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ORIGINAL RESEARCH

Off-diagonal condition

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Abstract
In this paper we are interested in maximal ideals of $\mathscr{C}^\infty$ functions with an off-diagonal condition. Their importance is related to solutions of nonlinear PDEs with singularities and also to applications in physics. Such ideals were first studied by E. E. Rosinger. Examples of such ideals are the ones originating from the Colombeau algebras of generalized functions. It is in this context that we determine maximal ideals with an off-diagonal condition.

Keywords Colombeau · Generalized functions · Sharp topology · Ultrametric · Ideal · Diagonal

Mathematics Subject Classification Primary 46F30; Secondary 46T20

1 Introduction

By introducing distribution theory, Schwartz was among the first to provide us with a well developed theory of generalized functions. This is a linear theory not admitting multiplication among the new objects. Several attempts were made to find a setting which permits such multiplications while maintaining the essence of Schwartz' theory. A general way of constructing these algebras is due to E. E. Rosinger. We refer the reader to [21] and its list of references for the importance and more details about these algebras. Below is a short description of Rosinger's construction.

We dedicate this paper to Professor Jorge Aragona of the University of São Paulo - Brazil, for his lifetime contributions in the field of Colombeau Generalized Functions. Professor Aragona was the mentor of the Brazilian group and made a lot of friends with persons in the field. His contributions and expertise will be missed as will be his presence.

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Let Λ be an infinite set, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $\Omega \subset \mathbb{K}^n$ an open subset and $\mathscr{C}(\Omega)$ the ring of continuous functions taking values in $\mathbb{K}$. Consider the commutative unital algebra $\mathscr{C}(\Omega)^\Lambda$ of nets $(f_\lambda)_{\lambda\in\Lambda}$ and its diagonal

$$\mathcal{U}_\Lambda(\Omega) = \{(f_\lambda)_{\lambda\in\Lambda} : f_\lambda = f_{\lambda_0}\},$$

with $\lambda_0 \in \Lambda$ a fixed index. We have that $\mathcal{U}_\Lambda(\Omega)$ is a subalgebra of $\mathscr{C}(\Omega)^\Lambda$ and there is a natural algebra isomorphism $u : \mathscr{C}(\Omega) \to \mathcal{U}_\Lambda(\Omega)$, defined by $u(f) = (f_\lambda)_{\lambda\in\Lambda}$; $f_\lambda = f$, $\forall\, \lambda \in \Lambda$. Given a subalgebra $\mathscr{A} \subset \mathscr{C}(\Omega)^\Lambda$ and an ideal $\mathfrak{I} \lhd \mathscr{A}$, we say that the pair $(\mathscr{A}, \mathfrak{I})$ satisfies an off-diagonal condition (it is an off-diagonal pair) if the following holds:

$$\mathfrak{I} \cap (\mathscr{A} \cap \mathcal{U}_\Lambda(\Omega)) = \{0\}.$$

The resulting quotient algebra $\mathscr{A}/\mathfrak{I}$ is called an off-diagonal quotient algebra. Of particular interest are the off-diagonal quotient algebras with $\mathscr{A} \subset \mathscr{C}^\infty(\Omega)^\Lambda \subset \mathscr{C}(\Omega)^\Lambda$ since, in this case, one can choose $\mathfrak{I}$ to be a differential ideal and hence, the resulting off-diagonal quotient is a differential algebra. The interest in these off-diagonal quotients is because they are natural candidates for environments in which multiplication of Schwartz's distributions can take place. Examples of such candidates are the Colombeau algebras of generalized functions, introduced in the eighties by J. F. Colombeau, in which distributions may be multiplied and hence are environments for solving nonlinear PDEs. The ideal of the off-diagonal pair can be adjusted to control the singularities that can be handled by the corresponding off-diagonal quotient.

As in [21], we are interested in the maximal ideals of these off-diagonal quotients. If we consider Λ with the discrete topology, the maximal ideals we are interested in satisfy certain vanishing conditions coming from the Stone-Čech compactification $\beta(\Lambda \times \Omega)$ of $\Lambda \times \Omega$. This is a well know construction, but may be highly non-trivial. Fortunately, in the case of Colombeau algebras, one can view the elements of the off-diagonal quotient as functions over a topological space taking values in an ultrametric ring. This permits the use of the ideas of the classical construction of Gillman and Jerison (see [15,16]). The needed topology was defined, but not used, by H. Biagioni, and studied in more details by D. Scarpalézos. As noted by M. Oberguggenberger, this topology was coined the sharp topology. Being crucial in further developments of the theory, it permitted, together with the key notion of generalized point values introduced by M. Kunzinger and M. Oberguggenberger, the development of a differential calculus which permits to see distributions as differentiable functions defined over an ultrametric space and taking values in an ultrametric ring (see [6]). This is where non-Archimedean analysis comes into the picture. Recall that non-Archimedean function theory was introduced by J. Tate when studying elliptic curves with bad reduction (see [11]). So it should not come as a surprise that non-Archimedean analysis comes into the picture when studying nonlinear PDEs with singularities. Other off-diagonal quotients were introduced by Todorov and Vernaeve. Classical texts containing the Theory of Colombeau generalized functions are [1,10,12,13,17–19] and [20]. A very interesting and completely different approach can be found in [25].

In [16], Khelif and Scarpalézos use a quotient algebra of the simplified algebra of Colombeau generalized functions to continue the study of its maximal ideals started in [9]. If $\mathfrak{m} \lhd \overline{\mathbb{K}}$ and $\mathbb{L} = \overline{\mathbb{K}}/\mathfrak{m}$, they prove that elements of the quotient algebra may be identified with continuous maps from a subset of $\mathbb{L}^n$ taking values in $\mathbb{L}$. This quotient algebra is also a differential algebra containing an embedding of the space of Schwartz distributions. Using a compactification process, and the methods developed in [9], they parametrize the maximal spectra. Our approach is a bit different and concerns both the simple and full Colombeau algebra.

In this paper, we pick up the string left in the paragraph after [9, Theorem 6.5], and study the maximal ideals of Colombeau algebras. This is where the link between ideals of Colombeau algebras and with Gillman and Jerison's classical construction is established. With this, we give one more solution for the embedding problem, i.e., the problem of embedding the Schwartz distributions in a differential algebra such that the image of the ring of $\mathscr{C}^\infty-$functions is a subalgebra and the embedding commutes with partial derivation, showing that there is an embedding of Schwartz' distributions in a ring of infinitely differentiable functions contained in $\mathscr{F}(\mathbb{L}^n, \mathbb{L})$, where $\mathbb{L}$ is as in the previous paragraph. The relationship with the ultrafilter construction of Todorov and Vernaeve is highlighted. This is done in the next section. It turned out that idempotents play a central role in the algebraic part of the theory. First they were totally identified for the simplified algebras and later this was also done for the ring of Colombeau full generalized numbers (see [5,7]). In all cases, it turns out that the Boolean algebra consists of characteristic functions defined on the parameter space of the defining algebra. Making use of the results of Section 4, we continue this line of research and determine the Boolean algebra of the full algebra of Colombeau generalized functions. To do so, we make use of results of [14] which generalizes the differential calculus of [6], defined in the context of the simplified algebra, to the context of the full algebra of Colombeau generalized functions. We also study traces of ideals, and filters in the context of the full algebra.

Notation is mostly standard, and we refer the reader to the classical texts for notation and definitions which we may have omitted.

2 Preliminary

We start recalling the construction of the Boolean algebra of Colombeau algebras. First determined in [7] for the simplified algebras, this was later done in [5] in the context of the full algebras. For $A \subset \mathcal{A}_0(\mathbb{K})$, A^c denotes the complement of A in $\mathcal{A}_0(\mathbb{K})$ and χ_A the characteristic function of A with domain $\mathcal{A}_0(\mathbb{K})$. Let $\mathcal{S} = \left\{S \subset I | 0 \in \bar{S} \cap \bar{S^c}\right\}$, where $\bar{S}$ denotes the topological closure of the set S in $I =]0, 1]$, and

$$\mathcal{S}_f = \left\{A \subset \mathcal{A}_0(\mathbb{K}) | \forall\, p \in \mathbb{N},\ \exists\, \varphi \in \mathcal{A}_p(\mathbb{K}), \{\varepsilon | \varphi_\varepsilon \in A\} \in \mathcal{S}\right\}.$$

$P_*(\mathcal{S}_f)$ denotes the set of all the subsets $\mathcal{F}$ of $P(\mathcal{S}_f)$ which satisfy: $\forall\, A \in \mathcal{S}_f$, either $A \in \mathcal{F}$ or $A^c \in \mathcal{F}$, and if $A, B \in \mathcal{F}$, then $A \cup B \in \mathcal{F}$.

For each $\mathcal{F} \in P_*(\mathcal{S}_f)$, $g_f(\mathcal{F})$ denotes the ideal of $\overline{\mathbb{K}}$ generated by the set of idempotents $\{\chi_A : A \in \mathcal{F}\}$. For each $\mathcal{F} \in P_*(\mathcal{S}_f)$ the ideal $g_f(\mathcal{F})$ is a proper prime ideal (see [5,7]).

