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EFFECT OF A VACCINATION ON AN EPIDEMIC MODEL

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ABSTRACT. A vaccination term is incorporated to the "$S \to I \to R$" epidemic model of K. Dietz, with oscillatory contact rate. It is shown that the features of the bifurcations of subharmonic solutions are preserved, regardless the loss of symmetry in the bifurcation diagram. It is also shown that a small vaccination can be used to control the bifurcation diagram.

KEY WORDS AND PHRASES: Epidemic model - Vaccine - Subharmonic solutions - Bifurcation

1. INTRODUCTION

K. Dietz [4] proposes a model for some epidemic diseases where oscillations are caused by a periodic contact rate. This is motivated by seasonal variations of contact rates occuring very often in several infectious endemic diseases. Besides, it is possible that endemic diseases become concomitantly epidemic, as it is known. Among these cases are mumps, chicken pox and measles, for instance. See Bailey [2], Grossman [6], London & Yorke [8]. The search for subharmonic solutions, which is done in Smith [9], is motivated by outbreaks of incidence, with period multiple of one year, observed in some cases.

After a rescaling of parameters and change of variables, Smith [9] reduces the model to a perturbed two-dimensional hamiltonian system, where the perturbation is a damping plus a periodic forcing term, with period one. Taking a fixed n-periodic solution of the unperturbed system, called *basic solution*, and considering parameters ε and δ as a measure of the damping and the forcing terms, respectively, he describes the local bifurcation diagram for n-periodic solutions close to the basic one, up to a phase shift, near $\varepsilon = \delta = 0$. These results are in the line of bifurcation near families, in the sense of Hale & Táboas [7], but Smith exploits the symmetries of the model to obtain symmetric bifurcation curves.

A question we deal with in this paper is the following: given a bifurcation diagram of the type described in [9], but not necessarily symmetric, add to the periodic part of the model a one-periodic term, meaning a vaccination, in such a way that the resulting differential equation has that bifurcation diagram.

In section 2 we describe the Dietz' model with the incorporation of a periodic term representing a vaccination. We use the same estimates for the order of magnitude of the biological paramenters considered by Smith in [9]

In section 3 we show that some results from [9] remain valid in the presence of a one-periodic vaccination term, including symmetries of bifurcation diagrams. The object of interest are the subharmonic solutions of order n (n-periodic) of the perturbed equation, near the basic solution, up to phase shifts. The local bifurcation curves are smooth curves in the $\varepsilon\delta$-plane, through the origin, defining sectorial regions where the number of these solutions is constant. The vaccination term breaks the symmetries of the model in the rescaled parameters, however, we use the procedure of [7], see also [10], to accomplish the bifurcation equation and solve it. It

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should be noticed that, although the presence of a vaccination causes a loss of symmetry in the rescaled parameters, the bifurcation diagrams in the original parameter space remain symmetric, even in this case.

The section 4 is concerned with an interesting question. There, when a given bifurcation diagram has two curves defining four sectorial regions, the inverse problem of obtaining the vaccination term starting from the bifurcation diagram is worked out. In order to make the problem treatable, we identify diagrams with bifurcation curves tangent at the origin. We find a vaccination, which depends periodically on the time, with the same frequency of the contact rate, originating the *apriori* given diagram with the above features. We need to impose bounds to the periodic modulation of the contact rate, which implies a boundedness condition on the vaccination.

2. The model

We assume the following hypotheses among the most common in the related literature:

(1) This is a "$S \to I \to R$" model, i.e., the population consists of susceptibles, infectives and recovered individuals. These individuals compose the S, I and R subpopulations, respectively, and are transferred from one to another according to the arrows.

(2) The birth and death rates are both equal to a constant μ, so that the population size remains constant. This is a reasonable simplification for most populations in western societies, according to Anderson & May [1].

(3) There is no latent period, i.e., the disease is contracted upon exposure.

(4) The contact rate $\beta(t)$ has a constant average part, β, and a periodic modulation, with period one year, representing seasonal variations:

$$\beta(t) = \beta(1 + \delta \cos 2\pi t), \quad |\delta| < 1.$$

(5) After an effective contact, an infective individual recovers at a recovery rate γ. By an effective contact is meant an encounter which will result in the infection of a susceptible individual by an infective one.

(6) After recovery, an individual stays lifelong immune.

(7) For simplicity, the vaccination is assumed to take place at the birth. The effectiveness of vaccination depends on the percentage of newborn who receive the vaccination and on the effectiveness of the vaccine itself.

(8) The vaccine gives lifelong protection.

(9) The vaccination is affected by the periodic term of the contact rate. It is a periodic function of time, with period one year, and is constant when the contact rate is constant.

