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**BESOV-ISH SPACES THROUGH ATOMIC DECOMPOSITION**



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We use the method of atomic decomposition to build new families of function spaces, similar to Besov spaces, in measure spaces with grids, a very mild assumption. Besov spaces with low regularity are considered in measure spaces with good grids, and we obtain results on multipliers and left compositions in this setting.

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## 1. Introduction

Besov spaces  $B_{p,q}^s(\mathbb{R}^n)$  were introduced in [Besov 1959]. This scale of spaces has been a favourite over the years, with thousands of references available. Perhaps two of its most interesting features are that many earlier classes of function spaces appear in this scale, as Sobolev spaces, and also that there are many equivalent ways to define  $B_{p,q}^s(\mathbb{R}^n)$ , so that you can pick the most suitable one for your purpose. The reader may see [Stein 1970; Peetre 1976; Triebel 1983] for an introduction of Besov spaces on  $\mathbb{R}^n$ . For a historical account on Besov spaces and related topics see [Triebel 1983] and the shorter but useful [Jaffard 2000; Yuan, Sickel and Yang 2010; Besov and Kalyabin 2003].

In the last decades there was a huge amount of activity on the generalisation of harmonic analysis (see [Deng and Han 2009]), and Besov spaces in particular, to less regular *phase spaces*, replacing  $\mathbb{R}^n$  by something with a poorer structure. It turns out that for *small*  $s > 0$  and  $p, q \geq 1$ , a proper definition demands strikingly weak assumptions. There is a large body of literature that provides a definition and properties of Besov spaces on *homogeneous spaces*, as defined in [Coifman and Weiss 1971]. Those are quasimetric spaces with a doubling measure, which includes in particular Ahlfors regular metric spaces. We refer to the pioneering works [Han and Sawyer 1994; Han, Lu and Yang 1999] for Besov spaces on homogeneous spaces and more recently [Alvarado and Mitrea 2015; Koskela, Yang and Zhou 2011] (in this case for metric measure spaces) and [Triebel 1997; 2005] for Besov spaces on fractals. There is a long list of examples of homogeneous spaces in [Coifman and Weiss 1977]. The  $d$ -sets as defined in [Triebel 1997] are also examples of homogeneous spaces.

*We give a very elementary (and yet practical) construction of Besov spaces  $B_{p,q}^s$ , with  $p, q \geq 1$  and  $0 < s < 1/p$ , for measure spaces endowed with a grid, that is, a sequence of finite partitions of measurable sets satisfying certain mild properties.*

This construction is close to the martingale Besov spaces as defined in [Gu and Taibleson 1992]; however, we deal with “nonisotropic splitting” in our grid without applying the Gu–Taibleson recalibration procedure to our grid, which simplifies the definition and allows a broader class of examples.

The primary tool in this work is the concept of *atomic decomposition*. An atomic decomposition represents each function in a space of functions as an (infinite) linear combination of fractions of the so-called *atoms*. The advantage of atomic decompositions is that the atoms are functions that are often far more regular than a typical element of the space. However, a distinctive feature (compared with Fourier series with either a Hilbert basis or unconditional basis) is that such atomic decomposition is in general *not* unique. Nevertheless, in a successful atomic decomposition of a normed space of functions, we can attribute a “cost” to each possible representation, and the norm of the space is equivalent to the minimal cost (or infimum) among all representations. A function represented by a single atom has norm at most 1, so the term “atom” seems to be appropriate.

Coifman [1974] introduced the atomic decomposition of the real Hardy space  $H^p(\mathbb{R})$  and Latter [1978] found an atomic decomposition for  $H^p(\mathbb{R}^n)$ . The influential work [Frazier and Jawerth 1985] gave us an atomic decomposition for the Besov spaces  $B_{p,q}^s(\mathbb{R}^n)$ . In the context of homogeneous spaces, we have results of [Han, Lu and Yang 1999] on the atomic decomposition of Besov spaces by Hölder atoms.

Closer to the spirit of this work we have the atomic decomposition of Besov space  $B_{1,1}^s([0, 1])$ , with  $s \in (0, 1)$ , of [de Souza 1985a] using special atoms, which we call *Souza's atoms* (see also [de Souza 1985b; de Souza, O'Neil and Sampson 1986]). A Souza's atom  $a_J$  on an interval  $J$  is quite simple, consisting of a function whose support is  $J$  and  $a_J$  is *constant* on  $J$ .

We also refer to the results on the B-spline atomic decomposition of the Besov space of the unit cube of  $\mathbb{R}^n$  in [DeVore and Popov 1988] (with [Ciesielski 1975] as a precursor), which in the case  $0 < s < 1/p$  reduces to an atomic decomposition by Souza's atoms, and the work [Oswald 1990; 1994] on finite element approximation in bounded polyhedral domains on  $\mathbb{R}^n$ .

On the study of Besov spaces on  $\mathbb{R}^n$  and smooth manifolds, Souza's atoms may seem to have setbacks that restrict its usefulness. They are not smooth, so it is fair to doubt the effectiveness of atomic decomposition by Souza's atoms to obtain a better understanding of a partial differential equation or to represent data faithfully/without artifacts, a constant concern in applications of smooth wavelets (see [Jaffard, Meyer and Ryan 2001]).

On the other hand, in the study of ergodic properties of piecewise smooth dynamical systems, *transfer operators* play a huge role. Those operators act on Lebesgue spaces  $L^1(m)$ , but they are more useful if one can show they have a (good) action on more regular function spaces. Unfortunately, in many cases the transfer operator does *not* preserve smoothness or even continuity of a function. Discontinuities are a fact of life in this setting, and they do not go away as in certain dissipative PDEs, since the positive 1-eigenvectors of this operator, of utmost importance in its study, often have discontinuities themselves. The works [Lasota and Yorke 1973; Hofbauer and Keller 1982] are landmark results in this direction. See also [Baladi 2000; Broise 1996] for more details. Atomic decomposition with Souza's atoms, being the simplest possible kind of atom with discontinuities, might come in handy in these cases. That was a major motivation for this work.

Besides this fact, in an *abstract* homogeneous space higher-order smoothness does not seem to be a very useful concept since we can define  $B_{p,q}^s$  just for small values of  $s$ , so atomic decompositions by Souza's atoms sound far more attractive.

Indeed, the simplicity of Souza's atoms allows us to throw away the necessity of a metric/topological structure on the phase space. We define *Besov spaces* on every measure space with a nonatomic finite measure, provided we endow it with a *good grid*. A *good grid* is just a sequence of finite partitions of measurable sets satisfying certain mild properties. We give the definition of Besov space on measure space with a (good) grid in Part II.

In the literature we usually see a known space of functions described using atomic decomposition. This typically comes after a careful study of such space, and it is often a challenging task. Rarer is the path we follow here. We *start* by *defining* the Besov spaces *through* atomic decomposition by Souza's atoms. This construction of the spaces and the study of its basic properties, as its completeness and compact inclusion in Lebesgue spaces, uses fairly general and simple arguments, and it does not depend on the particular nature of the atoms used, except for very mild conditions on its regularity. In Part I we describe this construction in full generality.

*We construct Besov-ish spaces. They are far more general than Besov spaces. In particular, the nature of the atoms may change with its location and scale in the phase space and the grid itself can be very irregular.*

The main result in Part I is Proposition 8.1. Due to the generality of the setting, its statement is hopelessly technical in nature; however, this very powerful result has a simple meaning. Suppose we have an atomic decomposition of a function. If we replace each of those atoms by a combination of atoms (possibly of a different class) in such way that we do not change the location of the support and also the “mass” of the representation is concentrated pretty much on the same original scale, then we obtain a new atomic decomposition, and the cost of this atomic decomposition can be estimated by the cost of the original atomic decomposition. We will use this result many times throughout this work. This is obviously connected with *almost diagonal operators* as in [Frazier and Jawerth 1985; 1990].

In Part II we also offer a detailed analysis of the Besov spaces defined there. Since we define it using combinations of Souza’s atoms, it is not clear a priori how rich are those spaces. So:

*We give a bunch of alternative characterisations of these Besov spaces. We show that using more flexible classes of atoms (piecewise Hölder atoms,  $p$ -bounded variation atoms and even Besov atoms with higher regularity), we obtain the same Besov space. This often allows us to easily verify if a given function belongs to  $B_{p,q}^s$ . We also have a mean oscillation characterisation in the spirit of [Dorronsoro 1985, Gu and Taibleson 1992], and we also obtained another one using Haar wavelets.*

Haar wavelets were introduced in [Haar 1910] in the real line. The elegant construction of unbalanced Haar wavelets in general measure spaces with a grid of [Girardi and Sweldens 1997] will play an essential role here. If in general homogeneous spaces, the Calderón reproducing formula appears to be the bit of harmonic analysis that survives in it and it allows to carry out the theory; in finite measure spaces with a good grid (and in particular *compact* homogenous spaces) full-blown Haar systems are alive and well. Recently a Haar system was used in [Kairema, Li, Pereyra and Ward 2016] to study dyadic versions of the Hardy and BMO spaces in homogeneous spaces.

We also provide a few applications in Part III. In particular, we study the boundedness of pointwise multipliers acting in the Besov space. Since it is effortless to multiply Souza’s atoms, the proofs of these results are concise and easy to understand, generalising some of the results for Besov spaces in  $\mathbb{R}^n$  of [Triebel 1977; Sickel 1999]. We also study the boundedness of left composition in Besov spaces of measure spaces with a grid, similar to some results for  $B_{p,q}^s(\mathbb{R}^n)$  in [Bourdaud and Kateb 1991] (see also [Bourdaud and Kateb 1990, 1995]).

It may come as a surprise to the reader that Besov spaces on compact homogeneous spaces as defined in [Han, Lu and Yang 1999] (and in particular Gu–Taibleson recalibrated martingale Besov spaces [1992]) are indeed *particular cases* of Besov spaces defined here, provided  $0 < s < 1/p$  and  $s$  is small. We show this in a forthcoming work [Smânia 2021].

## 2. Notation

We will use  $C_1, C_2, \dots$  for positive constants and  $\lambda_1, \lambda_2, \dots$  for positive constants smaller than 1. See Table 1 for other important notation.

symbol	description
$I$	phase space
$m$	finite measure in the phase space $I$
$a_P, b_P$	an atom supported on $P$
$\mathcal{A}$	a class of atoms
$\mathcal{A}(Q)$	set of atoms of class $\mathcal{A}$ supported on $Q$
$\mathcal{A}_{s,p}^{sz}$	class of $(s, p)$ -Souza's atoms
$\mathcal{A}_{s,\beta,p}^h$	class of $(s, \beta, p)$ -Hölder atoms
$\mathcal{A}_{s,\beta,p}^{bv}$	class of $(s, \beta, p)$ -bounded variation atoms
$\mathcal{A}_{s,\beta,p,q}^{bs}$	class of $(s, \beta, p, q)$ -Besov's atoms
$\mathcal{P}$	grid of $I$
$\mathcal{P}^k$	family subsets of $I$ at the $k$ -th level of $\mathcal{P}$
$\lambda_1 \leq \lambda_2$	describes geometry of the grid $\mathcal{P}$
$\mathcal{B}_{p,q}^s(\mathcal{A})$	$(s, p, q)$ -Banach space defined by the class of atoms $\mathcal{A}$
$\mathcal{B}_{p,q}^s$ or $\mathcal{B}_{p,q}^s(\mathcal{A}_{s,p}^{sz})$	$(s, p, q)$ -Banach space defined by Souza's atoms
$P, Q, W$	elements of the grid $\mathcal{P}$
$L^p$ or $L^p(m)$	Lebesgue spaces of $(I, \mathbb{A}, m)$
$ \cdot _p$	norm in $L^p$ , $p \in (0, \infty]$
$p'$	$1/p + 1/p' = 1$ , with $p \in [1, \infty]$
$\rho$	$\min\{1, p, q\}$
$\hat{t}$	$\max\{t, 1\}$

**Table 1.** Most important notation/symbols/constants.

### Part I. Divide and rule

In Part I, we are going to assume  $s > 0$ ,  $p \in (0, \infty)$  and  $q \in (0, \infty]$ .

#### 3. Measure spaces and grids

Let  $I$  be a measure space with a  $\sigma$ -algebra  $\mathbb{A}$  and  $m$  be a measure on  $(I, \mathbb{A})$ ,  $m(I) < \infty$ . Given a measurable set  $J \subset I$  let  $|J| = m(J)$ . We denote the Lebesgue spaces of  $(I, \mathbb{A}, m)$  by  $L^p$ . A *grid* is a sequence of finite families of measurable sets with positive measure  $\mathcal{P} = (\mathcal{P}^k)_{k \in \mathbb{N}}$  so that at least one of these families is nonempty and:

( $G_1$ ) Given  $Q \in \mathcal{P}^k$ , let

$$\Omega_Q^k = \{P \in \mathcal{P}^k : P \cap Q \neq \emptyset\}.$$

Then

$$C_1 = \sup_k \sup_{Q \in \mathcal{P}^k} \#\Omega_Q^k < \infty.$$

Define  $|\mathcal{P}^k| = \sup\{|\mathcal{Q}| : \mathcal{Q} \in \mathcal{P}^k\}$ . To simplify the notation we also assume that  $P \neq Q$  for every  $P \in \mathcal{P}^i$  and  $Q \in \mathcal{P}^j$  satisfying  $i \neq j$ . We often abuse notation using  $\mathcal{P}$  for both  $(\mathcal{P}^k)_{k \in \mathbb{N}}$  and  $\bigcup_k \mathcal{P}^k$ .

**Remark 3.1.** There are plenty of examples of spaces with grids. Perhaps the simplest one is obtained considering  $[0, 1)$  with the Lebesgue measure and the *dyadic grid*  $\mathcal{D} = (\mathcal{D}^k)_k$  given by

$$\mathcal{D}^k = \{[i/2^k, (i+1)/2^k) : 0 \leq i < 2^k\},$$

We can also consider the dyadic grid  $\mathcal{D}_n = (\mathcal{D}_n^k)_k$  of  $[0, 1)^n$ , endowed with the Lebesgue measure, given by

$$\mathcal{D}_n^k = \{J_1 \times \cdots \times J_n : \text{with } J_i \in \mathcal{D}^k\}.$$

and also the corresponding  $d$ -adic grids replacing  $2^k$  by  $d^k$  in the above definitions. The above grids are somehow special since they are nested sequences of partitions of the phase space  $I$  and all elements on the same level have exactly the same measure.

**Remark 3.2.** Indeed, any measure space with a finite nonatomic measure can be endowed with a grid made of a nested sequence of partitions and such that all elements on the same level have precisely the same measure since any such measure space is measure-theoretically the same as a finite interval with the Lebesgue measure.

**Remark 3.3.** If we consider a (quasi)-metric space  $I$  with a finite measure  $m$ , we would like to construct “nice” grids on  $(I, m)$ . It turns out that if  $(I, m)$  is a homogeneous space one can construct a nested sequence of partitions of the phase space  $I$  and all elements on the same level are open subsets and have “essentially” the same measure. This is an easy consequence of a remarkable result of [Christ 1990]. See [Smânia 2021].

**Remark 3.4.** One can construct grids for smooth compact manifolds and bounded polyhedral domains in  $\mathbb{R}^n$  using successive subdivisions of an initial triangulation of the domain (see for instance [Oswald 1990, 1994]).

#### 4. A bag of tricks

Following closely the notation of [Triebel 1997], consider the set  $\ell_q(\ell_p^{\mathcal{P}})$  of all indexed sequences

$$x = (x_P)_{P \in \mathcal{P}},$$

with  $x_P \in \mathbb{C}$ , satisfying

$$\|x\|_{\ell_q(\ell_p^{\mathcal{P}})} = \left( \sum_k \left( \sum_{P \in \mathcal{P}^k} |x_P|^p \right)^{q/p} \right)^{1/q} < \infty,$$

with the usual modification when  $q = \infty$ . Then  $(\ell_q(\ell_p^{\mathcal{P}}), |\cdot|_{\ell_q(\ell_p^{\mathcal{P}})})$  is a complex  $\rho$ -Banach space with  $\rho = \min\{1, p, q\}$ ; that is,  $d(x, y) = |x - y|_{\ell_q(\ell_p^{\mathcal{P}})}^\rho$  is a complete metric in  $\ell_q(\ell_p^{\mathcal{P}})$ .

The following is a pair of arguments we will use across this paper to estimate norms in  $\ell_p$  and  $\ell_q(\ell_p^{\mathcal{P}})$ . Those are very elementary, and we do not claim any originality. We collect them here to simplify the exposition. The reader can skip this for the cases  $p, q \geq 1$ , when the results below reduce to the familiar Hölder and Young inequalities. Their proofs were mostly based on [Frazier and Jawerth 1985, proof of Theorem 3.1]. Recall that for  $t \in (0, \infty]$  we defined  $\hat{t} = \max\{1, t\}$ .

**Proposition 4.1** (Hölder-like trick). *Let  $t \in (0, \infty)$  and  $q \in (0, \infty]$ . Let  $a = (a_k)_k$ ,  $b = (b_k)_k$ ,  $c = (c_k)_k$  be nonnegative sequences such that for every  $k$*

$$a_k^{1/\hat{t}} \leq C^{1/\hat{t}} b_k^{1/\hat{t}} c_k^{1/\hat{t}}.$$

Then if  $q < \infty$ , we have

$$\left( \sum_k a_k^{1/\hat{t}} \right)^{\hat{t}/t} \leq C^{1/t} C_2(t, q, b) \left( \sum_k c_k^{q/t} \right)^{1/q},$$

and if  $q = \infty$ , we have

$$\left( \sum_k a_k^{1/\hat{t}} \right)^{\hat{t}/t} \leq C^{1/t} C_2(t, q, b) \sup_k c_k^{1/t},$$

where:

- (A) If  $t \geq 1$  and  $q \geq 1$  then  $C_2(t, q, b) = \left( \sum_k b_k^{q'/t} \right)^{1/q'}$  if  $q > 1$ , and  $C_2(t, 1, b) = \sup_k b_k^{1/t}$  if  $q = 1$ .
- (B) If  $t \geq 1$  and  $q \leq 1$  then  $C_2(t, q, b) = \sup_k b_k^{1/t}$ .
- (C) If  $t < 1$  and  $q/t \geq 1$  then  $C_2(t, q, b) = \left( \sum_k b_k^{(q/t)'} \right)^{1/(t(q/t)' )}$  if  $q < t$ , and  $C_2(t, q, b) = \sup_k b_k^{1/t}$  if  $q = t$ .
- (D) If  $t < 1$  and  $q/t < 1$  then  $C_2(t, q, b) = \sup_k b_k^{1/t}$ .

*Proof.* Case A: If  $t \geq 1$  and  $q \geq 1$ , by the Hölder inequality for the pair  $(q, q')$

$$\sum_k a_k^{1/\hat{t}} = \sum_k a_k^{1/t} \leq C^{1/t} \left( \sum_k b_k^{q'/t} \right)^{1/q'} \left( \sum_k c_k^{q/t} \right)^{1/q}.$$

Case B: If  $t \geq 1$  and  $q \leq 1$  then the triangular inequality for  $|\cdot|^q$  implies

$$\left( \sum_k a_k^{1/\hat{t}} \right)^q = \left( \sum_k a_k^{1/t} \right)^q \leq C^{q/t} \left( \sum_k b_k^{1/t} c_k^{1/t} \right)^q \leq C^{q/t} \sum_k (b_k^{1/t} c_k^{1/t})^q \leq C^{q/t} \left( \sup_k b_k^{q/t} \right) \left( \sum_k c_k^{q/t} \right).$$

Case C: If  $t < 1$  and  $q/t \geq 1$  then  $\hat{t}/t = 1/t$  and by the Hölder inequality for the pair  $(q/t, (q/t)')$

$$\sum_k a_k^{1/\hat{t}} \leq C \left( \sum_k b_k^{(q/t)'} \right)^{1/(q/t)'} \left( \sum_k c_k^{q/t} \right)^{t/q}.$$

Case D: If  $t < 1$  and  $q/t < 1$  then using the triangular inequality for  $|\cdot|^{q/t}$

$$\left( \sum_k a_k^{1/\hat{t}} \right)^{q/t} \leq C^{q/t} \left( \sum_k b_k c_k \right)^{q/t} \leq C^{q/t} \sum_k (b_k c_k)^{q/t} \leq C^{q/t} \left( \sup_k b_k^{q/t} \right) \left( \sum_k c_k^{q/t} \right). \quad \square$$

**Proposition 4.2** (convolution trick). *Let  $p, q \in (0, \infty)$ . Let  $a = (a_k)_{k \in \mathbb{Z}}$ ,  $b = (b_k)_{k \in \mathbb{Z}}$ ,  $c = (c_k)_{k \in \mathbb{Z}} \geq 0$  be such that for every  $k$*

$$a_k^{1/\hat{p}} \leq C^{1/\hat{p}} \sum_{i \in \mathbb{Z}} b_{k-i}^{1/\hat{p}} c_i^{1/\hat{p}}.$$

Then

$$\left( \sum_k a_k^{q/p} \right)^{1/q} \leq C^{1/p} C_3(p, q, b) \left( \sum_k c_k^{q/p} \right)^{1/q},$$

where  $C_3 \geq 1$  satisfies:

- (A) If  $p \geq 1$  and  $q \geq 1$  then  $C_3(p, q, b) = \sum_{k \in \mathbb{Z}} b_k^{1/p}$ .  
 (B) If  $p \geq 1$  and  $q \leq 1$  then  $C_3(p, q, b) = \left( \sum_{k \in \mathbb{Z}} b_k^{q/p} \right)^{1/q}$ .  
 (C) If  $p < 1$  and  $q/p \geq 1$  then  $C_3(p, q, b) = \left( \sum_{k \in \mathbb{Z}} b_k \right)^{1/p}$ .  
 (D) If  $p < 1$  and  $q/p < 1$  then  $C_3(p, q, b) = \left( \sum_{k \in \mathbb{Z}} b_k^{q/p} \right)^{1/q}$ .

*Proof.* Case A: If  $p \geq 1$  and  $q \geq 1$ , by the Young inequality for the pair  $(1, q)$

$$\left( \sum_k a_k^{q/p} \right)^{1/q} \leq C^{1/p} \left( \sum_k b_k^{1/p} \right) \left( \sum_k c_k^{q/p} \right)^{1/q}.$$

Case B: If  $p \geq 1$  and  $q \leq 1$  then the triangular inequality for  $|\cdot|^q$  and the Young inequality for the pair  $(1, 1)$  imply

$$\sum_k a_k^{q/p} = C^{q/p} \sum_k \left( \sum_{i \in \mathbb{Z}} b_{k-i}^{1/p} c_i^{1/p} \right)^q \leq C^{q/p} \sum_k \sum_{i \in \mathbb{Z}} b_{k-i}^{q/p} c_i^{q/p} \leq C^{q/p} \left( \sum_k b_k^{q/p} \right) \left( \sum_k c_k^{q/p} \right).$$

Case C: If  $p < 1$  and  $q/p \geq 1$  then by the Young inequality for the pair  $(1, q/p)$

$$\left( \sum_k a_k^{q/p} \right)^{p/q} \leq C \left( \sum_k b_k \right) \left( \sum_k c_k^{q/p} \right)^{p/q}.$$

Case D: If  $p < 1$  and  $q/p < 1$  then using the triangular inequality for  $|\cdot|^{q/p}$  and the Young inequality for the pair  $(1, 1)$

$$\sum_k a_k^{q/p} \leq C^{q/p} \sum_k \left( \sum_{i \in \mathbb{Z}} b_{k-i} c_i \right)^{q/p} \leq C^{q/p} \sum_k \sum_{i \in \mathbb{Z}} b_{k-i}^{q/p} c_i^{q/p} \leq C^{q/p} \left( \sum_k b_k^{q/p} \right) \left( \sum_k c_k^{q/p} \right). \quad \square$$

## 5. Atoms

Let  $\mathcal{P}$  be a grid. Let  $p \in [1, \infty)$ ,  $u \in [1, \infty]$ , and  $s > 0$ . A family of *atoms* associated with  $\mathcal{P}$  of type  $(s, p, u)$  is an indexed family  $\mathcal{A}$  of pairs  $(\mathcal{B}(Q), \mathcal{A}(Q))_{Q \in \bigcup_k \mathcal{P}^k}$ , where:

- (A<sub>1</sub>)  $\mathcal{B}(Q)$  is a complex Banach space contained in  $L^{pu}$ .  
 (A<sub>2</sub>) If  $\phi \in \mathcal{B}(Q)$  then  $\phi(x) = 0$  for every  $x \notin Q$ .  
 (A<sub>3</sub>)  $\mathcal{A}(Q)$  is a convex subset of  $\mathcal{B}(Q)$  such that  $\phi \in \mathcal{A}(Q)$  if and only if  $\sigma \phi \in \mathcal{A}(Q)$  for every  $\sigma \in \mathbb{C}$  satisfying  $|\sigma| = 1$ .  
 (A<sub>4</sub>) We have

$$|\phi|_{pu} \leq |Q|^{s-1/(u'p)}$$

for every  $\phi \in \mathcal{A}(Q)$ .

