CONTINUITY OF THE UNBOUNDED ATTRACTORS FOR A FRACTIONAL PERTURBATION OF A SCALAR REACTION-DIFFUSION EQUATION

MAYKEL BELLUZI, MATHEUS C. BORTOLAN, UBIRAJARA CASTRO, AND JULIANA FERNANDES

Abstract. In this work we study the continuity (both upper and lower semicontinuity), defined using the Hausdorff semidistance, of the unbounded attractors for a family of fractional perturbations of a scalar reaction-diffusion equation with a non-dissipative nonlinear term.

MSC 2020. Primary: 5B41, 5B40, 47H20. Secondary: 5R11, 35K58.
Keywords. unbounded attractors, fractional perturbations, reaction-diffusion equation, upper semicontinuity, lower semicontinuity.

1. Introduction

In this work we consider the following family of scalar reaction–diffusion equations with Dirichlet boundary conditions and initial conditions:

$$\begin{cases} \partial_t u + (-\partial_{xx})^\alpha u = f(x,u), \quad x \in (0,\ell), \\ u(t,0) = u(t,\ell) = 0, \\ u(0,x) = u_0(x), \end{cases} \tag{1.1}$$

where $\alpha > 0$, $(-\partial_{xx})^\alpha$ represents the fractional power of the negative Laplacian operator (in the sense of [1]) and $f\colon (0,\ell)\times\mathbb{R}\to\mathbb{R}$ is a *non-dissipative nonlinearity* (we will make sense of this notion below).

We shall understand (1.1) as a perturbation of the limiting problem (that is, when $\alpha = 1$), as α varies in a suitable small neighborhood of 1. Those type of perturbed semilinear equations are often referred to as *fractional approximations*

[M. Belluzi] This study was financed in part by the Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP) # 2022/01439-5, Brasil

Address: Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo (USP), CEP 13566-590, São Carlos - SP, Brasil. e-mail: maykelbelluzi@icmc.usp.br, ORCiD: 0000-0003-2182-719X.

[M.C. Bortolan] This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001.

Address: Departamento de Matemática, Centro de Ciências Físicas e Matemáticas, Universidade Federal de Santa Catarina (UFSC), Campus Florianópolis, CEP 88040-090, Florianópolis - SC, Brasil. e-mail: m.bortolan@ufsc.br, ORCiD: 0000-0002-4838-8847.

[U. Castro] Address: Instituto Federal de Educação, Ciência e Tecnologia de Goiás (IFG), Rua 02, QD. 10, Lts. 1 a 15, Residencial Bauman, CEP:766000-000, Cidade de Goiás - GO, Brasil. e-mail: ubirajara.castro@ifg.edu.br, ORCiD: 0009-0002-0038-8255.

[J. Fernandes] This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001, by Fundação Carlos Chaga Filho de Amparo à Pesquisa do Estado do Rio de Janeiro (FAPERJ) # 010.002597/2019 and by Conselho Nacional de Desenvolvimento Científico e Tecnológico – CNPq #310760/2020-8.

Address: Instituto de Matemática, Centro de Tecnologia, Av. Athos da Silveira Ramos 149, Cidade Uniersitária da Universidade Federal do Rio de Janeiro, CEP 21941-909, Rio de Janeiro - RJ, Brasil. e-mail: jfernandes@im.ufrj.br, ORCiD: 0000-0001-9401-3168.

of the original one, and they represent, in this case, variations in the diffusion properties of the Laplacian operator. Fractional approximations are fairly recent in the literature and they have a useful physical meaning: as pointed in [19] and [21], there is an extensive set of physical applications where sub-diffusivity or super-diffusivity take place, such as heat diffusion in sub/super conductors. Those situations are frequently represented by mathematical models in which the linear term that governs the equation is a fractional power of the negative Laplacian, $(-\Delta)^\alpha$. In those cases, sub-diffusion or super-diffusion are represented by $0<\alpha<1$ or $\alpha>1$, respectively.

Understanding the continuity of the asymptotic structures under these perturbations can be useful when modeling a real life problem, where one cannot accurately state if the medium has properties of sub or super-diffusion. This take on fractional approximations of semilinear problems was previously considered in several papers, such as [3, 4, 6, 7, 8, 9, 14, 15]. So far, the literature deals with an abstract semilinear problem, in a Banach space X, of the form

$$\begin{cases} u_t + \Lambda^\alpha u = F(u), \ t>0, \\ u(0)=u_0, \end{cases} \tag{1.2}$$

where $\alpha \in (1-\delta, 1+\delta)$, for some small $\delta>0$, and Λ is a linear positive operator. In general, the outline of the existent works is the following:

∘ present results regarding the existence of unique maximal solutions for (1.2);

∘ by using a *dissipative assumption* on F, ensure that these solutions are globally defined, that is, defined for all $t \geqslant 0$;

∘ prove the existence of (compact) global attractors for (1.2), as well as their upper and lower semicontinuity with respect to α.

Our purpose goes in a similar direction, but rather than assuming a dissipative condition for the nonlinearity, we will deal with a class of *non-dissipative nonlinearities*. In those cases, whenever it is possible to prove the existence of global solutions, the object in the phase space that provides the asymptotic dynamics for the problem, since there is no bounded absorbing set, is the *unbounded attractor*, that is, compactness is no longer assumed. The theory of unbounded attractors has a little over 40 years and can be found, for instance, in [2, 5, 10, 12, 13, 23, 24].

We shall prove the existence of unbounded attractors associated to (1.1), namely $\mathcal{J}_\alpha$, and we analyze their continuity as $\beta \to \alpha$, for α in small a neighborhood of 1. To the best of our knowledge, there is no other work, so far, in the literature that proves a result on continuity of unbounded attractors in terms of Hausdorff semidistance. In [12] the authors study a perturbation of a non-dissipative problem and their effect on the unbounded attractors. They were able to prove that the gradient structure in the unbounded attractor remains under these perturbations. Nevertheless, continuity in terms of the Hausdorff semidistance is not guaranteed in that situation. The type of perturbation considered in our paper, given in terms of fractional powers, allows this take on continuity of the family $\{\mathcal{J}_\alpha\}_{\alpha\in(1-\delta,1+\delta)}$ of unbounded attractors.

Let us describe our setting a little further. It is known from the usual semigroup theory that, for each $u_0 \in L^2(0,\ell)$, (1.1) has a *local solution* if $f(x,u)$ is locally Lipschitz continuous with respect to u, uniformly in x (see [17, Theorem 3.3.3]). Usually, in order to ensure that there exists a bounded absorbing set for (1.1), one requires that f satisfies a certain dissipative condition, given in terms of the first eigenvalue λ_1 of $-\partial_{xx}$, namely

$$\limsup_{|u|\to\infty} \frac{f(x,u)}{|u|} < \lambda_1,$$

uniformly in x (see [16]). Since we want to treat exactly the opposite case, where there is no bounded absorbing set, we will restrict ourselves to the case where

$$f(x,u) = bu + g(x,u), \tag{1.3}$$

with $b > \lambda_1$ and $g(x,u)$ bounded and globally Lipschitz in the variable u, uniformly for $x \in (0,\ell)$. We see that in this case we have the opposite condition of dissipation, that is,

$$\limsup_{|u|\to\infty} \frac{f(x,u)}{|u|} = b > \lambda_1.$$

It is clear that with condition (1.3), for each given $u_0 \in L^2(\Omega)$, problem (1.1) has a unique globally defined solution. Our goals in this work are as follows:

∘ to prove the existence of an unbounded attractor $\mathcal{J}_\alpha$ for (1.1);

∘ to prove the upper and lower semicontinuity, using the Hausdorff semidistance, of the family $\{\mathcal{J}_\alpha\}_{\alpha\in(1-\delta,1+\delta)}$ for a suitable small $\delta > 0$.

