_\xi) \subseteq W \subseteq cl(W) \subseteq \bigcup_{i \leq m} U_i.$$

Since $E_\xi \subseteq W$, it is standard to show that $\{n \in \omega : D_n^\xi \subseteq W\} \in p$, so $\{n \in \omega : B_n^\xi \subseteq cl(W)\} \in p$. Then $\{n \in \omega : B_n^\xi \subseteq \bigcup_{i \leq m} U_i\} \in p$. Finally for each $k \leq m$ there is $x_k \in E_\xi \cap U_k$. As x_k is the p-limit of a sequence in $\mathcal{A}_\xi$, we have that $\{n \in \omega : D_n^\xi \cap U_k \neq \emptyset\} \in p$ for each $k \leq m$. Since D_n^ξ is dense in B_n^ξ for each $n \in \omega$,

$$\{n \in \omega : D_n^\xi \cap U_k \neq \emptyset\} = \{n \in \omega : B_n^\xi \cap U_k \neq \emptyset\}$$

and we obtain that $\{n \in \omega : B_n^\xi \cap U_k \neq \emptyset\} \in p$ for each $k \leq m$. Thus

$$\{n \in \omega : B_n^\xi \in \mathcal{U}\} =$$
$$\{n \in \omega : B_n^\xi \subseteq \bigcup_{i \leq m} U_i\} \cap \Big(\bigcap_{k \leq m} \{n \in \omega : B_n^\xi \cap U_k \neq \emptyset\} \Big) \in p.$$

Therefore $cl(E_\xi) = p - lim\ B_n^\xi$ and this ends the proof. □

We are ready to define our main example:

Example 4.4. Assume all the conditions from Theorem 3.1 and let X be an HFD space which satisfies $(1) - (3)$ from the same theorem. Then there exists a collectionwise normal, ultrapseudocompact space $Z \subseteq X$ such that $hd(Z) = \omega$, Z^λ is countably compact for every cardinal $\lambda < \kappa$ but Z is not p-compact for any $p \in \omega^*$.

Proof. Consider all definitions and notations from the previous sections. Fix a countable dense subset D of X, which is possible because every HFD space is hereditarily separable. Since X is dense in $\{0,1\}^{\omega_1}$, it follows that D is dense in $\{0,1\}^{\omega_1}$. Enumerate the family

$$\{S : S \text{ is a sequence of } X^\lambda \text{ for some } \lambda < \kappa\}$$

as $\{S_\xi : \xi < \kappa\}$ in such a way that $\{S_\xi(n) : n \in \omega\} \subseteq \{x_\mu : \mu < \xi\}^{\lambda_\xi}$ for every $\xi < \kappa$, where λ_ξ denote the cardinal under κ for which S_ξ is a sequence in X^{λ_ξ}. This enumeration is possible since $2^\lambda \leq \kappa$ for each $\lambda < \kappa$. For each $\xi < \kappa$, set $\{y_\xi(\beta, n) : \beta < \lambda_\xi \text{ and } n \in \omega\}$ the family of sequences in X so that $y_\xi(\beta, n) = S_\xi(n)(\beta)$ for every $\beta < \lambda_\xi$ and $n \in \omega$. Also enumerate all free ultrafilters as $\{r_\delta : 0 < \delta < \kappa\}$.

By transfinite recursion we will define, for each $\eta < \kappa$, B_η and C_η disjoint subsets of κ of cardinality smaller than κ such that:

(1) $B_0 = \rho_0$ and $C_0 = \emptyset$, where ρ_0 is the first ordinal such that $D \subseteq \{x_\mu : \mu < \rho_0\}$,
(2) $\bigcup_{\eta < \alpha} B_\eta \subseteq B_\alpha$ and $\bigcup_{\eta < \alpha} C_\eta \subseteq C_\alpha$ for every $\alpha < \kappa$,
(3) for every $\alpha < \kappa$, either $\{x_\rho : \rho \in \bigcup_{\eta < \alpha} B_\eta\}^\lambda$ is countably compact for every $\lambda < \kappa$ or $S_{\xi(\alpha)}$ has an accumulation point in $\{x_\rho : \rho \in B_\alpha\}^{\lambda_{\xi(\alpha)}}$ where $\xi(\alpha)$ is the first ordinal for which $S_{\xi(\alpha)}$ is contained in $\{x_\rho : \rho \in \bigcup_{\eta < \alpha} B_\eta\}^{\lambda_{\xi(\alpha)}}$ and $S_{\xi(\alpha)}$ does not have an accumulation point in $\{x_\rho : \rho \in \bigcup_{\eta < \alpha} B_\eta\}^{\lambda_{\xi(\alpha)}}$, and
(4) for every $0 < \alpha < \kappa$ there is a sequence $\sigma : \omega \to B_\alpha$ such that either the r_α-lim of $(x_{\sigma(n)})_{n \in \omega}$ does not exist in X, or the index of the r_α-lim of $(x_{\sigma(n)})_{n \in \omega}$ belongs to C_α (i.e. $\zeta \in C_\alpha$ where $x_\zeta = r_\alpha - lim\ x_{\sigma(n)}$).