We will say that  $\phi \in \mathcal{A}(Q)$  is an  $\mathcal{A}$ -atom of type  $(s, p, u)$  supported on  $Q$  and that  $\mathcal{B}(Q)$  is the local Banach space on  $Q$ . Sometimes we also assume one of the following:

- (A<sub>5</sub>) For every  $Q \in \mathcal{P}$  we have that  $\mathcal{A}(Q)$  is a compact subset in the strong topology of  $L^p$ .  
 (A<sub>6</sub>) We have  $p \in [1, \infty)$  and for every  $Q \in \mathcal{P}$  the set  $\mathcal{A}(Q)$  is a sequentially compact subset in the *weak* topology of  $L^p$ .

(A<sub>7</sub>) For every  $Q \in \mathcal{P}$  we have that  $\mathcal{B}(Q)$  is finite-dimensional and  $\mathcal{A}(Q)$  contains a neighbourhood of 0 in  $\mathcal{B}(Q)$ .

We provide examples of classes of atoms in Section 11.

### 6. Besov-ish spaces

Let  $p \in (0, \infty)$ ,  $u \in [1, \infty]$ ,  $q \in (0, \infty]$ ,  $s > 0$ ,  $\mathcal{P} = (\mathcal{P}^k)_{k \geq 0}$  be a grid and let  $\mathcal{A}$  be a family of atoms of type  $(s, p, u)$ . We will also assume that:

(G<sub>2</sub>) We have

$$C_4 = C_2(p, q, (|\mathcal{P}^k|^s)_k) < \infty$$

and

$$\lim_k |\mathcal{P}^k| = 0.$$

The *Besov-ish space*  $\mathcal{B}_{p,q}^s(I, \mathcal{P}, \mathcal{A})$  is the set of all complex-valued functions  $g \in L^p$  that can be represented by an absolutely convergent series on  $L^p$

$$g = \sum_{k=0}^{\infty} \sum_{Q \in \mathcal{P}^k} s_Q a_Q, \tag{6-1}$$

where  $a_Q$  is in  $\mathcal{A}(Q)$ ,  $s_Q \in \mathbb{C}$  and with finite cost

$$\left( \sum_{k=0}^{\infty} \left( \sum_{Q \in \mathcal{P}^k} |s_Q|^p \right)^{q/p} \right)^{1/q} < \infty. \tag{6-2}$$

Note that the inner sum in (6-1) is finite. By absolutely convergent in  $L^p$  we mean that

$$\sum_{k=0}^{\infty} \left| \sum_{Q \in \mathcal{P}^k} s_Q a_Q \right|_p^{p/\hat{p}} < \infty.$$

The series in (6-1) is called a  $\mathcal{B}_{p,q}^s(I, \mathcal{P}, \mathcal{A})$ -*representation* of the function  $g$ . Define

$$|g|_{\mathcal{B}_{p,q}^s(I, \mathcal{P}, \mathcal{A})} = \inf \left( \sum_{k=0}^{\infty} \left( \sum_{Q \in \mathcal{P}^k} |s_Q|^p \right)^{q/p} \right)^{1/q},$$

where the infimum runs over all possible representations of  $g$  as in (6-1).

Quite often throughout this work, when the choice of the measure space  $I$  and/or the grid  $\mathcal{P}$  is obvious we will write either  $\mathcal{B}_{p,q}^s(\mathcal{P}, \mathcal{A})$  or even  $\mathcal{B}_{p,q}^s(\mathcal{A})$  instead of  $\mathcal{B}_{p,q}^s(I, \mathcal{P}, \mathcal{A})$ . Whenever we write just  $\mathcal{B}_{p,q}^s$  it means that we choose the Souza's atoms  $\mathcal{A}_{s,p}^{s,z}$ , with parameters  $s$  and  $p$ , a class of atoms we properly define in Section 11A.

**Proposition 6.1.** *Assume G<sub>1</sub>–G<sub>2</sub> and A<sub>1</sub>–A<sub>4</sub>. Let  $t \in (0, \infty)$  be such that*

$$s - \frac{1}{p} + \frac{1}{t} \geq 0, \quad p \leq t \leq pu, \tag{6-3}$$

and suppose

$$C_5 = C_1^{1+1/t} C_2(t, q, (|\mathcal{P}^k|^{t(s-1/p+1/t)})_k) < \infty. \tag{6-4}$$

Then for all coefficients  $s_Q$  satisfying (6-2) and all  $\mathcal{A}$ -atoms  $a_Q$  on  $Q$  the series (6-1) converges absolutely on  $L^t$ . Indeed

$$\left| \sum_{Q \in \mathcal{P}^k} s_Q a_Q \right|_t \leq C_1^{1+1/t} |\mathcal{P}^k|^{s-1/p+1/t} \left( \sum_{Q \in \mathcal{P}^k} |s_Q|^p \right)^{1/p} \quad (6-5)$$

and

$$|g|_t \leq C_5 |g|_{\mathcal{B}_{p,q}^s(\mathcal{A})}. \quad (6-6)$$

*Proof.* Firstly note that if  $p \leq t \leq pu$ ,

$$\begin{aligned} |a_P|_t^t &= \int |a_P(x)|^t 1_P dm(x) \leq \|a_P\|_{(pu)/t}^t \|1_P\|_{(pu)/(pu-t)} \\ &\leq |a_P|_{pu}^t |P|^{(pu-t)/(pu)} \leq |P|^{t(s-1/p+1/t)} |P|^{(pu-t)/(pu)} = |P|^{t(s-1/p+1/t)}. \end{aligned}$$

Consequently

$$\begin{aligned} \left| \sum_{Q \in \mathcal{P}^k} s_Q a_Q \right|_t^t &\leq \int \left| \sum_{Q \in \mathcal{P}^k} s_Q a_Q \right|^t dm \leq \sum_{Q \in \mathcal{P}^k} \int_Q \left| \sum_{P \in \mathcal{P}^k} s_P a_P \right|^t dm \leq \sum_{Q \in \mathcal{P}^k} \int_Q \left| \sum_{P \in \Omega_Q^k} s_P a_P \right|^t dm \\ &\leq C_1^t \sum_{Q \in \mathcal{P}^k} \int_Q \sum_{P \in \Omega_Q^k} |s_P a_P|^t dm \leq C_1^t \sum_{Q \in \mathcal{P}^k} \sum_{P \in \Omega_Q^k} \int |s_P a_P|^t dm \\ &\leq C_1^t \sum_{Q \in \mathcal{P}^k} \sum_{P \in \Omega_Q^k} \int |s_P|^t |a_P|_t^t dm \leq C_1^t \sum_{Q \in \mathcal{P}^k} \sum_{P \in \Omega_Q^k} |P|^{t(s-1/p+1/t)} |s_P|^t \\ &\leq C_1^{t+1} \sum_{P \in \mathcal{P}^k} |P|^{t(s-1/p+1/t)} |s_P|^t \leq C_1^{t+1} |\mathcal{P}^k|^{t(s-1/p+1/t)} \sum_{P \in \mathcal{P}^k} |s_P|^t. \end{aligned}$$

By Proposition 4.1 (Hölder-like trick) and  $p \leq t$  we have

$$\begin{aligned} \left| \sum_k \sum_{Q \in \mathcal{P}^k} s_Q a_Q \right|_t &\leq \left( \sum_k \left( \left| \sum_{Q \in \mathcal{P}^k} s_Q a_Q \right|_t \right)^{1/\hat{t}} \right)^{\hat{t}/t} \\ &\leq C_1^{1+1/t} C_2(t, q, (|\mathcal{P}^k|^{t(s-1/p+1/t)})_k) \left( \sum_k \left( \sum_{P \in \mathcal{P}^k} |s_P|^t \right)^{q/t} \right)^{1/q} \\ &\leq C_1^{1+1/t} C_2(t, q, (|\mathcal{P}^k|^{t(s-1/p+1/t)})_k) \left( \sum_k \left( \sum_{P \in \mathcal{P}^k} |s_P|^p \right)^{q/p} \right)^{1/q}. \quad \square \end{aligned}$$

**Remark 6.2.** Note that due to  $G_2$  if  $t = p$  then  $C_5 < \infty$ . Sometimes it is convenient to use sharper estimates than (6-5) and (6-6) replacing  $|\mathcal{P}^k|$  by the sequence

$$C^k = \max\{|Q| : Q \in \mathcal{P}^k, s_Q \neq 0\}.$$

For instance, if  $s_Q = 0$  for every  $Q \in \mathcal{P}^k$  with  $k \leq N$ , then we can replace  $C_2(t, q, (|\mathcal{P}^k|^{t(s-1/p+1/t)})_k)$  by  $C_2(t, q, (|\mathcal{P}^k|^{t(s-1/p+1/t)} 1_{(N, \infty)}(k))_k)$  in (6-4). Here  $1_{(N, \infty)}$  denotes the indicator function of the set  $(N, \infty)$ .

**Proposition 6.3.** *Assume  $G_1$ – $G_2$  and  $A_1$ – $A_4$ . Then  $\mathcal{B}_{p,q}^s(\mathcal{A})$  is a complex linear space and  $|\cdot|_{\mathcal{B}_{p,q}^s(\mathcal{A})}$  is a  $\rho$ -norm, with  $\rho = \min\{1, p, q\}$ . Moreover the linear inclusion*

$$\iota : (\mathcal{B}_{p,q}^s(\mathcal{A}), |\cdot|_{\mathcal{B}_{p,q}^s(\mathcal{A})}) \rightarrow (L^p, |\cdot|_p)$$

is continuous.

*Proof.* Let  $f, g \in \mathcal{B}_{p,q}^s(\mathcal{A})$ . Then there are  $\mathcal{B}_{p,q}^s(\mathcal{A})$ -representations

$$f = \sum_{k=0}^{\infty} \sum_{Q \in \mathcal{P}^k} s'_Q a'_Q \quad \text{and} \quad g = \sum_{k=0}^{\infty} \sum_{Q \in \mathcal{P}^k} s_Q a_Q.$$

Let  $\text{sgn } 0 = 0$  and  $\text{sgn } z = z/|z|$  otherwise. Of course

$$\sum_{Q \in \mathcal{P}^k} c_Q b_Q = \sum_{Q \in \mathcal{P}^k} s'_Q a'_Q + \sum_{Q \in \mathcal{P}^k} s_Q a_Q, \quad (6-7)$$

where<sup>1</sup>

$$b_Q = \frac{|s'_Q|}{|s_Q| + |s'_Q|} \text{sgn}(s'_Q) a'_Q + \frac{|s_Q|}{|s_Q| + |s'_Q|} \text{sgn}(s'_Q) a_Q$$

and

$$c_Q = |s'_Q| + |s_Q|.$$

Note that  $\text{sgn}(s'_Q) a'_Q$  and  $\text{sgn}(s_Q) a_Q$  are atoms due to  $A_3$ . So by  $A_3$  we have that  $b_Q$  is also an atom, since it is a convex combination of atoms. In particular

$$\sum_k \sum_{Q \in \mathcal{P}^k} c_Q b_Q$$

converges absolutely in  $L^p$  to  $f + g$ . It remains to prove that this is a  $\mathcal{B}_{p,q}^s(\mathcal{A})$ -representation of  $f + g$ .

Indeed

$$\left( \sum_{k=0}^{\infty} \left( \sum_{Q \in \mathcal{P}^k} |c_Q|^p \right)^{q/p} \right)^{\rho/q} \leq \left( \sum_{k=0}^{\infty} \left( \sum_{Q \in \mathcal{P}^k} |s'_Q|^p \right)^{q/p} \right)^{\rho/q} + \left( \sum_{k=0}^{\infty} \left( \sum_{Q \in \mathcal{P}^k} |s_Q|^p \right)^{q/p} \right)^{\rho/q}.$$

Taking the infimum over all possible  $\mathcal{B}_{p,q}^s(\mathcal{A})$ -representations of  $f$  and  $g$  we obtain

$$|f + g|_{\mathcal{B}_{p,q}^s(\mathcal{A})}^{\rho} \leq |f|_{\mathcal{B}_{p,q}^s(\mathcal{A})}^{\rho} + |g|_{\mathcal{B}_{p,q}^s(\mathcal{A})}^{\rho}.$$

The identity  $|\gamma f|_{\mathcal{B}_{p,q}^s(\mathcal{A})} = |\gamma| |f|_{\mathcal{B}_{p,q}^s(\mathcal{A})}$  is obvious. By Proposition 6.1 we have that if  $|f|_{\mathcal{B}_{p,q}^s(\mathcal{A})} = 0$  then  $|f|_p = 0$ , so  $f = 0$ , so  $|\cdot|_{\mathcal{B}_{p,q}^s(\mathcal{A})}$  is a  $\rho$ -norm; moreover (6-6) tell us that  $\iota$  is continuous.  $\square$

**Proposition 6.4.** *Assume  $G_1$ – $G_2$  and  $A_1$ – $A_4$ . Suppose that  $g_n$  are functions in  $\mathcal{B}_{p,q}^s(\mathcal{A})$  with  $\mathcal{B}_{p,q}^s(\mathcal{A})$ -representations*

$$g_n = \sum_{k=0}^{\infty} \sum_{Q \in \mathcal{P}^k} s_Q^n a_Q^n,$$

<sup>1</sup>We don't need to worry so much if  $|s_Q| + |s'_Q| = 0$ , since in this case  $c_Q = 0$  and we can choose  $b_Q$  to be an arbitrary atom (for instance  $a_Q$ ) in such way that (6-7) holds. For this reason we are not going to explicitly deal with similar situations (which are going to appear quite often) throughout the paper.

where  $a_Q^n$  is a  $\mathcal{A}$ -atom supported on  $Q$ , satisfying:

(i) There is  $C$  such that for every  $n$

$$\left( \sum_{k=0}^{\infty} \left( \sum_{Q \in \mathcal{P}^k} |s_Q^n|^p \right)^{q/p} \right)^{1/q} \leq C. \quad (6-8)$$

(ii) For every  $Q \in \mathcal{P}$  we have that  $s_Q = \lim_n s_Q^n$  exists.

(iii) For every  $Q \in \mathcal{P}$  there is  $a_Q \in \mathcal{A}(Q)$  such that

- (1) either the sequence  $a_Q^n$  converges to  $a_Q$  in the strong topology of  $L^p$ , or
- (2) we have  $p \in [1, \infty)$  and  $a_Q^n$  weakly converges to  $a_Q$ .

Then  $g_n$  either strongly or weakly converges in  $L^p$ , respectively, to  $g \in \mathcal{B}_{p,q}^s(\mathcal{A})$ , where  $g$  has the  $\mathcal{B}_{p,q}^s(\mathcal{A})$ -representation

$$g = \sum_{k=0}^{\infty} \sum_{Q \in \mathcal{P}^k} s_Q a_Q \quad (6-9)$$

that satisfies

$$\left( \sum_{k=0}^{\infty} \left( \sum_{Q \in \mathcal{P}^k} |s_Q|^p \right)^{q/p} \right)^{1/q} \leq C. \quad (6-10)$$

*Proof.* By (6-8) it follows that (6-10) holds and that (6-9) is indeed a  $\mathcal{B}_{p,q}^s(\mathcal{A})$ -representation of a function  $g$ . It remains to prove that  $g_n$  converges to  $g$  in  $L^p$  in the topology under consideration. Given  $\epsilon > 0$ , fix  $N$  large enough such that

$$C_1^{1+1/p} C_2(p, q, (|\mathcal{P}^k|^{ps} 1_{[N,\infty)}(k))_k) (2C^\rho + 1)^{1/\rho} < (\epsilon/2)^{\hat{p}/p}.$$

We can write

$$g_n - g = \sum_{k \leq N} \sum_{Q \in \mathcal{P}^k} (s_Q^n a_Q^n - s_Q a_Q) + \sum_{k > N} \sum_{Q \in \mathcal{P}^k} c_Q^n b_Q^n,$$

where

$$b_Q^n = \frac{|s_Q^n|}{|s_Q^n| + |s_Q|} \operatorname{sgn}(s_Q^n) a_Q^n + \frac{|s_Q|}{|s_Q^n| + |s_Q|} \operatorname{sgn}(-s_Q) a_Q$$

is an atom in  $\mathcal{A}(Q)$ , and

$$c_Q^n = |s_Q^n| + |s_Q|.$$

Note that the series in the right-hand side converges absolutely in  $L^p$ . Of course

$$\left( \sum_{k=0}^{\infty} \left( \sum_{Q \in \mathcal{P}^k} |c_Q^n|^p \right)^{q/p} \right)^{1/q} \leq (2C^\rho + 1)^{1/\rho}.$$

So by (6-6) in Proposition 6.1 (see also Remark 6.2) we have

$$\left| \sum_{k > N} \sum_{Q \in \mathcal{P}^k} c_Q^n b_Q^n \right|_p \leq C_1^{1+1/p} C_2(p, q, (|\mathcal{P}^k|^{ps} 1_{[N,\infty)}(k))_k) (2C^\rho + 1)^{1/\rho} < (\epsilon/2)^{\hat{p}/p}.$$

In the case (ii.1), note that if  $n$  is large enough then

$$\left| \sum_{k \leq N} \sum_{Q \in \mathcal{P}^k} (s_Q^n a_Q^n - s_Q a_Q) \right|_p^{p/\hat{p}} < \epsilon/2,$$

and consequently  $|g - g_n|_p^{p/\hat{p}} < \epsilon$ . So  $g_n$  strongly converges to  $g$ .

In the case (ii.2), given  $\phi \in (L^p)^\star$ , with  $p \geq 1$  we have that for  $n$  large enough

$$\left| \phi \left( \sum_{k \leq N} \sum_{Q \in \mathcal{P}^k} (s_Q^n a_Q^n - s_Q a_Q) \right) \right| \leq |\phi|_{(L^p)^\star} \epsilon/2,$$

and of course

$$\left| \phi \left( \sum_{k > N} \sum_{Q \in \mathcal{P}^k} c_Q^n b_Q^n \right) \right| \leq |\phi|_{(L^p)^\star} \epsilon/2,$$

so  $g_n$  weakly converges to  $g$  in  $L^p$ . □

**Corollary 6.5.** *Assume  $G_1$ – $G_2$  and  $A_1$ – $A_4$ , and either*

- (1) *we have  $A_5$  or*
- (2) *we have  $p \geq 1$  and  $A_6$ .*

*Then:*

- (i) *Let  $g_n \in \mathcal{B}_{p,q}^s(\mathcal{A})$  be such that  $|g_n|_{\mathcal{B}_{p,q}^s(\mathcal{A})} \leq C$  for every  $n$ . Then there is a subsequence that converges either strongly or weakly in  $L^p$  to some  $g \in \mathcal{B}_{p,q}^s(\mathcal{A})$  with  $|g|_{\mathcal{B}_{p,q}^s(\mathcal{A})} \leq C$ .*
- (ii) *In both cases  $(\mathcal{B}_{p,q}^s(\mathcal{A}), |\cdot|_{\mathcal{B}_{p,q}^s(\mathcal{A})})$  is a complex  $\rho$ -Banach space, with  $\rho = \min\{1, p, q\}$ .*
- (iii) *If  $A_5$  holds then the inclusion*

$$\iota : (\mathcal{B}_{p,q}^s(\mathcal{A}), |\cdot|_{\mathcal{B}_{p,q}^s(\mathcal{A})}) \rightarrow (L^p, |\cdot|_p)$$

*is a compact linear inclusion.*

*Proof. Proof of (i):* There are  $\mathcal{B}_{p,q}^s(\mathcal{A})$ -representations

$$g_n = \sum_{k=0}^{\infty} \sum_{Q \in \mathcal{P}^k} s_Q^n a_Q^n,$$

where  $a_Q^n$  is a  $\mathcal{A}$ -atom supported on  $Q$  and

$$\left( \sum_{k=0}^{\infty} \left( \sum_{Q \in \mathcal{P}^k} |s_Q^n|^p \right)^{q/p} \right)^{1/q} \leq C + \epsilon_n, \quad (6-11)$$

and  $1 \geq \epsilon_n \rightarrow 0$ . In particular,  $|s_Q^n| \leq C + 1$ . Since the set  $\bigcup_k \mathcal{P}^k$  is countable, by the Cantor diagonal argument, taking a subsequence we can assume that  $s_Q^n \rightarrow_n s_Q$  for some  $s_Q \in \mathbb{C}$ . Due to  $A_5$  ( $A_6$ ) and the Cantor diagonal argument, we can suppose that  $a_Q^n$  strongly (weakly) converges in  $L^p$  to some  $a_Q \in \mathcal{A}(Q)$ . We set

$$g = \sum_{k=0}^{\infty} \sum_{Q \in \mathcal{P}^k} s_Q a_Q.$$

By Proposition 6.4 we conclude that  $g \in \mathcal{B}_{p,q}^s(\mathcal{A})$  with  $|g|_{\mathcal{B}_{p,q}^s(\mathcal{A})} \leq C$ , and that  $g_n$  converges to  $g$  in  $L^p$ .

Proof of (ii): Let  $g_n$  be a Cauchy sequence on  $\mathcal{B}_{p,q}^s(\mathcal{A})$ . By Proposition 6.1 we have that  $g_n$  is also a Cauchy sequence in  $L^p$ . Let  $g$  be its limit in  $L^p$ . By Corollary 6.5(i) we have that  $g \in \mathcal{B}_{p,q}^s(\mathcal{A})$ . Note that for large  $m$  and  $n$

$$\|g_n - g_m\|_{\mathcal{B}_{p,q}^s(\mathcal{A})} \leq \epsilon,$$

and  $g_n - g_m$  converges to  $g_n - g$  in  $L^p$ , so again by Corollary 6.5(i) we have that

$$\|g_n - g\|_{\mathcal{B}_{p,q}^s(\mathcal{A})} \leq \epsilon,$$

so  $g_n$  converges to  $g$  in  $\mathcal{B}_{p,q}^s(\mathcal{A})$ .

Proof of (iii): It follows from (i). □

The proof of the following result is quite similar.

**Corollary 6.6.** *Assume  $G_1$ – $G_2$  and  $A_1$ – $A_4$ , and either*

(1) *we have  $A_5$  or*

(2) *we have  $p \geq 1$  and  $A_6$ .*

*Then for every  $f \in \mathcal{B}_{p,q}^s(\mathcal{A})$  there is a  $\mathcal{B}_{p,q}^s(\mathcal{A})$ -representation*

$$f = \sum_k \sum_{P \in \mathcal{P}^k} c_P a_P$$

*such that*

$$\|f\|_{\mathcal{B}_{p,q}^s(\mathcal{A})} = \left( \sum_k \left( \sum_{P \in \mathcal{P}^k} |c_P|^p \right)^{q/p} \right)^{1/q}.$$

We refer to [Edmunds and Triebel 1996] for more information on compact linear transformations between quasi-Banach spaces.