To that end, we organize this paper as follows: Section 2 is dedicated to the abstract theory of unbounded attractors, where we review the main results already established in the literature. In Section 3 we prove that (1.1), under condition (1.3), possess an unbounded attractor $\mathcal{J}_\alpha$ (see Theorem 3.5). Lastly, Section 4 is dedicated to the results on upper and lower semicontinuity of the family of unbounded attractors $\{\mathcal{J}_\alpha\}_{\alpha\in(1-\delta,1+\delta)}$ (see Theorems 4.6 and 4.10).

2. Unbounded attractors

In this section, following [10], we briefly present the abstract theory of unbounded attractors for semigroups in Banach spaces. Let $T = \{T(t) \colon t \geqslant 0\}$ be a semigroup in a Banach space $(X, \|\cdot\|)$, that is,

∘ $T(0)x = x$ for all $x \in X$;

∘ $T(t+s) = T(t)T(s)$ for all $t, s \geqslant 0$;

∘ the map $[0,\infty) \times X \ni (t,x) \mapsto T(t)x \in X$ is continuous.

For nonempty sets $A, B \subset X$ we define the *Hausdorff semidistance* between A and B by

$$d_H(A,B) = \sup_{a\in A} \inf_{b\in B} \|a-b\|.$$

Definition 2.1. A closed set $\mathcal{U} \subset X$ is called an **unbounded attractor** for T if

- **(a)** $\mathcal{U}$ is *invariant* for T, that is, $T(t)\mathcal{U} = \mathcal{U}$ for all $t \geqslant 0$;
- **(b)** $\mathcal{U}$ *attracts bounded sets under the action of* T, that is, for each $B \subset X$ bounded we have

$$\lim_{t\to\infty} d_H(T(t)B, \mathcal{U}) = 0;$$

- **(c)** there is no proper closed subset of $\mathcal{U}$ satisfying both **(a)** and **(b)**.

Comparing this definition with the one of a (compact) global attractor (see [25]) for semigroups, one can see that we replace the compactness with the minimality condition **(c)**. This enlarges the class of dynamical systems that has its long-time dynamics represented by a particular set of the phase space. However, the lack of compactness allows situations where the characterization of the unbounded attractor is complicated, and the uniqueness of this object is not ensured. The absence of uniqueness is problematic if we wish to provide a set in the phase space that describe the asymptotic dynamics. If there is more than one unbounded attractor, which one will be used to describe the asymptotic dynamics?

Luckily, in certain situations, one can find a unique unbounded attractor and even characterize it in terms of global solutions. In [10], following the ideas of [13], the authors exhibited a guideline to prove that, under suitable conditions, the set of *bounded in the past global solutions*

$$\mathcal{J} = \{\xi(0)\colon \xi \text{ is a bounded in the past global solution of } T\},$$

is the unique unbounded attractor of T. Here, $\xi\colon \mathbb{R} \to X$ is a *bounded in the past global solution of* T if $T(t)\xi(s) = \xi(t+s)$ for all $t \geqslant 0$ and $s \in \mathbb{R}$, and the set $\xi((-\infty, 0])$ is bounded in X. This set $\mathcal{J}$ has the following properties:

Proposition 2.2. [10, Proposition 3]

- **(i)** *$\mathcal{J}$ is invariant,*
- **(ii)** *if $A \subset X$ is bounded and invariant then $A \subset W^u(A) \subset \mathcal{J}$, where the unstable set $W^u(A)$ is defined by*

$$W^u(A) = \Big\{\xi(0)\colon \xi \textit{ is a global solution of } T \textit{ such that } \lim_{s\to-\infty} d_H(\xi(s), A) = 0\Big\}.$$

Now we present the main definitions and results of [10] that will be later used in the text.

Definition 2.3. A semigroup T in a Banach space X is **u-asymptotically compact** if for each $B \subset X$ bounded there exists $t_0 = t_0(B) \geqslant 0$ such that for each $t \geqslant t_0$ there exists a compact set $K(t) \subset X$ with

$$\lim_{t\to\infty} d_H(T(t)B, K(t)) = 0.$$

We say that a set $G \subset X$ is **u-strongly absorbing** for T if

- **$(\mathbf{A}_1)$** G is positively invariant by T, that is, $T(t)G \subset G$ for all $t \geqslant 0$;
- **$(\mathbf{A}_2)$** for each $B \subset X$ bounded there exists $t_0 = t_0(B) \geqslant 0$ such that $T(t)B \subset G$ for all $t \geqslant t_0$;
- **$(\mathbf{A}_3)$** there exists a sequence of bounded sets $\{H_n\}_{n\in\mathbb{N}} \subset G$ such that:
  - $H_n \subset H_{n+1}$ for each $n \in \mathbb{N}$;
  - $G \setminus H_n$ is positively invariant by T for each $n \in \mathbb{N}$;
  - if $B \subset G$ is bounded there exists $n \in \mathbb{N}$ such that $B \subset H_n$.
- **$(\mathbf{A}_4)$** $\lim_{t\to\infty} d_H(T(t)G, \mathcal{J}) = 0$.

With that, the main result of [10] regarding the existence of the unbounded attractor is as follows.

Theorem 2.4. [10, Theorem 4 and Proposition 10] *If T is u-asymptotically compact and has an u-strongly absorbing set G, than $\mathcal{J}$ is the unique unbounded attractor for T. Moreover, $\mathcal{J} \subset G$ and $\mathcal{J}$ is bounded-compact, that is, $\mathcal{J} \cap F$ is compact for each closed and bounded subset F of X.*

Now we present the characterization of the unbounded attractors in terms of the unstable sets of its invariant bounded sets, as given in [10].

Definition 2.5. We say that a collection $\mathcal{E}$ of subsets of X is a *bounded disjoint collection of isolated invariant sets* for the semigroup T if

- each $E \in \mathcal{E}$ is bounded, invariant for T, and there exists $\varepsilon > 0$ such that E is the maximal invariant set in $\mathcal{O}_\varepsilon(E) = \{x \in X\colon d(x, e) < \varepsilon \text{ for some } e \in E\}$;
- there exists $\delta > 0$ such that for all $E, E^* \in \mathcal{E}$ with $E \neq E^*$ we have

$$\mathcal{O}_\delta(E) \cap \mathcal{O}_\delta(E^*) = \varnothing.$$

The letter u in front of the definitions stands for "unbounded", in order to distinguish them from the classical definitions from the dissipative case.

Let T be a semigroup such that the set $\mathcal{J}$ of bounded in the past global solution is the unique unbounded attractor, and assume that $\mathcal{E}$ a bounded disjoint collection of isolated invariant sets of T. We say that T is **$\mathcal{E}$-gradient** if there exists a continuous function $V\colon \mathcal{J} \to \mathbb{R}$ such that

- $\mathbb{R}^+ \ni t \mapsto V(T(t)x)$ decreases for any $x \in \mathcal{J}$.
- If $V(T(t)x) = V(x)$ for all $t \geqslant 0$, then $x \in E$ for some $E \in \mathcal{E}$.
- V is constant on the connected components of each $E \in \mathcal{E}$.

In this case, V is called an **$\mathcal{E}$-Lyapunov functional** for T.