Remark. We propose the hypothesis (9) considering that fluctuations of the contact rate should motivate variations in the vaccination process. It is natural to admit a constant vaccination when the contact rate is constant.

Let S, I and R denote the proportion of the susceptible, infective and immune parts of the population, respectively. According to (1), (2) and (6), we can assume $S + I + R = 1$.

Let $v(t)$ be a measure of the effectiveness of vaccination. By assumptiom (7), $\mu v(t)$ measures the amount of newborn who go into the recovered class and $(1 - v(t))\mu$ measures the amount of those who remain susceptible.

Under these assumptions we are led to the following equations:

$$\begin{aligned} \dot{S} &= (1 - v(t))\mu - \mu S - \beta(t) IS \\ \dot{I} &= \beta(t) IS - \gamma I - \mu I \\ \dot{R} &= \gamma I - \mu R + \mu v(t) \end{aligned} \tag{2.1}$$

Notice that the first two equations are sufficient to describe the model. So, we will refer to (2.1) as the system composed by the first two equations.

Two important paramenters from a biological point of view are the infectious period, $P = 1/(\mu+\gamma)$, and the reproduction rate of the infection, $Q = \beta P$. By the definition of β and P, Q represents the number of secondary cases produced by an infectious individual inside a susceptible population. The disease will remain within the population provided the reproduction rate satisfies $Q = \beta/(\mu+\gamma) \geq 1$. So, from now on we will assume that Q exceeds the unity, a threshold condition for endemicity.

When $\delta = 0$, the assumptions (4) and (9) imply that the system is autonomous and the condition $Q > 1$ guarantee the existence of a nontrivial positive equilibrium of (2.1), (S_0, I_0), where

$$S_0 = \frac{\mu+\gamma}{\beta} = \frac{1}{Q}, \qquad I_0 = \left(1 - \frac{1}{Q}\right)\mu P, \tag{2.2}$$

which is asymptotically stable. Another equilibrium is $\tilde{S}_0 = 1$, $\tilde{I}_0 = 0$.

According to Anderson & May [1], for most of the common diseases in developed countries, the infectious period P is of the order of a few days, 0.01 to 0.02 years, while the life expectancy, $1/\mu$, is of the order of 70 years. So that the assumption $\mu = 10^{-2}(\text{year})^{-1}$ is reasonable. The reproduction rate of the infection is of the order of 10.

These considerations suggest the introduction of a small parameter $\varepsilon \approx 10^{-2}$, such that, for a constant $\Delta > 0$,

$$\varepsilon^2 = \mu P \approx 10^{-4}, \qquad \gamma + \mu = \frac{\Delta}{\varepsilon} \approx 10^{-2}. \tag{2.3}$$

Introducing new variables, $\bar{S} = QS$, $\bar{I} = QI$, and dropping the bars out, the system (2.1) becomes

$$\begin{aligned} \dot{S} &= \frac{\Delta}{\varepsilon}\left[\varepsilon^2 Q - \varepsilon^2 Q v(t) - \varepsilon^2 S - (1 + \delta\cos 2\pi t) I S\right] \\ \dot{I} &= \frac{\Delta}{\varepsilon}\left[(1 + \delta\cos 2\pi t) S - 1\right] I, \end{aligned} \tag{2.4}$$

where, for $\delta = 0$, the equilibria are $(S_0, I_0) = \left(1, \varepsilon^2(Q-1)\right)$ and $(\tilde{S}_0, \tilde{I}_0) = (Q, 0)$.

In the absence of seasonal oscillations, by using (2.3), the characteristic roots of the linear part of (2.4), near (S_0, I_0), can be written as

$$\lambda = -\frac{\mu Q}{2} \pm i\sqrt{\frac{\mu}{P}(Q-1) - \left(\frac{\mu Q}{2}\right)^2}.$$

These considerations concerning the biological parameters, give $\mu(Q-1)/P \gg (\mu Q/2)^2$. Hence, $\Re\lambda = -\mu Q/2$ and $\nu := [\mu(Q-1)/P]^{1/2}$ mean, respectively, the damping and the free frequency. Moreover, one might expects that external excitations, as seasonally oscillating agents, could interact with the damping and bring about oscillatory behavior, provided the period of the contact rate were different from the characteristic period, i.e., $1 \neq 2\pi/\nu$ (or $1/n \neq 2\pi/\nu$, $n = 2, 3, \ldots$), in order to avoid simple (or subharmonic) resonance.

Let us define new variables x, y as introduced by Smith in [9]:

$$S = 1 + \varepsilon\sqrt{K}x, \qquad I = \varepsilon^2 K(1+y), \tag{2.5}$$

where $K = Q - 1$.