**Corollary 6.7.** *Assume  $G_1$ – $G_2$  and  $A_1$ – $A_4$ . If for every  $Q \in \mathcal{P}$  we have that  $\mathcal{B}(Q)$  is finite-dimensional and  $\mathcal{A}(Q)$  is a closed subset of  $\mathcal{B}(Q)$  then  $(\mathcal{B}_{p,q}^s(\mathcal{A}), \|\cdot\|_{\mathcal{B}_{p,q}^s(\mathcal{A})})$  is a  $\rho$ -Banach space, with  $\rho = \min\{1, p, q\}$ .*

*Proof.* Since all norms are equivalent in  $\mathcal{B}(Q)$  we have that  $A_4$  implies that  $\mathcal{A}(Q)$  is a closed and bounded subset of  $\mathcal{B}(Q)$ , so it is compact. By Corollary 6.5(ii) it follows that  $(\mathcal{B}_{p,q}^s(\mathcal{A}), \|\cdot\|_{\mathcal{B}_{p,q}^s(\mathcal{A})})$  is a  $\rho$ -Banach space. □

## 7. Scales of spaces

Note that a family of atoms  $\mathcal{A}$  of type  $(s, p, u)$  induces an one-parameter scale

$$\tilde{s} \rightarrow \mathcal{A}_{\tilde{s}, p},$$

where  $\mathcal{A}_{\tilde{s}, p}$  is the family of atoms of type  $(\tilde{s}, p, u)$  defined by

$$\mathcal{A}_{\tilde{s}, p}(Q) = \{|Q|^{\tilde{s}-s} a_Q : a_Q \in \mathcal{A}\}.$$

Moreover a family of atoms  $\mathcal{A}$  of type  $(s, p, \infty)$  induces a two-parameter scale

$$(\tilde{s}, \tilde{p}) \rightarrow \mathcal{A}_{\tilde{s}, \tilde{p}},$$

where  $\mathcal{A}_{\tilde{s}, \tilde{p}}$  is the family of atoms of type  $(\tilde{s}, \tilde{p}, \infty)$  defined by

$$\mathcal{A}_{\tilde{p}, \tilde{s}}(Q) = \{|Q|^{\tilde{s}-s+1/p-1/\tilde{p}} a_Q : a_Q \in \mathcal{A}\}.$$

**Proposition 7.1.** *Assume  $G_1$ – $G_2$ . Suppose that the  $(s, p, \infty)$ -atoms  $\mathcal{A}$  satisfy  $A_1$ – $A_4$ . Let  $0 \leq s < \tilde{s}$  and  $q, \tilde{q} \in [1, \infty]$ . Suppose*

$$\left( \sum_k |\mathcal{P}^k|^{q(\tilde{s}-s)} \right)^{1/q} < \infty.$$

Then:

- (A) We have  $\mathcal{B}_{p, \tilde{q}}^{\tilde{s}}(\mathcal{A}_{\tilde{s}, p}) \subset \mathcal{B}_{p, q}^s(\mathcal{A}_{s, p})$  and the inclusion is a continuous linear map.
- (B) Suppose they also satisfy  $A_5$ . Let  $g_n \in \mathcal{B}_{p, \tilde{q}}^{\tilde{s}}(\mathcal{A}_{\tilde{s}, p})$  be such that  $|g_n|_{\mathcal{B}_{p, \tilde{q}}^{\tilde{s}}(\mathcal{A}_{\tilde{s}, p})} \leq C$  for every  $n$ . Then there is a subsequence that converges in  $\mathcal{B}_{p, q}^s(\mathcal{A}_{s, p})$  to some  $g \in \mathcal{B}_{p, \tilde{q}}^{\tilde{s}}(\mathcal{A}_{\tilde{s}, p})$  with  $|g|_{\mathcal{B}_{p, \tilde{q}}^{\tilde{s}}(\mathcal{A}_{\tilde{s}, p})} \leq C$ .
- (C) Suppose they also satisfy  $A_7$ . The inclusion  $\iota : \mathcal{B}_{p, \tilde{q}}^{\tilde{s}}(\mathcal{A}_{\tilde{s}, p}) \mapsto \mathcal{B}_{p, q}^s(\mathcal{A}_{s, p})$  is a compact linear map.

*Proof.* Consider a  $\mathcal{B}_{p, \tilde{q}}^{\tilde{s}}(\mathcal{A}_{\tilde{s}, p})$ -representation

$$f = \sum_{k=0}^{\infty} \sum_{Q \in \mathcal{P}^k} s_Q a_Q.$$

Since  $a_Q$  is an  $\mathcal{A}_{\tilde{s}, p}$ -atom, we have that  $b_Q = a_Q |Q|^{\tilde{s}-s}$  is an  $\mathcal{A}_{s, p}$ -atom. In particular, we can write

$$f = \sum_{k=0}^{\infty} \sum_{Q \in \mathcal{P}^k} s_Q |Q|^{\tilde{s}-s} b_Q.$$

If  $k \geq k_0$  then

$$\left( \sum_{Q \in \mathcal{P}^k} |s_Q|^p |Q|^{p(\tilde{s}-s)} \right)^{1/p} \leq |\mathcal{P}^k|^{\tilde{s}-s} \left( \sum_{Q \in \mathcal{P}^k} |s_Q|^p \right)^{1/p} \leq |\mathcal{P}^k|^{\tilde{s}-s} \left( \sum_{k \geq k_0} \left( \sum_{Q \in \mathcal{P}^k} |s_Q|^p \right)^{\tilde{q}/p} \right)^{1/\tilde{q}},$$

so

$$\left( \sum_{k \geq k_0} \left( \sum_{Q \in \mathcal{P}^k} |s_Q|^p |Q|^{p(\tilde{s}-s)} \right)^{q/p} \right)^{1/q} \leq \left( \sum_{k \geq k_0} |\mathcal{P}^k|^{q(\tilde{s}-s)} \right)^{1/q} \left( \sum_{k \geq k_0} \left( \sum_{Q \in \mathcal{P}^k} |s_Q|^p \right)^{\tilde{q}/p} \right)^{1/\tilde{q}}. \quad (7-12)$$

Proof of (A): In particular, taking  $k_0 = 0$  we conclude that  $\mathcal{B}_{p, \tilde{q}}^{\tilde{s}}(\mathcal{A}_{\tilde{s}, p}) \subset \mathcal{B}_{p, q}^s(\mathcal{A}_{s, p})$  and

$$|f|_{\mathcal{B}_{p, q}^s(\mathcal{A}_{s, p})} \leq \left( \sum_k |\mathcal{P}^k|^{q(\tilde{s}-s)} \right)^{1/q} |f|_{\mathcal{B}_{p, \tilde{q}}^{\tilde{s}}(\mathcal{A}_{\tilde{s}, p})}.$$

Proof of (B): By definition, there exist  $s_Q^n \in \mathbb{C}$ , such that

$$g_n = \sum_{k=0}^{\infty} \sum_{Q \in \mathcal{P}^k} s_Q^n a_Q^n,$$

where  $a_Q^n$  is a  $\mathcal{A}_{\tilde{s}, p}$ -atom supported on  $Q$  and

$$\left( \sum_{k=0}^{\infty} \left( \sum_{Q \in \mathcal{P}^k} |s_Q^n|^p \right)^{\tilde{q}/p} \right)^{1/\tilde{q}} \leq C + \epsilon_n, \quad (7-13)$$

where  $\epsilon_n \rightarrow 0$ . In particular,  $|s_Q^n| \leq C + \epsilon_n$ . Since the set  $\bigcup_k \mathcal{P}^k$  is countable, by the Cantor diagonal argument, taking a subsequence we can assume that  $s_Q^n \rightarrow s_Q$  and (due to  $A_5$ ) that  $a_Q^n$  converges in  $\mathcal{B}(Q)$  and  $L^p$  to some  $a_Q \in \mathcal{A}_{\bar{s},p}$ . By Proposition 6.4 the sequence  $g_n$  converges in  $L^p$  to a function  $g$  such that  $|g|_{\mathcal{B}_{p,\bar{q}}^{\bar{s}}(\mathcal{A}_{\bar{s},p})} \leq C$  and with  $\mathcal{B}_{p,\bar{q}}^{\bar{s}}(\mathcal{A}_{\bar{s},p})$ -representation

$$g = \sum_{k=0}^{\infty} \sum_{Q \in \mathcal{P}^k} s_Q a_Q.$$

It remains to show that the convergence indeed occurs in the topology of  $\mathcal{B}_{p,q}^s(\mathcal{A}_{s,p})$ . For every  $k_0 \geq 0$  and  $\delta > 0$  we can write

$$g_n - g = \sum_{k < k_0} \sum_{Q \in \mathcal{P}^k} \delta d_Q^n + \sum_{k \geq k_0} \sum_{Q \in \mathcal{P}^k} |Q|^{\bar{s}-s} c_Q^n b_Q^n,$$

where

$$d_Q^n = \frac{1}{\delta} (s_Q^n a_Q^n - s_Q a_Q),$$

and with  $b_Q^n \in \mathcal{A}_{s,p}$  given by

$$b_Q^n = \frac{|Q|^{s-\bar{s}} |s_Q^n|}{|s_Q^n| + |s_Q|} \operatorname{sgn}(s_Q^n) a_Q^n + \frac{|Q|^{s-\bar{s}} |s_Q|}{|s_Q^n| + |s_Q|} \operatorname{sgn}(-s_Q) a_Q,$$

and

$$c_Q^n = |s_Q^n| + |s_Q|.$$

Note that  $b_Q^n \in \mathcal{A}_{s,p}$ . Given  $\epsilon > 0$ , choose  $k_0$  such that  $b_Q^n \in \mathcal{A}_{s,p}$ . Given  $\epsilon > 0$ , choose  $k_0$  such that

$$\left( \sum_{k \geq k_0} |\mathcal{P}^k|^{q(\bar{s}-s)} \right)^{1/q} (2C + 1) \leq (\epsilon/2)^{1/\rho}.$$

By (7-12) and (7-13) for each  $n$  large enough we have

$$\left( \sum_{k \geq k_0} \left( \sum_{Q \in \mathcal{P}^k} |c_Q^n|^p |Q|^{p(\bar{s}-s)} \right)^{q/p} \right)^{1/q} \leq \left( \sum_{k \geq k_0} |\mathcal{P}^k|^{q(\bar{s}-s)} \right)^{1/q} (2C + 1) < (\epsilon/2)^{1/\rho}.$$

In particular

$$\left| \sum_{k \geq k_0} \sum_{Q \in \mathcal{P}^k} |Q|^{\bar{s}-s} c_Q^n b_Q^n \right|_{\mathcal{B}_{p,q}^s(\mathcal{A}_{s,p})}^\rho < \epsilon/2.$$

Choose  $\delta > 0$  such that

$$\left( \sum_{k < k_0} \left( \sum_{Q \in \mathcal{P}^k} \delta^p \right)^{q/p} \right)^{\rho/q} < \epsilon/2.$$

Proof of (C): Due to  $A_7$  there is  $\eta > 0$  such that for every  $Q \in \mathcal{P}^k$ , with  $k < k_0$ , if  $h \in \mathcal{B}(Q)$  satisfies  $|h|_{\mathcal{B}(Q)} \leq \eta$  then  $h \in \mathcal{A}_{p,s}(Q)$ . Since  $\lim_n s_Q^n a_Q^n = s_Q a_Q$  in  $\mathcal{B}(Q)$  we conclude that for  $n$  large enough we have

$$d_Q^n = \frac{1}{\delta} (s_Q^n a_Q^n - s_Q a_Q) \in \mathcal{A}_{s,p}(Q)$$

for every  $Q \in \mathcal{P}^k$ , with  $k < k_0$ . In particular

$$\left| \sum_{k < k_0} \sum_{Q \in \mathcal{P}^k} \delta d_Q^n \right|_{\mathcal{B}_{p,q}^s(\mathcal{A}_{s,p})}^\rho < \epsilon/2.$$

We conclude that

$$|g_n - g|_{\mathcal{B}_{p,q}^s(\mathcal{A}_{s,p})}^\rho < \epsilon$$

for  $n$  large enough, so the sequence  $g_n$  converges (by Corollary 6.5) to  $g$  in the topology of  $\mathcal{B}_{p,q}^s(\mathcal{A}_{s,p})$ .  $\square$

### 8. Transmutation of atoms

It turns out that sometimes a Besov-ish space can be obtained using different classes of atoms. The key result in Part I is the following (see Figure 1):

**Proposition 8.1** (transmutation of atoms). *Assume:*

(I) *Let  $\mathcal{A}_2$  be a class of  $(s, p, u_2)$ -atoms for a grid  $\mathcal{W}$ , satisfying  $A_1$ – $A_4$  and  $G_1$ – $G_2$ . Let  $\mathcal{G}$  be also a grid satisfying  $G_1$ – $G_2$ .*

(II) *Let  $k_i \in \mathbb{N}$  for  $i \geq 0$  be a sequence such that there is  $\alpha > 0$  and  $A, B \in \mathbb{R}$  satisfying*

$$\alpha i + A \leq k_i \leq \alpha i + B$$

*for every  $i$ .*

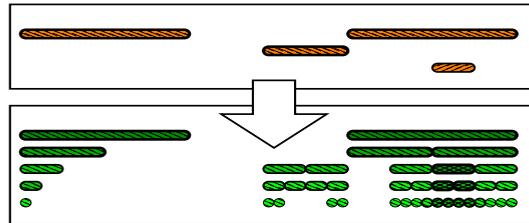
(III) *There is  $\lambda \in (0, 1)$  such that the following holds. For every  $Q \in \mathcal{G}$  and  $P \in \mathcal{W}$  satisfying  $P \subset Q$  there are atoms  $b_{P,Q} \in \mathcal{A}_2(P)$  and corresponding  $s_{P,Q} \in \mathbb{C}$  such that*

$$h_Q = \sum_k \sum_{P \in \mathcal{W}^k, P \subset Q} s_{P,Q} b_{P,Q}$$

*is a  $\mathcal{B}_{p,q}^s(\mathcal{A}_2)$ -representation of a function  $h_Q$ , with  $s_{P,Q} = 0$  for every  $Q \in \mathcal{G}^i$ ,  $P \in \mathcal{W}^k$  with  $k < k_i$  and moreover*

$$\sum_{P \in \mathcal{W}^k, P \subset Q} |s_{P,Q}|^p \leq C_6 \lambda^{k-k_i} \tag{8-14}$$

*for every  $k \geq k_i$ .*



**Figure 1.** In the transmutation of atoms we replace atoms whose supports are represented in orange (supports with distinct scales are drawn in distinct lines for a better understanding) by a linear combination of atoms (whose supports are represented in green), in such way that we do not change too much the location of the support and also most of the “mass” of the representation concentrates on the same original scale.

Let

$$\mathcal{H}^k = \bigcup_{Q \in \mathcal{G}} \{P \subset Q : P \in \mathcal{W}^k \text{ and } s_{P,Q} \neq 0\}.$$

Then:

(A) For all coefficients  $(c_Q)_{Q \in \mathcal{G}}$  such that

$$\left( \sum_i \left( \sum_{Q \in \mathcal{G}^i} |c_Q|^p \right)^{q/p} \right)^{1/q} < \infty$$

we have that the sequence

$$N \mapsto \sum_{i \leq N} \sum_{Q \in \mathcal{G}^i} c_Q h_Q \quad (8-15)$$

converges in  $L^p$  to a function in  $\mathcal{B}_{p,q}^s(\mathcal{A}_2)$  that has a  $\mathcal{B}_{p,q}^s(\mathcal{A}_2)$ -representation

$$\sum_k \sum_{P \in \mathcal{H}^k} m_P d_P, \quad (8-16)$$

where  $m_P \geq 0$  for every  $P$  and

$$\left( \sum_k \left( \sum_{\substack{P \in \mathcal{W}^k \\ P \in \mathcal{H}}} |m_P|^p \right)^{q/p} \right)^{1/q} \leq C_1 C_6^{1/p} \lambda^{-B/p} C_3(p, q, b) C_7^{1/q} \left( \sum_i \left( \sum_{Q \in \mathcal{G}^i} |c_Q|^p \right)^{q/p} \right)^{1/q}. \quad (8-17)$$

Here  $C_7 = \max\{\ell \in \mathbb{N}, \ell < \alpha\} + 1$  and  $b = (b_n)_{n \in \mathbb{Z}}$  is defined by

$$b_n = \begin{cases} \lambda^{\alpha n} & \text{if } n > A/\alpha - 1, \\ 0 & \text{if } n \leq A/\alpha - 1. \end{cases}$$

(B) Suppose that the assumptions of (A) hold and that  $s_{P,Q}$  are nonnegative real numbers and  $b_{P,Q} > 0$  on  $P$  for every  $P, Q$ . Then  $m_P \neq 0$  and  $d_P \neq 0$  on  $P$  imply that  $P \subset \text{supp } h_Q$  for some  $Q \in \mathcal{W}^k$  satisfying  $c_Q \neq 0$  and  $s_{P,Q} > 0$ . If we additionally assume that  $c_Q \geq 0$  for every  $Q$  then  $m_P \neq 0$  also implies  $d_P > 0$  on  $P$ .

(C) Let  $\mathcal{A}_1$  be a class of  $(s, p, u_1)$ -atoms for the grid  $\mathcal{G}$  satisfying  $A_1$ – $A_4$ . Suppose that there is  $\lambda < 1$  such that for every atom  $a_Q \in \mathcal{A}_1(Q)$  we can find  $s_{P,Q}$  and  $b_{P,Q}$  in (III) such that  $h_Q = a_Q$ . Then

$$\mathcal{B}_{p,q}^s(\mathcal{A}_1) \subset \mathcal{B}_{p,q}^s(\mathcal{A}_2)$$

and this inclusion is continuous. Indeed

$$|\phi|_{\mathcal{B}_{p,q}^s(\mathcal{A}_2)} \leq C_1 C_6^{1/p} \lambda^{-B/p} C_3(p, q, b) C_7^{1/q} |\phi|_{\mathcal{B}_{p,q}^s(\mathcal{A}_1)}$$

for every  $\phi \in \mathcal{B}_{p,q}^s(\mathcal{A}_1)$ .

*Proof.* For every  $P \in \mathcal{H}^k$ , with  $k \in \mathbb{N}$ , and  $N \in \mathbb{N} \cup \{\infty\}$  define

$$m_{P,N} = \sum_{i \leq N} \sum_{\substack{Q \in \mathcal{G}^i \\ P \subset Q}} |c_Q s_{P,Q}| = \sum_{\substack{k_i \leq k \\ i \leq N}} \sum_{\substack{Q \in \mathcal{G}^i \\ P \subset Q}} |c_Q s_{P,Q}|.$$

Due to  $G_2$  this sum has a finite number of terms. If this sum has zero terms, define  $m_{P,N} = 0$  and let  $d_{P,N}$  be the zero function. Otherwise define

$$d_{P,N} = \frac{1}{m_{P,N}} \sum_{i \leq N} \sum_{\substack{Q \in \mathcal{G}^i \\ P \subset Q}} c_Q s_{P,Q} b_{P,Q}. \quad (8-18)$$

We have that  $d_{P,N}$  is an  $\mathcal{A}_2(P)$ -atom.

**Claim I.** *We claim that for  $N \in \mathbb{N}$*

$$\sum_k \sum_{P \in \mathcal{H}^k} m_{P,N} d_{P,N} = \sum_{i \leq N} \sum_{Q \in \mathcal{G}^i} c_Q h_Q.$$

Note that if  $Q \in \mathcal{G}^i$  then due to (6-5), with  $t = p$  and (8-14) we have

$$\sum_k \left| \sum_{\substack{P \in \mathcal{H}^k \\ P \subset Q}} s_{P,Q} b_{P,Q} \right|_p^{p/\hat{p}} < \infty.$$

Consequently we can do the following manipulation in  $L^p$ :

$$\sum_{i \leq N} \sum_{Q \in \mathcal{G}^i} c_Q h_Q = \sum_{i \leq N} \sum_{Q \in \mathcal{G}^i} \sum_k \sum_{\substack{P \in \mathcal{H}^k \\ P \subset Q}} s_{P,Q} b_{P,Q} = \sum_k \sum_{P \in \mathcal{H}^k} \sum_{i \leq N} \sum_{Q \in \mathcal{G}^i} \sum_{P \subset Q} s_{P,Q} b_{P,Q} = \sum_k \sum_{P \in \mathcal{H}^k} m_{P,N} d_{P,N}.$$

This concludes the proof of Claim I.