Proposition 2.6. [10, Theorem 8 and Corollary 9] *Let T be a semigroup such that the set $\mathcal{J}$ of bounded in the past global solution is the unique unbounded attractor, and assume that $\mathcal{E}$ is a bounded disjoint collection of isolated invariant sets of T. If T is $\mathcal{E}$-gradient and $\mathcal{J}$ is bounded-compact then*

$$\mathcal{J} = W^u\Big(\bigcup_{E\in\mathcal{E}} E\Big).$$

If $\mathcal{E} = \{E_1, \dots, E_n\}$ then

$$\mathcal{J} = \bigcup_{i=1}^{n} W^u(E_i).$$

It is of particular interest the case where the collection $\mathcal{E}$ of bounded disjoint isolated invariant sets is formed by a finite number of isolated equilibria, that is, $\mathcal{E} = \{u_1, \dots, u_n\}$. If T is $\mathcal{E}$-gradient, we have

$$\mathcal{J} = \bigcup_{i=1}^{n} W^u(u_i).$$

3. Existence of unbounded attractors

We deal with (1.1) in the non-dissipative case, that is, we shall assume (1.3). Let

$$A := -\partial_{xx}\colon D(A) \subset L^2(0,\ell) \to L^2(0,\ell)$$

be the negative Laplacian with Dirichlet boundary conditions and domain $D(A) = H^2(0,\ell)\cap H^1_0(0,\ell)$. In this case, A is a densely defined positive self-adjoint operator, with compact resolvent and spectrum $\sigma(A) = \sigma_p(A) = \{\lambda_j\}_{j\geqslant 1}$ satisfying

$$\lambda_1 > 0, \quad \lambda_j \leqslant \lambda_{j+1} \text{ for all } n \in \mathbb{N} \quad \text{and} \quad \lambda_j \to \infty,$$

and to each λ_j we have an associate eigenvector $\varphi_j \in D(A)$. We assume that there exists $N \in \mathbb{N}$ such that

$$\lambda_N < b < \lambda_{N+1}, \tag{3.1}$$

and define

$$\sigma = \frac{1}{2}\min\{b - \lambda_N,\ \lambda_{N+1} - b\} > 0. \tag{3.2}$$

Problem (1.1) can be rewritten as an abstract evolution equation in $X = L^2(0,\ell)$ of the form

$$\begin{cases} u_t = (-A^\alpha + bI)u + \tilde{g}(u),\ t > 0, \\ u(0) = u_0 \in X, \end{cases} \tag{3.3}$$

where $\alpha > 0$, and $\tilde{g}(u)(x) = g(x, u(x))$ for $u \in X$ and $x \in (0,\ell)$. Note that $\tilde{g}\colon X \to X$ is bounded and globally Lipschitz in X, since $g(x,u)$ is bounded and globally Lipschitz on u, uniformly for $x \in (0,\ell)$. We denote

$$L_\alpha := -A^\alpha + bI, \tag{3.4}$$

and we have the following properties:

Recall that a point $u \in X$ is an equilibrium for T if $T(t)u = u$ for all $t \geqslant 0$.

Lemma 3.1. *Let L_α be the linear operator defined in (3.4).*

(1) $\sigma(L_\alpha) = \sigma_p(L_\alpha) = -[\sigma(A)]^\alpha + b$. *Furthermore, φ_j is the eigenvalue of L_α associated to $-\lambda_j^\alpha + b$, where φ_j is the eigenvector of A associated with λ_j, for each $j \in \mathbb{N}$;*

(2) *there exists $\delta > 0$ such that for $\alpha \in (1-\delta, 1+\delta)$, L_α is the infinitesimal generator of an analytic semigroup $\{e^{L_\alpha t} \colon t \geqslant 0\}$ in $X = L^2(0,\ell)$. Moreover, $e^{L_\alpha t}$ is a compact operator for each $t > 0$;*

(3) *problem (3.3) has unique solution $u_\alpha(\cdot, u_0) \colon [0,\infty) \to X$. Setting $T_\alpha(t)u_0 = u_\alpha(t,u_0)$ for $t \geqslant 0$, $u_0 \in X$ and $\alpha > 0$, $T_\alpha = \{T_\alpha(t) \colon t \geqslant 0\}$ defines a compact semigroup in X for each $\alpha > 0$, and for each $t \geqslant 0$ we have*

$$T_\alpha(t)u_0 = e^{L_\alpha t}u_0 + \int_0^t e^{L_\alpha (t-s)}\tilde{g}(T_\alpha(s)u_0)ds.$$

Proof. The proof of **(1)** follows from [20, Theorem 5.3.1] and the fact that the eigenvectors of A^α associated to an λ_j^α are the same as the eigenvector of A associated to an λ_j. Using [18, Theorem 2] we obtain item **(2)**. Item **(3)** follows from [17, Theorem 3.3.3 and Corollary 3.3.5] and the compactness of $e^{L_\alpha t}$ for $t > 0$.

□

We define

$$E_N = \operatorname{span}\{\varphi_1, \ldots, \varphi_N\} \quad \text{and} \quad F_N = E_N^\perp.$$

Clearly $X = E_N \oplus F_N$ and we denote any element $u \in X$ as $u = p + q \in E_N \oplus F_N$. Let $P_N \colon X \to X$ denote the orthonormal projection over E_N. With $\delta > 0$ as in Lemma 3.1, we can assume, without loss of generality, that it is small enough so that

$$\sigma < \min\{b - \lambda_N^\alpha,\ \lambda_{N+1}^\alpha - b\} \quad \text{for all } \alpha \in (1-\delta, 1+\delta), \tag{3.5}$$

where $\sigma > 0$ is given by (3.2). Also, we choose $\eta > 0$ such that

$$-\lambda_1^\alpha + b < \eta \quad \text{for all } \alpha \in (1-\delta, 1+\delta). \tag{3.6}$$

Using the classical theory of spectral decomposition of sectorial operators presented in [18, Theorem III.6.17] and [22, (2.5.14)] we have the following result.

Lemma 3.2. *With these assumptions, for $\alpha \in (1-\delta, 1+\delta)$ we have*

(1) $L_\alpha P_N = P_N L_\alpha$ *and* $e^{L_\alpha t}P_N = P_N e^{L_\alpha t}$ *for all* $t \geqslant 0$*;*

(2) $L_\alpha^+ = L_\alpha|_{E_N} = L_\alpha P_N \in \mathcal{L}(E_N)$, $\|L_\alpha^+\| \leqslant \eta$ *and*

$$\langle L_\alpha^+ p, p\rangle \geqslant \sigma \|p\|^2 \quad \text{for all } p \in E_N;$$

(3) $L_\alpha^- = L_\alpha|_{F_N} = L_\alpha(I - P_N)$ *is a sectorial operator in F_N, hence it generates an analytic semigroup $\{e^{L_\alpha^- t} \colon t \geqslant 0\}$ in F_N, and*

$$\langle L_\alpha^- q, q\rangle \leqslant -\sigma \|q\|^2 \quad \text{for all } q \in F_N \cap D(L_\alpha);$$

(4) *there exists $M \geqslant 1$ such that for all $t \geqslant 0$ and $\alpha \in (1-\delta, 1+\delta)$ we have*

$$\|e^{L_\alpha^+ t}p\| \geqslant Me^{\sigma t}\|p\| \quad \text{for all } p \in E_N,$$

$$\|e^{L_\alpha^- t}q\| \leqslant Me^{-\sigma t}\|q\| \quad \text{for all } q \in F_N,$$

and for all $t > 0$ and $\alpha \in (1-\delta, 1+\delta)$ we have

$$\|L_\alpha^- e^{L_\alpha^- t}q\| \leqslant Mt^{-1}e^{-\sigma t}\|q\| \quad \text{for all } q \in F_N.$$

In order to prove the existence of a unique unbounded attractor for (3.3), we shall assume an additional condition on the nonlinearity $\tilde{g}$. We assume that for $p \in E_N$ and $q \in F_N$ we have

$$\|\tilde{g}(p+q)\| \to 0 \quad \text{as } \|p\| \to \infty, \tag{3.7}$$

uniformly for q in bounded subsets of F_N.