Since we wish to see the periodic part of the system (2.4) as a perturbation, the ratio δ/ε must be small. This sugests to replace δ by $\varepsilon\bar{\delta}$. If, in addition, we insert (2.5) in (2.4) and drop out the bar from $\bar{\delta}$, we obtain the following equations for x and y, with small parameters $\varepsilon > 0$ and δ:

$$(2.6\text{-}\varepsilon\delta)\quad \begin{aligned} \dot{x} &= -\nu\left[y + \frac{\varepsilon}{\sqrt{K}}x(1+K+Ky) + \left(1+\frac{1}{K}\right)v(t) + (1+\varepsilon\sqrt{K}x)\varepsilon\delta(1+y)\cos 2\pi t\right] \\ \dot{y} &= \nu\left[x(1+y) + \frac{\delta}{\sqrt{K}}(1+y)\cos 2\pi t + \varepsilon\delta x(1+y)\cos 2\pi t\right]. \end{aligned}$$

Although from an epidemiological standpoint, the meaningful semi-plane is $\varepsilon > 0$, we shall do our mathematical study of (2.6-$\varepsilon\delta$) allowing $\varepsilon \leq 0$.

3. The bifurcation diagram.

In this section we investigate the existence and bifurcation of n-periodic solutions of (2.6-$\varepsilon\delta$), for (ε,δ) in a small neighborhood of (0,0), near a basic one-periodic solution of the reduced equation, (2.6-00), up to phase shift.

Let us write (2.6-$\varepsilon\delta$) in the short form

$$(3.1\text{-}\varepsilon\delta)\qquad \dot{u} = \nu f(u) + \nu G(u,\varepsilon,\delta,t),$$

where $u = \operatorname{col}(x,y)$, $f(u) = \operatorname{col}(-y, x(1+y))$ and, according to the assumption (9), G is periodic in t, with period one, satisfying $G(u,0,0,t) \equiv 0$. Besides, suppose $\bar{u}(t) = (\bar{x}(t), \bar{y}(t))$ is a fixed n-periodic solution of the unperturbed equation

$$(3.1\text{-}00)\qquad \dot{u} = \nu f(u)$$

satisfying $\bar{x}(0) = 0$, $\bar{y}(0) > 0$, where n is a positive integer.

The function $V(x,y) = x^2 + 2y - 2\ln|1+y|$ is a first integral of (3.1-00) and this implies that any orbit near (0,0) is periodic and surround this point, i.e., the origin is a center. Considering the polar coordinates $(x(t), y(t)) = r(t)(\cos\theta(t), \sin\theta(t))$ of the solutions, the period T of such an orbit is given by

$$T = \int_{-\pi/2}^{3\pi/2} \frac{d\theta}{\nu(1 + r_0\cos^2\theta\sin\theta)}, \qquad 0 < r_0 < 1,$$

where $r_0 = r(0)$. Expanding in series,

$$T = \frac{2\pi}{\nu}\left(1 + \frac{1}{16}{r_0}^2 + \dots\right).$$

Furthermore, since the solution $(x,y) = (\nu t, -1)$ can be saw as having infinite period, we may take the integer n in the interval $2\pi/\nu < n < \infty$.

One whishes to characterize the number of n-periodic solutions of (3.1-$\varepsilon\delta$), which are near $\bar{u}(t-\alpha)$, for some α, $0 \leq \alpha < n$, when (ε,δ) varies on a neighborhood of $(0,0) \in \mathbb{R}^2$. We follow the same approach of Hale & Táboas [7], see also [10].

Let $\Gamma = \{\bar{u}(t) : 0 \leq t < n\}$, $u^{\perp}(t) := \operatorname{col}(\dot{y}(t), -\dot{x}(t))$, and consider a tubular neighborhood W of Γ. To each point $x \in W$ are associated coordinates (τ,σ) such that

$$x = \bar{u}(\tau) + \sigma u^{\perp}(\tau).$$

For each n-periodic solution $u(t)$ of (3.1-$\varepsilon\delta$) which remains in W, there exists a unique α, $0 \leq \alpha < n$, such that the coordinates of $u(\alpha) \in W$ are of the form $(\tau,\sigma) = (0,\sigma)$. Thus, such a solution can be represented by

$$(3.2)\qquad u(t) = \bar{u}(t-\alpha) + r(t-\alpha), \quad r(0)\cdot\dot{\bar{u}}(0) = 0,$$

where "$\cdot$" is the usual inner product in $\mathbb{R}^2$.