**Claim II.** *For every  $N \in \mathbb{N} \cup \{\infty\}$  we claim that*

$$\sum_k \sum_{P \in \mathcal{H}^k} m_{P,N} d_{P,N} \quad (8-19)$$

is a  $\mathcal{B}_{p,q}^s(\mathcal{A}_2)$ -representation and

$$\left( \sum_k \left( \sum_{P \in \mathcal{H}^k} |m_{P,N}|^p \right)^{q/p} \right)^{1/q} \leq C_1 C_6^{1/p} \lambda^{-B/p} C_3(p, q, b) C_7^{1/q} \left( \sum_i \left( \sum_{Q \in \mathcal{G}^i} |c_Q|^p \right)^{q/p} \right)^{1/q}. \quad (8-20)$$

Indeed

$$\begin{aligned} & \left( \sum_{P \in \mathcal{H}^k} |m_{P,N}|^p \right)^{1/\hat{p}} \\ &= \left( \sum_{P \in \mathcal{W}^k} \left( \sum_{\substack{k_i \leq k \\ i \leq N}} \sum_{\substack{Q \in \mathcal{G}^i \\ P \subset Q}} |c_Q s_{P,Q}| \right)^p \right)^{1/\hat{p}} \leq \sum_{\substack{k_i \leq k \\ i \leq N}} \left( \sum_{P \in \mathcal{W}^k} \left( \sum_{\substack{Q \in \mathcal{G}^i \\ P \subset Q}} |c_Q s_{P,Q}| \right)^p \right)^{1/\hat{p}} \\ &\leq C_1^{p/\hat{p}} \sum_{\substack{k_i \leq k \\ i \leq N}} \left( \sum_{P \in \mathcal{W}^k} \sum_{\substack{Q \in \mathcal{G}^i \\ P \subset Q}} |c_Q|^p |s_{P,Q}|^p \right)^{1/\hat{p}} \leq C_1^{p/\hat{p}} \sum_{\substack{k_i \leq k \\ i \leq N}} \left( \sum_{Q \in \mathcal{G}^i} |c_Q|^p \sum_{\substack{P \in \mathcal{W}^k \\ P \subset Q}} |s_{P,Q}|^p \right)^{1/\hat{p}} \\ &\leq C_1^{p/\hat{p}} C_6^{1/\hat{p}} \sum_{\substack{k_i \leq k \\ i \leq N}} \lambda^{(k-k_i)/\hat{p}} \left( \sum_{Q \in \mathcal{G}^i} |c_Q|^p \right)^{1/\hat{p}} \leq C_1^{p/\hat{p}} C_6^{1/\hat{p}} \sum_{\alpha i + A \leq k} \lambda^{(k-\alpha i - B)/\hat{p}} \left( \sum_{Q \in \mathcal{G}^i} |c_Q|^p \right)^{1/\hat{p}}. \quad (8-21) \end{aligned}$$

If  $\alpha = 1$  and  $A = B = 0$  then this is a convolution, so we can use Proposition 4.2 (the convolution trick) and (8-17) easily follows. In the general case, consider

$$u_k = \sum_{P \in \mathcal{H}^k} |m_{P,N}|^p \quad \text{and} \quad c_i = \sum_{Q \in \mathcal{G}^i} |c_Q|^p.$$

Every  $k \in \mathbb{N}$  can be written in a *unique* way as  $k = \alpha j_k + \ell_k + r_k$ , with  $j_k \in \mathbb{N}$ ,  $\ell_k \in \mathbb{N}$ ,  $\ell_k + r_k < \alpha$  and  $r_k \in [0, 1)$ . Fix  $\ell \in [0, \alpha) \cap \mathbb{N}$  and  $j \in \mathbb{N}$ . Then there is at most one  $k' \in \mathbb{N}$  such that  $\ell_{k'} = \ell$  and  $j_{k'} = j$ . Indeed, if  $k' = \alpha j' + \ell + r'$  and  $k'' = \alpha j'' + \ell + r''$ , with  $r', r'' \in [0, 1)$  and  $\ell + r'$  and  $\ell + r''$  smaller than  $\alpha$ , then  $k' - k'' = r' - r'' \in (-1, 1)$ , so  $k' = k''$  and  $r' = r''$ . If such a  $k'$  exists, set  $k(\ell, j) = k'$  and  $r(\ell, j) = k(\ell, j) - \alpha j - \ell$  and  $a_{\ell,j} = u_{k(\ell,j)}$ . Otherwise let  $a_{\ell,j} = 0$ . Then (8-21) implies

$$\begin{aligned} a_{\ell,j}^{1/\hat{p}} &\leq C_1^{p/\hat{p}} C_6^{1/\hat{p}} \sum_{\alpha i + A \leq \alpha j + \ell + r(\ell,j)} \lambda^{(\alpha j + \ell + r(\ell,j) - \alpha i - B)/\hat{p}} c_i^{1/\hat{p}} \\ &\leq C_1^{p/\hat{p}} C_6^{1/\hat{p}} \lambda^{-B/\hat{p}} \sum_{i \leq j + (-A + \ell + r(\ell,j))/\alpha} \lambda^{\alpha(j-i)/\hat{p}} c_i^{1/\hat{p}} \\ &\leq C_1^{p/\hat{p}} C_6^{1/\hat{p}} \lambda^{-B/\hat{p}} \sum_{i < j + -A/\alpha + 1} \lambda^{\alpha(j-i)/\hat{p}} c_i^{1/\hat{p}} \\ &\leq C_1^{p/\hat{p}} C_6^{1/\hat{p}} \lambda^{-B/\hat{p}} \sum_{i \in \mathbb{Z}} b_{j-i}^{1/\hat{p}} c_i^{1/\hat{p}}. \end{aligned}$$

Here  $b_n = \lambda^{\alpha n}$  if  $n > A/\alpha - 1$ , and  $b_n = 0$  otherwise. Fixing  $\ell \in \mathbb{N}$ ,  $\ell < \alpha$ , Proposition 4.2 (the convolution trick) gives us

$$K_\ell = \left( \sum_{\substack{k \in \mathbb{N} \\ \ell_k = \ell}} u_k^{q/p} \right)^{1/q} = \left( \sum_j a_{\ell,j}^{q/p} \right)^{1/q} \leq C_1 C_6^{1/p} \lambda^{-B/p} C_3(p, q, b) \left( \sum_i \left( \sum_{Q \in \mathcal{G}^i} |c_Q|^p \right)^{q/p} \right)^{1/q}$$

and

$$\begin{aligned} \left( \sum_k \left( \sum_{P \in \mathcal{H}^k} |m_{P,N}|^p \right)^{q/p} \right)^{1/q} &= \left( \sum_k u_k^{q/p} \right)^{1/q} = \left( \sum_{0 \leq \ell < \alpha} \sum_{\substack{k \in \mathbb{N} \\ \ell_k = \ell}} u_k^{q/p} \right)^{1/q} = \left( \sum_{0 \leq \ell < \alpha} K_\ell^q \right)^{1/q} \\ &\leq C_1 C_6^{1/p} \lambda^{-B/p} C_3(p, q, b) C_7^{1/q} \left( \sum_i \left( \sum_{Q \in \mathcal{G}^i} |c_Q|^p \right)^{q/p} \right)^{1/q}. \end{aligned} \quad (8-22)$$

This implies in particular that the sum in (8-19) is a  $\mathcal{B}_{p,q}^s(\mathcal{A}_2)$ -representation. This proves Claim II.

**Claim III.** *We have that in the strong topology of  $L^p$*

$$\lim_{N \rightarrow \infty} \sum_k \sum_{P \in \mathcal{H}^k} m_{P,N} d_{P,N} = \sum_k \sum_{P \in \mathcal{H}^k} m_{P,\infty} d_{P,\infty}.$$

For each  $P \in \mathcal{H}$  the sequence

$$N \mapsto m_{P,N} \quad (8-23)$$

is eventually constant and therefore convergent. The same happens with

$$N \mapsto d_{P,N}. \quad (8-24)$$

Estimate (8-20) and Proposition 6.4 imply that (8-15) converges in  $L^p$  to a function with  $\mathcal{B}_{p,q}^s(\mathcal{A}_2)$ -representation (8-19) with  $N = \infty$ . This concludes the proof of Claim III.

Then Claim I, II, and III imply (A) taking  $m_p = m_{p,\infty}$  and  $d_p = d_{p,\infty}$ . We have that (C) is an immediate consequence of (A). Note that (8-18) and (A) give (B).  $\square$

### 9. Good grids

A  $(\lambda_1, \lambda_2)$ -good grid, with  $0 < \lambda_1 < \lambda_2 < 1$ , is a grid  $\mathcal{P} = (\mathcal{P}^k)_{k \in \mathbb{N}}$  with the following properties:

(G<sub>3</sub>) We have  $\mathcal{P}^0 = \{I\}$ .

(G<sub>4</sub>) We have  $I = \bigcup_{Q \in \mathcal{P}^k} Q$  (up to a set of zero  $m$ -measure).

(G<sub>5</sub>) The elements of the family  $\{Q\}_{Q \in \mathcal{P}^k}$  are pairwise disjoint.

(G<sub>6</sub>) For every  $Q \in \mathcal{P}^k$  and  $k > 0$  there exists  $P \in \mathcal{P}^{k-1}$  such that  $Q \subset P$ .

(G<sub>7</sub>) We have

$$\lambda_1 \leq \frac{|Q|}{|P|} \leq \lambda_2$$

for every  $Q \subset P$  satisfying  $Q \in \mathcal{P}^{k+1}$  and  $P \in \mathcal{P}^k$  for some  $k \geq 0$ .

(G<sub>8</sub>) The family  $\bigcup_k \mathcal{P}^k$  generate the  $\sigma$ -algebra  $\mathbb{A}$ .

### 10. Induced spaces

Consider a Besov-ish space  $\mathcal{B}_{p,q}^s(I, \mathcal{P}, \mathcal{A})$ , where  $\mathcal{P}$  is a good grid. Given  $Q \in \mathcal{P}^{k_0}$ , we can consider the sequence of finite families of subsets  $\mathcal{P}_Q = (\mathcal{P}_Q^i)_{i \geq 0}$  of  $Q$  given by

$$\mathcal{P}_Q^i = \{P \in \mathcal{P}^{k_0+i} : P \subset Q\}.$$

Let  $\mathcal{A}_Q$  be the restriction of the indexed family  $\mathcal{A}$  of pairs  $(\mathcal{B}(P), \mathcal{A}(P))_{P \in \mathcal{P}}$  to indices belonging to  $\mathcal{P}_Q$ . Then we can consider the *induced* Besov-ish space  $\mathcal{B}_{p,q}^s(Q, \mathcal{P}_Q, \mathcal{A}_Q)$ . Of course the inclusion

$$i : \mathcal{B}_{p,q}^s(Q, \mathcal{P}_Q, \mathcal{A}_Q) \rightarrow \mathcal{B}_{p,q}^s(I, \mathcal{P}, \mathcal{A})$$

is well-defined and it is a weak contraction; that is,

$$|f|_{\mathcal{B}_{p,q}^s(I, \mathcal{P}, \mathcal{A})} \leq |f|_{\mathcal{B}_{p,q}^s(Q, \mathcal{P}_Q, \mathcal{A}_Q)}.$$

Under the degree of generality we are considering here, the *restriction* transformation

$$r : \mathcal{B}_{p,q}^s(I, \mathcal{P}, \mathcal{A}) \rightarrow L^p$$

given by  $r(f) = f \cdot 1_Q$ , is a bounded linear transformation; however, it is easy to find examples of Besov-ish spaces where  $f1_Q \notin \mathcal{B}_{p,q}^s(Q, \mathcal{P}_Q, \mathcal{A}_Q)$ .

### 11. Examples of classes of atoms

There are many classes of atoms one may consider. We list here just a few of them.

**11A. Souza's atoms.** Let  $Q \in \mathcal{P}$ . An  $(s, p)$ -Souza's atom supported on  $Q$  is a function  $a : I \rightarrow \mathbb{C}$  such that  $a(x) = 0$  for every  $x \notin Q$  and  $a$  is constant on  $Q$ , with

$$|a|_\infty \leq |Q|^{s-1/p}.$$

The set of Souza's atoms supported on  $Q$  will be denoted by  $\mathcal{A}_{s,p}^{sz}(Q)$ . A *canonical Souza's atom* on  $Q$  is the Souza's atom such that  $a(x) = |Q|^{s-1/p}$  for every  $x \in Q$ . Souza's atoms are  $(s, p, \infty)$ -type atoms.

**11B. Hölder atoms.** Suppose that  $I$  is a quasimetric space with a quasidistance  $d(\cdot, \cdot)$ , such that every  $Q \in \mathcal{P}$  is a bounded set and there is  $\lambda_3, \lambda_4 \in (0, 1)$  such that

$$\lambda_3 \leq \frac{\text{diam } P}{\text{diam } Q} \leq \lambda_4$$

for every  $P \subset Q$  with  $P \in \mathcal{P}^{k+1}$  and  $Q \in \mathcal{P}^k$ . Additionally assume there are  $C_8 \geq 0$  and  $D \geq 0$  such that

$$\frac{1}{C_8^D} |Q| \leq (\text{diam } Q)^D \leq C_8^D |Q|.$$

Let

$$0 < s < \frac{1}{p}, \quad s < \beta.$$

For every  $Q \in \mathcal{P}$ , let  $C^\alpha(Q)$  be the Banach space of all functions  $\phi$  such that  $\phi(x) = 0$  for  $x \notin Q$ , and

$$|\phi|_{C^\alpha(Q)} = |\phi|_\infty + \sup_{\substack{x, y \in Q \\ x \neq y}} \frac{|\phi(x) - \phi(y)|}{d(x, y)^\alpha} < \infty.$$

Let  $\mathcal{A}_{s,\beta,p}^h(Q) \subset C^{\beta D}(Q)$  be the convex subset of all functions  $\phi$  satisfying

$$\sup_{\substack{x, y \in Q \\ x \neq y}} \frac{|\phi(x) - \phi(y)|}{d(x, y)^{D\beta}} \leq |Q|^{s-1/p-\beta} \text{ and } |\phi|_\infty \leq |Q|^{s-1/p}.$$

We say that  $\mathcal{A}_{s,\beta,p}^h(Q)$  is the set of  $(s, \beta, p)$ -Hölder atoms supported on  $Q$ . Of course  $\mathcal{A}_{s,\beta,p}^h$ -atoms are  $(s, p, \infty)$ -type atoms and  $\mathcal{A}_{s,p}^{sz}(Q) \subset \mathcal{A}_{s,\beta,p}^h(Q)$ .

**11C. Bounded variation atoms.** Now suppose that  $I$  is an interval of  $\mathbb{R}$  with length 1,  $m$  is the Lebesgue measure on it and the partitions in the grid  $\mathcal{P}$  are partitions by intervals. Let  $Q$  be an interval and  $s \leq \beta$ ,  $p \in [1, \infty)$ . An  $(s, \beta, p)$ -bounded variation atom on  $Q$  is a function  $a : \mathbb{R} \rightarrow \mathbb{C}$  such that  $a(x) = 0$  for every  $x \notin Q$ ,

$$\begin{aligned} |a|_\infty &\leq |Q|^{s-1/p}, \\ \text{var}_{1/\beta}(a, Q) &\leq |Q|^{s-1/p}. \end{aligned}$$

Here  $\text{var}_{1/\beta}(\cdot, Q)$  is the pseudonorm

$$\text{var}_{1/\beta}(a, Q) = \sup \left( \sum_i |a(x_{i+1}) - a(x_i)|^{1/\beta} \right)^\beta,$$

where the sup runs over all possible sequences  $x_1 < x_2 < \dots < x_n$ , with  $x_i$  in the interior of  $Q$ . We will denote the set of bounded variation atoms on  $Q$  as  $\mathcal{A}_{s,p,\beta}^{bv}(Q)$ . Bounded variation atoms are also  $(s, p, \infty)$ -type atoms.

## Part II. Spaces defined by Souza's atoms

In Part II, we suppose that  $s > 0$ ,  $p \in [1, \infty)$  and  $q \in [1, \infty]$ .

### 12. Besov spaces in a measure space with a good grid

We will study the Besov-ish spaces  $\mathcal{B}_{p,q}^s(\mathcal{P}, \mathcal{A}_{s,p}^{sz})$  associated with the measure space with a good grid  $(I, \mathcal{P}, m)$ . We let  $\mathcal{B}_{p,q}^s = \mathcal{B}_{p,q}^s(\mathcal{P}, \mathcal{A}_{s,p}^{sz})$ . Note that  $\mathcal{A}_{s,p}^{sz}$  satisfies  $A_1$ – $A_7$ . Note that by Proposition 6.1 there is  $\beta > 1$  such that  $\mathcal{B}_{p,q}^s \subset L^\beta$ .

If  $p \in [1, \infty)$ ,  $q \in [1, \infty]$  and  $0 < s < 1/p$  we will say that  $\mathcal{B}_{p,q}^s$  is a *Besov space*.

### 13. Positive cone

We say that  $f$  is  $\mathcal{B}_{p,q}^s$ -positive if there is a  $\mathcal{B}_{p,q}^s$ -representation

$$f = \sum_k \sum_{P \in \mathcal{P}^k} c_P a_P,$$

where  $c_P \geq 0$  and  $a_P$  is the standard  $(s, p)$ -Souza's atom supported on  $P$ . The set of all  $\mathcal{B}_{p,q}^s$ -positive functions is a convex cone in  $\mathcal{B}_{p,q}^s$ , denoted by  $\mathcal{B}_{p,q}^{s+}$ . We can define a “norm” on  $\mathcal{B}_{p,q}^{s+}$  as

$$|f|_{\mathcal{B}_{p,q}^{s+}} = \inf \left( \sum_k \left( \sum_{P \in \mathcal{P}^k} c_P^p \right)^{q/p} \right)^{1/q},$$

where the infimum runs over all possible  $\mathcal{B}_{p,q}^s$ -positive representations of  $f$ . Of course for every  $f, g \in \mathcal{B}_{p,q}^{s+}$  and  $\alpha \geq 0$  we have

$$|\alpha f|_{\mathcal{B}_{p,q}^{s+}} = \alpha |f|_{\mathcal{B}_{p,q}^{s+}}, \quad |f + g|_{\mathcal{B}_{p,q}^{s+}} \leq |f|_{\mathcal{B}_{p,q}^{s+}} + |g|_{\mathcal{B}_{p,q}^{s+}}, \quad |f|_{\mathcal{B}_{p,q}^{s+}} \leq |f|_{\mathcal{B}_{p,q}^s}.$$

Moreover if  $f \in \mathcal{B}_{p,q}^s$  is a real-valued function then one can find  $f_+, f_- \in \mathcal{B}_{p,q}^{s+}$  such that  $f = f_+ - f_-$  and

$$|f_+|_{\mathcal{B}_{p,q}^{s+}} \leq |f|_{\mathcal{B}_{p,q}^s} \quad \text{and} \quad |f_-|_{\mathcal{B}_{p,q}^{s+}} \leq |f|_{\mathcal{B}_{p,q}^s}.$$

An obvious but important observation is:

**Proposition 13.1.** *If  $f \in \mathcal{B}_{p,q}^{s+}$  then its support*

$$\text{supp } f = \{x \in I : f(x) \neq 0\}$$

*is (up to a set of zero measure) a countable union of elements of  $\mathcal{P}$ .*

### 14. Unbalanced Haar wavelets

Let  $\mathcal{P} = (\mathcal{P}^k)_k$  be a good grid. For every  $Q \in \mathcal{P}^k$  let  $\Omega_Q = \{P_1^Q, \dots, P_{n_Q}^Q\}$ ,  $n_Q \geq 2$ , be the family of elements  $\mathcal{P}^{k+1}$  such that  $P_i^Q \subset Q$  for every  $i$ , and ordered in some arbitrary way. The elements of  $\Omega_Q$  will be called *children* of  $Q$ . Note that every  $Q \in \mathcal{P}$  has at least two children. We will use one of the methods described (type I tree with the logarithmic subtrees construction) in [Girardi and Sweldens 1997] to construct an unconditional basis of  $L^\beta$  for every  $1 < \beta < \infty$ . See Figure 2.

Let  $\mathcal{H}_Q$  be the family of pairs  $(S_1, S_2)$ , with  $S_i \subset \Omega_Q$  and  $S_1 \cap S_2 = \emptyset$ , defined as

$$\mathcal{H}_Q = \bigcup_{j \in \mathbb{N}} \mathcal{H}_{Q,j},$$

where  $\mathcal{H}_{Q,j}$  are constructed recursively in the following way. Let  $\mathcal{H}_{Q,0} = \{(A, B)\}$ , where  $A = \{P_1^Q, \dots, P_{[n_Q/2]}^Q\}$  and  $B = \{P_{[n_Q/2]+1}^Q, \dots, P_{n_Q}^Q\}$ . Here  $[x]$  denotes the integer part of  $x \geq 0$ . Suppose that we have defined  $\mathcal{H}_{Q,j}$ . For each element  $(S_1, S_2) \in \mathcal{H}_{Q,j}$ , fix an ordering  $S_1 = \{R_1^1, \dots, R_{n_1}^1\}$  and  $S_2 = \{R_1^2, \dots, R_{n_2}^2\}$ . For each  $i = 1, 2$  such that  $n_i \geq 2$ , define  $T_i^1 = \{R_1^i, \dots, R_{[n_i/2]}^i\}$  and  $T_i^2 = \{R_{[n_i/2]+1}^i, \dots, R_{n_i}^i\}$  and add  $(T_i^1, T_i^2)$  to  $\mathcal{H}_{Q,j+1}$ . This defines  $\mathcal{H}_{Q,j+1}$ .

Note that since  $\mathcal{P}$  is a good grid we have  $\mathcal{H}_{Q,j} = \emptyset$  for large  $j$  and indeed

$$\sup_{Q \in \mathcal{P}} \#\mathcal{H}_Q < \infty.$$

Define  $\mathcal{H} = \bigcup_{Q \in \mathcal{P}} \mathcal{H}_Q$ . For every  $S = (S_1, S_2) \in \mathcal{H}$  define

$$\phi_S = \frac{1}{m_{(S_1, S_2)}} \left( \frac{\sum_{P \in S_1} 1_P}{\sum_{P \in S_1} |P|} - \frac{\sum_{R \in S_2} 1_R}{\sum_{R \in S_2} |R|} \right),$$

where

$$m_{(S_1, S_2)} = \left( \frac{1}{\sum_{P \in S_1} |P|} + \frac{1}{\sum_{R \in S_2} |R|} \right)^{1/2}.$$

Note that

$$\int_Q \phi_S dm = 0.$$

Since  $1 \leq \#S_i \leq 1/\lambda_1$  we have

$$\lambda_1 |Q| \leq \sum_{P \in S_i} |P| \leq \frac{\lambda_2}{\lambda_1} |Q|,$$

so

$$\left( \frac{2\lambda_1}{\lambda_2} \right)^{1/2} \frac{1}{|Q|^{1/2}} \leq m_{(S_1, S_2)} \leq \left( \frac{2}{\lambda_1} \right)^{1/2} \frac{1}{|Q|^{1/2}}.$$

Consequently

$$\begin{aligned} \frac{C_9}{|Q|^{1/2}} &\leq \frac{1}{m_{(S_1, S_2)}} \min \left\{ \frac{1}{\sum_{P \in S_1} |P|}, \frac{1}{\sum_{R \in S_2} |R|} \right\} \\ &\leq |\phi_S(x)| \leq \frac{1}{m_{(S_1, S_2)}} \max \left\{ \frac{1}{\sum_{P \in S_1} |P|}, \frac{1}{\sum_{R \in S_2} |R|} \right\} \leq \frac{C_{10}}{|Q|^{1/2}} \end{aligned}$$

for every  $x \in \bigcup_{P \in S_1 \cup S_2} P$ . Here

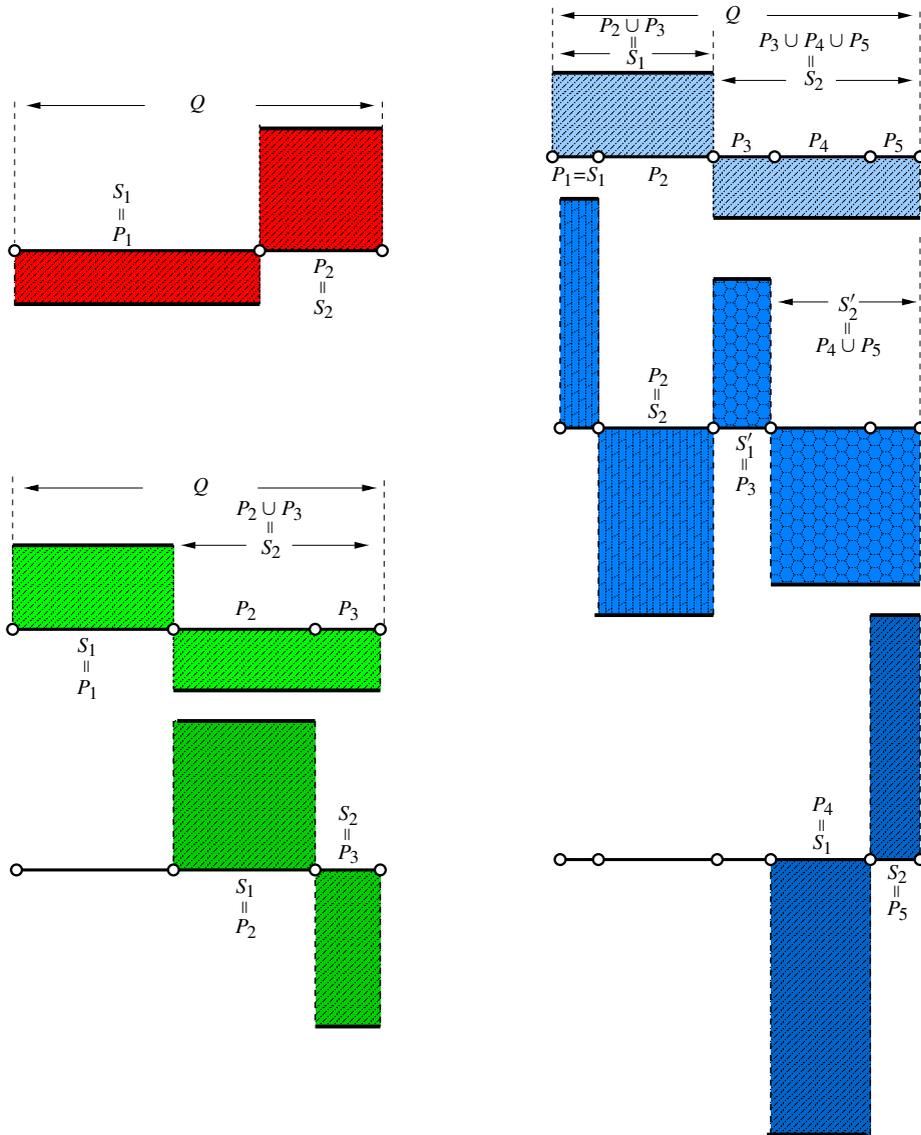
$$C_9 = \frac{\lambda_1^{3/2}}{\sqrt{2}\lambda_2} \text{ and } C_{10} = \frac{\lambda_2^{1/2}}{\sqrt{2}\lambda_1^{3/2}} + 1.$$

Let

$$\widehat{\mathcal{H}} = \{I\} \cup \mathcal{H}$$

and define

$$\phi_I = \frac{1_I}{|I|^{1/2}}.$$



**Figure 2.** Construction of the unbalanced Haar basis, following [Girardi and Sweldens 1997] in the case when  $\mathcal{P}$  is a grid of intervals. Every  $Q \in \mathcal{P}$  gives origin to a family of bounded functions indexed by the subtree  $\mathcal{H}_Q$ . We give examples of the logarithmic subtree construction when  $Q$  has 2 (red), 3 (green) and 5 (blue) children. Then  $\mathcal{H}_Q$  have 1, 2 and 4 elements, respectively.