Now we show that, under all these conditions, Theorem 2.4 can be applied to ensure that

$$\mathcal{J}_\alpha = \{\xi(0)\colon \xi \text{ is a bounded in the past global solution of } T_\alpha\}$$

is the unique unbounded attractor of the problem (3.3), for each $\alpha \in (1-\delta, 1+\delta)$.

Proposition 3.3. *There exists a constant $D > 0$, independent of $\alpha \in (1-\delta, 1+\delta)$, such that the set*

$$G := \{u = p + q \in E_N \oplus F_N \colon \|q\| \leqslant D\}$$

is u-strongly absorbing for T_α for all $\alpha \in (1-\delta, 1+\delta)$.

Proof. Since $\tilde{g}$ is bounded and globally Lipschitz, there exists $C > 0$ such that $\|\tilde{g}(u)\| \leqslant C$ and $\|\tilde{g}(u) - \tilde{g}(v)\| \leqslant C\|u - v\|$ for all $u, v \in X$. Define $D = \frac{C\sqrt{2\ell}}{\sigma} > 0$. It follows from [10, Lemma 17] that $T_\alpha(t)G \subset G$ and for each $B \subset X$ bounded there exists $t_0 = t_0(B) \geqslant 0$ such that $T(t)B \subset G$ for all $t \geqslant t_0$.

Fix $\kappa > 0$ and (see [10, (17)]) for $n \in \mathbb{N}$ set

$$H_n = \{u = p + q \in G \colon (1+\kappa)\|p\|^2 - \|q\|^2 \leqslant n^2\}.$$

It follows from [10] that $H_n \subset H_{n+1}$ for all $n \in \mathbb{N}$, $G \setminus H_n$ is positively invariant for T_α for all n sufficiently large, and if $B \subset G$ is bounded then $B \subset H_n$ for some $n \in \mathbb{N}$. Lastly, it follows from [10, Page 12] that $d_H(T_\alpha(t)G, \mathcal{J}_\alpha) \to 0$ as $t \to \infty$, hence G is u-strongly absorbing for T_α for all $\alpha \in (1-\delta, 1+\delta)$. ☐

Remark 3.4. Using the results of [10], we note that the sets H_n have the following property: for each $\alpha \in (1-\delta, 1+\delta)$ and $x \in X$, if $T_\alpha(\tau)x \in H_n$ for some $\tau > 0$ then $x \in H_n$.

With that we present the main result of this section, which shows that $\mathcal{J}_\alpha$ is indeed the unique unbounded attractor for T_α, for each $\alpha \in (1-\delta, 1+\delta)$.

Theorem 3.5. *Consider (1.1) and assume that (1.3), (3.1) and (3.7) hold. Let $\delta > 0$ be small enough such that (3.5) and (3.6) hold for $\alpha \in (1-\delta, 1+\delta)$. Then $\mathcal{J}_\alpha$ is the unique unbounded attractor of T_α, $\mathcal{J}_\alpha \subset G$ and $\mathcal{J}_\alpha$ is bounded-compact.*

Proof. Using Theorem 2.4, it just remains to note that T_α is u-asymptotically compact, which is trivial since $T_\alpha(t)$ is a compact map for each $t > 0$. ☐

4. Continuity of the unbounded attractors

Up to now we are able to say that each problem (1.1), for $\alpha \in (1-\delta, 1+\delta)$, has $\mathcal{J}_\alpha$ as its unique unbounded attractor, which are all subsets of G and are bounded-compact. In this section we shall prove that the family $\{\mathcal{J}_\alpha\}_{\alpha\in(1-\delta,1+\delta)}$ remains close in terms of the Hausdorff semidistance as $\beta \to \alpha$, for each $\alpha \in (1-\delta, 1+\delta)$. Throughout this section, we shall assume that conditions required on Theorem 3.5 hold.

4.1. Upper semicontinuity.

We recall that the family $\{\mathcal{J}_\alpha\}_{\alpha\in(1-\delta,1+\delta)}$ is said to be **upper semicontinuous** at $\alpha \in (1-\delta, 1+\delta)$ if

$$\lim_{\beta\to\alpha} d_H(\mathcal{J}_\beta, \mathcal{J}_\alpha) = 0.$$

In what follows, we prove this property. Fix $R > 0$ and for $\alpha \in (1-\delta, 1+\delta)$ we define the set

$$J_{\alpha,R} = \mathcal{J}_\alpha \cap \{p + q \in E_N \oplus F_N \colon \|p\| \leqslant R\}. \tag{4.1}$$

Since $\mathcal{J}_\alpha \subset G$, we have

$$J_{\alpha,R} = \mathcal{J}_\alpha \cap \{p + q \in E_N \oplus F_N \colon \|p\| \leqslant R \text{ and } \|q\| \leqslant D\},$$

and hence $J_{\alpha,R}$ is compact (from the bounded-compactness of $\mathcal{J}_\alpha$).

From now on, we assume that $\delta > 0$ obtained in Lemma 3.1 **(2)** is such that $\delta < \frac{1}{2}$. Hence, $(1-\delta, 1+\delta) \subset (\frac{1}{2}, \frac{3}{2})$. We also denote $|u|$ the $H^1_0(0,\ell)$-norm of $u \in H^1_0(0,\ell)$.

Lemma 4.1. *Given $0 < \xi < \sigma$, there exists $c_1 \geqslant 0$ such that for all $q \in F_N$ and $\alpha \in (1-\delta, 1+\delta)$ we have*

$$|e^{L^-_\alpha t} q| \leqslant c_1 t^{-\frac{1}{2\alpha}} e^{-\xi t}\|q\| \quad \text{for all } t > 0.$$

Proof. We know for all $t > 0$ and $q \in F_N$ we have

$$\|e^{L^-_\alpha t} q\| \leqslant M e^{-\sigma t}\|q\| \quad \text{and} \quad \|L^-_\alpha e^{L_\alpha t} q\| \leqslant M t^{-1} e^{-\sigma t}\|q\|.$$

Since $L^-_\alpha = L_\alpha = -A^\alpha + bI$ in F_N, we also have

$$\begin{aligned}\|A^\alpha e^{L^-_\alpha t} q\| &\leqslant \|L_\alpha e^{L_\alpha t} q\| + b\|e^{L_\alpha t} q\| \leqslant M t^{-1} e^{-\sigma t}\|q\| + M b e^{-\sigma t}\|q\| \\ &\leqslant M t^{-1} e^{-\xi t}\|q\| + M b t e^{-(\sigma-\xi)t} t^{-1} e^{-\xi t}\|q\| \\ &= M(1 + t b e^{-(\sigma-\xi)t}) t^{-1} e^{-\xi t}\|q\|,\end{aligned}$$

and since $[0,\infty) \ni t \mapsto t e^{-(\sigma-\xi)t}$ is bounded, it follows that for some $c > 0$ we have

$$\|A^\alpha e^{L^-_\alpha t} q\| \leqslant c t^{-1} e^{-\xi t}\|q\| \quad \text{for all } t > 0 \text{ and } q \in F_N.$$