$\mathrm{Inv}(\overline{\mathbb{K}})$ denotes the group of units of $\overline{\mathbb{K}}$ with subgroup $H = \{\dot{\alpha}_r : r \in \mathbb{R}\}$, where $\dot{\alpha}_r(\varphi) = (\mathrm{diam}(\mathrm{supp}(\varphi)))^r$ or $\dot{\alpha}_r(\varphi) = (i(\varphi))^r$, and $i(\varphi)$ denotes the diameter of the support of $\varphi \in \mathcal{A}_0(\mathbb{K})$. Recall that, for the simplified algebra, α_r, with $r \in \mathbb{R}$, denotes the class of the element $(\varepsilon \longrightarrow \varepsilon^r)$.

We refer the reader to the classical references for the definitions of the Colombeau algebras $\overline{\mathbb{K}}$ and $\mathcal{G}(\Omega)$. When working in the context of the full algebra, to avoid confusion, we will use the subscript $_f$ to distinguish the simplified and full setting, if necessary. When not explicit, the proof or assertion hold for both the simplified and full algebra.

The differential calculus developed in [6] permits to view generalized functions as $\mathscr{C}^\infty$-functions defined on a subset $\tilde{\Omega}_c$ contained in $\overline{\mathbb{K}}^n$. This was generalized to the context of the full algebra in [14]. $\overline{\mathbb{K}}^n$ is a Haussdorff space, with the sharp topology, containing $\tilde{\Omega}_c$ as a clopen subset and $\mathbb{K}^n$ is discretely embedded (see [6]). This differential calculus has most of the standard properties, and features of classical differential calculus. There exist non-constant functions with zero derivative, but restricting to Colombeau generalized functions, a function with zero derivative is constant.

Given $\xi \in \tilde{\Omega}_c$, define $\nu_\xi : \mathcal{G}(\Omega) \to \overline{\mathbb{K}}$ by $\nu_\xi(f) = f(\xi)$. If $Y \subset \tilde{\Omega}_c$ and $\mathfrak{I} \lhd \overline{\mathbb{K}}$ is an ideal, then

$$\mathcal{G}_{Y,\mathfrak{I}}(\Omega) = \{f \in \mathcal{G}(\Omega) : \nu_\xi(f) \in \mathfrak{I}\ \forall\ \xi \in Y\}$$

is an ideal of $\mathcal{G}(\Omega)$. These ideals have been studied in [9] for the simplified algebra. Here we shall study them in the context of the full algebra. Given an ideal $\mathfrak{J}$ of $\mathcal{G}(\Omega)$, its *generalized trace* is defined as $\mathrm{GTr}(\mathfrak{J}) = \{\xi \in \tilde{\Omega}_c | \nu_\xi(\mathfrak{J}) \neq \overline{\mathbb{K}}\}$, and $\mathrm{GTr}(\mathfrak{J}) \cap \Omega$ is called its *trace* (see [9]). In the simplified algebra, the set of compactly supported generalized functions, $\mathcal{G}_c(\Omega) = \{f \in \mathcal{G}(\Omega) : \mathrm{supp}(f) \subset\subset \Omega\}$, is a dense ideal of $\mathcal{G}(\Omega)$ whose generalized trace is empty (see [9]). In the case of the full algebra, this ideal is also dense and plays an important role in proving the existence of generalized solutions for a certain nonlinear parabolic equation (see [4]).

Given a non trivial zero divisor $x \in \overline{\mathbb{K}}$, its zero set, is given by $Z(\hat{x}) \in \mathcal{S}_f$, where $\hat{x}$ is any representatives of x. This gives rise to a unique element of the Boolean algebra $\mathcal{B}(\overline{\mathbb{K}})$ and we shall denote it by $\chi_{Z(\hat{x})}$. Note that the zero set of an invertible element gives rise to $1 \in \mathcal{B}(\overline{\mathbb{K}})$. So we do not need to fix a specific representative when working with zero sets of elements. The Fundamental Theorem of $\overline{\mathbb{K}}$ (see [5,9]) states that $x \in \mathrm{Inv}(\overline{\mathbb{K}})$ if, and only if, the zero set of x, $Z(\hat{x}) \notin \mathcal{S}_f$ for all representatives $\hat{x}$ of x, and $x \notin \mathrm{Inv}(\overline{\mathbb{K}})$ if, and only if, $\exists\, e \in \mathcal{B}(\overline{\mathbb{K}}) - \{0\}$ such that $x \cdot e = 0$. In particular, if $x \in \overline{\mathbb{K}} \setminus \{0\}$, and $x \notin \mathrm{Inv}(\overline{\mathbb{K}})$, then x is a zero divisor. One also has that $x \in \mathrm{Inv}(\overline{\mathbb{K}})$ if, and only if, there exists $r \in \mathbb{R}$ such that $\alpha_r \leq |x|$ and that $\mathcal{B}(\overline{\mathbb{K}}) = \{\chi_T : T \in \mathcal{S}\}$. The latter is a discrete subset of $\overline{\mathbb{K}}$ (see [5,6]).

Another important property that holds in all these algebras is what we call *Convexity of Ideals*: all these algebras are partially ordered rings and if $\mathcal{I}$ is an ideal, $y \in \mathcal{I}$ and $|x| \leq |y|$ then $x \in \mathcal{I}$. For the sake of completeness, we recall that, for example, for the simplified Colombeau ring of generalized numbers the norm of an element x is defined by $\|x\| = e^{-V(x)}$, where $V(x) = sup\{r \in \mathbb{R} : |x(\varepsilon)| = o(\varepsilon^r)\}$. For example, $\|\alpha_r\| = e^{-r}$ and thus as $r \longrightarrow \infty$ we have that $\alpha_r \longrightarrow 0$. We refer the reader to [5] for the definition of the norm in the full case and to [5,9] for convexity of ideals.

The subring of elements of $\overline{\mathbb{K}}$ whose limit when $\varepsilon \longrightarrow 0$ is equal to 0, we shall denote by $\overline{\mathbb{K}}_0$ (see [7,14]). This subring contains the infinitesimals of $\overline{\mathbb{K}}$ and thus contains all the elements whose norm are less than one. Hopefully, this does not create any confusion with similar notions in the text representing residual class fields. The meaning will be clear from the context.

3 Aragona algebras

In this section, we shall be working in both the simplified and full Colombeau algebra. The proofs we shall give work in both cases and, where needed, we shall highlight the results we are using to give the proofs in both the simple and full case. Unless explicit, we shall use the notation of the simplified algebra to keep notation simple. In this section, $\Omega \subset \mathbb{K}^n$ will denote an open and connected subset and $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. We will be using the Embedding Theorem, $\kappa : \mathcal{G}(\Omega) \to \mathscr{F}(\tilde{\Omega}_c, \overline{\mathbb{K}})$, and also the differential calculus introduced in [6] and extended in [14]. The reference [6, Proposition 4.4] will be of particular importance in this section. We shall show how this leads to new algebras which are nearer to standard calculus than the ones given in the embedding theorem in [6,14].

The starting point of this section is the paragraph after [9, Theorem 6.5]. There, the similarity of some maximal ideals in $\mathcal{G}(\Omega)$ with maximal ideals in rings of continuous functions is noted. We shall show how to develop this in the full and simplified Colombeau algebras.

Let $\mathfrak{I} \lhd \overline{\mathbb{K}}$ be an ideal, $\mathbb{L} = \overline{\mathbb{K}}/\mathfrak{I}$, and let $\pi : \overline{\mathbb{K}} \to \mathbb{L}$ be the canonical map. We extend this map from $\overline{\mathbb{K}}^n$ to $\mathbb{L}^n$, applying it in each coordinate, and still denoting it by π.