Let B_0 and C_0 be defined by (1).

Suppose that B_η and C_η are defined satisfying (1) – (4) for each $\eta < \alpha$. Let $B'_\alpha = \bigcup_{\eta<\alpha} B_\eta$ and $C'_\alpha = \bigcup_{\eta<\alpha} C_\eta$. If $\{x_\rho : \rho \in \bigcup_{\eta<\alpha} B'_\alpha\}^\lambda$ is countably compact for every $\lambda < \kappa$, then let $B^*_\alpha = B'_\alpha$, otherwise let $\xi(\alpha)$ the first ordinal such that $S_{\xi(\alpha)}$ is a sequence in $\{x_\rho : \rho \in \bigcup_{\eta<\alpha} B'_\alpha\}^{\lambda_{\xi(\alpha)}}$ and $S_{\xi(\alpha)}$ does not have an accumulation point in $\{x_\rho : \rho \in \bigcup_{\eta<\alpha} B'_\alpha\}^{\lambda_{\xi(\alpha)}}$. By item (2) from Theorem 3.1, there is $\theta \in L$ such that $\min(\{\theta\} \cup K_\theta) > \sup(B'_\alpha \cup C'_\alpha)$ and $S_{\xi(\alpha)}$ has p_θ-lim in $(\{x_\xi : \xi \in B'_\alpha \cup K_\theta\})^{\lambda_{\xi(\alpha)}}$. Set $B^*_\alpha = B'_\alpha \cup K_\theta$ in this case. Now by item (3) from Theorem 3.1, there exist $\zeta \in L \setminus \big(\sup(B^*_\alpha \cup C'_\alpha) + 1\big)$ such that, the r_α-limit of $\{x_{\zeta+n} : n \in \omega\}$ either does not exist in X or is not an element of $\{x_\mu : \mu < \sup(B^*_\alpha \cup C'_\alpha)\}$. Let $B_\alpha = B^*_\alpha \cup [\zeta, \zeta + \omega[$ and $C_\alpha = C'_\alpha$ in the case that $\{x_{\zeta+n} : n \in \omega\}$ does not have r_α-limit in X and otherwise let $x_\chi = r_\alpha - lim\ x_{\zeta+n}$, $B_\alpha = B^*_\alpha \cup ([\zeta, \zeta + \omega[\setminus\{\chi\})$ and $C_\alpha = C'_\alpha \cup \{\chi\}$. It is clear that B_α and C_α satisfy all the requirements.

We claim that $Z = \{x_\xi : \xi \in \bigcup_{\alpha<\kappa} B_\alpha\}$ is a space as required. Note that Z is collectionwise normal and hereditarily separable because it is a subspace of the HFD space X. Now suppose by contradiction that not every power under κ of Z is countably compact and let $\xi < \kappa$ be the first ordinal such that S_ξ does not have an accumulation point in Z^{λ_ξ} and let $\alpha < \kappa$ be the first ordinal such that $S_\xi(\beta)(n) \in \{x_\rho : \rho \in \bigcup_{\eta<\alpha} B_\eta\}$ for each $n \in \omega$ and $\beta < \lambda_\xi$. Then by item (3) we know that S_ξ has an accumulation point in $\{x_\rho : \rho \in B_\alpha\}^{\lambda_\xi} \subseteq Z^{\lambda_\xi}$ a contradiction. So Z^λ is countably compact for every $\lambda < \kappa$.

In particular Z is pseudocompact and dense in $\{0,1\}^{\omega_1}$, hence by Lemma 0.4, Z is ultrapseudocompact. Finally by item (4) we know that for every $p \in \omega^*$ there is a sequence in Z which does not have p-limit in Z. □

Some natural questions that are related to our example are:

Question 4.5. Is there consistently a topological group or a homogeneous space X such that $\mathcal{CL}(X)$ is countably compact, but X is not p-compact for any ultrafilter p?

Question 4.6. Is there consistently a space X such $\mathcal{CL}(X)$ is countably compact, but $X^{\mathfrak{c}}$ is not countably compact?

Question 4.7. Is there some model where X is p-compact for some $p \in \omega^*$ provided $\mathcal{CL}(X)$ is countably compact?

Two more questions related with p-compactness and p-pseudocompactness.

Question 4.8. Is there some model where every p-pseudocompact normal space is p-compact?

Question 4.9. Let $f : X \to Y$ be a perfect onto function where Y is p-pseudocompact. Is X p-pseudocompact?

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