Then by [Girardi and Sweldens 1997] we have that

$$\{\phi_s\}_{s \in \widehat{\mathcal{H}}}$$

is an unconditional basis of  $L^\beta$  for every  $\beta > 1$ .

### 15. Alternative characterizations, I: Messing with norms

We are going to describe three norms that are equivalent to  $|\cdot|_{\mathcal{B}_{p,q}^s}$ . Their advantage is that they are far more concrete, in the sense that we do not need to consider arbitrary atomic decompositions to define them.

**15A. Haar representation.** For every  $f \in L^\beta$ ,  $\beta > 1$ , the series

$$f = \sum_{S \in \tilde{\mathcal{H}}} d_S^f \phi_S \quad (15-25)$$

converges unconditionally in  $L^\beta$ , where  $d_S^f = \int f \phi_S dm$ . We will call the right-hand side of (15-25) the *Haar representation* of  $f$ . Define

$$N_{\text{haar}}(f) = |I|^{1/p-s-1/2} |d_I^f| + \left( \sum_k \left( \sum_{Q \in \mathcal{P}^k} |Q|^{1-sp-p/2} \sum_{S \in \mathcal{H}_Q} |d_S^f|^p \right)^{q/p} \right)^{1/q}.$$

**15B. Standard atomic representation.** Note that

$$k_I^f a_I = d_I^f \phi_I,$$

where  $k_I = |I|^{1/p-s-1/2} d_I^f$  and  $a_I$  is the canonical Souza's atom on  $I$ . Let  $S \in \mathcal{H}$ . Then  $S \in \mathcal{H}_Q$ , with  $S = (S_1, S_2)$  and some  $Q \in \mathcal{P}^k$ , with  $k \geq 0$ . It is easy to see that for every  $P \in S_1 \cup S_2$  the function

$$a_{S,P} = \frac{|Q|^{1/2}}{C_{10}} |P|^{s-1/p} \phi_S 1_P$$

is a Souza's atom on  $P$ . Choose

$$c_{S,P}^f = C_{10} |Q|^{-1/2} |P|^{1/p-s} d_S^f.$$

Note that

$$|c_{S,P}^f| \leq C_{10} \max\{\lambda_2^{1/p-s}, \lambda_1^{1/p-s}\} |Q|^{1/p-s-1/2} |d_S^f|. \quad (15-26)$$

For every child  $P$  of  $Q \in \mathcal{P}^k$ ,  $k \geq 0$ , define

$$\tilde{a}_P^f = \frac{1}{\tilde{k}_P^f} \sum_{S=(S_1, S_2) \in \mathcal{H}_Q} \sum_{P \in S_1 \cup S_2} c_{S,P}^f a_{S,P},$$

where

$$\tilde{k}_P^f = \sum_{S=(S_1, S_2) \in \mathcal{H}_Q} \sum_{P \in S_1 \cup S_2} |c_{S,P}^f|. \quad (15-27)$$

The (finite) number of terms on this sum depends only on the geometry of  $\mathcal{P}$ . Then  $\tilde{a}_P^f$  is a Souza's atom on  $P$  and

$$\sum_{S \in \mathcal{H}_Q} d_S^f \phi_S = \sum_{\substack{P \in \mathcal{P}^{k+1} \\ P \subset Q}} \tilde{k}_P^f \tilde{a}_P^f.$$

Let  $a_P$  be the canonical  $(s, p)$ -Souza's atom on  $P$  and choose  $x_P \in P$ . Define

$$k_P^f = \frac{\tilde{a}_P^f(x_P)}{|P|^{s-1/p}} \tilde{k}_P^f = \frac{1}{|P|^{s-1/p}} \sum_{S \in \mathcal{H}_Q} d_S^f \phi_S(x_P) = \frac{1}{|P|^{s-1/p}} \int f \sum_{S \in \mathcal{H}_Q} \phi_S(x_P) \phi_S dm.$$

In particular, for every  $P$

$$f \mapsto k_P^f$$

extends to a bounded linear functional in  $L^1$ . We have  $|k_P^f| \leq \tilde{k}_P^f$  and

$$f = k_I^f a_I + \sum_k \sum_{Q \in \mathcal{P}^k} \sum_{\substack{P \in \mathcal{P}^{k+1} \\ P \subset Q}} k_P^f a_P = \sum_i \sum_{Q \in \mathcal{P}^i} k_Q^f a_Q, \quad (15-28)$$

where this series converges unconditionally in  $L^B$ . Here  $a_P$  is the canonical Souza's atom. We will call the right-hand side of (15-28) the *standard atomic representation* of  $f$ . Let

$$N_{\text{st}}(f) = |k_I^f| + \left( \sum_{k \geq 1} \left( \sum_{Q \in \mathcal{P}^k} |k_Q^f|^p \right)^{q/p} \right)^{1/q}.$$

**15C. Mean oscillation.** Define for  $p \in [1, \infty)$

$$\begin{aligned} \text{osc}_p(f, Q) &= \inf_{c \in \mathbb{C}} \left( \int_Q |f(x) - c|^p dm(x) \right)^{1/p}, \\ \text{osc}_\infty(f, Q) &= \inf_{c \in \mathbb{C}} |f - c|_{L^\infty(Q)}. \end{aligned}$$

Define for every  $p \in [1, \infty)$  and  $q \in [1, \infty]$

$$\text{osc}_{p,q}^s(f) = \left( \sum_k \left( \sum_{Q \in \mathcal{P}^k} |Q|^{-sp} \text{osc}_p(f, Q)^p \right)^{q/p} \right)^{1/q}, \quad (15-29)$$

with the obvious adaptation for  $q = \infty$ . Let

$$N_{\text{osc}}(f) = |I|^{-s} |f|_p + \text{osc}_{p,q}^s(f).$$

**15D. These norms are equivalent.** We have:

**Theorem 15.1.** *Suppose  $s > 0$ ,  $p \in [1, \infty)$  and  $q \in [1, \infty]$ . Each one of the norms  $|f|_{\mathcal{B}_{p,q}^s}$ ,  $N_{\text{st}}(f)$ ,  $N_{\text{haar}}(f)$ ,  $N_{\text{osc}}(f)$  is finite if and only if  $f \in \mathcal{B}_{p,q}^s$ . Furthermore these norms are equivalent on  $\mathcal{B}_{p,q}^s$ . Indeed*

$$|f|_{\mathcal{B}_{p,q}^s} \leq N_{\text{st}}(f), \quad (15-30)$$

$$N_{\text{st}}(f) \leq C_{11} N_{\text{haar}}(f), \quad (15-31)$$

$$N_{\text{haar}}(f) \leq C_{10} N_{\text{osc}}(f), \quad (15-32)$$

$$N_{\text{osc}}(f) \leq C_{12} |f|_{\mathcal{B}_{p,q}^s}, \quad (15-33)$$

where

$$\begin{aligned} C_{11} &= 1 + C_{10} \max\{\lambda_2^{1/p-s}, \lambda_1^{1/p-s}\} \lambda_1^{-2-1/p}, \\ C_{12} &= C_1^{1+1/p} C_2(t, q, (|\mathcal{P}^k|^{sp})_k) |I|^{-s} + \frac{1}{1 - \lambda_2^s}. \end{aligned}$$

*Proof.* The inequality (15-30) is obvious. To simplify the notation we write  $d_S, k_P$  instead of  $d_S^f, k_P^f$ .

Proof of (15-31): The number of terms in the right-hand side of (15-27) depends only on the geometry of  $\mathcal{P}$ . Indeed

$$\sup_{Q \in \mathcal{P}} \sum_{S=(S_1, S_2) \in \mathcal{H}_Q} \#(S_1 \cup S_2) \leq \frac{1}{\lambda_1^2}. \quad (15-34)$$

Consider the standard atomic representation of  $f$  given by (15-28). Note that by (15-26)

$$\begin{aligned} \sum_{Q \in \mathcal{P}^k} \sum_{\substack{P \in \mathcal{P}^{k+1} \\ P \subset Q}} |k_P|^p &\leq \sum_{Q \in \mathcal{P}^k} \left( \sum_{S \in \mathcal{H}_Q} \sum_{\substack{P \in S_1 \cup S_2 \\ S=(S_1, S_2)}} |c_{S,P}| \right)^p \\ &\leq C_{10}^p \max\{\lambda_2^{1-sp}, \lambda_1^{1-sp}\} \sum_{Q \in \mathcal{P}^k} |Q|^{1-sp-p/2} \left( \sum_{S \in \mathcal{H}_Q} \sum_{\substack{P \in S_1 \cup S_2 \\ S=(S_1, S_2)}} |d_S| \right)^p \\ &\leq C_{10}^p \max\{\lambda_2^{1-sp}, \lambda_1^{1-sp}\} \lambda_1^{-2p} \sum_{Q \in \mathcal{P}^k} |Q|^{1-sp-p/2} \sum_{\substack{S \in \mathcal{H}_Q \\ S=(S_1, S_2)}} \sum_{P \in S_1 \cup S_2} |d_S|^p \\ &\leq C_{10}^p \max\{\lambda_2^{1-sp}, \lambda_1^{1-sp}\} \lambda_1^{-2p-1} \sum_{Q \in \mathcal{P}^k} |Q|^{1-sp-p/2} \sum_{S \in \mathcal{H}_Q} |d_S|^p \end{aligned}$$

for every  $k$ . Consequently

$$\begin{aligned} |k_I| + \left( \sum_{k \geq 1} \left( \sum_{P \in \mathcal{P}^k} |k_P|^p \right)^{q/p} \right)^{1/q} \\ \leq |I|^{1/p-s-1/2} |d_I| + C_{10} \max\{\lambda_2^{1/p-s}, \lambda_1^{1/p-s}\} \lambda_1^{-2-1/p} \left( \sum_k \left( \sum_{Q \in \mathcal{P}^k} |Q|^{1-sp-p/2} \sum_{S \in \mathcal{H}_Q} |d_S|^p \right)^{q/p} \right)^{1/q}. \end{aligned}$$

This completes the proof of (15-31).

Proof of (15-32): Note that

$$|d_I| \leq \int_I |f| |\phi_I| dm \leq |f|_p |I|^{1/2-1/p}.$$

Given  $\epsilon > 0$  and  $Q \in \mathcal{P}$ , choose  $c_Q \in Q$  such that

$$\left( \int_Q |f - c_Q|^p dm \right)^{1/p} \leq (1 + \epsilon) \text{osc}_p(f, Q).$$

Since  $\phi_S$  has zero mean on  $Q$  for every  $S \in \mathcal{H}_Q$  we have

$$\begin{aligned} \left( \sum_{Q \in \mathcal{P}^k} |Q|^{1-sp-p/2} \sum_{S \in \mathcal{H}_Q} |d_S|^p \right)^{1/p} &\leq \left( \sum_{Q \in \mathcal{P}^k} |Q|^{1-sp-p/2} \sum_{S \in \mathcal{H}_Q} \left| \int f \phi_S dm \right|^p \right)^{1/p} \\ &\leq \left( \sum_{Q \in \mathcal{P}^k} |Q|^{1-sp-p/2} \sum_{S \in \mathcal{H}_Q} \left| \int_Q f \phi_S - c_Q \phi_S dm \right|^p \right)^{1/p} \\ &\leq \left( \sum_{Q \in \mathcal{P}^k} |Q|^{1-sp-p/2} \sum_{S \in \mathcal{H}_Q} \left( \int_Q |f - c_Q| |\phi_S| dm \right)^p \right)^{1/p} \end{aligned}$$

$$\begin{aligned}
 &\leq C_{10} \left( \sum_{\substack{Q \in \mathcal{P}^k \\ Q \subset J}} |Q|^{1-sp-p/2} \sum_{S \in \mathcal{H}_Q} ((1+\epsilon) \operatorname{osc}_p(f, Q) |Q|^{1/p'-1/2})^p \right)^{1/p} \\
 &\leq C_{10} (1+\epsilon) \left( \sum_{Q \in \mathcal{P}^k} |Q|^{1-sp+p/p'-p} \sum_{S \in \mathcal{H}_Q} \operatorname{osc}_p(f, Q)^p \right)^{1/p} \\
 &\leq C_{10} (1+\epsilon) \left( \sum_{Q \in \mathcal{P}^k} |Q|^{-sp} \sum_{S \in \mathcal{H}_Q} \operatorname{osc}_p(f, Q)^p \right)^{1/p}.
 \end{aligned}$$

Since  $\epsilon$  is arbitrary, this concludes the proof of (15-32).

Proof of (15-33): Finally note that if  $f \in \mathcal{B}_{p,q}^s$  and  $\epsilon > 0$  then there is a  $\mathcal{B}_{p,q}^s$ -representation of  $f$

$$f = \sum_{P \in \mathcal{P}} k_P a_P$$

such that

$$\left( \sum_{k=0}^{\infty} \left( \sum_{Q \in \mathcal{P}^k} |k_Q|^p \right)^{q/p} \right)^{1/q} \leq (1+\epsilon) |f|_{\mathcal{B}_{p,q}^s}.$$

For each  $J \in \mathcal{P}^{k_0}$ , choose  $x_J \in J$ . Then

$$\begin{aligned}
 \left( \sum_{J \in \mathcal{P}^{k_0}} |J|^{-sp} \operatorname{osc}_p(f, J)^p \right)^{1/p} &\leq \left( \sum_{J \in \mathcal{P}^{k_0}} |J|^{-sp} \int_J \left| f(x) - \sum_{\substack{Q \in \mathcal{P} \\ J \subset Q}} k_Q a_Q(x_J) \right|^p dm \right)^{1/p} \\
 &\leq \left( \sum_{J \in \mathcal{P}^{k_0}} |J|^{-sp} \int \left| \sum_{\substack{k > k_0 \\ R \subset J}} \sum_{R \in \mathcal{P}^k} k_R a_R \right|^p dm \right)^{1/p} \\
 &\leq \left( \int \left| \sum_{k > k_0} \sum_{J \in \mathcal{P}^{k_0}} |J|^{-s} \sum_{\substack{R \in \mathcal{P}^k \\ R \subset J}} k_R a_R \right|^p dm \right)^{1/p} \\
 &\leq \sum_{k > k_0} \left( \int \left| \sum_{J \in \mathcal{P}^{k_0}} |J|^{-s} \sum_{\substack{R \in \mathcal{P}^k \\ R \subset J}} k_R a_R \right|^p dm \right)^{1/p} \\
 &\leq \sum_{k > k_0} \left( \sum_{J \in \mathcal{P}^{k_0}} |J|^{-sp} \int \left| \sum_{\substack{R \in \mathcal{P}^k \\ R \subset J}} k_R a_R \right|^p dm \right)^{1/p} \\
 &\leq \sum_{k > k_0} \left( \sum_{J \in \mathcal{P}^{k_0}} \left( \frac{\sup\{|R| : R \in \mathcal{P}^k, R \subset J\}}{|J|} \right)^{sp} \sum_{\substack{R \in \mathcal{P}^k \\ R \subset J}} |k_R|^p \right)^{1/p} \\
 &\leq \sum_{k > k_0} \left( \lambda_2^{sp(k-k_0)} \sum_{J \in \mathcal{P}^{k_0}} \sum_{\substack{R \in \mathcal{P}^k \\ R \subset J}} |k_R|^p \right)^{1/p} \\
 &\leq \sum_{k > k_0} \lambda_2^{s(k-k_0)} \left( \sum_{R \in \mathcal{P}^k} |k_R|^p \right)^{1/p}.
 \end{aligned} \tag{15-35}$$

This is a convolution, so

$$\left( \sum_{k_0} \left( \sum_{J \in \mathcal{P}^{k_0}} |J|^{-sp} \operatorname{osc}_p(f, J)^p \right)^{q/p} \right)^{1/q} \leq \frac{1+\epsilon}{1-\lambda_2^s} |f|_{\mathcal{B}_{p,q}^s},$$

and since  $\epsilon > 0$  is arbitrary, by Proposition 6.1 we obtain

$$|I|^{-s} |f|_p + \left( \sum_{k_0} \left( \sum_{J \in \mathcal{P}^{k_0}} |J|^{-sp} \operatorname{osc}_p(f, J)^p \right)^{q/p} \right)^{1/q} \leq \left( C_1^{1+1/p} C_2(t, q, (|\mathcal{P}^k|^{sp})_k) |I|^{-s} + \frac{1}{1-\lambda_2^s} \right) |f|_{\mathcal{B}_{p,q}^s}.$$

This proves (15-33).  $\square$

The following is an important consequence of this section.

**Corollary 15.2.** *For each  $P \in \mathcal{P}$  there exists a linear functional in  $L^1$*

$$f \mapsto k_P^f$$

with the following property. The so-called standard  $\mathcal{B}_{p,q}^s$ -representation of  $f \in \mathcal{B}_{p,q}^s$  given by

$$f = \sum_k \sum_{P \in \mathcal{P}^k} k_P^f a_P$$

satisfies

$$\left( \sum_k \left( \sum_{P \in \mathcal{P}^k} |k_P^f|^p \right)^{q/p} \right)^{1/p} \leq C_{10} C_{11} C_{12} |f|_{\mathcal{B}_{p,q}^s}.$$

## 16. Alternative characterizations, II: Messing with atoms

Here we move to alternative descriptions of  $\mathcal{B}_{p,q}^s$ , which are quite different from those in Section 15. Instead of choosing a definitive representation of elements of  $\mathcal{B}_{p,q}^s$ , we indeed give atomic decompositions of  $\mathcal{B}_{p,q}^s$  using far more general classes of atoms.

**16A. Using Besov's atoms.** The advantage of Besov's atoms is that they are a wide and general class of atoms, which includes even unbounded functions. They can be considered in *every* measure space endowed with a good grid as in Section 3. Moreover in appropriate settings they contain Hölder and bounded variations atoms, which will be quite useful in the other characterizations of  $\mathcal{B}_{p,q}^s(\mathcal{P}, \mathcal{A}_{s,p}^{sz})$ . The atomic decompositions of  $\mathcal{B}_{p,q}^s(\mathbb{R}^n)$  by Besov's atoms were considered in [Triebel 2003] in the case  $s > 0$ ,  $p = q \in [1, \infty]$  and in [Schneider and Vybíral 2013] in the case  $s > 0$ ,  $p, q \in [1, \infty]$ . They are referred there as “nonsmooth atomic decompositions”.

Let  $s < \beta$  and  $\tilde{q} \in [1, \infty]$ . An  $(s, \beta, p, \tilde{q})$ -Besov atom on the interval  $Q$  is a function  $a \in \mathcal{B}_{p,\tilde{q}}^\beta(Q, \mathcal{P}_Q, \mathcal{A}_{\beta,p}^{sz})$  such that  $a(x) = 0$  for  $x \notin Q$ , and

$$|a|_{\mathcal{B}_{p,\tilde{q}}^\beta(Q, \mathcal{P}_Q, \mathcal{A}_{\beta,p}^{sz})} \leq \frac{1}{C_{13}} |Q|^{s-\beta},$$

where

$$C_{13} = C_1^{1+1/p} \left( \sum_k \lambda_2^{k\beta\tilde{q}} \right)^{1/\tilde{q}} \geq 1.$$

The family of  $(s, \beta, p, \tilde{q})$ -Besov atoms supported on  $Q$  will be denoted by  $\mathcal{A}_{s,\beta,p,\tilde{q}}^{bv}(Q)$ . Naturally  $\mathcal{A}_{s,p}^{sz}(Q) \subset \mathcal{A}_{s,\beta,p,\tilde{q}}^{bs}(Q)$ . By Proposition 6.1 we have

$$\begin{aligned} |a|_p &\leq \frac{C_1^{1+1/p}}{C_{13}} \left( \sum_k |\mathcal{P}_Q^k|^{\beta\tilde{q}} \right)^{1/\tilde{q}} |a|_{\mathcal{B}_{p,\tilde{q}}^\beta(Q, \mathcal{P}_Q, \mathcal{A}_{\beta,p}^{sz})} \\ &\leq |Q|^{s-\beta} |Q|^\beta = |Q|^s = |Q|^{s-1/\infty}, \end{aligned} \quad (16-36)$$

so an  $(s, \beta, p, \tilde{q})$ -Besov atom is an atom of type  $(s, p, 1)$ .

The following result says there are many ways to define  $\mathcal{B}_{p,q}^s$  using various classes of atoms.

**Proposition 16.1** (Souza's atoms and Besov's atoms). *Let  $\mathcal{P}$  be a good grid. Let  $\mathcal{A}$  be a class of  $(s, p, u)$ -atoms, with  $u \geq 1$ , such that for some  $s < \beta$ ,  $\tilde{q} \in [1, \infty]$ , and  $C_{14}, C_{15} \geq 0$  we have that for every  $Q \in \mathcal{P}$*

$$\frac{1}{C_{14}} \mathcal{A}_{s,p}^{sz}(Q) \subset \mathcal{A}(Q) \subset C_{15} \mathcal{A}_{s,\beta,p,\tilde{q}}^{bs}(Q). \quad (16-37)$$

Then

$$\mathcal{B}_{p,q}^s(\mathcal{P}, \mathcal{A}_{s,p}^{sz}) = \mathcal{B}_{p,q}^s(\mathcal{P}, \mathcal{A}) = \mathcal{B}_{p,q}^s(\mathcal{P}, \mathcal{A}_{s,\beta,p,\tilde{q}}^{bs}).$$

Moreover

$$|f|_{\mathcal{B}_{p,q}^s(\mathcal{A})} \leq C_{14} |f|_{\mathcal{B}_{p,q}^s(\mathcal{A}_{s,p}^{sz})} \quad \text{and} \quad |f|_{\mathcal{B}_{p,q}^s(\mathcal{A}_{s,p}^{sz})} \leq \frac{C_{15}}{1 - \lambda_2^{\beta-s}} |f|_{\mathcal{B}_{p,q}^s(\mathcal{A})}.$$

*Proof.* The first inequality is obvious. To prove the second inequality, recall that due to Proposition 8.1 it is enough to show the following claim.