Given $f \in \mathcal{G}(\Omega)$, denoting $\kappa(f)$ still by f, $x \in \tilde{\Omega}_c$ and $h \in \mathfrak{I}^n$ such that $x+h \in \tilde{\Omega}_c$, there exists $z \in \tilde{\Omega}_c$ such that

$$f(x+h) - f(x) = \langle \nabla f(z) \mid h \rangle \in \mathfrak{I}.$$

It follows that, in $\mathbb{L}$, $\pi(f(x+h)) = \pi(f(x))$. Note that by [6, Proposition 3.2] if $\|h\| < 1$, then $x + h \in \tilde{\Omega}_c$. Consider the algebra of functions $\mathscr{F}(\mathbb{L}^n, \mathbb{L})$, and its subalgebra $\mathscr{A} = \mathscr{F}(X, \mathbb{L})$, with $X = \pi(\tilde{\Omega}_c)$. We consider all these spaces with the quotient topology. By what we just proved, there exists an algebra homomorphism

$$\Psi : \mathcal{G}(\Omega) \to \mathscr{A}$$

defined by

$$\Psi(f)(\pi(x)) = \pi(f(x)),$$

recalling that we denote $\kappa(f)$ still by f. In case $\mathfrak{J} = \mathfrak{m}$ is a maximal ideal, then $\mathbb{L}$ is the residual field of $\mathfrak{m}$, and we denote it by $\mathbb{L} = \overline{\mathbb{K}}_{\mathfrak{m}}$. We also denote by $X(n, \mathbb{L}) = \pi(\tilde{\Omega}_c)$ and $\mathscr{A}(\mathfrak{m}, \mathbb{L}) = \mathscr{A} = \mathscr{F}(X(n, \mathbb{L}), \mathbb{L})$ the algebra of functions defined in $X(n, \mathbb{L})$ and taking values in $\mathbb{L}$. We call $X(n, \mathbb{L}) \subset \overline{\mathbb{K}}_{\mathfrak{m}}^n$ an *Aragona n-Space* and $\mathscr{A}(\mathfrak{m}, \mathbb{L})$ an *Aragona Algebra*. These names are given in honor of Jorge Aragona who contributed much to the development of the theory of Colombeau generalized functions. In the case of an Aragona Algebra we have:

Lemma 1 *The kernel of* Ψ *is given by*

$$\mathfrak{J} = \ker(\Psi) = \{f \in \mathcal{G}(\Omega) : \Im m(f) \subset \mathfrak{m}\}.$$

Moreover, $\mathfrak{J}$ *is a differential ideal of* $\mathcal{G}(\Omega)$*, and if* $F \in \mathcal{G}(\Omega)$ *is such that* $\frac{\partial F}{\partial x_j} \in \mathfrak{J}$, $\forall\, 1 \leq j \leq n$*, then there exists a constant* $c \in \overline{\mathbb{K}}$ *such that* $c + F \in \mathfrak{J}$*. In particular, if* $n = 1$*, this means that each* $f \in \mathfrak{J}$ *has a primitive in* $\mathfrak{J}$*.*

Proof We first suppose that $n = 1$. Since $\mathfrak{m}$ is closed (see [5,7]), we have that if $f \in \ker(\Psi)$, then, by [6,14],

$$f'(x) = \lim_{\mathbb{N} \ni k \to \infty} \frac{f(x + \alpha_k) - f(x)}{\alpha_k} \in \mathfrak{m}.$$

As noted after the embedding theorem in [6,14], all elements $f \in \mathcal{G}(\Omega)$ have a primitive F and we may suppose $0 \in \Omega$ and $F(0) = 0$. Hence, if $f \in \mathfrak{J}$ it follows that $F(x) = F(x) - F(0) = F'(c)(x - 0) = f(c) \cdot x \in \mathfrak{m}$, $\forall\, x \in \tilde{\Omega}_c$. It follows that $F \in \mathfrak{J}$. From this it follows that, in general, if $f \in \mathfrak{J}$, then $\partial^\alpha f \in \mathfrak{J}$, $\forall\, \alpha$, multi-index. Using Leibniz's rule, this proves that $\ker(\Psi)$ is a differentiable ideal of $\mathcal{G}(\Omega)$. Since $F(x + h) - F(x) = \langle \nabla F(z) \mid h \rangle \in \mathfrak{J}$, because the coordinates of the gradient are in the kernel, the last part of the Lemma follows. □

It follows from Lemma 1 that the differential algebra $\mathcal{G}(\Omega)/\ker(\Psi)$ (see also [16] for the latter algebra) is contained in an Aragona algebra, and the former is a quotient algebra of a Colombeau algebra.

If $f \in \mathscr{C}^\infty(\Omega)$ is non-zero then, since $\mathscr{C}^\infty(\Omega)$ is canonically embedded as a subalgebra in $\mathcal{G}(\Omega)$, it follows that $\Im m(f)$ contains units, and hence $f \notin \ker(\Psi)$. Consequently, we have that $\mathscr{C}^\infty(\Omega) \cap \ker(\Psi) = \{0\}$, which is an off-diagonal condition.

If $0 \in \Omega$ and δ is the Dirac distribution, then $\delta(0) \in \alpha_{-1} \cdot \mathbb{K}^* \subset \mathrm{Inv}(\overline{\mathbb{K}})$ and hence $\delta \notin \ker(\Psi)$. Our aim is to prove that $\ker(\Psi) \cap \mathscr{D}'(\Omega) = \{0\}$ meaning that we may embed the Schwartz distributions in an Aragona algebra (see Theorem 2, the Embedding Theorem, for the equivalent statement: $Ker(\Psi) \cap \mathscr{E}'(\Omega) = \{0\}$). It is easily seen that Ψ is injective when restricted to $\mathscr{D}'(\Omega)$ if, and only if, Ψ is injective when restricted to $\mathscr{E}'(\Omega)$, the subspace of distributions of compact support. In fact, if $w \in \mathfrak{J} \cap \mathscr{D}'(\Omega)$ is non-zero, then there exists a compact subset $K \subset \Omega$ such that the restriction $\chi_K \cdot w \neq 0$ and $\Psi(\chi_K \cdot w) = \Psi(\chi_K) \cdot \Psi(w) = 0$, where, in this specific case, χ_K denotes the characteristic function of the set K. This notation should not cause any misinterpretation when dealing with idempotents.

If $\mathbb{K} = \mathbb{R}$, then $\overline{\mathbb{R}}_{\mathfrak{m}}$ is a real closed field. The facts that $\overline{\mathbb{R}}$ is partially ordered and $\mathfrak{m}$ is a convex ideal, imply that $\overline{\mathbb{R}}_{\mathfrak{m}}$ is a totally ordered field. We refer the reader to [9] and also to a, yet, unpublished paper of Aragona-Juriaans-Martins ([8]) (written also as a result of interactions with D. Scarpalézos), where the study of the residue fields was undertaken and several, now folklore, results were proved. Some of them are recalled and used below.

In case $\mathbb{K} = \mathbb{R}$, we have that $\mathbb{L} = \overline{\mathbb{R}}_{\mathfrak{m}}$ is a real closed field from which it follows that $\overline{\mathbb{C}}_{\mathfrak{m}}$ is an algebraically closed field and does not depend on the maximal ideal $\mathfrak{m}$, that is, all these fields are isomorphic. The last claim follows from a classical result of Steinitz on real closed fields since it can be shown that polynomials of odd degree and coefficients in $\overline{\mathbb{R}}_{\mathfrak{m}}$ always have a zero in $\overline{\mathbb{R}}_{\mathfrak{m}}$ (see [8] fore more details). All these fields are ultrametric fields, where the ultrametric is given by

$$\|\pi(\alpha)\|_{\mathfrak{m}} = \inf\{\|\alpha + h\| : h \in \mathfrak{m}\}.$$

Moreover, this ultrametric induces the quotient topology on these fields, in particular, π is a Lipschitz function with Lipschitz constant equal to 1. We also have that $\|ab\|_{\mathfrak{m}} \leq \|a\|_{\mathfrak{m}} \cdot \|b\|_{\mathfrak{m}}$. We shall be using these facts in the rest of the section.

We first prove a result which relates the norm defined on $\mathbb{L}$ with the ultrafilter, $\mathcal{F}$, defined by $\mathfrak{m}$. We refer the reader to [5,7] and [24] for more details on this. We will be using freely results contained in these references.

Lemma 2 *Let $\mathfrak{m} \lhd \overline{\mathbb{K}}$ be a maximal ideal, $\mathbb{L} = \overline{\mathbb{K}}_{\mathfrak{m}}$, and $\mathcal{F}$ the ultrafilter associated with $\mathfrak{m}$. If $\alpha \in \overline{\mathbb{K}}$, then*

$$\|\pi(\alpha)\|_{\mathfrak{m}} = \inf\{\|\alpha \cdot \chi_A\| : A \in \mathcal{F}\}.$$

Proof Note first that $\pi(\chi_A) = 1, \ \forall \ A \in \mathcal{F}$. So if $\alpha \notin \mathfrak{m}$, then $\pi(\alpha \cdot \chi_A) = \pi(\alpha)$. We also have that $\|x\| = \max\{\|x \cdot \chi_A\|, \ \|x \cdot \chi_{A^c}\|\} \geq \|x \cdot \chi_A\|, \ \forall \ x \in \overline{\mathbb{K}}$. From this it follows that

$$\|\pi(\alpha)\|_{\mathfrak{m}} = \inf\{\|\alpha + h\| : h \in \mathfrak{m}\} \geq \inf\{\|\alpha \cdot \chi_{Z(h)}\| : h \in \mathfrak{m}\}.$$

But $\alpha \cdot \chi_{Z(h)} = \alpha - \alpha \cdot \chi_{Z(h)^c} = \alpha + h_1, \ h_1 \in \mathfrak{m}$, and hence

$$\inf\{\|\alpha \cdot \chi_{Z(h)}\| : h \in \mathfrak{m}\} \geq \inf\{\|\alpha + h\| : h \in \mathfrak{m}\} = \|\pi(\alpha)\|_{\mathfrak{m}}.$$

This proves that equality holds. □

Note that Lemma 2 shows clearly the relation that exist between the non-standard construction of $\mathbb{L}$ given in [24] and the one using the quotient map. Since $\mathcal{F}$ has the finite intersection property, Lemma 2 actually says that α is totally "peeled" of by the elements of $\mathcal{F}$. If one could picture $\pi(\alpha)$ in $\mathbb{L}$ it would look totally fragmented (looking like nothing was left over of α). Our next result uses key ideas of Non-Archimedean Analysis for which we refer the interested reader to [11].