Since $\alpha > \frac{1}{2}$, from [17, Theorem 1.4.4] there exists $K > 0$ such that

$$|e^{L^-_\alpha t} q| = \|A^{\frac{1}{2}} e^{L^-_\alpha t} q\| \leqslant K \|A^\alpha e^{L^-_\alpha t} q\|^{\frac{1}{2\alpha}} \|e^{L^-_\alpha t} q\|^{1-\frac{1}{2\alpha}}.$$

Using the previous estimates, we obtain

$$|e^{L^-_\alpha t} q| \leqslant K c^{\frac{1}{2\alpha}} M^{1-\frac{1}{2\alpha}} t^{-\frac{1}{2\alpha}} e^{-\xi t}\|q\| \leqslant c_1 t^{-\frac{1}{2\alpha}} e^{-\xi t}\|q\|,$$

where $c_1 = \sup_{\alpha \in (1-\delta,1+\delta)} K c^{\frac{1}{2\alpha}} M^{1-\frac{1}{2\alpha}} < \infty$. □

Proposition 4.2. *If $\{x_n\}_{n\in\mathbb{N}} \subset G$, $t_n \to \infty$, $\alpha_n \to \alpha \in (1-\delta, 1+\delta)$, and $\{T_{\alpha_n}(t_n)x_n\}_{n\in\mathbb{N}}$ is bounded, then $\{T_{\alpha_n}(t_n)x_n\}_{n\in\mathbb{N}}$ has a convergent subsequence. Furthermore, given $\{x_n\}_{n\in\mathbb{N}} \subset X$ bounded with $x_n \in \mathcal{J}_{\alpha_n}$ for each $n \in \mathbb{N}$ then $\{x_n\}_{n\in\mathbb{N}}$ has a convergent subsequence in X.*

Proof. Since $\alpha \in (1-\delta, 1+\delta)$, we can assume, without loss of generality, that $\alpha_n \in (1-\delta, 1+\delta)$ for all $n \in \mathbb{N}$. We write $x_n = p_n + q_n \in E_N + F_N$, $p_n(t) = P_N T_{\alpha_n}(t) x_n$ and $q_n(t) = (I - P_N) T_{\alpha_n}(t) x_n$ for all $n \in \mathbb{N}$ and $t \geqslant 0$. Since $\{T_{\alpha_n}(t_n)x_n\}_{n\in\mathbb{N}}$ is bounded we have $\{p_n(t_n)\}_{n\in\mathbb{N}} \subset E_N$ bounded, and since E_N is finite dimensional, it possess a convergent subsequence, which we name the same. Using Lemma 4.1, the fact that $\|q_n\| \leqslant D$ (since $\{x_n\}_{n\in\mathbb{N}} \subset G$), the fact that the function $\tilde{g}$ is bounded, and that

$$q_n(t) = e^{L_{\alpha_n} t} q_n + \int_0^t e^{L_{\alpha_n}(t-s)} \tilde{g}(T_{\alpha_n}(s)x_n) ds \quad \text{for } t > 0,$$

we obtain

$$|q_n(t)| \leqslant c_1 t^{-\frac{1}{2\alpha_n}} e^{-\xi t} + C \int_0^t (t-s)^{-\frac{1}{2\alpha_n}} e^{-\xi(t-s)} ds.$$

For $t \geqslant 1$ we have

$$|q_n(t)| \leqslant c_1 + \frac{2C\alpha_n}{2\alpha_n - 1} + \frac{Ce^{-\xi}}{\xi},$$

which, in particular, implies that $\{q_n(t_n)\}_{n\in\mathbb{N}}$ is bounded in $H^1_0(0,\ell)$ and, hence, it has a convergent subsequence in $L^2(0,\ell)$. Since $T_{\alpha_n}(t_n)x_n = p_n(t_n) + q_n(t_n)$, the first part of the result is proved.

For the second statement, since $x_n \in \mathcal{J}_{\alpha_n}$ and $\mathcal{J}_{\alpha_n}$ is T_{α_n}-invariant, there exists $z_n \in \mathcal{J}_{\alpha_n} \subset G$ such that $T_{\alpha_n}(n)z_n = x_n$. Since $\{z_n\}_{n\in\mathbb{N}} \subset G$, the result follows directly from the first part. ☐

Lemma 4.3. *For compact sets $K \subset X$, $J \subset [0,\infty)$, and $\alpha \in (1-\delta, 1+\delta)$, we have*

$$\sup_{(t,x)\in J\times K} \|T_\beta(t)x - T_\alpha(t)x\| \to 0 \qquad \text{as } \beta \to \alpha.$$

Also, given $(t_n, x_n) \to (t, x)$ and $\alpha_n \to \alpha \in (1-\delta, 1+\delta)$ we have $T_{\alpha_n}(t_n)x_n \to T_\alpha(t)x$ as $n \to \infty$.

Proof. First statement on the semigroup convergence as $\beta \to \alpha$ over compact subsets of $[0,\infty) \times X$ follows similarly to [8, Theorem 5.6], where the authors proved the convergence of the nonlinear semigroups as $\beta \to 1$ (no significant change appears when we consider $\beta \to \alpha$, for $\alpha \in (1-\delta, 1+\delta)$). For the second part, set $K = \{x_n : n \in \mathbb{N}\} \cup \{x\}$ and $J = \{t_n : n \in \mathbb{N}\} \cup \{t\}$, which are compact. Given $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that if $n \geqslant n_0$ then

$$\sup_{(s,y)\in J\times K} \|T_{\alpha_n}(s)y - T_\alpha(s)y\| < \frac{\varepsilon}{2}.$$

From the continuity properties of T_α, there exists $\mu > 0$ such that if $|t-s| < \mu$ and $\|y - x\| < \mu$ then

$$\|T_\alpha(t)x - T_\alpha(s)y\| < \frac{\varepsilon}{2}.$$

Thus, if $n \in \mathbb{N}$ is sufficiently large we have

$$\|T_{\alpha_n}(t_n)x_n - T_\alpha(t)x\| \leqslant \|T_{\alpha_n}(t_n)x_n - T_\alpha(t_n)x_n\| + \|T_\alpha(t_n)x_n - T_\alpha(t)x\| < \varepsilon,$$

and the conclusion holds. ☐

Proposition 4.4. *Given $\{x_n\}_{n\in\mathbb{N}} \subset X$ bounded with $x_n \in \mathcal{J}_{\alpha_n}$ for each $n \in \mathbb{N}$ then $\{x_n\}_{n\in\mathbb{N}}$ has a convergent subsequence to a point $x \in \mathcal{J}_\alpha$. In particular, for each $R > 0$ and $\alpha \in (1-\delta, 1+\delta)$ we have*

$$\lim_{\beta\to\alpha} d_H(J_{\beta,R}, J_{\alpha,R}) = 0, \tag{4.2}$$

where $J_{\beta,R}$ and $J_{\alpha,R}$ are as in (4.1).