**Claim.** *Let  $b_J$  be an  $(s, \beta, p, \tilde{q})$ -Besov atom on  $J \in \mathcal{P}^j$ . Then for every  $P \subset J$  with  $P \in \mathcal{P}$  there is  $m_P \in \mathbb{C}$  such that*

$$b_J = \sum_{P \subset J} m_P a_P,$$

where  $a_P$  is the canonical  $(s, p)$ -Souza's atom on  $P$  and

$$\left( \sum_{P \in \mathcal{P}^k, P \subset J} |m_P|^p \right)^{1/p} \leq \frac{2}{C_{13}} \lambda_2^{(k-j)(\beta-s)}.$$

Indeed, since

$$|b_J|_{\mathcal{B}_{p,\tilde{q}}^\beta(J, \mathcal{P}, \mathcal{A}_{\beta,p}^s)} \leq \frac{1}{C_{13}} |J|^{s-\beta},$$

there exists a  $\mathcal{B}_{p,\tilde{q}}^\beta(J, \mathcal{P}, \mathcal{A}_{\beta,p}^s)$ -representation

$$b_J = \sum_{P \in \mathcal{P}, P \subset J} c_P d_P,$$

where  $d_P$  is the canonical  $(\beta, p)$ -Souza's atom on  $P$  and

$$\left( \sum_{P \in \mathcal{P}^k, P \subset J} |c_P|^p \right)^{1/p} \leq \left( \sum_i \left( \sum_{P \in \mathcal{P}^i, P \subset J} |c_P|^p \right)^{\tilde{q}/p} \right)^{1/\tilde{q}} \leq \frac{2}{C_{13}} |J|^{s-\beta}.$$

Then

$$a_P = |P|^{s-\beta} d_P$$

is an  $(s, p)$ -Souza's atom and

$$b_J = \sum_{P \in \mathcal{P}, P \subset J} m_P a_P,$$

with  $m_P = c_P |P|^{\beta-s}$  and

$$\begin{aligned} \left( \sum_{P \in \mathcal{P}^k, P \subset J} |m_P|^p \right)^{1/p} &= \left( \sum_{P \in \mathcal{P}^k, P \subset J} \left( \frac{|P|}{|J|} \right)^{p(\beta-s)} |J|^{p(\beta-s)} |c_P|^p \right)^{1/p} \\ &\leq \lambda_2^{(k-j)(\beta-s)} |J|^{\beta-s} \left( \sum_{P \in \mathcal{P}^k, P \subset J} |c_P|^p \right)^{1/p} \leq \frac{2}{C_{13}} \lambda_2^{(k-j)(\beta-s)}. \quad \square \end{aligned}$$

**16B. Using Hölder atoms.** Suppose that  $I$  is a quasimetric space with a quasidistance  $d(\cdot, \cdot)$  and a good grid satisfying the assumptions in Section 11B.

**Proposition 16.2.** *Suppose*

$$0 < s < \beta < \tilde{\beta},$$

$p \in [1, \infty)$  and  $\tilde{q} \in [1, \infty]$ . For every  $Q \in \mathcal{P}$  we have

$$C_{16} \mathcal{A}_{s, \tilde{\beta}, p}^h(Q) \subset \mathcal{A}_{s, \beta, p, \tilde{q}}^{bs}(Q) \quad (16-38)$$

for some  $C_{16} > 0$ . Moreover

$$C_{17} \mathcal{A}_{s, \beta, p}^h(Q) \subset \mathcal{A}_{s, \beta, p, \infty}^{bs}(Q) \quad (16-39)$$

for some  $C_{17} > 0$ . In particular

$$\mathcal{B}_{p, q}^s = \mathcal{B}_{p, q}^s(\mathcal{A}_{s, \tilde{\beta}, p}^h) = \mathcal{B}_{p, q}^s(\mathcal{A}_{s, \beta, p}^h(Q)) \quad (16-40)$$

and the corresponding norms are equivalent.

*Proof.* Let  $\phi \in \mathcal{A}_{s, \tilde{\beta}, p}^h(Q)$ . Then  $\phi$  has a continuous extension to  $\bar{Q}$ . So firstly we assume that  $\phi \geq 0$  has a continuous extension to  $\bar{Q}$ . Define

$$c_Q = \min \phi(Q) |Q|^{1/p-\beta}$$

and for every  $P \subset W \subset Q \in \mathcal{P}^j$  with  $P \in \mathcal{P}^{k+1}$  and  $W \in \mathcal{P}^k$  define

$$c_P = (\inf \phi(P) - \inf \phi(W)) |P|^{1/p-\beta}. \quad (16-41)$$

Of course in this case  $c_P \geq 0$  and

$$\begin{aligned} |c_P| &\leq (\inf \phi(P) - \inf \phi(W)) |P|^{1/p-\beta} \leq |Q|^{s-1/p-\tilde{\beta}} (\text{diam } W)^{\tilde{\beta}D} |P|^{1/p-\beta} \\ &\leq \frac{C_8^{D\tilde{\beta}}}{\lambda_3^{\tilde{\beta}}} \left( \frac{|P|}{|Q|} \right)^{1/p} |Q|^{s-\tilde{\beta}} |P|^{\tilde{\beta}-\beta} \leq \frac{C_8^{D\tilde{\beta}}}{\lambda_3^{\tilde{\beta}}} C_{18} \left( \frac{|P|}{|Q|} \right)^{1/p} |Q|^{s-\beta} \left( \frac{|P|}{|Q|} \right)^{\tilde{\beta}-\beta}. \end{aligned}$$

Here  $C_{18} = \sup_{Q \in \mathcal{P}} |Q|^{\tilde{\beta} - \beta}$ . Consequently

$$\begin{aligned} \sum_{\substack{P \in \mathcal{P}^{k+1} \\ P \subset Q}} |c_P|^p &\leq |Q|^{p(s-\beta)} \lambda_2^{(k+1-j)p(\tilde{\beta}-\beta)} \sum_{\substack{P \in \mathcal{P}^{k+1} \\ P \subset Q}} \frac{C_{18}^p C_8^{D\tilde{\beta}p}}{\lambda_3^{p\tilde{\beta}}} \frac{|P|}{|Q|} \\ &\leq |Q|^{p(s-\beta)} \frac{C_{18}^p C_8^{D\tilde{\beta}p}}{\lambda_3^{p\tilde{\beta}}} \lambda_2^{(k+1-j)p(\tilde{\beta}-\beta)}, \end{aligned} \quad (16-42)$$

so

$$\left( \sum_k \left( \sum_{\substack{P \in \mathcal{P}^{k+1} \\ P \subset Q}} |c_P|^p \right)^{\tilde{q}/p} \right)^{1/\tilde{q}} \leq \frac{C_{18} C_8^{D\tilde{\beta}}}{\lambda_3^\beta} \left( \frac{1}{1 - \lambda_2^{\tilde{q}(\tilde{\beta}-\beta)}} \right)^{1/\tilde{q}} |Q|^{s-\beta}.$$

This implies that

$$\tilde{\phi} = \sum_{\substack{P \in \mathcal{P} \\ P \subset Q}} c_P a_P,$$

where  $a_P$  is the canonical  $(\beta, p)$ -atom on  $P$ , is a  $\mathcal{B}_{p,\tilde{q}}^\beta(Q, \mathcal{P}_Q \mathcal{A}_{\beta,p}^{sz})$ -representation of a function  $\tilde{\phi}$ . From (16-41) it follows that

$$\sum_{k \leq N} \sum_{P \in \mathcal{P}^k} c_P a_P(x) = \min \phi(W)$$

for every  $x \in W \in \mathcal{P}^N$ . In particular

$$\lim_{N \rightarrow \infty} \sum_{k \leq N} \sum_{P \in \mathcal{P}^k} c_P a_P(x) = \phi(x)$$

for almost every  $x$ , so  $\tilde{\phi} = \phi$ . So

$$|C_{16} \phi|_{\mathcal{B}_{p,\tilde{q}}^\beta(Q, \mathcal{P}_Q \mathcal{A}_{\beta,p}^{sz})} \leq \frac{1}{2} |Q|^{s-\beta}, \quad (16-43)$$

where

$$C_{16} = \left( 2 \frac{C_{18} C_8^{D\tilde{\beta}}}{\lambda_3^\beta} \left( \frac{1}{1 - \lambda_2^{\tilde{q}(\tilde{\beta}-\beta)}} \right)^{1/\tilde{q}} \right)^{-1}.$$

In the general case, note that  $\phi_+(x) = \max\{\phi, 0\}$ ,  $\phi_-(x) = \min\{-\phi, 0\} \in \mathcal{A}_{s,\tilde{\beta},p}^h(Q)$  and  $\phi = \phi_+ - \phi_-$ . Applying (16-43) to  $\phi_+$  and  $\phi_-$  we obtain (16-38).

The second inclusion (16-39) in the proposition can be obtained taking  $\tilde{\beta} = \beta$  in (16-42) and making a few modifications in the above argument. By Proposition 16.1 we have (16-40).  $\square$

**16C. Using bounded variation atoms.** Now suppose that  $I$  is an interval of  $\mathbb{R}$  of length 1,  $m$  is the Lebesgue measure on it and the partitions in  $\mathcal{P}$  are partitions by intervals.

**Proposition 16.3.** *If*

$$0 < s \leq \beta \leq \frac{1}{p}$$

then

$$C_{20} \mathcal{A}_{s,\beta,p}^{bv}(Q) \subset \mathcal{A}_{s,\beta,p,\infty}^{bs}(Q)$$

for every  $Q \in \mathcal{P}$ . If

$$0 < s \leq \beta < \tilde{\beta} \leq \frac{1}{p}$$

then

$$C_{21} \mathcal{A}_{s, \tilde{\beta}, p}^{bv}(Q) \subset \mathcal{A}_{s, \beta, p, q}^{bs}(Q)$$

for every  $q \in [1, \infty]$ . In particular

$$\mathcal{B}_{p, q}^s = \mathcal{B}_{p, q}^s(\mathcal{A}_{s, \beta, p}^{bv}) = \mathcal{B}_{p, q}^s(\mathcal{A}_{s, \tilde{\beta}, p}^{bv})$$

and the corresponding norms are equivalents.

*Proof.* Suppose

$$0 < s \leq \beta \leq \tilde{\beta} \leq 1.$$

Let  $Q \in \mathcal{P}^j$  and  $a_Q \in \mathcal{A}_{s, \tilde{\beta}, p}^{bv}(Q)$ . We have

$$|a_Q|_p \leq (|Q| |Q|^{sp-1})^{1/p} = |Q|^s \leq C_{19} |Q|^{s-\beta},$$

where  $C_{19} = \sup_{Q \in \mathcal{P}} |Q|^\beta$ . Note that for every  $k \geq j$

$$\begin{aligned} \sum_{W \in \mathcal{P}^k} |W|^{-\beta p} \text{osc}_p(a_Q, W)^p &\leq \sum_{W \subset Q, W \in \mathcal{P}^k} |W|^{1-\beta p} \text{osc}_\infty(a_Q, W)^p \\ &\leq \left( \sum_{W \subset Q, W \in \mathcal{P}^k} |W|^{(1-\beta p)/(1-\tilde{\beta} p)} \right)^{1-\tilde{\beta} p} \left( \sum_{W \subset Q, W \in \mathcal{P}^k} \text{osc}_\infty(a_Q, W)^{1/\tilde{\beta}} \right)^{\tilde{\beta} p} \\ &\leq \left( \max_{W \subset Q, W \in \mathcal{P}^k} |W|^{((\tilde{\beta}-\beta)p)/(1-\tilde{\beta} p)} \right)^{1-\tilde{\beta} p} \left( \sum_{W \subset Q, W \in \mathcal{P}^k} |W| \right)^{1-\tilde{\beta} p} (\text{var}_{1/\tilde{\beta}}(a_Q, Q))^p \\ &\leq \lambda_2^{(k-j)(\tilde{\beta}-\beta)p} |Q|^{(\tilde{\beta}-\beta)p} |Q|^{1-\tilde{\beta} p} |Q|^{sp-1} \leq \lambda_2^{(k-j)(\tilde{\beta}-\beta)p} |Q|^{(s-\beta)p}. \end{aligned}$$

Note that in the case  $\tilde{\beta} p = 1$  the argument above needs a simple modification. For  $k < j$  let  $W_Q \in \mathcal{P}^k$  be such that  $Q \subset W_Q$ . Then

$$\begin{aligned} \sum_{W \in \mathcal{P}^k} |W|^{-\beta p} \text{osc}_p(a_Q, W)^p &\leq |W_Q|^{-\beta p} |Q| |Q|^{sp-1} \leq |W_Q|^{-\beta p} |Q|^{sp} \\ &= \left( \frac{|Q|}{|W_Q|} \right)^{\beta p} |Q|^{(s-\beta)p} \leq \lambda_2^{(j-k)\beta p} |Q|^{(s-\beta)p}. \end{aligned}$$

By Theorem 15.1 we have that if  $\beta = \tilde{\beta}$  then  $C_{20} \mathcal{A}_{s, \beta, p}^{bv}(Q) \subset \mathcal{A}_{s, \beta, p, \infty}^{bs}(Q)$  for some  $C_{20} > 0$ , and if  $\beta < \tilde{\beta}$  we have that for every  $q \in [1, \infty)$

$$|I|^{-s} |a_Q|_p + \left( \sum_k \left( \sum_{W \in \mathcal{P}^k} |W|^{-sp} \text{osc}_p(a_Q, W)^p \right)^{q/p} \right)^{1/q} \leq \left( C_{19} |I|^{-s} + \left( \frac{1}{1-\lambda^q(\tilde{\beta}-\beta)} + \frac{1}{1-\lambda_2^{q\beta}} \right)^{1/q} \right) |Q|^{s-\beta},$$

so  $C_{20} \mathcal{A}_{s, \tilde{\beta}, p}^{bv}(Q) \subset \mathcal{A}_{s, \beta, p, q}^{bs}(Q)$  for some  $C_{21} > 0$ .  $\square$

### 17. Dirac's approximations

We will use the Haar basis and notation defined by Section 14. For every  $x_0 \in I$  and  $k_0 \in \mathbb{N}$  define the finite family

$$\mathcal{S}_{x_0}^{k_0} = \left\{ S : S = (S_1, S_2) \in \mathcal{H}(Q), \text{ with } Q \in \mathcal{P}^k, k < k_0, x_0 \in \bigcup_{a=1,2} \bigcup_{P \in S_a} P \right\}.$$

Let  $N(x_0, k_0) = \#\mathcal{S}_{x_0}^{k_0}$ . Then we can enumerate the elements

$$S^1, S^2, \dots, S^{N(x_0, k_0)}$$

of  $\mathcal{S}_{x_0}^{k_0}$  such that  $S^i = (S_1^i, S_2^i)$  satisfies

$$x_0 \in \bigcup_{P \in S_{a_i}^i} P$$

for some  $a_i \in \{1, 2\}$  and

$$\bigcup_{P \in S_1^{i+1} \cup S_2^{i+1}} P \subset \bigcup_{Q \in S_1^i \cup S_2^i} Q$$

for every  $i$ . Let

$$\psi_0 = \frac{1_I}{|I|}$$

and define for  $i > 0$

$$\psi_i = (-1)^{a_i+1} \frac{\sum_{R \in S_{3-a_i}^i} |R|}{\sum_{Q \in S_{a_i-1}^{i-1}} |Q|} \left( \frac{\sum_{P \in S_1^i} 1_P}{\sum_{P \in S_1^i} |P|} - \frac{\sum_{R \in S_2^i} 1_R}{\sum_{R \in S_2^i} |R|} \right) = (-1)^{a_i+1} \frac{\sum_{R \in S_{3-a_i}^i} |R|}{\sum_{Q \in S_{a_i-1}^{i-1}} |Q|} m_{S^i} \phi_i.$$

One can prove by induction on  $j$  that for  $j > 0$

$$\sum_{i=0}^j \psi_i = \frac{\sum_{P \in S_{a_j}^j} 1_P}{\sum_{P \in S_{a_j}^j} |P|},$$

and in particular

$$\sum_{i=0}^{N(x_0, k_0)} \psi_i(x_0) = \frac{1_{Q_{k_0}}}{|Q_{k_0}|},$$

where  $x_0 \in Q_{k_0} \in \mathcal{P}^{k_0}$ . In other words

$$\frac{1}{|I|^{1/2}} \phi_I + \sum_{i=1}^{N(x_0, k_0)} (-1)^{a_i+1} \left( \frac{\sum_{R \in S_{3-a_i}^i} |R|}{\sum_{Q \in S_{a_i-1}^{i-1}} |Q|} \right) m_{S^i} \phi_{S^i} = \frac{1_{Q_{k_0}}}{|Q_{k_0}|}. \quad (17-44)$$

Note that

$$\begin{aligned} \left( \frac{\sum_{R \in S_{3-a_i}^i} |R|}{\sum_{Q \in S_{a_i-1}^{i-1}} |Q|} \right) m_{S^i} &= \left( \frac{\sum_{R \in S_{3-a_i}^i} |R|}{\sum_{Q \in S_{a_i-1}^{i-1}} |Q|} \right) \left( \frac{\sum_{R \in S_2^i} |R| + \sum_{P \in S_1^i} |P|}{(\sum_{R \in S_2^i} |R|)(\sum_{P \in S_1^i} |P|)} \right)^{1/2} \\ &= \left( \frac{\sum_{R \in S_{3-a_i}^i} |R|}{\sum_{P \in S_{a_i}^i} |P|} \right)^{1/2} \frac{1}{(\sum_{Q \in S_{a_i-1}^{i-1}} |Q|)^{1/2}}. \end{aligned} \quad (17-45)$$

Multiplying (17-44) by  $f$ , integrating it term by term, and using (17-45) we obtain

$$\frac{d_I}{|I|^{1/2}} + \sum_{i=1}^{N(x_0, k_0)} (-1)^{a_i+1} \left( \frac{\sum_{R \in \mathcal{S}_{3-a_i}^i} |R|}{\sum_{P \in \mathcal{S}_{a_i}^i} |P|} \right)^{1/2} \frac{1}{(\sum_{Q \in \mathcal{S}_{a_i-1}^{i-1}} |Q|)^{1/2}} d_{S^i} = \int f \cdot \frac{1_{Q_{k_0}}}{|Q_{k_0}|} dm.$$

If  $f \in L^\beta$ , with  $\beta > 1$ , it can be written as

$$f = \sum_{S \in \widehat{\mathcal{H}}(\mathcal{P})} d_S \phi_S,$$

with  $d_S = \int f \phi_S dm$ , where this series converges unconditionally on  $L^\beta$ . Let

$$f_{k_0} = d_I \phi_I + \sum_{k < k_0} \sum_{Q \in \mathcal{P}^k} \sum_{S \in \widehat{\mathcal{H}}(Q)} d_S \phi_S. \quad (17-46)$$

Then

$$\begin{aligned} f_{k_0}(x_0) &= d_I \phi_I + \sum_{i=1}^{N(x_0, k_0)} d_{S^i} \phi_{S^i}(x_0). \\ &= \frac{d_I}{|I|^{1/2}} + \sum_{i=1}^{N(x_0, k_0)} (-1)^{a_i+1} d_{S^i} \frac{1}{m_{S^i}} \frac{1}{\sum_{P \in \mathcal{S}_{a_i}^i} |P|} \\ &= \frac{d_I}{|I|^{1/2}} + \sum_{i=1}^{N(x_0, k_0)} (-1)^{a_i+1} d_{S^i} \left( \frac{(\sum_{R \in \mathcal{S}_2^i} |R|)(\sum_{P \in \mathcal{S}_1^i} |P|)}{\sum_{R \in \mathcal{S}_2^i} |R| + \sum_{P \in \mathcal{S}_1^i} |P|} \right)^{1/2} \frac{1}{\sum_{P \in \mathcal{S}_{a_i}^i} |P|} \\ &= \frac{d_I}{|I|^{1/2}} + \sum_{i=1}^{N(x_0, k_0)} (-1)^{a_i+1} d_{S^i} \left( \frac{\sum_{R \in \mathcal{S}_{3-a_i}^i} |R|}{\sum_{P \in \mathcal{S}_{a_i}^i} |P|} \right)^{1/2} \frac{1}{(\sum_{Q \in \mathcal{S}_{a_i-1}^{i-1}} |Q|)^{1/2}} = \int f \cdot \frac{1_{Q_{k_0}}}{|Q_{k_0}|} dm. \end{aligned}$$

Let

$$f = \sum_k \sum_{P \in \mathcal{P}^k} k_P a_P \quad (17-47)$$

be the series given by (15-28). Note that

$$f_{k_0} = \sum_{k \leq k_0} \sum_{P \in \mathcal{P}^k} k_P a_P. \quad (17-48)$$

Consequently:

**Proposition 17.1** (Dirac's approximations). *Let  $f \in L^\beta$ , with  $\beta > 1$ . Let*

$$f = \sum_k \sum_{P \in \mathcal{P}^k} k_P a_P.$$

(A) *If this representation is either as in (15-28) or  $k_P \geq 0$  for every  $P$  then we have for every  $Q \in \mathcal{P}$*

$$\left| \sum_{J \in \mathcal{P}, Q \subset J} k_J |J|^{s-1/p} \right| \leq |f 1_Q|_\infty.$$

(B) *In the case of the representation (15-28) we also have*

$$\sum_{J \in \mathcal{P}, Q \subset J} k_J |J|^{s-1/p} = \int f \cdot \frac{1_Q}{|Q|} dm.$$

*Proof.* We have that (A) is obvious if  $k_P \geq 0$  for every  $P$ . In the other case note that for every  $x_0 \in Q \in \mathcal{P}^{k_0}$  we have

$$f_{k_0}(x_0) = \sum_{J \in \mathcal{P}, Q \subset J} k_J |J|^{s-1/p} = \int f \cdot \frac{1_Q}{|Q|} dm,$$

so (A) and (B) follow. □

### Part III. Applications

In Part III, we suppose that  $0 < s < 1/p$ ,  $p \in [1, \infty)$  and  $q \in [1, \infty]$ .

#### 18. Pointwise multipliers acting on $\mathcal{B}_{p,q}^s$

Here we will apply the previous sections to study pointwise multipliers of  $\mathcal{B}_{p,q}^s$ . To be more precise, let  $g : I \rightarrow \mathbb{C}$  be a measurable function. We say that  $g$  is a pointwise multiplier acting on  $\mathcal{B}_{p,q}^s$  if the transformation

$$G(f) = gf$$

defines a bounded operator in  $\mathcal{B}_{p,q}^s$ . We denote the set of pointwise multipliers by  $M(\mathcal{B}_{p,q}^s)$ . We can consider the norm on  $M(\mathcal{B}_{p,q}^s)$  given by

$$|g|_{M(\mathcal{B}_{p,q}^s)} = \sup\{|gf|_{\mathcal{B}_{p,q}^s} : |f|_{\mathcal{B}_{p,q}^s} \leq 1\}.$$

Of course a necessary condition for a function to be a multiplier is that

$$\mathcal{B}_{p,q,\text{selfs}}^s = \{g \in \mathcal{B}_{p,q}^s : \sup_{a_Q \in \mathcal{A}_{s,p}^{sz}} |ga_Q|_{\mathcal{B}_{p,q}^s} < \infty\}.$$

Define

$$|g|_{\mathcal{B}_{p,q,\text{selfs}}^s} = \sup_{a_Q \in \mathcal{A}_{s,p}^{sz}} |ga_Q|_{\mathcal{B}_{p,q}^s}.$$

The linear space  $\mathcal{B}_{p,q,\text{selfs}}^s$  endowed with  $|\cdot|_{\mathcal{B}_{p,q,\text{selfs}}^s}$  is a normed space introduced in [Triebel 2003]. We have

$$|g|_{\mathcal{B}_{p,q,\text{selfs}}^s} \leq |g|_{M(\mathcal{B}_{p,q}^s)}.$$

In the following three propositions we see that many results of [Triebel 2003; Schneider and Vybíral 2013] for Besov spaces in  $\mathbb{R}^n$  can be easily moved to our setting. The simplest case occurs when  $p = q = 1$ .