The 1-1 correspondence between solutions $u(t)$ of (3.1-$\varepsilon\delta$) and $u(t+\alpha)$ of

$$\text{(3.3-}\varepsilon\delta\alpha\text{)} \qquad \dot{u} = \nu f(u) + \nu G(u, \varepsilon, \delta, t+\alpha),$$

makes the problem equivalent to studying the n-periodic solutions of (3.3-$\varepsilon\delta\alpha$) of the form

$$\text{(3.4)} \qquad u(t) = \bar{u}(t) + r(t), \quad r(0)\cdot\dot{\bar{u}}(0) = 0,$$

with small r. This relation and (3.3-$\varepsilon\delta\alpha$) lead to the problem of searching small n-periodic solutions $r(t)$ of

$$\text{(3.5-}\varepsilon\delta\alpha\text{)} \qquad \dot{r} = \nu A(t)r + \nu\bar{G}(r, \varepsilon, \delta, , t, \alpha), \quad r(0)\cdot\dot{\bar{u}}(0) = 0,$$

$A(t) := \frac{d}{du} f(\bar{u}(t))$ and $\bar{G}(r,\varepsilon,\delta,t,\alpha) := G(\bar{u}(t)+r, \varepsilon, \delta, t+\alpha) + f(\bar{u}(t)+r) - f(\bar{u}(t)) - A(t)r$.

The function $\dot{\bar{u}}$ is a nontrivial n-periodic solution of the linear variational equation associated to $(3.1-00)$, around $\bar{u}(t)$,

$$\text{(3.6)} \qquad \dot{u} = \nu A(t)u.$$

The following hypothesis, which is regularly assumed from now on,

(H1) *The space of n-periodic solutions of* (3.6) *is spanned by* $\dot{\bar{u}}$.

holds generically, that is, the set of amplituds r_0 where (H1) is not verified is finite in any bounded interval $[0, \bar{r}]$, see Smith [9, sec. 3]. As a consequence of (H1), the space of n-periodic solutions of the adjoint equation to (3.6),

$$\text{(3.7)} \qquad \dot{w} = -\nu w A(t),$$

where w is a row-vector, is one-dimensional. By using the first integral $V(x,y)$ of (3.1-00), it can be shown that $\bar{w}(t) := \left(\bar{x}(t), \frac{\bar{y}(t)}{1+\bar{y}(t)}\right)$ is a nontrivial n-periodic solution to (3.7) which, therefore, spans the space of such solutions to (3.7).

Let $\mathcal{P}_n$ be the space of continuous n-periodic maps from $\mathbb{R}$ to $\mathbb{R}^2$, with the supremum norm, and $\mathcal{P}_n^1$ its algebraic subspace of the C^1 maps endowed with a C^1 norm.

Let us consider the projections $\bar{P} : \mathcal{P}_n^1 \to \mathcal{P}_n^1$, $\bar{Q} : \mathcal{P}_n \to \mathcal{P}_n$, given by

$$\bar{P}\phi = |\dot{\bar{u}}(0)|^2 (\phi(0)\cdot\dot{\bar{u}}(0))\dot{\bar{u}}, \quad \phi \in \mathcal{P}_n^1,$$
$$\bar{Q}\phi = \eta\left[\int_0^n \phi(t)\cdot\bar{w}(t)\,dt\right]\bar{w}, \quad \phi \in \mathcal{P}_n,$$

where $\eta = \left[\int_0^n \bar{w}^2(t)\,dt\right]^{-1}$.

If the maps $L : \mathcal{P}_n^1 \to \mathcal{P}_n$, $N : \mathcal{P}_n^1 \times \mathbb{R}^3 \to \mathcal{P}_n$ are defined by $Lr = \dot{r} - \nu A(t)r$, $N(r,\varepsilon,\delta,\alpha) = \nu\bar{G}(r,\varepsilon,\delta,t,\alpha)$, the problem (3.5-$\varepsilon\delta\alpha$) can be rewritten as

$$\text{(3.8)} \qquad \begin{aligned} Lr &= N(r,\varepsilon,\delta,\alpha), \\ \bar{P}r &= 0. \end{aligned}$$

In order to apply the Liapunov-Schmidt reduction to this problem, notice that

$$L\mathcal{P}_n^1 = (I - \bar{Q})\mathcal{P}_n,$$

where I denotes the identity operator. This fact is, indeed, a formulation of the Fredholm Alternative, see Hale [6], for instance.

Furthermore, an application of the Closed Graph Theorem gives the existence of a bounded linear operator $K : (I - \bar{Q})\mathcal{P}_n \to (I - \bar{P})\mathcal{P}_n^1$ which is a right inverse of L, that is, $LK = I$ in $(I - \bar{Q})\mathcal{P}_n$ and $KL = (I - \bar{P})\mathcal{P}_n^1$ in $\mathcal{P}_n^1$.