Proof. Let $\xi_n : \mathbb{R} \to X$ be a bounded in the past global solution of T_{α_n} with $\xi_n(0) = x_n$. Using the definition of G, there exists H_m such that $\{x_n\}_{n\in\mathbb{N}} \subset H_m$. Hence (see Remark 3.4) $\xi_n(t) \in H_m$ for all $t \leqslant 0$ and $n \in \mathbb{N}$. From Theorem 4.2, $\{x_n\}_{n\in\mathbb{N}}$ has a convergent subsequence, which we call the same, to a point $x \in X$. Again, using the previous lemma, $\{\xi_n(-1)\}_{n\in\mathbb{N}}$ has a convergent subsequence, which we call the same, to a point $x_{-1} \in X$. From Lemma 4.3, we have

$$\xi_n(0) = T_{\alpha_n}(1)\xi_n(-1) \to T_\alpha(1)x_{-1},$$

and since $\xi_n(0) \to x$, we obtain $T_\alpha(1)x_{-1} = x$. For each $k \in \mathbb{N}$ we obtain a subsequence such that $\xi_n(-j) \to x_{-j} \in X$ for $j = 0, \dots, k$, and such that $T_\alpha(1)x_{-k} = x_{-k+1}$ for each $k \geqslant 1$.

Setting $\xi(t) = T(t)x$ for $t \geqslant 0$ and $\xi(t) = T(t+k)x_{-k}$ for $t \in [-k, -k+1)$, for each $k \in \mathbb{N}$, then it is simple to see that ξ is a global solution of T_α with $\xi(0) = x$. Since the set H_m is closed, it is easy to see that $\xi(t) \in H_m$ for all $t \leqslant 0$, which implies that ξ is bounded in the past. Hence, $x \in \mathcal{J}_\alpha$.

Now assume that (4.2) does not hold. Then there exists $\varepsilon > 0$, $\alpha_n \to \alpha$, $x_n \in J_{\alpha_n,R}$ for each $n \in \mathbb{N}$ such that

$$d_H(x_n, J_{\alpha,R}) \geqslant \varepsilon \qquad \text{for all } n \in \mathbb{N}. \tag{4.3}$$

From what we have just proved, since $x_n \in \mathcal{J}_{\alpha_n}$ and $\{x_n\}_{n\in\mathbb{N}}$ is bounded, up to a subsequence, $x_n \to x \in \mathcal{J}_\alpha$. Since P_N is continuous, $P_N x_n \to P_N x$. As

$\|P_N x_n\| \leqslant R$ for all n we obtain $\|P_N x\| \leqslant R$, and hence $x \in J_{\alpha,R}$ and we obtain a contradiction with (4.3).

□

Using [10, Lemma 18 and Proposition 20] we have the following:

Lemma 4.5. *Given $\varepsilon > 0$ there exists $R > 0$ such that for all $\alpha \in (1-\delta, 1+\delta)$ we have*

$$\mathcal{J}_\alpha \cap \{p + q \in E_N \oplus F_N \colon \|p\| > R\} \subset \mathcal{O}_\varepsilon(E_N).$$

Moreover, for each $p \in E_N$ and $\alpha \in (1-\delta, 1+\delta)$ there exists $q = q(\alpha) \in F_N$ such that $p + q \in \mathcal{J}_\alpha$.

Theorem 4.6 (Upper semicontinuity)**.** *For each $\alpha \in (1-\delta, 1+\delta)$ we have*

$$\lim_{\beta \to \alpha} d_H(\mathcal{J}_\beta, \mathcal{J}_\alpha) = 0.$$

Proof. If that is not the case, there exists $\varepsilon_0 > 0$, $\alpha_n \to \alpha$ and $x_n \in \mathcal{J}_{\alpha_n}$ such that

$$d_H(x_n, \mathcal{J}_\alpha) \geqslant \varepsilon_0 \quad \text{for all } n \in \mathbb{N}. \tag{4.4}$$

If $\{x_n\}_{n\in\mathbb{N}}$, up to a subsequence, is bounded, then there exists $R > 0$ such that $x_n \in J_{\alpha_n,R}$ for all $n \in \mathbb{N}$, and it follows from Proposition 4.4 that $\{x_n\}_{n\in\mathbb{N}}$, up to a subsequence, converges to some $x \in J_{\alpha,R} \subset \mathcal{J}_\alpha$, which contradicts (4.4). It remains to treat the case where $\|x_n\| \to \infty$ as $n \to \infty$. Since $\{x_n\}_{n\in\mathbb{N}} \subset G$, setting $p_n = P_N x_n$, we must have $\|p_n\| \to \infty$ as $n \to \infty$. From Lemma 4.5, there exists $R > 0$ such that for all $\alpha \in (1-\delta, 1+\delta)$ we have

$$\mathcal{J}_\alpha \cap \{p + q \in E_N \oplus F_N \colon \|p\| > R\} \subset \mathcal{O}_{\frac{\varepsilon_0}{4}}(E_N).$$

For all n sufficiently large we have $\|p_n\| > R$ and hence $x_n \in \mathcal{O}_{\frac{\varepsilon_0}{4}}(E_N)$. Thus, for $q_n = x_n - p_n$, we obtain

$$\|q_n\| = \|x_n - p_n\| = \inf_{p \in E_N} \|x_n - p\| = d_H(x_n, E_N) < \frac{\varepsilon_0}{4}.$$

Also, from Lemma 4.5, there exists $q_n^\alpha \in F_N$ such that $y_n := p_n + q_n^\alpha \in \mathcal{J}_\alpha$, and since $\|p_n\|$ is large, we obtain

$$\|q_n^\alpha\| = \|y_n - p_n\| \overset{(*)}{=} \inf_{p \in E_N} \|y_n - p\| = d_H(y_n, E_N) < \frac{\varepsilon_0}{4},$$

where in $(*)$ we used the fact that $P_N y_n = p_n$. This implies that for n sufficiently large we have

$$d_H(x_n, \mathcal{J}_\alpha) \leqslant \|x_n - y_n\| = \|q_n - q_n^\alpha\| \leqslant \|q_n\| + \|q_n^\alpha\| < \frac{\varepsilon_0}{2},$$

and contradicts (4.4). This completes the proof. □

4.2. **Lower semicontinuity.** Similarly to the upper semicontinuity, we say that the family $\{\mathcal{J}_\alpha\}_{\alpha \in (1-\delta,1+\delta)}$ is **lower semicontinuous** at $\alpha \in (1-\delta, 1+\delta)$ if

$$\lim_{\beta \to \alpha} d_H(\mathcal{J}_\alpha, \mathcal{J}_\beta) = 0.$$

Although alike in definition, the obtainment of the lower semicontinuity is far more troublesome, and requires additional information on the internal structures of the unbounded attractors $\mathcal{J}_\alpha$. We first note that from the regularizing properties of the equation, $\mathcal{J}_\alpha \subset H^1_0(0,\ell)$ for $\alpha \in (1-\delta, 1+\delta)$. Furthermore, (3.3) has a Lyapunov function $V_\alpha \colon \mathcal{J}_\alpha \to \mathbb{R}$ given by

$$V_\alpha(u) = \|A^{\frac{\alpha}{2}} u\|^2 - b\|u\|^2 - 2\int_0^\ell \widetilde{G}(u)dx, \tag{4.5}$$

where $\widetilde{G}(u)(x) = \int_0^{u(x)} g(x,s)ds$ for $x \in (0,\ell)$. Indeed, taking the inner product in $L^2(0,\ell)$ of (3.3) by u_t and using the fact that $A^\alpha = (-\partial_{xx})^\alpha$ is self-adjoint, we obtain

$$\begin{aligned}\|u_t\|^2 &= \langle u_t, u_t\rangle = \langle -A^\alpha u, u_t\rangle + \langle bu, u_t\rangle + \langle g(x,u), u_t\rangle \\ &= -\frac{1}{2}\frac{d}{dt}\left[\|A^{\frac{\alpha}{2}}u\|^2 - b\|u\|^2 - 2\int_0^\ell \widetilde{G}(u)dx\right]\end{aligned}$$

and consequently, for V_α defined by (4.5), we obtain that $\frac{d}{dt}V_\alpha(u) = -2\|u_t\|^2$ along solutions of (3.3). Therefore, V_α is strictly decreasing over non-constant trajectories and it is constant on the equilibria of (3.3). Henceforth, if $\mathcal{E}_\alpha$ is the set of equilibria of T_α, then T_α is $\mathcal{E}_\alpha-$gradient, and Proposition 2.6 implies that $\mathcal{J}_\alpha = W^u(\mathcal{E}_\alpha)$.