Corollary 1 *Let* $\mathfrak{m} \lhd \overline{\mathbb{K}}$ *be a maximal ideal,* $\mathbb{L} = \overline{\mathbb{K}}_m$, $r \in \mathbb{R}$, $\mathbb{L}^0 = \{x \in \mathbb{L} : \|x\|_{\mathfrak{m}} \leq 1\}$ *and* $\check{\mathbb{L}} = \{x \in \mathbb{L} : \|x\|_{\mathfrak{m}} < 1\}$. *Then we have*

1. *If* $\pi(x) < \pi(y)$ *in* $\mathbb{L}$, *then* $\|\pi(x)\|_{\mathfrak{m}} \leq \|\pi(y)\|_{\mathfrak{m}}$, $\forall\, x, y \in \overline{\mathbb{K}}$.
2. *All elements of* $\mathbb{L}$ *are power multiplicative, i.e.,* $\|\pi(x^n)\|_{\mathfrak{m}} = \|\pi(x)\|^n_{\mathfrak{m}}$, $\forall\, n \in \mathbb{N}$ *and* $x \in \overline{\mathbb{K}}$.
3. $\mathbb{L}^0$ *is a subring and* $\check{\mathbb{L}}$ *is a prime ideal in* $\mathbb{L}^0$.
4. $\pi(\alpha_r)$ *is a multiplicative element of* $\mathbb{L}$, *i.e., in* $\mathbb{L}$, *we have that*

$$\|\pi(\alpha_r \cdot x)\|_{\mathfrak{m}} = \|\pi(\alpha_r)\|_{\mathfrak{m}} \cdot \|\pi(x)\|_{\mathfrak{m}}, \ \forall\, x \in \overline{\mathbb{K}}.$$

Proof To prove the first item, we may choose $0 < x < y$ in $\overline{\mathbb{K}}$. For each $A \in \mathcal{F}$, we then have that $0 < x \cdot \chi_A < y \cdot \chi_A$. Taking infimum, the result follows.

To prove the second item we use Lemma 2, and the fact that $\|x^n\| = \|x\|^n$ in $\overline{\mathbb{K}}$.

For the third item, if $x, y \in \mathbb{L}^0$, then $\|x \cdot y\|_{\mathfrak{m}} \leq \|x\|_{\mathfrak{m}} \cdot \|y\|_{\mathfrak{m}} \leq 1$, and $\|x + y\|_{\mathfrak{m}} \leq \max\{\|x\|_{\mathfrak{m}}, \|y\|_{\mathfrak{m}}\}$, showing that $\mathbb{L}^0$ is a subring of $\mathbb{L}$. In the same way, if follows that $\check{\mathbb{L}}$ is a subring too, and contained in $\mathbb{L}^0$. Now choose $x, y \in \mathbb{L}^0$ such that $x \cdot y \in \check{\mathbb{L}}$. Since $\check{\mathbb{L}}$ is a subring, and $\mathbb{L}$ is totally ordered, we may suppose that $0 < x < y$ and so $\|x\|_{\mathfrak{m}} \leq \|y\|_{\mathfrak{m}}$. Next, multiplying with x, we have that $0 < x^2 < x \cdot y$. Using the second item, we have that $\|x\|^2_{\mathfrak{m}} = \|x^2\|_{\mathfrak{m}} \leq \|x \cdot y\|_{\mathfrak{m}} < 1$. So we must have that $x \in \check{\mathbb{L}}$, proving the primeness of this ideal.

To prove the last item, we recall from [5,7] that the elements α_r are multiplicative elements of $\overline{\mathbb{K}}$ and that $\|\chi_A\| = 1$. Using this and Lemma 2, we have that

$$\|\pi(\alpha_r)\|_{\mathfrak{m}} = \inf\{\|\alpha_r \cdot \chi_A\| : A \in \mathcal{F}\} = \|\alpha_r\| \cdot \inf\{\|\chi_A\| : A \in \mathcal{F}\} = \|\alpha_r\|.$$

From this we obtain that

$$\begin{aligned} \|\pi(\alpha_r \cdot x)\|_{\mathfrak{m}} &= \inf\{\|\alpha_r \cdot x \cdot \chi_A\| : A \in \mathcal{F}\} \\ &= \|\alpha_r\| \cdot \inf\{\|x \cdot \chi_A\| : A \in \mathcal{F}\} \\ &= \|\alpha_r\| \cdot \|\pi(x)\|_{\mathfrak{m}} \\ &= \|\pi(\alpha_r)\|_{\mathfrak{m}} \cdot \|\pi(x)\|_{\mathfrak{m}}. \end{aligned}$$

□

It follows that the resulting norm is always a *pm*-norm, i.e., $\|a^n\| = \|a\|^n$ for $a \in \mathbb{L}$, and $\mathbb{L}^0/\check{\mathbb{L}}$ is always an integral domain. The reader may find more on power-multiplicative norms, *pm*-norms, in [11, Page 30]. Consequently, the following result holds for all residue class fields obtained form Colombeau algebras.

Theorem 1 *Let* $\mathfrak{m} \lhd \overline{\mathbb{K}}$ *be a maximal ideal and* $\mathbb{L} = \overline{\mathbb{K}}_{\mathfrak{m}}$. *Then* $\|\cdot\|_{\mathfrak{m}}$ *is a valuation on* $\mathbb{L}$.

Proof To prove this result, one applies Corollary 1 and [11, 1.5.3, Proposition 1], which states that for a norm to be a valuation it is sufficient to have certain multiplicative elements and for $\mathbb{L}^0/\check{\mathbb{L}}$ to be an integral domain. □

We are now in position to apply the Theory of Non-Archimedean Analysis (see [11]). To do so, we shall use Theorem 1 to introduce a Differential Calculus in $\mathscr{F}(\mathbb{L}^n, \mathbb{L})$ and thus setting the stage to embed the space of Schwartz distributions in an Aragona algebra.

Since $\mathbb{L}$ is a field, the definition of a derivation on $\mathbb{L}$ is the standard one given in Ultrametric Calculus: we say that a function $f : U \subset \mathbb{L}^n \to \mathbb{L}$ is differentiable at a point x_0 if there exists $a \in \mathbb{L}^n$ such that

$$\lim_{x \to x_0} \frac{f(x) - f(x_0) - \langle a, x - x_0 \rangle}{\|x - x_0\|_{\mathfrak{m}}} = 0.$$

Let $f \in \mathcal{G}(\Omega)$ and consider $\Psi(f)$. Our next result shows that $\Psi(f)$ is a differentiable function defined on an Aragona space and taking values in the field $\mathbb{L}$.

Lemma 3 *Let $y_0 = \pi(x_0) \in X(n, \mathbb{L})$ and $f \in \mathcal{G}(\Omega)$. The function $\Psi(f)$ is differentiable at y_0.*

Proof We use the end part of the proof of the embedding theorem, [6, Theorem 4.1] (see also [14]). For any $x \in \tilde{\Omega}$ and $h \in \mathfrak{m}^n$, we have that there exists a constant $C(h)$ such that

$$\begin{aligned} \|\pi(f(x+h) - f(x_0) - f'(x_0)(x+h-x_0))\|_{\mathfrak{m}} &\leq \\ \leq \|f(x+h) - f(x_0) - f'(x_0)(x+h-x_0)\| &\leq \\ \leq C\|x - x_0 + h\|^2&. \end{aligned}$$

From this it follows that

$$\begin{aligned} \|\Psi(f)(\pi(x)) - \Psi(f)(\pi(x_0)) - \Psi(f')(\pi(x_0))(\pi(x) - \pi(x_0))\|_{\mathfrak{m}} &\leq \\ \leq C\|\pi(x) - \pi(x_0)\|_{\mathfrak{m}}^2&. \end{aligned}$$

From this and the fact that $\|\cdot\|_{\mathfrak{m}}$ is a valuation on $\mathbb{L}$, the result follows. □

Using induction and partial derivatives, it follows that an element f of $\mathcal{G}(\Omega)$ induces a $\mathscr{C}^\infty$−map defined on an Aragona space. Since Ψ has a non-zero kernel, the resulting map may be the zero map. However, we shall prove that the Schwartz distributions are embedded in Aragona algebras. The whole point here is that Aragona algebras are nice algebras, where everything behaves just like in standard calculus. If one considers $\mathbb{K} = \mathbb{C}$ then, by an already mentioned classical result of Steinitz, the fields $\mathbb{L} = \overline{\mathbb{K}_m}$ are all isomorphic (see [8]) and, therefore, for each n there is a unique Aragona Space and a unique Aragona algebra.