Decomposing the first equation of (3.8) in its components in the suplementary subspaces $\bar{Q}\mathcal{P}_n$ and $(I - \bar{Q})\mathcal{P}_n$, taking also into account the properties of K combined with the second equation, one arrives to the following pair of equations equivalent to (3.8):

$$(3.9) \qquad \begin{aligned} &(a) \quad r = K(I - \bar{Q})N(r, \varepsilon, \delta, \alpha), \\ &(b) \quad 0 = \bar{Q}N(r, \varepsilon, \delta, \alpha). \end{aligned}$$

The compactness of the interval $[0, n]$ implies, after a finite number of applications of the Implicit Function Theorem, there exists a neighborhood $V \subset \mathcal{P}_n^1$ of $r = 0$, a neighborhood U of $(0,0) \in \mathbb{R}^2$ and a C^1 function $r^*(\varepsilon, \delta, \alpha)$ satisfying (3.9(a)), for all $(\varepsilon, \delta, \alpha) \in U \times [0, n]$, and $r^*(0, 0, \alpha) = 0$, $0 \le \alpha \le n$.

Thus, for any $(\varepsilon, \delta, \alpha) \in U \times [0, n]$, there exists a n-periodic solution of (3.1-$\varepsilon\delta$), close to $\bar{u}(t - \alpha)$, satisfying (3.2), if and only if, $(\varepsilon, \delta, \alpha)$ satisfies the bifurcation equation, (3.9(b)), with r replaced by $r^*(\varepsilon, \delta, \alpha)$. According to the former definitions, (3.9(b)) is

$$(3.10) \qquad \int_0^n \left[\bar{w}(t) \cdot N(r^*(\varepsilon, \delta, \alpha), \varepsilon, \delta, \alpha)(t))\right] dt = 0.$$

Since G is 1-periodic in t, it suffices to consider solutions $(\varepsilon, \delta, \alpha) \in U \times [0, 1]$ of (3.10), for if it solves (3.10), then so will $(\varepsilon, \delta, \alpha + j)$, $j = 1, 2, \ldots, n - 1$.

By defining the one-periodic function

$$(3.11) \qquad A(\alpha) = \int_0^n \bar{w}(t) \left[\frac{\partial G}{\partial \sigma}(\bar{u}(t), 0, 0, t + \alpha)\right] dt,$$

where $\sigma = (\varepsilon, \delta)$, the equation (3.10) can be rewritten as

$$(3.12) \qquad A(\alpha) \cdot \sigma + R(\sigma, \alpha) = 0,$$

where $R(\sigma, \alpha) = O(|\sigma|^2)$, as $\sigma \to 0$.

We need the following hypothesis on the vaccination term:

$$(\text{H2}) \qquad \left.\frac{\partial v(t)}{\partial \varepsilon}\right|_{(\bar{u}(t),0,0,t-\alpha)} = 0, \qquad \left.\frac{\partial v(t)}{\partial \delta}\right|_{(\bar{u}(t),0,0,t-\alpha)} \neq 0.$$

Remark. In the next section we restrict ourselves to cases where $v(t)$ is independent of ε, so that half (H2) is *apriori* satisfyed.

Considering the above definitions and assuming (H2), is a matter of some calculation to see that $A(\alpha)$ is given by

$$(3.13) \qquad A(\alpha) = (-\gamma_1, \gamma_2 \cos 2\pi\alpha + \gamma_3(\alpha)),$$

where

$$\gamma_1 := \left(\frac{1}{\sqrt{K}} + \sqrt{K}\right) \int_0^n \bar{x}^2(t)\, dt > 0$$

$$\gamma_2 := \frac{1}{\sqrt{K}} \int_0^n \bar{y}(t) \cos 2\pi t\, dt$$

$$\gamma_3 := -\left(1 + \frac{1}{K}\right) \int_0^n \bar{x}(t) \left.\frac{\partial v(t)}{\partial \delta}\right|_{(\bar{u}(t),0,0,t-\alpha)} dt.$$

Therefore, the closed curve $A(\alpha)$ in the $\varepsilon\delta$-plane is a vertical segment, with abscissa $-\gamma_1$, according to the Figure 3.1.

Remark. **According to Smith [9], the conditon $\gamma_2 \neq 0$ holds generically in ν.**
Another hypothesis needed on $v(t)$ is that the 1-periodic function h, given by

$$h(\alpha) := \gamma_2 \cos 2\pi\alpha + \gamma_3(\alpha), \qquad 0 \leq \alpha < 1,$$

satisfies the following condition:

$$\text{(H3)} \qquad \begin{aligned} & h \in C^2, \\ & h'(\alpha) = 0, \text{ if and only if } \alpha = \alpha_i, \quad i = 1, \dots, n, \\ & h''(\alpha_i) \neq 0, \text{ and } i \neq j \text{ implies } h(\alpha_i) \neq h(\alpha_j), \quad i, j = 1, \dots, n. \end{aligned}$$

Let $\alpha_m, \alpha_M \in [0,1)$ be the absolute minimum and maximum of h, respectively. Since $R(\sigma,\alpha) = O(|\sigma|^2)$, as $\sigma \to 0$, we can ensure that the solutions $\sigma = (\varepsilon, \delta)$ of (3.12) in some neighborhood U_0 of the origin, are in a sectorial subset K^* of U_0, which is a perturbation of the sectors $K = \{\sigma \in U_0 :$ there exists $\alpha \in [0,1)$, with $A(\alpha) \cdot \sigma = 0\}$.
We keep this notation for the statement of the next theorem.