**Proposition 18.1.** *We have that  $M(\mathcal{B}_{1,1}^s) = \mathcal{B}_{1,1,\text{selfs}}^s$ .*

*Proof.* Let  $g \in \mathcal{B}_{1,1,\text{selfs}}^s$ . Given  $f \in \mathcal{B}_{1,1}^s$  and  $\epsilon > 0$  one can find a  $\mathcal{B}_{1,1}^s$ -representation

$$f = \sum_k \sum_{Q \in \mathcal{P}^k} c_Q a_Q,$$

where  $a_Q$  is an  $(s, 1)$ -Souza's atom and

$$\sum_k \sum_{Q \in \mathcal{P}^k} |c_Q| < (1 + \epsilon) |f|_{\mathcal{B}_{1,1}^s},$$

so

$$\sum_k \sum_{Q \in \mathcal{P}^k} |c_Q g a_Q|_{\mathcal{B}_{1,1}^s} < (1 + \epsilon) |g|_{\mathcal{B}_{1,1,\text{self}}^s} |f|_{\mathcal{B}_{1,1}^s},$$

and consequently

$$|gf|_{\mathcal{B}_{1,1}^s} = \left| \sum_k \sum_{Q \in \mathcal{P}^k} c_Q g a_Q \right|_{\mathcal{B}_{1,1}^s} \leq (1 + \epsilon) |g|_{\mathcal{B}_{1,1,\text{self}}^s} |f|_{\mathcal{B}_{1,1}^s}.$$

Since  $\epsilon$  is arbitrary we get

$$|gf|_{\mathcal{B}_{1,1}^s} \leq |g|_{\mathcal{B}_{1,1,\text{self}}^s} |f|_{\mathcal{B}_{1,1}^s}. \quad \square$$

**Lemma 18.2.** *Let  $W \in \mathcal{P}$ . The restriction application*

$$r : \mathcal{B}_{p,q}^s(I, \mathcal{P}, \mathcal{A}_{s,p}^{sz}) \rightarrow \mathcal{B}_{p,q}^s(W, \mathcal{P}_W, \mathcal{A}_{s,p}^{sz})$$

given by  $r(f) = 1_W f$  is continuous. Indeed there is  $C_{22} \geq 1$  that does not depend on  $W$  such that:

(A) For every  $f \in \mathcal{B}_{p,q}^s$  we have

$$|1_W f|_{\mathcal{B}_{p,q}^s(W, \mathcal{P}_W, \mathcal{A}_{s,p}^{sz})} \leq C_{22} |f|_{\mathcal{B}_{p,q}^s}. \quad (18-49)$$

In particular  $1_W \in M(\mathcal{B}_{p,q}^s)$ .

(B) For every  $\mathcal{B}_{p,q}^{s+}$ -representation

$$f = \sum_k \sum_{Q \in \mathcal{P}^k} c_Q a_Q$$

one can find a  $\mathcal{B}_{p,q}^{s+}(W, \mathcal{P}_W, \mathcal{A}_{s,p}^{sz})$ -representation

$$1_W f = \sum_k \sum_{Q \in \mathcal{P}^k} d_Q a_Q$$

such that

$$\left( \sum_k \left( \sum_{Q \in \mathcal{P}^k} (d_Q)^p \right)^{q/p} \right)^{1/q} \leq C_{22} \left( \sum_k \left( \sum_{Q \in \mathcal{P}^k} (c_Q)^p \right)^{q/p} \right)^{1/q}. \quad (18-50)$$

Moreover  $d_Q \neq 0$  implies  $Q \subset \text{supp } f$ .

*Proof.* Let  $Q \in \mathcal{P}$ . Denote by  $a_Q$  the canonical  $(s, p)$ -Souza's atom supported on  $Q$ . If  $W \cap Q = \emptyset$ , we can write  $1_W a_Q = 0 a_Q$ . If  $Q \subset W$  then  $1_W a_Q = 1 a_Q$ . If  $W \subset Q$  then

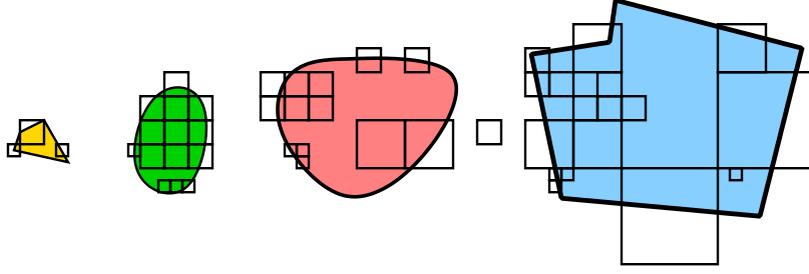
$$1_W a_Q = \left( \frac{|W|}{|Q|} \right)^{1/p-s} a_W,$$

where

$$\left( \frac{|W|}{|Q|} \right)^{1/p-s} \leq \lambda_2^{(k_0(W) - k_0(Q))(1/p-s)}.$$

In every case we can write

$$h_Q = 1_W a_Q = \sum_k \sum_{\substack{P \in \mathcal{P}^k \\ P \subset Q}} s_{P,Q} a_P,$$



**Figure 3.** (Non-Archimedean behaviour) Illustration for Proposition 18.4. The filled regions are the supports of the functions  $g_i$ . The squares are the elements of the grid on which there are atoms contributing to the representation of  $f$ . Every square intercepts at most one support, so each atom “sees” only the function whose support is nearby.

with

$$\sum_{\substack{P \in \mathcal{P}^k \\ P \subset Q}} |s_{P,Q}|^p \leq \lambda_2^{(k-k_0(Q))(1-sp)}.$$

By Proposition 8.1(A) and (B) there is  $C_{22}$  such that (A) and (B) hold. □

**Proposition 18.3.** *We have that  $\mathcal{B}_{p,q,\text{self}s}^s \subset L^\infty$  and this inclusion is continuous.*

*Proof.* Let  $g \in \mathcal{B}_{p,q,\text{self}s}^s$ . Then  $g a_Q \in \mathcal{B}_{p,q}^s$  and by Lemma 18.2 we have

$$|g 1_Q|_{\mathcal{B}_{p,q}^s(Q, \mathcal{P}_Q, \mathcal{A}_{s,p}^{sz})} \leq C_{22} |g 1_Q|_{\mathcal{B}_{p,q}^s} \leq C_{22} |g|_{\mathcal{B}_{p,q,\text{self}s}^s}.$$

By Proposition 6.1 (taking  $t = p$ ) we have  $(s, p)$ -Souza’s atom  $a_Q$ , and we have

$$|g a_Q|_p \leq C_1^{1+1/p} C_2(p, q, (|\mathcal{P}_Q^k|^{sp})_k) C_{22} |g|_{\mathcal{B}_{p,q,\text{self}s}^s} < C_{23} |Q|^s |g|_{\mathcal{B}_{p,q,\text{self}s}^s}$$

for some constant  $C_{23}$ . In other words

$$\left( |Q|^{sp-1} \int_Q |g|^p dm \right)^{1/p} \leq C_{23} |Q|^s |g|_{\mathcal{B}_{p,q,\text{self}s}^s},$$

so

$$\frac{1}{|Q|} \int_Q |g|^p dm \leq C_{23}^p |g|_{\mathcal{B}_{p,q,\text{self}s}^s}^p$$

for every  $Q \in \mathcal{P}$ . Due to the fact that  $\bigcup_k \mathcal{P}^k$  generates the  $\sigma$ -algebra  $\mathbb{A}$ , by Lévy’s upward theorem (see [Williams 1991]) for almost every  $x \in I$  the following holds. If  $x \in Q_k \in \mathcal{P}^k$  then

$$\lim_k \frac{1}{|Q_k|} \int_{Q_k} |g|^p dm = |g(x)|^p.$$

So

$$|g|_\infty \leq C_{23} |g|_{\mathcal{B}_{p,q,\text{self}s}^s}. \quad \square$$

**18A. Non-Archimedean behaviour in  $\mathcal{B}_{p,\tilde{q},\text{self}s}^\beta$ .** If we have a sequence  $g_i \in M(\mathcal{B}_{p,q}^s)$  we can get the naive estimate

$$\left| \left( \sum_i g_i \right) f \right|_{\mathcal{B}_{p,q}^s} \leq \left( \sum_i |g_i|_{M(\mathcal{B}_{p,q}^s)} \right) |f|_{\mathcal{B}_{p,q}^s}.$$

Nevertheless, remarkably, sometimes one can get a far better estimate. To state the result we need to define

$$|g|_{\mathcal{B}_{p,q,\text{self}s}^{s,t}} = \sup_{\substack{a_Q \in \mathcal{A}_{s,p}^{s,z} \\ Q \in \mathcal{P}^j \\ j \geq t}} |ga_Q|_{\mathcal{B}_{p,q}^s}.$$

It is easy to see that this norm is equivalent to  $|\cdot|_{\mathcal{B}_{p,q,\text{self}s}^s}$ .

**Proposition 18.4.** *Let  $\beta > s$ . There is  $C_{24}$  with the following property. Let  $g_i \in \mathcal{B}_{p,\tilde{q},\text{self}s}^{\beta,t_i}$ , with  $i \in \Lambda \subset \mathbb{N}$ , and  $t_i \in \mathbb{N}$ .*

*Consider a function  $f$  with a  $\mathcal{B}_{p,q}^s$ -representation*

$$f = \sum_k \sum_{Q \in \mathcal{P}^k} c_Q a_Q$$

*satisfying:*

(A) *We have*

$$\sup_{\substack{Q \in \mathcal{P} \\ c_Q \neq 0}} \sum_{Q \cap \text{supp } g_i \neq \emptyset} |g_i|_{\mathcal{B}_{p,\tilde{q},\text{self}s}^{\beta,t_i}} \leq N.$$

(B) *If  $Q \in \mathcal{P}^k$  satisfies  $c_Q \neq 0$  and  $Q \cap \text{supp } g_i \neq \emptyset$  then  $k \geq t_i$ .*

*Then we can find a  $\mathcal{B}_{p,q}^s$ -representation*

$$\left( \sum_i g_i \right) f = \sum_k \sum_{P \in \mathcal{P}^k} d_Q a_Q \tag{18-51}$$

*such that*

$$\left( \sum_k \left( \sum_{P \in \mathcal{P}^k} |d_Q|^p \right)^{q/p} \right)^{1/q} \leq C_{24} N \left( \sum_k \left( \sum_{P \in \mathcal{P}^k} |c_Q|^p \right)^{q/p} \right)^{1/q}. \tag{18-52}$$

*Proof.* It is enough to prove the result for the case when  $\Lambda$  is finite. Let  $Q \in \mathcal{P}^{k_0}$  with  $c_Q \neq 0$ . There is  $\{i_1, \dots, i_j\} \subset \Lambda$  such that

$$\left( \sum_i g_i \right) a_Q = \sum_{\ell \leq j} g_{i_\ell} a_Q \tag{18-53}$$

and  $Q \cap \text{supp } g_{i_\ell} \neq \emptyset$  for every  $\ell$ . In particular  $k_0 \geq \max_\ell t_{i_\ell}$ . By Lemma 18.2 for each  $\ell \leq j$  we can find a  $\mathcal{B}_{p,q}^\beta$ -representation

$$g_{i_\ell} a_Q = \sum_k \sum_{\substack{P \in \mathcal{P}^k \\ P \subset Q}} \tilde{s}_{P,Q}^\ell b_P \tag{18-54}$$

such that  $b_P$  is the canonical  $(\beta, p)$ -Souza's atom supported on  $P$  and

$$\left( \sum_k \left( \sum_{\substack{P \in \mathcal{P}^k \\ P \subset Q}} |\tilde{s}_{P,Q}^\ell|^p \right)^{\tilde{q}/p} \right)^{1/\tilde{q}} \leq 2C_{22} |g_{i_\ell}|_{\mathcal{B}_{p,\tilde{q},\text{self}s}^{\beta,t_{i_\ell}}}.$$

Since  $b_P = |P|^{\beta-s} a_P$ , where  $a_P$  is the canonical  $(s, p)$ -Souza's atoms supported on  $P$ , we can write

$$g_{i_\ell} a_Q = \sum_k \sum_{\substack{P \in \mathcal{P}^k \\ P \subset Q}} s_{P,Q}^\ell a_P,$$

with  $s_{P,Q}^\ell = \tilde{s}_{P,Q}^\ell |P|^{\beta-s}$  satisfying

$$\left( \sum_{\substack{P \in \mathcal{P}^k \\ P \subset Q}} |s_{P,Q}^\ell|^p \right)^{1/p} \leq 2C_{22} |g_{i_\ell}|_{\mathcal{B}_{p,\tilde{q},\text{self}s}^{\beta,i_\ell}} \lambda_2^{(\beta-s)(k-k_0)} \sup_{Q \in \mathcal{P}} |Q|^{\beta-s},$$

so we can write

$$\left( \sum_i g_i \right) a_Q = \sum_k \sum_{\substack{P \in \mathcal{P}^k \\ P \subset Q}} s_{P,Q} a_P,$$

with

$$s_{P,Q} = \sum_\ell s_{P,Q}^\ell$$

satisfying

$$\left( \sum_{\substack{P \in \mathcal{P}^k \\ P \subset Q}} |s_{P,Q}|^p \right)^{1/p} \leq 2NC_{22} \lambda_2^{(\beta-s)(k-k_0)} \sup_{Q \in \mathcal{P}} |Q|^{\beta-s}.$$

By Proposition 8.1(A) we can find a  $\mathcal{B}_{p,q}^s$ -representation (18-51) satisfying (18-52).  $\square$

**Remark 18.5.** If  $g$  is  $\mathcal{B}_{p,q}^\beta$ -positive we can define

$$|g|_{\mathcal{B}_{p,q,\text{self}s}^{s+t}} = \sup_{\substack{a_Q \in \mathcal{A}_{s,p}^{s,t} \\ Q \in \mathcal{P}^j \\ j \geq t}} |ga_Q|_{\mathcal{B}_{p,q}^{s+t}}.$$

If we assume additionally that  $g_i$  are  $\mathcal{B}_{p,q}^{\beta,i}$ -positive, Proposition 18.4 remains true if we replace all the instances of  $|\cdot|_{\mathcal{B}_{p,q,\text{self}s}^{s,t}}$  by  $|\cdot|_{\mathcal{B}_{p,q,\text{self}s}^{s+t}}$  in its statement. Moreover by Proposition 8.1(B) and Lemma 18.2(B) we can conclude that:

- (i) If  $c_Q \geq 0$  for every  $Q$  then  $d_Q \geq 0$  for every  $Q$ .
- (ii) If  $Q$  is such that  $d_Q \neq 0$  then  $Q \subset \text{supp } g_i$  for some  $i \in \Lambda$ .

**Corollary 18.6.** For every  $\beta > s$  and  $\tilde{q} \in [1, \infty]$  we have  $\mathcal{B}_{p,\tilde{q},\text{self}s}^\beta \subset M(\mathcal{B}_{p,q}^s)$ . Moreover this inclusion is continuous.

**18B. Strongly regular domains.** We may wonder under which conditions the characteristic function of a set  $\Omega$  is a pointwise multiplier in  $\mathcal{B}_{p,q}^s$ .

**Definition 18.7.** A measurable set  $\Omega \subset I$  is a  $(\alpha, C_{25}, C_{26})$ -strongly regular domain if for every  $Q \in \mathcal{P}^j$ , with  $j \geq C_{26}$ , there is family  $\mathcal{F}^k(Q \cap \Omega) \subset \mathcal{P}^k$  such that:

- (i) We have  $Q \cap \Omega = \bigcup_k \bigcup_{P \in \mathcal{F}^k(Q \cap \Omega)} P$ .
- (ii) If  $P, W \in \bigcup_k \mathcal{F}^k(Q \cap \Omega)$  and  $P \neq W$  then  $P \cap W = \emptyset$ .

(iii) We have

$$\sum_{P \in \mathcal{F}^k(Q \cap \Omega)} |P|^\alpha \leq C_{25} |Q|^\alpha. \quad (18-55)$$

The following result can be associated with results in [Triebel 2003] for  $B_{p,p}^s(\mathbb{R}^n)$ , especially when we consider the setting of Besov spaces in compact homogenous spaces. See Section 18B for details. See also [Schneider and Vybíral 2013].

**Proposition 18.8.** *If  $\Omega$  is a  $(1 - \beta p, C_{25}, C_{26})$ -strongly regular domain then*

$$|1_\Omega|_{\mathcal{B}_{p,\infty,\text{self}}^{\beta+, C_{26}}} \leq C_{25}^{1/p}. \quad (18-56)$$

*Proof.* Given  $Q \in \mathcal{P}^j$ , with  $j \geq C_{26}$ , we can write

$$1_\Omega a_Q = \sum_k \sum_{P \in \mathcal{F}^k(Q \cap \Omega)} \left( \frac{|P|}{|Q|} \right)^{1/p-\beta} a_P, \quad (18-57)$$

where  $a_P$  is a  $(\beta, p)$ -atom. Note that

$$\left( \sum_{P \in \mathcal{F}^k(Q \cap \Omega)} \left( \frac{|P|}{|Q|} \right)^{1-\beta p} \right)^{1/p} \leq C_{25}^{1/p},$$

so (18-56) holds.  $\square$

**Proposition 18.9** (pointwise multipliers I). *There is  $C_{27}$  with the following property. Suppose that  $\Omega_i$  are  $(1 - \beta p, K_i, t_i)$ -strongly regular domains,  $i \in \Lambda \subset \mathbb{N}$ , and  $\Theta_i > 0$  for every  $i \in \Lambda_i$ . Consider a function  $f$  with a  $\mathcal{B}_{p,q}^s$ -representation*

$$f = \sum_k \sum_{Q \in \mathcal{P}^k} c_Q a_Q$$

satisfying:

(A) *We have*

$$\sup_{\substack{Q \in \mathcal{P} \\ c_Q \neq 0}} \sum_{Q \cap \Omega_i \neq \emptyset} \Theta_i K_i^{1/p} \leq N.$$

(B) *If  $Q \in \mathcal{P}^k$  satisfies  $c_Q \neq 0$  and  $Q \cap \Omega_i \neq \emptyset$  then  $k \geq t_i$ .*

*Then we can find a  $\mathcal{B}_{p,q}^s$ -representation*

$$\left( \sum_i \Theta_i 1_{\Omega_i} \right) f = \sum_k \sum_{P \in \mathcal{P}^k} d_Q a_Q \quad (18-58)$$

such that

$$\left( \sum_k \left( \sum_{P \in \mathcal{P}^k} |d_Q|^p \right)^{q/p} \right)^{1/q} \leq C_{27} N \left( \sum_k \left( \sum_{P \in \mathcal{P}^k} |c_Q|^p \right)^{q/p} \right)^{1/q}. \quad (18-59)$$

Moreover:

(i) *If  $Q$  satisfies  $d_Q \neq 0$  then  $Q \subset \Omega_i$  for some  $i \in \Lambda$ .*

(ii) *If  $c_Q \geq 0$  for every  $Q$  then  $d_Q \geq 0$  for every  $Q$ .*

*Proof.* It follows from Proposition 18.4, Proposition 18.8 and Remark 18.5.  $\square$

**18C. Functions on  $\mathcal{B}_{p,\infty}^{1/p} \cap L^\infty$ .** We want to give explicit examples of multipliers in  $\mathcal{B}_{p,q}^s$ . One should compare the following result with [Triebel 1977] of the regularity of the multiplication on Besov spaces. See also [Mazyra and Shaposhnikova 2009] for more information on multipliers in classical Besov spaces.

**Proposition 18.10** (pointwise multipliers II). *Let  $g \in \mathcal{B}_{p,\infty}^{1/p} \cap L^\infty$ . Then the multiplier operator*

$$G : \mathcal{B}_{p,q}^s \rightarrow \mathcal{B}_{p,q}^s$$

*defined by  $G(f) = gf$  is a well-defined and bounded operator acting on  $(\mathcal{B}_{p,q}^s, |\cdot|_{\mathcal{B}_{p,q}^s})$ . Indeed*

$$|G|_{\mathcal{B}_{p,q}^s} \leq C_{11}C_{12} \frac{|g|_{\mathcal{B}_{p,\infty}^{1/p}}}{1 - \lambda_2^{1/p-s}} + |g|_\infty,$$

where  $C_{11} = C_{11}(1/p, p, \infty)$  and  $C_{12} = C_{12}(1/p, p, \infty)$  are as in Corollary 15.2.

**Remark 18.11.** We can get a similar result replacing  $\mathcal{B}_{p,b}^a$  by  $\mathcal{B}_{p,b}^{a+}$  everywhere.

*Proof.* Let  $a_Q = |Q|^{s-1/p} 1_Q$  be the canonical  $(s, p)$ -Souza's atom on  $Q$  and  $b_J = 1_J$  be the canonical  $(1/p, p)$ -Souza's atom on  $J$ . Given  $\epsilon > 0$ , let

$$f = \sum_k \sum_{Q \in \mathcal{P}^k} c_Q a_Q$$

be a  $\mathcal{B}_{p,q}^s$ -representation of  $f$  such that

$$\left( \sum_k \left( \sum_{Q \in \mathcal{P}^k} |c_Q|^p \right)^{q/p} \right)^{1/q} \leq (1 + \epsilon) |f|_{\mathcal{B}_{p,q}^s}$$

and let

$$g = \sum_k \sum_{J \in \mathcal{P}^k} e_J b_J$$

be a  $\mathcal{B}_{p,\infty}^{1/p}$ -representation of  $g$  given by Corollary 15.2 (in the case of Remark 18.11 we can consider an optimal  $\mathcal{B}_{p,\infty}^{1/p}$ -positive representation of  $g$ ). We claim that

$$u_1 = \sum_j \sum_{J \in \mathcal{P}^j} \left( \sum_{J \subset Q, Q \neq J, Q \in \mathcal{P}} \left( \frac{|J|}{|Q|} \right)^{1/p-s} c_Q e_J \right) a_J \quad \text{and} \quad u_2 = \sum_k \sum_{Q \in \mathcal{P}^k} \left( \sum_{Q \subset J, J \in \mathcal{P}} c_Q e_J \right) a_Q$$

are  $\mathcal{B}_{p,q}^s$ -representations of functions  $u_i \in \mathcal{B}_{p,q}^s$ . Firstly note that the inner sums are finite. Moreover if  $J \in \mathcal{P}^j$  we denote by  $Q_k(J)$  the unique element of  $\mathcal{P}^k$ , with  $k \leq j$ , that satisfies  $J \subset Q_k(J)$  then

$$\begin{aligned} \left( \sum_{J \in \mathcal{P}^j} \left| \sum_{k \leq j} \left( \frac{|J|}{|Q_k(J)|} \right)^{1/p-s} c_{Q_k(J)} e_J \right|^p \right)^{1/p} &\leq \sum_{k \leq j} \lambda_2^{(j-k)(1/p-s)} \left( \sum_{J \in \mathcal{P}^j} |c_{Q_k(J)} e_J|^p \right)^{1/p} \\ &\leq \left( \sum_{J \in \mathcal{P}^j} |e_J|^p \right)^{1/p} \sum_{k \leq j} \lambda_2^{(j-k)(1/p-s)} \max_{Q \in \mathcal{P}^k} |c_Q| \\ &\leq \left( \max_j \left( \sum_{J \in \mathcal{P}^j} |e_J|^p \right)^{1/p} \right) \sum_{k \leq j} \lambda_2^{(j-k)(1/p-s)} \left( \sum_{Q \in \mathcal{P}^k} |c_Q|^p \right)^{1/p} \\ &\leq C_{11}C_{12} |g|_{\mathcal{B}_{p,\infty}^{1/p}} \sum_{k \leq j} \lambda_2^{(j-k)(1/p-s)} \left( \sum_{Q \in \mathcal{P}^k} |c_Q|^p \right)^{1/p}. \end{aligned}$$

The right-hand side is a convolution, so we can easily get

$$|u_1|_{\mathcal{B}_{p,q}^s} \leq (1 + \epsilon) C_{11} C_{12} \frac{|g|_{\mathcal{B}_{p,\infty}^{1/p}}}{1 - \lambda_2^{1/p-s}} |f|_{\mathcal{B}_{p,q}^s}.$$

Moreover by Proposition 17.1.B, with  $s = 1/p$ , we obtain

$$\left( \sum_{Q \in \mathcal{P}^k} \left| \sum_{J \in \mathcal{P}, Q \subset J} c_Q e_J \right|^p \right)^{1/p} \leq \left( \sum_{Q \in \mathcal{P}^k} |c_Q|^p \left| \sum_{J \in \mathcal{P}, Q \subset J} e_J \right|^p \right)^{1/p} \leq \left( \sum_{Q \in \mathcal{P}^k} |c_Q|^p \right)^{1/p} |g|_\infty. \quad (18-60)$$

So

$$|u_1|_{\mathcal{B}_{p,q}^s} \leq (1 + \epsilon) |f|_{\mathcal{B}_{p,q}^s} |g|_\infty.$$

We claim that  $gf = u_1 + u_2$ . Indeed let

$$f_{k_0} = \sum_{k < k_0} \sum_{Q \in \mathcal{P}^k} c_Q a_Q \quad \text{and} \quad g_{k_0} = \sum_{k < k_0} \sum_{J \in \mathcal{P}^k} e_J b_J.$$

By Proposition 6.1 we have

$$\lim_{k_0} |g_{k_0} - g|_{p'} = 0 \quad \text{and} \quad \lim_{k_0} |f_{k_0} - f|_p = 0.$$

So

$$\lim_{k_0} |f_{k_0} g_{k_0} - fg|_1 = 0.$$

Note that:

(A) If  $Q \subset J$  then  $a_Q b_J = a_Q$ .