Recall that if $T(t)$ is Fréchet differentiable in X for each $t \geqslant 0$, then an equilibrium $u_0 \in X$ is called **hyperbolic** if the spectrum $\sigma(D_xT(t)u_0)$ of the linear operator $D_xT(t)u_0 \in \mathcal{L}(X)$ does not intersect the unitary circle S^1 of $\mathbb{C}$ for each $t > 0$, that is, $\sigma(D_xT(t)u_0) \cap S^1 = \varnothing$ for each $t > 0$.

From now on we assume moreover that

$$\tilde{g}\colon H^1_0(0,\ell) \to L^2(0,\ell) \text{ is continuously Frechét differentiable,}$$

which implies, since $\mathcal{J}_\alpha \subset H^1_0(0,\ell)$, that $T_\alpha(t)|_{\mathcal{J}_\alpha}\colon \mathcal{J}_\alpha \to \mathcal{J}_\alpha$ is Frechét differentiable in $H^1_0(0,\ell)$ for each $t \geqslant 0$ and $\alpha \in (1-\delta, 1+\delta)$. This is true, for instance, when $\frac{\partial g}{\partial u}(x,u)$ is globally Lipschitz in u, uniformly for $x \in (0,\ell)$.

Proposition 4.7. *Assume that there exists $\alpha \in (1-\delta, 1+\delta)$ such that $\mathcal{E}_\alpha$ consists exactly of n hyperbolic equilibria $e_{\alpha,1}, \dots, e_{\alpha,n}$. Then there exists $\mu_\alpha > 0$ such that for $\beta \in (\alpha-\mu_\alpha, \alpha+\mu_\alpha) \subset (1-\delta, 1+\delta)$, $\mathcal{E}_\beta$ consists exactly of n hyperbolic equilibria $e_{\beta,1}, \dots, e_{\beta,n}$ with*

$$\max_{i=1,\dots,n} \|e_{\beta,i} - e_{\alpha,i}\| \to 0 \quad \text{as } \beta \to \alpha.$$

Proof. If $\alpha = 1$, the result follows from [8, Proposition 12] For the general case, consider the operators

$$B = A^\alpha,\ \gamma = \frac{\beta}{\alpha},\ B^\gamma = (A^\alpha)^{\frac{\beta}{\alpha}},$$

and see that $\gamma \to 1$ as $\beta \to \alpha$. Hence the result follows from the case $\alpha = 1$. □

From this point forward, we shall assume that for any $\alpha \in (1-\delta, 1+\delta)$, the set of equilibria $\mathcal{E}_\alpha$ for (3.3) consists of n hyperbolic equilibria $e_{\alpha,1}, \dots, e_{\alpha_n}$. Therefore, we have $\mathcal{J}_\alpha = \cup_{i=1}^n W^u(e_{\alpha,i})$.

Remark 4.8. In [11, Theorem 1.1] the authors proved that for (1.1) with $\alpha = 1$, $\mathcal{E}_1$ *generically* consists only of hyperbolic equilibria, that is, this property holds for any f in a residual subset of $C^2([0,\ell] \times \mathbb{R}, \mathbb{R})$.

For $\rho > 0$ and $e_{\alpha,i} \in \mathcal{E}_\alpha$, we define the **$\rho$-local unstable manifold** of $e_{\alpha,i}$ as the set

$$\begin{aligned}W^u_\rho(e_{\alpha,i}) = \{\xi(0)\colon\ &\xi \text{ is a global solution of } T_\alpha \text{ with } \|\xi(t) - e_{\alpha,i}\| < \rho \text{ for all } t \leqslant 0 \\ &\text{and } \xi(t) \to e_{\alpha,i} \text{ as } t \to -\infty\}.\end{aligned}$$

In [8] the authors proved that the local unstable manifolds of equilibria behaves continuously as $\alpha \to 1^-$. With minor changes in the proof, which we omit, we can state the following result.

Proposition 4.9. [8, Theorem 6.9] *There exists a $\rho > 0$ sufficiently small such that*

$$\max_{i=1,\dots,n} \Big(d_H(W^u_\rho(e_{\beta,i}), W^u_\rho(e_{\alpha,i})) + d_H(W^u_\rho(e_{\alpha,i}), W^u_\rho(e_{\beta,i})) \Big) \to 0 \quad \text{as } \beta \to \alpha,$$

for each $\alpha \in (1-\delta, 1+\delta)$.

With that we can prove the lower semicontinuity of the family $\{\mathcal{J}_\alpha\}_{\alpha\in(1-\delta,1+\delta)}$, as we state and prove next.

Theorem 4.10 (Lower semicontinuity)**.** *For each $\alpha \in (1-\delta, 1+\delta)$, we have*

$$\lim_{\beta\to\alpha} d_H(\mathcal{J}_\alpha, \mathcal{J}_\beta) = 0.$$

Proof. It is enough to prove that given any $\varepsilon > 0$, there exists a $\mu = \mu(\varepsilon) > 0$ such that for any $x_\alpha \in \mathcal{J}_\alpha$ and $\beta \in (\alpha - \mu, \alpha + \mu)$, we can find $x_\beta \in \mathcal{J}_\beta$ such that

$$\|x_\beta - x_\alpha\| < \varepsilon.$$

Let $R = R(\varepsilon) > 0$ be the constant obtained in Lemma 4.5 such that, for all $\beta \in (1-\delta, 1+\delta)$,

$$\mathcal{J}_\beta \cap \{p + q \in E_N \oplus F_N \colon \|p\| > R\} \subset \mathcal{O}_{\frac{\varepsilon}{2}}(E_N).$$

For $x_\alpha = p + q_\alpha \in E_N \oplus F_N$, with $x_\alpha \in \mathcal{J}_\alpha$ and $\|p\| > R$, then from Lemma 4.5, $\|q_\alpha\| < \frac{\varepsilon}{2}$, and for any $\beta \in (1-\delta, 1+\delta)$, there exists $q_\beta \in F_N$ with $\|q_\beta\| < \frac{\varepsilon}{2}$ and such that $x_\beta = p + q_\beta \in \mathcal{J}_\beta$. Hence

$$\|x_\alpha - x_\beta\| = \|q_\alpha - q_\beta\| < \varepsilon.$$

Assume now that $x_\alpha = p + q_\alpha \in \mathcal{J}$ with $\|p\| \leqslant R$, that is, $x_\alpha \in \mathcal{J}_{\alpha,R}$ (see (4.1)). From Theorem 3.5, $\mathcal{J}_{\alpha,R}$ is compact and there exists $x_{\alpha,1}, ..., x_{\alpha,m} \in \mathcal{J}_{\alpha,R}$ such that

$$\mathcal{J}_{\alpha,\mathcal{R}} \subset \bigcup_{j=1}^{m} \mathcal{O}_{\frac{\varepsilon}{2}}(x_{\alpha,j}).$$