FIGURE 3.1

Theorem 3.1. *Suppose the hypotheses (H1–3) are satisfied. Then there exists a neighborhood $V \subset \mathcal{P}_n^1$ of $\bar{u}_\alpha$, $0 \leq \alpha < 1$, where $\bar{u}_\alpha(t) := \bar{u}(t - \alpha)$, a ball U_0, with center in $\sigma = 0$, in the σ-plane, and a set $K^* \subset U_0$, close to K (in a sense clarified below), such that:*

1. *If $\sigma \in \mathrm{int}(K^*)$, there are at least two n-periodic solutions of (3.1-$\varepsilon\delta$) in V.*
2. *If $\sigma \in U_0 - K^*$, there is no n-periodic solution of (3.1-$\varepsilon\delta$) in V.*
3. *If $\sigma \in \partial K^*$, there is a unique n-periodic solution of (3.1-$\varepsilon\delta$) in V, where ∂ denotes the boundary in U_0.*

Moreover, for each α_i, $i = 1, \dots, n$, specified in the hypothesis (H3), there exists exactly one curve $C_i \subset U_0$, tangent to the line $\gamma_1\varepsilon = h(\alpha_i)\delta$, at $\sigma = 0$, which divides U_0 in two connected subsets such that, the number of n-periodic solutions of (3.1-$\varepsilon\delta$) in V changes by two, when σ, $\sigma \neq 0$, crosses each C_i, $i = 1, \dots, n$.

We mean the set K^* is close to K in the sense that its boundary in U_0 is given by curves tangent, at the origin, to the lines $\gamma_1\varepsilon = h(\alpha_m)\delta$ and $\gamma_1\varepsilon = h(\alpha_M)\delta$.
A proof of the Theorem 3.1 can be found, with minor changes, in [7] or [10], for instance.

FIGURE 3.2

The Figure 3.2 shows a case where the function h has precisely one maximum and one minimum. The numbers in the parenthesis indicate the number of n-periodic solutions in V, when σ is in the corresponding region.

4. An Inverse Problem

In this section we are concerned with bifurcation diagrams of the type described in the Theorem 3.1, defined by a pair of bifurcation curves, C_1 and C_2, such that the number of solutions either change from zero to two or vice-versa.

It follows from the Figure 3.1 that a diagram of this type is determined by two curves through the origin, transversal to both of the axes. The sectors containing a segment of the ε-axis being the regions where there is no n-periodic solution of (3.1-$\varepsilon\delta$) in V.

If C_1, C_2 are the bifurcation curves of such a diagram $\mathcal{D}$, we denote $\mathcal{D} = (C_1, C_2)$

Definition 4.1. We say that two diagrams of the type just described, (C_1, C_2) and $(\tilde{C}_1, \tilde{C}_2)$, are equivalent, if C_i is tangent, at the origin, to $\tilde{C}_i$, $i = 1, 2$. The equivalence class of a diagram $\mathcal{D}$ will be represented by (ℓ_1, ℓ_2), where ℓ_1 and ℓ_2 are the lines tangent to the bifurcation curves at the origin.

Now we are in a position to formulate precisely the problem we are concerned with.

Let $\bar{u}(t)$ be a n-periodic solution of the equation (3.1-00) and ℓ_1, ℓ_2, two transversal lines through the origin of the $\varepsilon\delta$-plane, both being transversal to the axes. Find out a vaccination term $v(t)$ such that, if $\mathcal{D}$ is the bifurcation diagram of n-periodic solutions of (3.1-$\varepsilon\delta$) near $\bar{u}_\alpha$, for some $\alpha \in [0, 1)$, then $\mathcal{D} \in (\ell_1, \ell_2)$.

Given an equivalence class (ℓ_1, ℓ_2), the following theorem shows how a vaccination v can be chosen in order to have a bifurcation diagram $\mathcal{D} \in (\ell_1, \ell_2)$.

In order to solve this problem, we need to vaccinate obeying the coditions assumed before on the vaccination. As we pointed out in the hypothesis (9), it is natural to take a constant vaccination in case of constant contact rate. Besides being periodic, with period one year, $v(t)$, should satisfy $0 \leq v(t) < 1$.