Theorem 2 (Embedding Theorem) *Given an open subset $\Omega \subset \mathbb{R}^n$ there exists an ultrametric field $\mathbb{L}$, with the ultrametric given by a valuation, an Aragona space $X(n, \mathbb{L}) \subset \mathbb{L}^n$, an Aragona algebra $\mathscr{A}(\mathfrak{m}, \mathbb{L})$ and an embedding*

$$\tau : \mathscr{D}'(\Omega) \to \mathscr{A}(\mathfrak{m}, \mathbb{L}) \cap \mathscr{C}^\infty(X(n, \mathbb{L}), \mathbb{L})$$

such that

$$\tau(\partial^\alpha T) = D^\alpha(\tau(T)),\ \forall\, T \in \mathscr{D}'(\Omega).$$

Proof The map $\tau = \Psi \circ \kappa \circ \iota$, where κ is the map from the embedding theorem [6, Theorem 4.1] (see also [14]) and ι is the map defined in [17, Theorem 1.2.10, Theorem 1.7.15]. This is obviously a linear map and so we just have to prove that it is injective. As seen above, this means that we must prove that $\ker(\Psi) \cap \mathscr{E}'(\Omega) = \{0\}$. Let $w \in \ker(\Psi) \cap \mathscr{E}'(\Omega)$ and $V \subset \Omega$ a relatively compact subset of Ω containing the support of w, $supp(w)$. Choose $\phi_0 \in D(\Omega)$ such that $\langle w, \phi_0\rangle \in \mathbb{R}^*$. It follows from [17, Theorem 1.5.8, Theorem 1.7.28] and [2, Proposition 3] that

$$\langle w, \phi_0\rangle = \int\limits_V \iota(w)(x)\iota(\phi_0)(x)dx = \iota(w)(p_0)\iota(\phi_0)(p_0)\mu(V)$$

for some $p_0 \in \tilde{V} \subset \tilde{\Omega}_c$ and where $\mu(V)$ is the Lesbesgue measure of V. It readily follows that $\iota(w)(p_0) \in \mathrm{Inv}(\overline{\mathbb{K}}) \bigcap \mathfrak{m}$, a contradiction. That the embedding commutes with derivation follows directly from the definition of the maps, and from the embedding theorem of [6,14]. □

The interested reader may find an interesting and different approach in [22,23]. The theorem allows us to see distributions as functions seen in courses of standard calculus. We will use this theorem to study maximal ideals of the algebra of Colombeau generalized functions $\mathcal{G}(\Omega)$.

Let $\Omega \subset \mathbb{R}^n$ and $0 \neq T \in \mathscr{E}'(\Omega)$ a distribution of compact support and let V be an open relatively compact subset of Ω containing the support of T. Then there exists $r > 0$ such that $\forall\, x \in \tilde{V} \subset \tilde{\Omega}_c$, we have that $\alpha_r \cdot T(x) \in \overline{\mathbb{K}}_0$. Hence the point values of $\alpha_r T$ are all in $\overline{\mathbb{K}}_0$.

To finish this section we shall have a closer look at the Aragona spaces. Using the non-standard analogue, for $x_0 \in \overline{\mathbb{K}}$ the halo of x_0 is the set $\mathrm{halo}(x_0) = \{x \in \overline{\mathbb{K}} : x - x_0 \in \overline{\mathbb{K}}_0\} = x_0 + \mathrm{halo}(0)$ (see [7,14] for the definition of the ring $\overline{\mathbb{K}}_0$). By [6, Proposition 3.2], we have that if $x_0 \in \tilde{\Omega}_c$, then $\mathrm{halo}(x_0) \subset \tilde{\Omega}_c$.

This proves that all elements of $\tilde{\Omega}_c$ are well inside $\tilde{\Omega}_c$ and the latter is the galaxy of all its points. This is what made it possible to define a differential calculus in $\overline{\mathbb{K}}$ in general and, in particular, on the Colombeau algebras (see [6,14]). The fact that maximal ideals $\mathfrak{m} \lhd \overline{\mathbb{K}}$ are closed, convex and uniquely determine an ultrafilter on $\mathcal{P}(\Lambda)$, where Λ is the parameter set of the Colombeau algebra, allowed us to map this structure to $\mathbb{L} = \overline{\mathbb{K}}_{\mathfrak{m}}$ maintaining the same claims for the Aragona space $\pi(\tilde{\Omega}_c)$.

Recall that an element $b \in \mathbb{L} - \mathbb{R}$ is limited if $r < b < s$ for real numbers $r, s \in \mathbb{R}$. So all elements of an Aragona space are limited elements, because all their coordinates are limited (recall that their representatives are contained in compact subsets). Since $\mathbb{R}$ is embedded in $\mathbb{L}$ as a discrete set, it follows that if $b \in \mathbb{L} - \mathbb{R}$ is limited, say $r < b < s$ with $r, s \in \mathbb{R}$, then the set $A_0 = \{x \in \mathbb{R} : x < b\}$ is bounded above in $\mathbb{R}$ by s and $A_1 = \{x \in \mathbb{R} : b < x\}$ is bounded below in $\mathbb{R}$ by r. Hence $b_0 = \sup(A_0) \in \mathbb{R}$ and $b_1 = \inf(A_1) \in \mathbb{R}$. Clearly all elements of A_1 are upper bounds for A_0 and hence $b_0 \leq b_1$. If they were not equal then there would exist $n \in \mathbb{N}$ such that $b_0 < b_0 + \frac{1}{n} < b_1$. Since $\mathbb{L}$ is totally ordered and $b \notin \mathbb{R}$, we have that either $b_0 + \frac{1}{n} < b$ or $b < b_0 + \frac{1}{n}$. In the latter case, we have that $b_0 + \frac{1}{n} \in A_1$ and hence $b_0 + \frac{1}{n} \geq \inf(A_1) = b_1$, a contradiction. In the former case we would have that $b_0 + \frac{1}{n} \in A_0$ and hence $b_0 + \frac{1}{n} \leq \sup(A_0) = b_0$, a contradiction and hence $b_0 = b_1$. Since $b \neq b_0 \in \mathbb{R}$, we may suppose that $b - b_0 > 0$. If $b - b_0$ were not an infinitesimal there would exist $n \in \mathbb{N}$ such that $b - b_0 > \frac{1}{n}$. This would imply that $b_0 + \frac{1}{n} \in A_0$ an hence $b_0 + \frac{1}{n} \leq \sup(A_0) = b_1 = b_0$, a contradiction. This proves that for a limited element of $b \in \mathbb{L}$ there exists $b_0 \in \mathbb{R}$ such that $b \in \text{halo}(b_0)$. This element $b_0 \in \mathbb{R}$ must be unique because two such real numbers would differ by an infinitesimal, and hence, must be equal. Recalling that the coordinates of all elements of an Aragona space are limited this proves the following lemma.

Lemma 4 *Let $X(n, \mathbb{L})$ be an Aragona space. Then it is the disjoint union of the halo of its points, i.e.,*

$$X(n, \mathbb{L}) = \bigcup_{x \in \Omega} \text{halo}(x),$$

where

$$\text{halo}(x) = x + \{y \in \mathbb{L} : \|y\|_{\mathfrak{m}} < 1\}.$$

The Lemma suggests that we may not be aware of much what is going on around us and that continuity as we know it might be mere illusion. It also permits the following: given an element $f \in \mathscr{A}(\mathfrak{m}, \mathbb{L})$ we associate to it a map $\tilde{f} \in \mathscr{F}(\Omega, \mathbb{L})$ given by

$$\tilde{f}(x) = f(x).$$

We say that an element f of an Aragona algebra is *limited* provided $f(x)$ is limited, for all $x \in X(n, \mathbb{L})$, where the latter is an Aragona space. In particular such a function is bounded in the common sense provided $\Omega \subset \mathbb{R}^n$ is bounded. In this case we can define

$$\tilde{f}(x) = \text{shadow}[f(x)],$$

where $\text{shadow}[f(x)] \in \mathbb{R}$ is the unique real number such that $f(x)$ is in the $\text{halo}(\text{shadow}[f(x)])$. As seen in [6], such elements are related to the composition of generalized functions, and we have that $\tilde{f} = 0$ if, and only if, $\Im m(f) \subset \text{halo}(0)$. We say that a function f of an Aragona algebra is *bounded* if there exists $r > 0$ such that $\alpha_r \cdot f$ is limited. Any limited function is bounded and the set of bounded functions form a subring of $\mathscr{A}(\mathfrak{m}, \mathbb{L})$. If a function is not bounded or limited, we say that it is *unbounded*. For example $f_0 = \delta$, the Delta Dirac function, is not limited, since its value at 0 belongs to $\alpha_{-1} \cdot \mathbb{K}^*$. On the other hand, a distribution of compact support is a bounded element since, as seen before, for some α_r its image is contained in $\alpha_r \cdot \overline{\mathbb{K}}_0$. Since there exist distributions of unbounded order, it follows that there exist unbounded functions in Aragona algebras. As noted in [15], the existence of such functions are related to the existence of maximal ideals which are not fixed, thus indicating the existence of non-fixed maximal ideals in Aragona algebras.