For each $j = 1, \dots, m$, since $\mathcal{J}_\alpha = \bigcup_{i=1}^n W^u(e_{\alpha,i})$, $x_{\alpha,j} \in W^u(e_{\alpha,i})$ for some $i = 1, \dots, n$. Let ξ be the global solution of T_α such that $x_{\alpha,j} = \xi(0)$ as in the definition of $W^u(e_{\alpha,i})$. Thus, for some $t_0 < 0$, we have $z_{\alpha,j} = \xi(t_0) \in W^u_\rho(e_{\alpha,i})$. From the continuity of the unstable manifolds of equilibria, we can find $z_{\beta,j} \in W^u_\rho(e_{\beta,i})$ arbitrarily close to $z_{\alpha,j}$, as long as β is arbitrarily close to α. The continuity of the functions $y \mapsto T_\alpha(t_0)y \in X$ and $\beta \mapsto T_\beta(t_0)y$ allow us to obtain, for β in a small neighborhood of α, such that $\|z_{\alpha,j} - z_{\beta,j}\|$ is sufficiently small, hence

$$\begin{aligned}\|x_{\alpha,j} - x_{\beta,j}\| &= \|T_\alpha(t_0)z_{\alpha,i} - T_\beta(t_0)z_{\beta,j}\| \\ &\leqslant \|T_\alpha(t_0)z_{\alpha,j} - T_\alpha(t_0)z_{\beta,j}\| + \|T_\alpha(t_0)z_{\beta,j} - T_\beta(t_0)z_{\beta,j}\| < \frac{\varepsilon}{2}.\end{aligned}$$

Since $\|x_\alpha - x_{\alpha,j}\| < \frac{\varepsilon}{2}$ for some $j = 1, \dots, m$, we obtain

$$\|x_\alpha - x_{\beta,j}\| \leqslant \|x_\alpha - x_{\alpha_j}\| + \|x_{\alpha_j} - x_{\beta,j}\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2},$$

proving the theorem. □

Conclusion. Joining these results, we have the following: there exists $\delta > 0$ sufficiently small such that for each $\alpha \in (1-\delta, 1+\delta)$ equation (1.1) has $\mathcal{J}_\alpha$ (the set of bounded in the past global solutions of (1.1)) as its unique unbounded attractor and

$$\lim_{\beta\to\alpha} \left[d_H(\mathcal{J}_\beta, \mathcal{J}_\alpha) + d_H(\mathcal{J}_\alpha, \mathcal{J}_\beta) \right] = 0.$$

References

[1] Amann, H. *Linear and quasilinear parabolic problems. Vol. I*, vol. 89 of *Monographs in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1995. Abstract linear theory.

[2] Banaśkiewicz, J., Carvalho, A. N., Garcia-Fuentes, J., and Kalita, P. Autonomous and non-autonomous unbounded attractors in evolutionary problems. *Journal of Dynamics and Differential Equations* (2022), 1–54.

[3] Belluzi, M., Bezerra, F. D. M., and Nascimento, M. J. D. On spectral and fractional powers of damped wave equations. *Commun. Pure Appl. Anal. 21*, 8 (2022), 2739–.

[4] Belluzi, M., Nascimento, M. J. D., and Schiabel, K. On a cascade system of Schrödinger equations. Fractional powers approach. *J. Math. Anal. Appl. 506*, 1 (2022), Paper No. 125644.

[5] Ben-Gal, N. *Grow-up solutions and heteroclinics to infinity for scalar parabolic PDEs*. ProQuest LLC, Ann Arbor, MI, 2010. Thesis (Ph.D.)–Brown University.

[6] Bezerra, F. D. M., Carvalho, A. N., Cholewa, J. W., and Nascimento, M. J. D. Parabolic approximation of damped wave equations via fractional powers: fast growing nonlinearities and continuity of the dynamics. *J. Math. Anal. Appl. 450*, 1 (2017), 377–405.

[7] Bezerra, F. D. M., Carvalho, A. N., Dlotko, T., and Nascimento, M. J. D. Fractional Schrödinger equation; solvability and connection with classical Schrödinger equation. *J. Math. Anal. Appl. 457*, 1 (2018), 336–360.

[8] Bezerra, F. D. M., Carvalho, A. N., and Nascimento, M. J. D. Fractional approximations of abstract semilinear parabolic problems. *Discrete Contin. Dyn. Syst. Ser. B 25*, 11 (2020), 4221–4255.

[9] Bezerra, F. D. M., and Santos, L. A. Fractional powers approach of operators for abstract evolution equations of third order in time. *J. Differential Equations 269*, 7 (2020), 5661–5679.

[10] Bortonal, M., and Fernandes, J. Sufficient conditions for the existence and uniqueness of maximal attractors for autonomous and nonautonomous dynamical systems. *Journal of Dynamics and Differential Equations 1* (2022).

[11] Brunovský, P., and Poláčik, P. The Morse-Smale structure of a generic reaction-diffusion equation in higher space dimension. *J. Differential Equations 135*, 1 (1997), 129–181.

[12] Carvalho, A. N., and Pimentel, J. F. S. Autonomous and non-autonomous unbounded attractors under perturbations. *Proc. Roy. Soc. Edinburgh Sect. A 149*, 4 (2019), 877–903.

[13] Chepyzhov, V. V., and Goritskiĭ, A. Y. Unbounded attractors of evolution equations. In *Properties of global attractors of partial differential equations*, vol. 10 of *Adv. Soviet Math.* Amer. Math. Soc., Providence, RI, 1992, pp. 85–128.

[14] Dlotko, T. Navier-Stokes equation and its fractional approximations. *Appl. Math. Optim. 77*, 1 (2018), 99–128.

[15] Dlotko, T., Liang, T., and Wang, Y. Critical and super-critical abstract parabolic equations. *Discrete Contin. Dyn. Syst. Ser. B 25*, 4 (2020), 1517–1541.

[16] Hale, J. K. *Asymptotic behavior of dissipative systems*, vol. 25 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1988.

[17] Henry, D. *Geometric theory of semilinear parabolic equations*, vol. 840 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1981.

[18] Kato, T. *Perturbation theory for linear operators*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1980 edition.

[19] Kilbas, A. A., Srivastava, H. M., and Trujillo, J. J. Fractional differential equations: a emergent field in applied and mathematical sciences. In *Factorization, singular operators and related problems (Funchal, 2002)*. Kluwer Acad. Publ., Dordrecht, 2003, pp. 151–173.

[20] Martínez Carracedo, C., and Sanz Alix, M. *The theory of fractional powers of operators*, vol. 187 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam, 2001.

[21] Micu, S., and Zuazua, E. On the controllability of a fractional order parabolic equation. *SIAM J. Control Optim. 44*, 6 (2006), 1950–1972.

[22] Pazy, A. *Semigroups of linear operators and applications to partial differential equations*, vol. 44 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1983.

[23] Pimentel, J., and Rocha, C. Noncompact global attractors for scalar reaction-diffusion equations. *São Paulo J. Math. Sci. 9*, 2 (2015), 299–310.

[24] Pimentel, J., and Rocha, C. A permutation related to non-compact global attractors for slowly non-dissipative systems. *J. Dynam. Differential Equations 28*, 1 (2016), 1–28.

[25] Robinson, J. C. *Infinite-dimensional dynamical systems*. Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge, 2001. An introduction to dissipative parabolic PDEs and the theory of global attractors.