Theorem 4.1. *Let ℓ_1, ℓ_2 two lines through the origin of the $\varepsilon\delta$-plane, transversal to the axes, with slopes $(-m_1)^{-1}$, $(-m_2)^{-1}$, respectively, and consider the equivalence class (ℓ_1, ℓ_2), accord-*

ing to Definition 4.1. Suppose the hypothesis (H1) is satisfied and define

$$A := \frac{\pi\sqrt{K}}{\nu Q \gamma_2}, \quad a := \left(\frac{m_1 - m_2}{2}\right)\gamma_1 - \gamma_2.$$

Let δ, $|\delta| < 1$, be allowed to vary in the interval $(-1/4Aa, 1/4Aa)$ and define the vaccination v by $v(t) := \delta(2Aa\sin 2\pi t - b\sqrt{K}x/\gamma_1) + c$, where $b = -\gamma_1\frac{m_1+m_2}{2}$, and c is choosen in such a way that $0 \leq v(t) < 1$.

Then, if $\mathcal{D}$ is the bifurcation diagram of n-periodic solutions of (3.1-$\varepsilon\delta$) near the family $\bar{u}_\alpha$, $\alpha \in [0,1)$, then $\mathcal{D} \in (\ell_1, \ell_2)$.

Proof. Let r_1, r_2 be the lines through the origin, orthogonal to ℓ_1, ℓ_2 and, therefore, with slopes m_1, m_2, respectively. So that the line $s : \varepsilon = -\gamma_1$ intersect r_1 and r_2 in the points $P_1 = (-\gamma_1, -m_1\gamma_1)$ and $P_2 = (-\gamma_1, -m_2\gamma_1)$, respectively, according to the Figure 4.1.

FIGURE 4.1

Let us define the one-periodic map $\alpha \in \mathbb{R} \to (-\gamma_1, b + (\gamma_2 + a)\cos 2\pi\alpha) \in s$. Since $(-\gamma_1, b)$ is the medium point of the segment $[P_1, P_2] \subset s$, the choice of a implies the range of this map is $[P_1, P_2]$.

The theorem will be proved if we show that this map is precisely $A(\alpha)$, given in (3.11), for the vaccination v specified in the statement of the theorem. According to the definition of γ_3, we have:

$$\gamma_3 = -\left(1 + \frac{1}{K}\right)\left[2Aa\int_0^n \bar{x}(t)\sin 2\pi(t+\alpha)\,dt + b\frac{\sqrt{K}}{\gamma_1}\int_0^n \bar{x}^2(t)\,dt\right].$$

Taking into account the definition of γ_1, the symmetries of $\bar{x}$, and noticing that an integration by parts gives

$$\int_0^n \bar{x}(t)\sin \pi t\,dt = -\frac{\nu\sqrt{K}}{2\pi}\gamma_2,$$

after some calculations we arrive to

$$A(\alpha) = \big(-\gamma_1, (a + \gamma_2)\cos 2\pi\alpha + b\big),$$

which completes the proof. □

Remarks. (1) The symmetric case, $m_1 = -m_2$, corresponds to $b = 0$, where the vaccination is given by $v(t) = \delta 2Aa \sin 2\pi t + c$, leading to a closed segment, $A(\alpha) = (-\gamma_1, (a + \gamma_2)\cos 2\pi\alpha)$, symmetric with respect to the ε axis. The case investigated by Smith [9] corresponds to $a = c = 0$, where there is no vaccination.

(2) Let us recall that, motivated by the need of considering $|\delta/\varepsilon|$ small, we have made a rescaling, $(\varepsilon, \delta) \leftrightarrow (\varepsilon, \varepsilon\bar{\delta})$, and then dropped the bar out. In returning to the initial setting the remark below may be useful.

Now retaking the bar, given a pair of lines through the origin, $\ell_j : \bar{\delta} = k_j\varepsilon$, $j = 1, 2$, a diagram $\mathcal{D} \in (\ell_1, \ell_2)$, in the way back to the original parameters is led to a diagram, whose bifurcation curves are tangent to the parabolas $\delta = k_j\varepsilon^2$, $j = 1, 2$, at the origin of the (ε, δ)-plane axis. If $k_1, k_2 > 0$, the region K, defined before the theorem 3.1, goes back to the region in the upper semi-plane whose boundary is given by the parabolas $\delta = k_j\varepsilon^2$, $j = 1, 2$. If $k_1 > 0$ and $k_2 < 0$, K goes goes back to $\{(\varepsilon, \delta) | \delta \geq k_1\varepsilon^2\} \cup \{(\varepsilon, \delta) | \delta \leq k_2\varepsilon^2\}$. See the figure 4.2. The shaded regions corresponds to the locus where there are precisely two subharmonic solutions of (2.1), of order n, close to the basic solution, up to a phase shift.