Fixing $q = \pi(p) \in X$, the maximal ideal $\mathcal{G}_{p,\mathfrak{m}}(\Omega)$ is the inverse image of the maximal ideal $\{f \in \mathscr{A} : \Psi(f)(q) = 0\}$. Note that $\ker(\Psi)$ is the intersection

$$\ker(\Psi) = \bigcap_{p\in\tilde{\Omega}_c} \mathcal{G}_{p,\mathfrak{m}}.$$

Although Ψ has a kernel, i.e., Ψ is not injective, we may use Aragona algebras to reduce any question in a Colombeau algebra to an Aragona algebra. The reason for this is that the Jacobson Radical, $\mathrm{Rad}(\overline{\mathbb{K}}) = \{0\}$ (see [5,7]) and hence we may choose the embedding such that it separates two given points of $\tilde{\Omega}_c$. Moreover, the fact that $\mathrm{Rad}(\overline{\mathbb{K}}) = \{0\}$ implies that we may embed $\overline{\mathbb{K}}$ in a direct product of fields and hence we may embed $\mathcal{G}(\Omega)$ in a subdirect product of Aragona Algebras. We shall use this in the next section.

4 Filters and idempotents

In this section we shall be dealing with the full Colombeau algebra. In [24] it is proved that if $\mathfrak{p}$ is a prime ideal in the ring of Colombeau generalized numbers, then $\mathcal{L} = \{A \subset (0, 1) : \chi_A \in \mathfrak{p}\}$ is an ultrafilter on $(0, 1)$ containing $J_0 = \{]\eta, 1[: \eta \in]0, 1[\}$. In the full algebra, define

$$\mathcal{J}_f = \left\{A \subset \mathcal{A}_0(\mathbb{K}) | \forall\, p \in \mathbb{N},\ \exists\, \varphi \in \mathcal{A}_p(\mathbb{K}) \text{ such that } \{\varepsilon|\varphi_\varepsilon \in A\} \in J_0\right\}.$$

Lemma 5 *In $\overline{\mathbb{K}}$ the following hold.*

1. *If $\mathcal{F} \in P_*(\mathcal{S}_f)$, then*

$$\mathcal{L}_f = \left\{A \subset \mathcal{A}_0(\mathbb{K}) : \chi_A \in g_f(\mathcal{F})\right\} = \mathcal{F} \cup \{A \subset \mathcal{A}_0(\mathbb{K}) : \chi_A \notin \mathcal{J}_f\}$$

   is a maximal co-filter on $\mathcal{A}_0(\mathbb{K})$.
2. *If $\mathcal{L}_f$ is a maximal co-filter on $\mathcal{A}_0(\mathbb{K})$ containing $\mathcal{J}_f$, then*

$$\mathcal{F} = \mathcal{L}_f \cap \mathcal{S}_f \in P_*(\mathcal{S}_f).$$

Proof As already noted, $g_f(\mathcal{F})$ is a prime ideal of $\overline{\mathbb{K}}$ and hence

$$\left\{A \subset \mathcal{A}_0(\mathbb{K}) : \chi_A \in g_f(\mathcal{F})\right\}$$

is an ultrafilter. It follows that $\mathcal{L}_f$ is a maximal co-filter. This proves the first item.

Since $\mathcal{L}_f$ is a maximal co-filter, it is closed under finite union and $\forall A \in \mathcal{S}_f$, either A or A^c is in $\mathcal{L}_f$. The proof will be completed if proved that $\mathcal{F}$ is closed under finite union. In fact, note that we have the disjoint union $\mathcal{L}_f = (\mathcal{S}_f \cap \mathcal{L}_f) \cup (\mathcal{J}_f \cap \mathcal{L}_f)$ and, clearly, $\mathcal{S}_f \cap \mathcal{J}_f = \emptyset$. If $A \in \mathcal{S}_f$, then either A or $A^c \in \mathcal{L}_f$, and hence either A or $A^c \in \mathcal{L}_f \cap \mathcal{S}_f = \mathcal{F}$. Finally, if $A, B \in \mathcal{F}$, then $A \cup B \in \mathcal{L}_f$ and, since we wrote $\mathcal{L}_f$ as a disjoint union, we need to show that $A \cup B \in \mathcal{S}_f \cap \mathcal{L}_f$. If this were not the case, then $A \cup B \in \mathcal{J}_f \cap \mathcal{L}_f$. From this we have that $\chi_A + \chi_B - \chi_{A \cap B} = 0$. Multiplying the latter equation with χ_A, we have that $\chi_A = 0$, and hence $A \in \mathcal{J}_f$, a contradiction. □

Corollary 2 *If* $\mathfrak{p} \lhd \overline{\mathbb{K}}$ *is a prime ideal, then* $g_{\mathfrak{p}} = \{A \subset \mathcal{A}_0(\mathbb{K}) : \chi_A \in \mathfrak{p}\}$ *is an ultrafilter on* $\mathcal{A}_0(\mathbb{K})$ *containing* $\mathcal{J}_f$.

We now determine the Boolean algebra of $\mathcal{G}(\Omega)$. We shall give two different proofs: one using the Calculus defined on Colombeau algebras and the other using the Calculus defined on Aragona algebras.

Theorem 3 *For an open connected subset* Ω *of* $\mathbb{R}^n$*, we have that the Boolean algebra* $\mathcal{B}(\mathcal{G}(\Omega)) = \mathcal{B}(\overline{\mathbb{K}})$.

Proof To prove that $\mathcal{B}(\overline{\mathbb{K}}) \subset \mathcal{B}(\mathcal{G}(\Omega))$, we just have to consider $\mathcal{B}(\overline{\mathbb{K}})$ embedded in $\mathcal{G}(\Omega)$ as constant functions.

Let $f \in \mathcal{B}(\mathcal{G}(\Omega))$ be a non-trivial idempotent. Then, since f is a differentiable function on $\tilde{\Omega}_c$ and is also an idempotent, we have that $f(x)$, is idempotent in $\overline{\mathbb{K}}$ for all $x \in \tilde{\Omega}_c$. Consequently $f(x) \in \mathcal{B}(\overline{\mathbb{K}})$. Since the Boolean algebra of $\overline{\mathbb{K}}$ is a discrete subset, Ω is connected and f is differentiable, it follows that f is constant on $\tilde{\Omega}_c$ (see the analogue of [6, Proposition 4.7] in [14]). Hence there exists $T \in \mathcal{S}_f$, such that $f = \chi_T$. □

We now give a proof using Aragona algebras. Let $f \in \mathcal{B}(\mathcal{G}(\Omega))$ be a non-trivial idempotent and choose points, $x_1 \neq x_0$ in $\tilde{\Omega}_c$ with $f(x_0) - f(x_1) \neq 0$. Since the Jacobson radical $Rad(\overline{\mathbb{K}}) = 0$ (see [5,7]), there exists a maximal ideal $\mathfrak{m} \lhd \overline{\mathbb{K}}$ such that $f(x_0) - f(x_1) \notin \mathfrak{m}$. Using the Aragona algebra related to $\mathfrak{m}$, we have that $\Psi(f)$ is a non-constant idempotent. But the only idempotents in $\mathbb{L} = \overline{\mathbb{K}}_{\mathfrak{m}}$ are 0 and 1. Since $\Psi(f)$ is continuous, we have a contradiction.

5 Traces of ideals

We recall the Generalized Cauchy-Schwartz Inequality and the Mean Value Theorem proved in [6] in the simplified setting which were generalized in the setting of the full algebras in [14]. Let Ω be a connected open subset of $\mathbb{R}^n$ and $f \in \mathcal{G}(\Omega)$. For $x, y \in \tilde{\Omega}_c$ there exists $z \in \tilde{\Omega}_c$ such that

$$f(x) - f(y) = \langle \nabla f(z) \mid x - y \rangle$$

and, for $x, y \in \overline{\mathbb{K}}$, we have that $|\langle x \mid y \rangle| \le [x]_2 \, [y]_2$, where $x = (x_1, \dots, x_n)$ and $[x]_2 = \left(\sum_{i=1}^n |x_i|^2\right)^{\frac{1}{2}}$.

Proposition 1 *Let* Ω *be an open subset of* $\mathbb{R}^n$ *and* $\xi \in \tilde{\Omega}_c$*. The function* $\nu_\xi : \mathcal{G}(\Omega) \to \overline{\mathbb{K}}$ *is a continuous epimorphism of* $\overline{\mathbb{K}}$*-algebras.*

Proof Since ν_ξ is an evaluation map, and the elements of $\overline{\mathbb{K}}$ are identified with the constant functions of $\mathcal{G}(\Omega)$, we have that ν_ξ is an epimorphism which is clearly $\overline{\mathbb{K}}$-linear.

To prove the continuity, we shall use notation and results from [3] and [4] on the topology of $\mathcal{G}(\Omega)$. Of particular importance here is [4, Theorem 2.7] which states that if $\Omega_1 \subset \Omega$ is relatively compact, then $\{W_{\beta,r} \;:\; \beta \in \mathbb{N}^n, r > 0\}$ is a filter basis of the topology on $\mathcal{G}(\Omega_1)$, where $W_{\beta,r} = \{f \in \mathcal{G}(\Omega_1) : \|\partial^\sigma f\|_{\Omega_1} \leq \dot{\alpha}_r, \forall\, 0 \leq \sigma \leq \beta\}$. Recall also that the sets $V_r[0] = \{x \in \overline{\mathbb{K}} \;:\; |x| < \dot{\alpha}_r\}$, with $r > 0$, generate the sharp topology on $\overline{\mathbb{K}}$ (see [2,3]).