(B) If  $J \subset Q$  then

$$a_Q b_J = \left( \frac{|J|}{|Q|} \right)^{1/p-s} a_J.$$

So

$$\begin{aligned} f_{k_0} g_{k_0} &= \sum_{k < k_0} \sum_{Q \in \mathcal{P}^k} \sum_{i < k_0} \sum_{J \in \mathcal{P}^i} e_J c_Q a_Q b_J \\ &= \sum_{k < k_0} \sum_{Q \in \mathcal{P}^k} \left( \sum_{\substack{J \in \mathcal{P} \\ Q \subset J}} e_J c_Q \right) a_Q + \sum_{k < k_0} \sum_{Q \in \mathcal{P}^k} \sum_{i < k_0} \sum_{\substack{J \in \mathcal{P}^i \\ J \subset Q \\ J \neq Q}} e_J c_Q \left( \frac{|J|}{|Q|} \right)^{1/p-s} a_J \\ &= \sum_{k < k_0} \sum_{Q \in \mathcal{P}^k} \left( \sum_{\substack{J \in \mathcal{P} \\ Q \subset J}} e_J c_Q \right) a_Q + \sum_{i < k_0} \sum_{J \in \mathcal{P}^i} \left( \sum_{\substack{Q \in \mathcal{P} \\ J \subset Q \\ J \neq Q}} e_J c_Q \left( \frac{|J|}{|Q|} \right)^{1/p-s} \right) a_J = u_{1,k_0} + u_{2,k_0}. \end{aligned}$$

Note that

$$\lim_{k_0} |u_{r,k_0} - u_r|_1 = 0 \quad \text{and} \quad |u_{r,k_0}|_{\mathcal{B}_{p,q}^s} \leq |u_r|_{\mathcal{B}_{p,q}^s} \quad \text{for } r = 1, 2.$$

Now we can use Corollary 6.5(i) to conclude the proof.  $\square$

### 19. $\mathcal{B}_{p,q}^s \cap L^\infty$ is a quasia algebra

Multipliers in  $\mathcal{B}_{p,q}^s \cap L^\infty$  are indeed much easier to come by.

**Proposition 19.1** (pointwise multipliers III). *Let  $g, f \in \mathcal{B}_{p,q}^s \cap L^\infty$ . Then  $g \cdot f \in \mathcal{B}_{p,q}^s \cap L^\infty$  and*

$$|f \cdot g|_{\mathcal{B}_{p,q}^s} + |f \cdot g|_\infty \leq C_{11} C_{12} (|f|_{\mathcal{B}_{p,q}^s} + |f|_\infty) (|g|_{\mathcal{B}_{p,q}^s} + |g|_\infty).$$

So  $\mathcal{B}_{p,q}^s \cap L^\infty$  is a quasi-Banach algebra. Here  $C_{11} = C_{11}(s, p, q)$  and  $C_{12} = C_{12}(s, p, q)$  are as in Corollary 15.2.

*Proof.* Of course  $|f \cdot g|_\infty \leq |f|_\infty |g|_\infty$ . Let  $a_Q = |Q|^{s-1/p} 1_Q$  be the canonical  $(s, p)$ -Souza's atom on  $Q$ . Let

$$f = \sum_k \sum_{Q \in \mathcal{P}^k} c_Q a_Q \quad \text{and} \quad g = \sum_k \sum_{J \in \mathcal{P}^k} e_J a_J$$

be  $\mathcal{B}_{p,q}^s$ -representations of  $f$  and  $g$  given by Corollary 15.2. We claim that

$$u_1 = \sum_k \sum_{Q \in \mathcal{P}^k} \left( \sum_{Q \subset J, J \in \mathcal{P}} |J|^{s-1/p} c_Q e_J \right) a_Q \quad \text{and} \quad u_2 = \sum_k \sum_{J \in \mathcal{P}^k} \left( \sum_{J \subset Q, Q \neq J, Q \in \mathcal{P}} |Q|^{s-1/p} c_Q e_J \right) a_J$$

are  $\mathcal{B}_{p,q}^s$ -representations of functions  $u_i \in \mathcal{B}_{p,q}^s$ . Moreover by Proposition 17.1.A we have

$$\left( \sum_{Q \in \mathcal{P}^k} \left| \sum_{J \in \mathcal{P}, Q \subset J} |J|^{s-1/p} c_Q e_J \right|^p \right)^{1/p} \leq \left( \sum_{Q \in \mathcal{P}^k} |c_Q|^p \left| \sum_{J \in \mathcal{P}, Q \subset J} |J|^{s-1/p} e_J \right|^p \right)^{1/p} \leq \left( \sum_{Q \in \mathcal{P}^k} |c_Q|^p \right)^{1/p} |g|_\infty. \quad (19-61)$$

So

$$|u_1|_{\mathcal{B}_{p,q}^s} \leq C_{11} C_{12} |g|_\infty |f|_{\mathcal{B}_{p,q}^s},$$

and by an analogous argument

$$|u_2|_{\mathcal{B}_{p,q}^s} \leq C_{11} C_{12} |f|_\infty |g|_{\mathcal{B}_{p,q}^s}.$$

Define  $f_{k_0}$  and  $g_{k_0}$  as in the proof of Proposition 18.10. By Proposition 17.1 we have  $|f_{k_0}| \leq |f|_\infty$  and  $|g_{k_0}| \leq |g|_\infty$ . Since  $\lim_{k_0} f_{k_0} = f$  and  $\lim_{k_0} g_{k_0} = g$  in  $L^p$ , we can assume, taking a subsequence if necessary, that  $f_{k_0} g_{k_0}$  converges pointwise to  $f g$ . So by the theorem of dominated convergence we have  $\lim_{k_0} f_{k_0} g_{k_0} = f g$  in  $L^1$ . Finally note that if  $J \subset Q$  then

$$a_Q a_J = |Q|^{s-1/p} a_J.$$

Now we can use the same argument as in the proof of Proposition 18.10 to conclude that  $g f = u_1 + u_2$ .  $\square$

**19A. Regular domains.** Here we will give sufficient conditions for the characteristic function of a set to define a bounded pointwise multiplier either on  $\mathcal{B}_{p,q}^s \cap L^\infty$ . For every set  $\Omega$ , let

$$k_0(\Omega) = \min\{k \geq 0 : \text{there exists } P \in \mathcal{P}^k \text{ such that } P \subset \Omega\}.$$

**Definition 19.2.** We say that a countable family of pairwise disjoint measurable sets  $\{\Omega_r\}_{r \in \Lambda}$  is an  $(\alpha, C_{28}, \lambda_5)$ -regular family if one can find families  $\mathcal{F}^k(\Omega_r) \subset \mathcal{P}^k$ ,  $k \geq k_0(\Omega_r)$ , such that:

(A) We have  $\Omega_r = \bigcup_{k \geq k_0(\Omega_r)} \bigcup_{Q \in \mathcal{F}^k(\Omega_r)} Q$ .

(B) If  $P, Q \in \bigcup_{k \geq k_0(\Omega_r)} \mathcal{F}^k(\Omega_r)$  and  $P \neq Q$  then  $P \cap Q = \emptyset$ .

(C) We have

$$\sum_{r \in \Lambda} \sum_{Q \in \mathcal{F}^k(\Omega_r)} |Q|^\alpha \leq C_{28} \lambda_5^{k-k_0(\bigcup_r \Omega_r)} \left| \bigcup_r \Omega_r \right|^\alpha. \quad (19-62)$$

We say that a measurable set  $\Omega$  is an  $(\alpha, C_{28}, \lambda_5)$ -regular domain if  $\{\Omega\}$  is an  $(\alpha, C_{28}, \lambda_5)$ -regular family.

**Proposition 19.3.** *Let  $\beta > s$ . Every  $(1 - \beta p, C, 0)$ -strongly regular domain is a  $(1 - sp, C', \lambda_2^{(\beta-s)p})$ -regular domain for some  $C'$ .*

*Proof.* Consider a  $(1 - \beta p, C_{25}, 0)$ -strongly regular domain  $\Omega$ . There are at most  $\lambda_1^{-k_0(\Omega)}$  elements in  $\mathcal{P}^{k_0(\Omega)}$  and

$$\left( \frac{\lambda_2}{\lambda_1} \right)^{-k_0(\Omega)} \leq \frac{|Q|}{|W|} \leq \left( \frac{\lambda_2}{\lambda_1} \right)^{k_0(\Omega)}$$

for every  $Q, W \in \mathcal{P}^{k_0(\Omega)}$ . Consequently

$$\left( \frac{\lambda_2}{\lambda_1} \right)^{-k_0(\Omega)} \leq \frac{|\Omega|}{|Q|} \leq \lambda_1^{-k_0(\Omega)} \left( \frac{\lambda_2}{\lambda_1} \right)^{k_0(\Omega)}$$

for each  $Q \in \mathcal{P}^{k_0(\Omega)}$ . For every  $Q \in \mathcal{P}^{k_0(\Omega)}$  there is a family  $\mathcal{F}^k(Q \cap \Omega)$  such that

$$\sum_k \sum_{P \in \mathcal{F}^k(Q \cap \Omega)} P = Q \cap \Omega \quad \text{and} \quad \sum_{P \in \mathcal{F}^k(Q \cap \Omega)} |P|^{1-\beta p} \leq C |Q|^{1-\beta p}.$$

Let

$$\mathcal{F}^k(\Omega) = \bigcup_{Q \in \mathcal{P}^{k_0(\Omega)}} \mathcal{F}^k(Q \cap \Omega).$$

We have

$$\begin{aligned} \sum_{Q \in \mathcal{P}^{k_0(\Omega)}} \sum_{P \in \mathcal{F}^k(Q \cap \Omega)} |P|^{1-sp} &= \sum_{P \in \mathcal{F}^k(Q \cap \Omega)} |P|^{1-\beta p} |P|^{(\beta-s)p} \\ &\leq \sum_{Q \in \mathcal{P}^{k_0(\Omega)}} \left( \max_{P \in \mathcal{F}^k(Q \cap \Omega)} |P|^{(\beta-s)p} \right) \sum_{P \in \mathcal{F}^k(Q \cap \Omega)} |P|^{1-\beta p} \\ &\leq \lambda_2^{(k-k_0(\Omega))(\beta-s)p} \sum_{Q \in \mathcal{P}^{k_0(\Omega)}} |Q|^{(\beta-s)p} \sum_{P \in \mathcal{F}^k(Q \cap \Omega)} |P|^{1-\beta p} \\ &\leq C \lambda_2^{(k-k_0(\Omega))(\beta-s)p} \sum_{Q \in \mathcal{P}^{k_0(\Omega)}} |Q|^{(\beta-s)p} |Q|^{1-\beta p} \\ &\leq C \lambda_2^{(k-k_0(\Omega))(\beta-s)p} \sum_{Q \in \mathcal{P}^{k_0(\Omega)}} |Q|^{1-sp} \\ &\leq C \lambda_1^{-k_0(\Omega)} \left( \frac{\lambda_2}{\lambda_1} \right)^{k_0(\Omega)(1-sp)} \lambda_2^{(k-k_0(\Omega))(\beta-s)p} |\Omega|^{1-sp}. \quad \square \end{aligned}$$

**Remark 19.4.** Suppose that there is  $C_{29}$  such that for every  $k$  and every  $Q, W \in \mathcal{P}^k$  we have

$$\frac{1}{C_{29}} \leq \frac{|Q|}{|W|} \leq C_{29}$$

and

$$\#\{P \in \mathcal{P}^{k_0(\Omega)} : P \cap \Omega \neq \emptyset\} \leq C_{30}.$$

Then it is easy to see that one can choose  $C' = C_{30}C_{29}C$ .

The following result is similar to results for Sobolev spaces of [Faraco and Rogers 2013]. See also [Sickel 1999].

**Corollary 19.5.** *If  $\{\Omega_r\}_{r \in \Lambda}$  is a  $(1 - ps, C_{28}, \lambda_5)$ -regular family then there is  $C_{31}$  such that for every  $g \in \mathcal{B}_{p,q}^s \cap L^\infty$  and  $r \in \Lambda$  we can find a  $\mathcal{B}_{p,q}^s$ -representation*

$$g \cdot 1_{\Omega_r} = \sum_k \sum_{Q \in \mathcal{P}^k, Q \subset \Omega_r} d_Q^r a_Q \quad (19-63)$$

such that

$$\left( \sum_j \left( \sum_r \sum_{\substack{Q \in \mathcal{P}^j \\ Q \subset \Omega_r}} |d_Q^r|^p \right)^{q/p} \right)^{1/q} \leq C_{31} |g|_{\mathcal{B}_{p,q}^s}. \quad (19-64)$$

Note that

$$\Omega = \bigcup_r \Omega_r$$

is a  $(1 - ps, C_{28}, \lambda_5)$ -regular domain and  $F(g) = g1_\Omega$  is a bounded operator in  $\mathcal{B}_{p,q}^s \cap L^\infty$  satisfying

$$|F|_{\mathcal{B}_{p,q}^s \cap L^\infty} \leq C_{11}C_{12} \left( 1 + \frac{C_{28}^{1/p}}{(1 - \lambda_5^{q/p})^{1/q}} |\Omega|^{1/p-s} \right). \quad (19-65)$$

Moreover

$$|1_\Omega|_{\mathcal{B}_{p,q}^s} \leq \frac{C_{28}^{1/p}}{(1 - \lambda_5^{q/p})^{1/q}} |\Omega|^{1/p-s}. \quad (19-66)$$

*Proof.* Notice that

$$f = 1_{\bigcup_r \Omega_r} = \sum_k \sum_{Q \in \mathcal{P}^k} c_Q a_Q,$$

where  $c_Q = |Q|^{1/p-s}$  for every  $Q \in \bigcup_k \bigcup_r \mathcal{F}^k(\Omega_r)$  and  $c_Q = 0$  otherwise. Let

$$g = \sum_k \sum_{J \in \mathcal{P}^k} e_J a_J$$

be  $\mathcal{B}_{p,q}^s$ -representations  $g$  given by Corollary 15.2. Consider  $u_1, u_2$  as in the proof of Proposition 19.1. By Proposition 17.1 we can get exactly the same estimate as in the proof of Proposition 19.1.

Note that those  $Q \in \mathcal{P}^k$  for which the corresponding atom  $a_Q$  has a nonvanishing coefficient in the definition of  $u_1$  belong to  $\bigcup_r \bigcup_j \mathcal{F}^j(\Omega_r)$ , and moreover every  $J \in \mathcal{P}^k$  for which the corresponding

atom  $a_J$  has nonvanishing coefficients in the definition of  $u_2$  is contained in some  $Q \in \mathcal{F}^j(\Omega_r)$ , for some  $j$  and  $r$ . In particular  $J \subset \Omega_r$ . So (19-63) holds, with

$$d_Q^r = \left( \sum_{Q \subset J, J \in \mathcal{P}} |J|^{s-1/p} c_{Qe_J} \right) + \left( \sum_{Q \subset J, J \neq Q, J \in \mathcal{P}} |J|^{s-1/p} c_{Je_Q} \right)$$

for every  $Q \subset \Omega_r$ .

Note also that

$$\left( \sum_k \left( \sum_r \sum_{Q \in \mathcal{F}^k(\Omega_r)} |Q|^{1-sp} \right)^{q/p} \right)^{1/q} \leq C_{28}^{1/p} \left( \sum_{k \geq k_0(\cup_r \Omega_r)} \lambda_5^{(k-k_0(\Omega))q/p} \right)^{1/q} |\Omega|^{1/p-s} \leq \frac{C_{28}^{1/p}}{(1-\lambda_5^{q/p})^{1/q}} |\Omega|^{1/p-s},$$

so (19-66) and consequently (19-65) hold.  $\square$

**Remark 19.6.** Using the methods in [Faraco and Rogers 2013] one can show that quasiballs in  $[0, 1]^n$  (and in particular quasidisks in  $[0, 1]^2$ , that is, domains delimited by quasicircles) give examples of regular domains in  $[0, 1]^n$  endowed with the good grid of dyadic  $n$ -cubes and the Lebesgue measure  $m$ .

## 20. A remarkable description of $\mathcal{B}_{1,1}^s$

When  $p = q = 1$  (and  $s > 0$  small), something curious happens. We can skip the good grid and characterise the Besov space  $\mathcal{B}_{1,1}^s$  of a homogeneous space using regular domains. Fix  $C_{28} \geq 1$  and  $\lambda_5 \in (0, 1)$ . Let  $\mathcal{W}$  be the family of all  $(1-s, C_{28}, \lambda_5)$ -regular domains. Of course  $\mathcal{P} \subset \mathcal{W}$ . Let  $\widehat{\mathcal{W}}$  be a family of sets satisfying

$$\mathcal{P} \subset \widehat{\mathcal{W}} \subset \mathcal{W}.$$

Define  $B^{1-s}$  as the set of all functions  $f \in L^{1/(1-s)}$  that can be written as

$$f = \sum_{i=0}^{\infty} c_i \frac{1_{A_i}}{|A_i|^{1-s}}, \quad (20-67)$$

where  $A_i \in \widehat{\mathcal{W}}$  for every  $i \in \mathbb{N}$  and

$$\sum_i |c_i| < \infty.$$

It is easy to see that

$$|f|_{1/(1-s)} \leq \sum_i |c_i|.$$

Define

$$|f|_{B^{1-s}} = \inf \sum_i |c_i|,$$

where the infimum runs over all possible representations (20-67). One can see that  $(B^{1-s}, |\cdot|_{B^{1-s}})$  is a normed vector space.

**Proposition 20.1.** *We have that  $B^{1-s} = \mathcal{B}_{1,1}^s(\mathcal{P})$  and the corresponding norms are equivalent.*

*Proof.* Note that (19-66) says that there is  $C$  such that if  $A \in \mathcal{W}$  then  $1_A \in \mathcal{B}_{1,1}^s(\mathcal{P})$  and

$$|1_A|_{\mathcal{B}_{1,1}^s(\mathcal{P})} \leq C|A|^{1-s}.$$

In particular, if  $f$  has a representation (20-67) we conclude that

$$|f|_{\mathcal{B}_{1,1}^s(\mathcal{P})} \leq C|f|_{B^{1-s}}.$$

In particular  $B^{1-s} \subset \mathcal{B}_{1,1}^s(\mathcal{P})$ . On the other hand if  $g \in \mathcal{B}_{1,1}^s(\mathcal{P})$ . then we can write

$$g = \sum_{k=0}^{\infty} \sum_{Q \in \mathcal{P}^k} s_Q \frac{1_Q}{|Q|^{1-s}},$$

$$\sum_{P \in \mathcal{P}} |s_Q| = \sum_{k=0}^{\infty} \sum_{Q \in \mathcal{P}^k} |s_Q| < \infty,$$

and  $|g|_{\mathcal{B}_{1,1}^s(\mathcal{P})}$  is the infimum of  $\sum_{P \in \mathcal{P}} |s_Q|$  over all possible representations. In particular  $g \in B^{1-s}$  and

$$|g|_{B^{1-s}} \leq |g|_{\mathcal{B}_{1,1}^s(\mathcal{P})}. \quad \square$$

**Remark 20.2.** Let  $I = [0, 1]$  with the dyadic grid  $\mathcal{D}$  and the Lebesgue measure  $m$ . We prove in Part IV that  $\mathcal{B}_{1,1}^s(\mathcal{D})$ , with  $0 < s < 1$ , is the Besov space  $B_{1,1}^s([0, 1])$ , and its norms are equivalent. Note that every interval  $[a, b] \subset [0, 1]$  is a  $(1-s, 2, 2^{s-1})$ -regular domain. So we can apply Proposition 20.1 with  $\widehat{\mathcal{W}} = \{[a, b] : 0 \leq a < b \leq 1\}$ . That is,  $f$  belongs to  $B_{1,1}^s([0, 1])$  if and only if it can be written as in (20-67), where every  $A_i$  is an interval and  $\sum_i |c_i| < \infty$ , and the norm in  $B_{1,1}^s([0, 1])$  is equivalent to the infimum of  $\sum_i |c_i|$  over all possible such representations. This characterisation of the Besov space  $B_{1,1}^s([0, 1])$  was first obtained in [de Souza 1985a].

## 21. Left compositions

The following result generalises a well-known result on left composition operators acting on Besov spaces of  $\mathbb{R}^n$ . See [Bourdaud and Kateb 1990; 1991; 1995] for recent developments on the study of left compositions on Besov spaces of  $\mathbb{R}^n$ .

**Proposition 21.1.** *Let*

$$g : I \rightarrow \mathbb{C}$$

*be a Lipschitz function such that  $g(0) = 0$ . Then the left composition*

$$L_g : \mathcal{B}_{p,q}^s \rightarrow \mathcal{B}_{p,q}^s$$

*defined by  $L_g(f) = g \circ f$  is well-defined and*

$$|g \circ f|_p + \text{osc}_{p,q}^s(g \circ f) \leq K(|f|_p + \text{osc}_{p,q}^s(f)),$$

*where  $K$  is the Lipschitz constant of  $g$ . Consequently there exists  $C$  such that*

$$|L_g(f)|_{\mathcal{B}_{p,q}^s} \leq C|f|_{\mathcal{B}_{p,q}^s}$$

*for every  $f \in \mathcal{B}_{p,q}^s$ .*

*Proof.* Note that

$$\begin{aligned} \operatorname{osc}_p(g \circ f, Q) &= \inf_{a \in \mathbb{C}} \left( \int_Q |g(f(x)) - a|^p dm(x) \right)^{1/p} \\ &\leq \inf_{a \in \mathbb{C}} \left( \int_Q |g(f(x)) - g(a)|^p dm(x) \right)^{1/p} \\ &\leq K \inf_{a \in \mathbb{C}} \left( \int_Q |f(x) - a|^p dm(x) \right)^{1/p} = K \operatorname{osc}_p(f, Q). \end{aligned} \quad (21-68)$$

So it easily follows that  $\operatorname{osc}_{p,q}^s(g \circ f) \leq K \operatorname{osc}_{p,q}^s(f)$ . Of course  $|g \circ f|_p \leq K|f|_p$ . In particular  $g \circ f \in \mathcal{B}_{p,q}^s$ .  $\square$

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