Since ν_ξ is a homomorphism, we only have to prove its continuity at 0. Let $(f_n) \subset \mathcal{G}(\Omega)$ converge to zero. Since $\xi = [(\xi_\varphi)]$ is compactly supported, there exists $\Omega_1 \subset\subset \Omega$ and $p \in \mathbb{N}$ such that $\xi_\varphi \in \Omega_1,\ \forall\, \varphi \in \mathcal{A}_p(\mathbb{K})$. Since the restriction homomorphism $j : \mathcal{G}(\Omega) \to \mathcal{G}(\Omega_1)$ is continuous, given an element $W_{\beta,r}$ of the filtered basis of $\mathcal{G}(\Omega_1)$ there exists $n_0 \in \mathbb{N}$ such that $j(f_n) \in W_{\beta,r},\ \forall\, n \geq n_0$. This translates into $|\partial^\sigma f_n(z)| \leq \dot{\alpha}_r,\ \forall\, z \in \tilde{\Omega}_1,\ \forall\, \sigma \leq \beta$. In particular, $|f_n(\xi)| \leq \dot{\alpha}_r,\ \forall\, n \geq n_0$, and hence $f_n(\xi) \in V_r[0],\ \forall\, n \geq 0$, which proves that $f_n(\xi)$ converges to 0, if $r \longrightarrow \infty$. □

As we have seen in the preliminaries, *convexity of ideals* implies that if $\mathfrak{J}$ is an ideal of $\overline{\mathbb{R}}$, and $x, y \in \overline{\mathbb{R}}$, then $x \in \mathfrak{J}$ if, and only if, $|x| \in \mathfrak{J}$ and if $|y| \leq |x|$, then $y \in \mathfrak{J}$.

Theorem 4 *Let* $\xi, \zeta \in \tilde{\Omega}_c,\ z = [\xi - \zeta]_2\,,\ \mathfrak{I},\ \mathfrak{J} \lhd \overline{\mathbb{K}}$ *ideals and* $\mathfrak{L} \lhd \mathcal{G}(\Omega)$ *an ideal.*

1. *If* $z \in \mathfrak{J}$, *then* $\mathcal{G}_{\xi,\mathfrak{J}}(\Omega) = \mathcal{G}_{\zeta,\mathfrak{J}}(\Omega)$*;*
2. *Let* $\mathfrak{J}$ *be prime. Then* $z \in \mathfrak{J}$ *if, and only if,* $\mathcal{G}_{\xi,\mathfrak{J}}(\Omega) = \mathcal{G}_{\zeta,\mathfrak{J}}(\Omega)$*;*
3. $\mathrm{Tr}(\mathfrak{L}) = \mathrm{Tr}(\overline{\mathfrak{L}})$ *and* $\mathrm{GTr}(\mathfrak{L}) = \mathrm{GTr}(\overline{\mathfrak{L}})$*;*
4. *If* $x_0 \in \mathrm{GTr}(\mathfrak{L})$ *and* $L = \nu_{x_0}(\mathfrak{L})$, *then* $\tilde{\Omega}_c \cap (x_0 + L^n) \subset \mathrm{GTr}(\mathfrak{L})$, *where* $x_0 + L^n = \{x_0 + x | x \in L^n\}$*;*
5. *If* $\mathfrak{I} \neq \mathfrak{J}$, *then* $\mathcal{G}_{\xi,\mathfrak{I}}(\Omega) \neq \mathcal{G}_{\xi,\mathfrak{J}}(\Omega)$.

Proof Let $f \in \mathcal{G}_{\xi,\mathfrak{J}}(\Omega)$. As already mentioned in the beginning of this section, there exists $z \in \tilde{\Omega}_c$ such that $f(\xi) - f(\zeta) = \langle \nabla f(z) \mid \xi - \zeta \rangle$. From this we have that,

$$|f(\xi) - f(\zeta)| = |\langle \nabla f(z) \mid \xi - \zeta \rangle| \leq [\nabla f(z)]_2\, [\xi - \zeta]_2 = [\nabla f(z)]_2 \cdot z$$

Convexity of ideals and the hypothesis imply that $f(\zeta) \in \mathfrak{J}$. Consequently, $\mathcal{G}_{\xi,\mathfrak{J}}(\Omega) = \mathcal{G}_{\zeta,\mathfrak{J}}(\Omega)$, thus proving the first item.

Let $\mathfrak{J}$ be prime and let $\hat{f}(\varepsilon, x) := \langle x - \xi_\varepsilon \mid x - \xi_\varepsilon \rangle$. This is obviously a moderate function and hence its image $f \in \mathcal{G}(\Omega)$. Clearly $f(\xi) = 0$ and thus $f \in \mathcal{G}_{\xi,\mathfrak{J}}(\Omega)$. We also have that $f(\zeta) = |[\xi - \zeta]_2|^2 = z^2 \in \mathfrak{J}$, and hence, by primeness of $\mathfrak{J}$ it follows that $z \in \mathfrak{J}$. Since the previous item gives us the converse, the second item is proved.

Let $\mathfrak{L}$ be an ideal of $\mathcal{G}(\Omega)$ and $x_0 \in \mathrm{GTr}(\mathfrak{L})$. If $x_0 \notin \mathrm{GTr}(\overline{\mathfrak{L}})$, then there exists an element $f \in \overline{\mathfrak{L}}$ such that $f(x_0) = 1$. Choose $(f_n) \subset \mathfrak{L}$ converging to f. The continuity of ν_{x_0} implies that $(\nu_{x_0}(f_n))$ converges to $\nu_{x_0}(f) \in \overline{\mathbb{K}}$. Since $\mathrm{Inv}(\overline{\mathbb{K}})$ is open in $\overline{\mathbb{K}}$ and $\nu_{x_0}(f) = 1 \in \mathrm{Inv}(\overline{\mathbb{K}})$, there exists $n_0 \in \mathbb{N}$ such that $f_n(x_0) \in \mathrm{Inv}(\overline{\mathbb{K}})$. From this it follows that $x_0 \notin \mathrm{GTr}(\mathfrak{L})$, a contradiction. This proves that $\mathrm{GTr}(\mathfrak{L}) \subset \mathrm{GTr}(\overline{\mathfrak{L}})$. Since the converse is clear, the third item is proved.

Let $x_0 \in \mathrm{GTr}(\mathfrak{L})$, $L = \nu_{x_0}(\mathfrak{L})$ and $x \in \tilde{\Omega}_c \cap (x_0 + L^n)$. We have $x = x_0 + h$, $h = (h_1, \ldots, h_n)$, $h_i \in L$, $\forall\, i = 1, \ldots, n$. By the choice of x_0, we have that $L \neq \overline{\mathbb{K}}$. Choosing $f \in \mathfrak{L}$, as before, there exists $z \in \tilde{\Omega}_c$ such that

$$f(x) - f(x_0) = \langle \nabla f(z) \mid x - x_0 \rangle = \langle \nabla f(z) \mid h \rangle.$$

Hence

$$|f(x) - f(x_0)| = |\langle \nabla f(z) \mid h \rangle| \leq [\nabla f(z)]_2\, [h]_2\,.$$

By convexity of ideal, and because $h = x - x_0 \in L^n$, we have that $f(x) - f(x_0) \in L$. Consequently, $f(x) \in L$, and thus $\nu_x(\mathfrak{L}) \subset L \neq \overline{\mathbb{K}}$. Hence, $x \in \mathrm{GTr}(\mathfrak{L})$. This proves that $\tilde{\Omega}_c \cap (x_0 + L^n) \subset \mathrm{GTr}(\mathfrak{L})$.

To prove the last item, let $f \in \mathcal{G}_{\xi,\mathfrak{J}}(\Omega) \cap \overline{\mathbb{K}}$. Then f is a constant function and thus $f = f(\xi) \in \mathfrak{J}$. From this it follows that $\mathcal{G}_{\xi,\mathfrak{J}}(\Omega) \cap \overline{\mathbb{K}} = \mathfrak{J}$. □

Theorem 5 *Let $\mathfrak{J}$ and $\mathfrak{M}$ be ideals of $\mathcal{G}(\Omega)$. Then we have:*

1. $\mathrm{GTr}(\mathfrak{J})$ *is closed;*
2. *If $\mathfrak{M}$ is maximal, $\xi \in \mathrm{GTr}(\mathfrak{M})$ and $\mathfrak{m} = \nu_\xi(\mathfrak{M})$. Then*

   a. $\mathfrak{M} \cap \overline{\mathbb{K}} = \mathfrak{m}$*;*
   b. $\nu_x(\mathfrak{M}) = \mathfrak{m}$ *and* $\mathfrak{M} = \mathcal{G}_{x,\mathfrak{m}}(\Omega)$ *for all* $x \in \mathrm{GTr}(\mathfrak{M})$*;*
   c. $\tilde{\Omega}_c \cap \{\xi + h | h \in \mathfrak{m}^n\} = \mathrm{GTr}(\mathfrak{M})$.

Proof Consider a sequence $(\xi_n) \subset \mathrm{GTr}(\mathfrak{J})$ such that $\xi_n \to \xi$. If $\xi \notin \mathrm{GTr}(\mathfrak{J})$, then $\nu_\xi(\mathfrak{J}) = \overline{\mathbb{K}}$, and hence, there exists $f \in \mathfrak{J}$ such that $f(\xi) = 1$. By continuity of f, we have that $f(\xi_n) \to f(\xi) = 1$. Since $\mathrm{Inv}(\overline{\mathbb{K}})$ is open, there exists $n_0 \in \mathbb{N}$ such that $f(\xi_n) \in \mathrm{Inv}(\overline{\mathbb{K}})\ \forall\, n > n_0$. Hence, $\xi_n \notin \mathrm{GTr}(\mathfrak{J})$, a contradiction.

Let $f \in \mathfrak{M} \cap \overline{\mathbb{K}}$. We have that $f = f(\xi) = \nu_\xi(f) \in \mathfrak{m}$. On the other hand, if $\mathfrak{m} \neq \mathfrak{M} \cap \overline{\mathbb{K}}$, there exists $f_0 \in \mathfrak{m}$, but $f_0 \notin \mathfrak{M}$, By maximality of $\mathfrak{M}$, we have that $f_0 \cdot \mathcal{G}(\Omega) + \mathfrak{M} = \mathcal{G}(\Omega)$. Applying ν_ξ, we obtain that $\nu_\xi(f_0 \cdot \mathcal{G}(\Omega) + \mathfrak{M}) = \nu_\xi(\mathcal{G}(\Omega)) = \overline{\mathbb{K}}$. But $\nu_\xi(f_0 \cdot \mathcal{G}(\Omega) + \mathfrak{M}) = \nu_\xi(f_0 \cdot \mathcal{G}(\Omega)) + \nu_\xi(\mathfrak{M}) = f_0 \cdot \overline{\mathbb{K}} + \mathfrak{m} = \mathfrak{m} \neq \overline{\mathbb{K}}$, a contradiction.

We proceed by proving that $\mathfrak{m}$ is maximal. In fact, if this were not the case then there exists $x_0 \in \overline{\mathbb{K}}$ such that $x_0 \cdot \overline{\mathbb{K}} + \mathfrak{m} \neq \overline{\mathbb{K}}$. The maximality of $\mathfrak{M}$ gives that $x_0 \cdot \mathcal{G}(\Omega) + \mathfrak{M} = \mathcal{G}(\Omega)$. Reasoning as in the previous paragraph, we obtain that $x_0 \cdot \overline{\mathbb{K}} + \mathfrak{m} = \overline{\mathbb{K}}$, a contradiction.

Let $x \in \mathrm{GTr}(\mathfrak{J})$, from what we already proved, it follows that $\nu_x(\mathfrak{M}) = \mathfrak{M} \cap \overline{\mathbb{K}} = \mathfrak{m}$. Supposing that $\mathcal{G}_{x,\mathfrak{m}}(\Omega)$ is not contained in $\mathfrak{M}$ we may choose $f_0 \in \mathcal{G}_{x,\mathfrak{m}}(\Omega)$, with $f_0 \notin \mathfrak{M}$. By maximality of $\mathfrak{M}$, we have that $f_0 \cdot \mathcal{G}(\Omega) + \mathfrak{M} = \mathcal{G}(\Omega)$. Applying ν_x to this equation, we obtain that $\overline{\mathbb{K}} = \nu_x(f_0) \cdot \overline{\mathbb{K}} + \mathfrak{m} \subset \mathfrak{m} \cdot \overline{\mathbb{K}} + \mathfrak{m} = \mathfrak{m}$, a contradiction. Hence $\mathcal{G}_{x,\mathfrak{m}}(\Omega) \subset \mathfrak{M}$. Since the former is a maximal ideal, we have equality.

To prove the last item, note that, by the Theorem 4, $\tilde{\Omega}_c \cap \{\xi + h | h \in \mathfrak{m}^n\} \subset \mathrm{GTr}(\mathfrak{M})$. Let $x \in \mathrm{GTr}(\mathfrak{M})$, $h = x - \xi = (h_1, \ldots, h_n)$ and $z = [x - \xi]_2$. By what we already proved, we have that $\mathcal{G}_{x,\mathfrak{m}}(\Omega) = \mathfrak{M} = \mathcal{G}_{\xi,\mathfrak{m}}(\Omega)$. Since $\mathfrak{M}$ is maximal, and hence prime, it follows by the second item of Theorem 4 that $z \in \mathfrak{m}$. Since $|h_1| \leq z$, it follows by convexity of ideals that $h_i \in \mathfrak{m}, \ \forall\, i = 1, \ldots, n$. From this we have that $h \in \mathfrak{m}^n$, and $x = \xi + h \in \xi + \mathfrak{m}^n$. This, together with item 4 of Theorem 4 finishes the proof. ◻

Corollary 3 *Let $\mathfrak{M} \lhd \mathcal{G}(\Omega)$ be a maximal ideal. Then*

1. $\mathrm{Tr}(\mathfrak{M})$ *has at most one element;*
2. $\mathrm{Tr}(\mathfrak{M})$ *is non-empty if, and only if,* $\mathfrak{m} = \mathfrak{M} \cap \overline{\mathbb{K}}$ *is a maximal ideal and there exists a unique* $\xi \in \Omega$ *such that* $\mathfrak{M} = \mathcal{G}_{\xi,\mathfrak{m}}(\Omega)$*;*
3. $\mathrm{GTr}(\mathfrak{M})$ *is non-empty if, and only if,* $\mathfrak{m} = \mathfrak{M} \cap \overline{\mathbb{K}}$ *is a maximal ideal and there exists* $\xi \in \tilde{\Omega}_c$ *such that* $\mathfrak{M} = \mathcal{G}_{\xi,\mathfrak{m}}(\Omega)$*;*
4. *If* $\mathfrak{M}$ *is dense, then* $\mathrm{GTr}(\mathfrak{M}) = \emptyset$*.*

Proof Let $x, y \in \mathrm{Tr}(\mathfrak{M}) \subset \Omega$. By item 2.$c$ of the Theorem 5, we have that $x - y \in \mathfrak{m}^n$, where $\mathfrak{m} = \mathfrak{M} \cap \overline{\mathbb{K}}$ is maximal. If $x \neq y$, then one of the coordinates of $x - y \in \mathbb{K}$ is a non-zero real number belonging to $\mathfrak{m}$, a contradiction since $\mathfrak{m}$ is a proper ideal of $\overline{\mathbb{K}}$.

To prove the second item, note that if $\mathrm{Tr}(\mathfrak{M}) \neq \emptyset$ then, by the previous item, it has a unique element. The second item of the Theorem 5 gives us the desired conclusion.

If $\mathfrak{M}$ is maximal and dense then it can not be closed. Suppose that $\mathrm{GTr}(\mathfrak{M}) \neq \emptyset$ then, by the Theorem 5, $\mathfrak{M} = \mathcal{G}_{x,\mathfrak{m}}(\Omega)$, for $x \in \mathrm{GTr}(\mathfrak{M})$. Since the map ν_x is continuous and maximal ideals of $\overline{\mathbb{K}}$ are closed, it follows that $\mathfrak{M}$ is closed what contradicting the fact that it is a proper ideal. ◻

Let $\mathfrak{M} = \mathcal{G}_{x_0,\mathfrak{m}}(\Omega) \lhd \overline{\mathbb{K}}$ with $\mathfrak{m} \lhd \overline{\mathbb{K}}$ a maximal ideal. Given $f \in \mathcal{G}(\Omega)$ and $g = f - f(x_0)$, we have that $g(x_0) = 0 \in \mathfrak{m}$, and hence $g \in \mathfrak{M}$. From this it follows that $f + \mathfrak{M} = f(x_0) + \mathfrak{M}$, with $f(x_0) \in \overline{\mathbb{K}}$. This proves that the residue class fields $\frac{\mathcal{G}(\Omega)}{\mathfrak{M}} \simeq \frac{\overline{\mathbb{K}}}{\mathfrak{m}}$. Hence there arise no new residue class fields coming from maximal primes with non-empty generalized trace. We shall call them *fixed maximal ideals*. If a maximal ideal is not fixed we shall say that it is a *free ideal*. So an ideal is fixed if in an Aragona algebra the intersection of the zero sets of its elements is a point. Note that this is the classical idea of a maximal ideal in the space $\mathscr{C}(X) = \{f : X \to \mathbb{R} : f \text{ continuous}\}$, where X is a compact topological space (see [15]). This is exactly what was suggested in [9], as mentioned before. This is clearly a statement about the maximal Z-filters in such spaces. In our case, an Aragona space is not compact, since it contains some open $\Omega \subset \mathbb{R}^n$ as a discrete subset.

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