FIGURE 4.2

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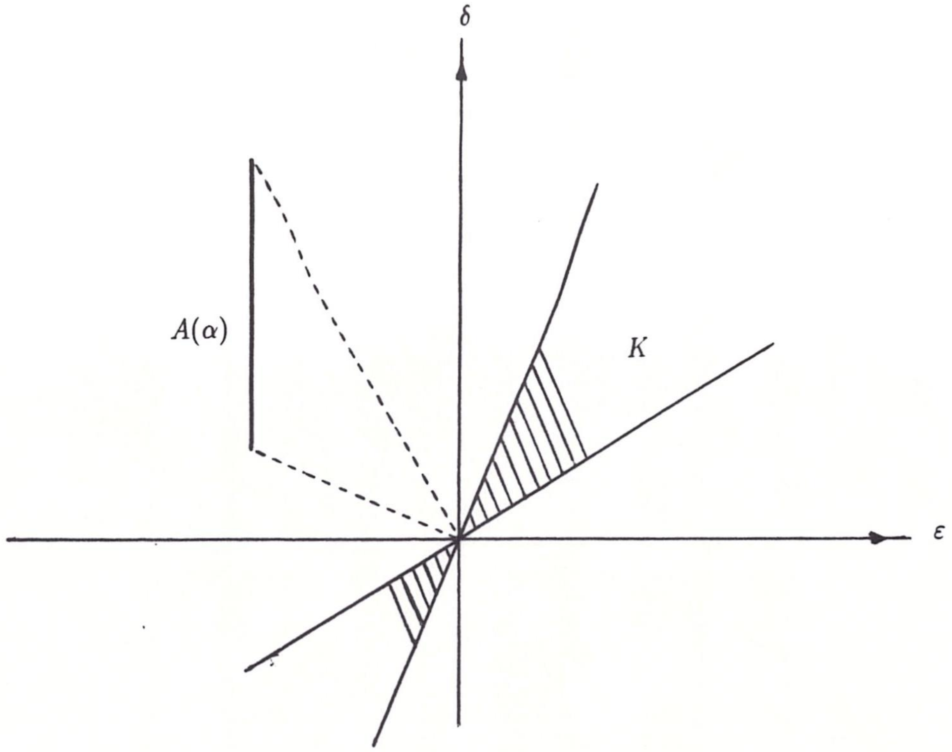

FIGURE 3.1

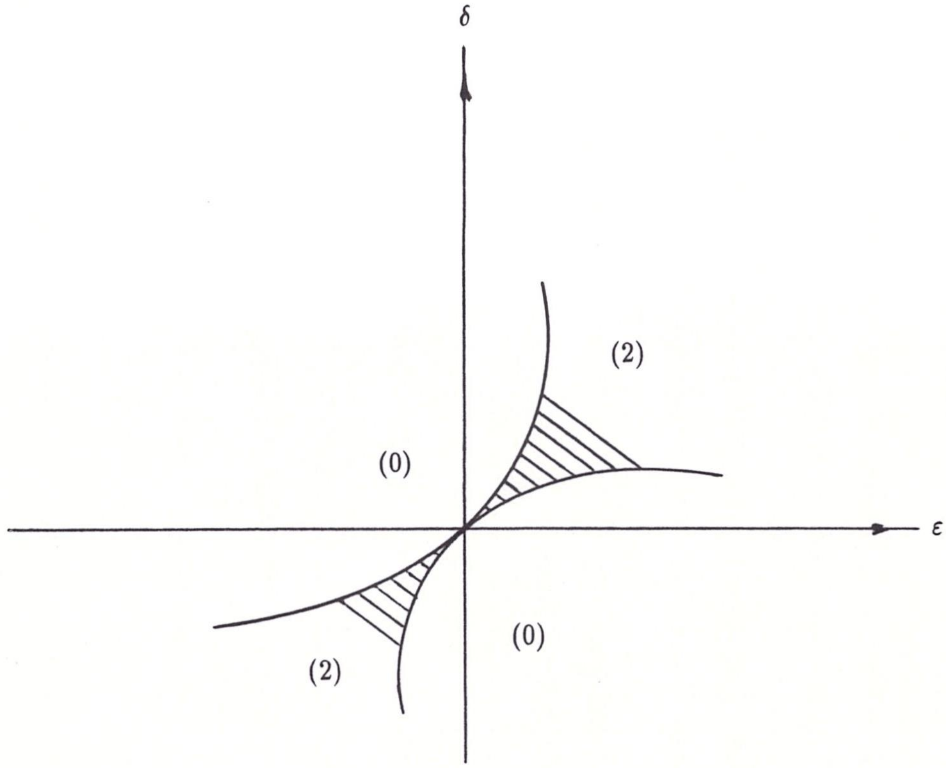

FIGURE 3.2

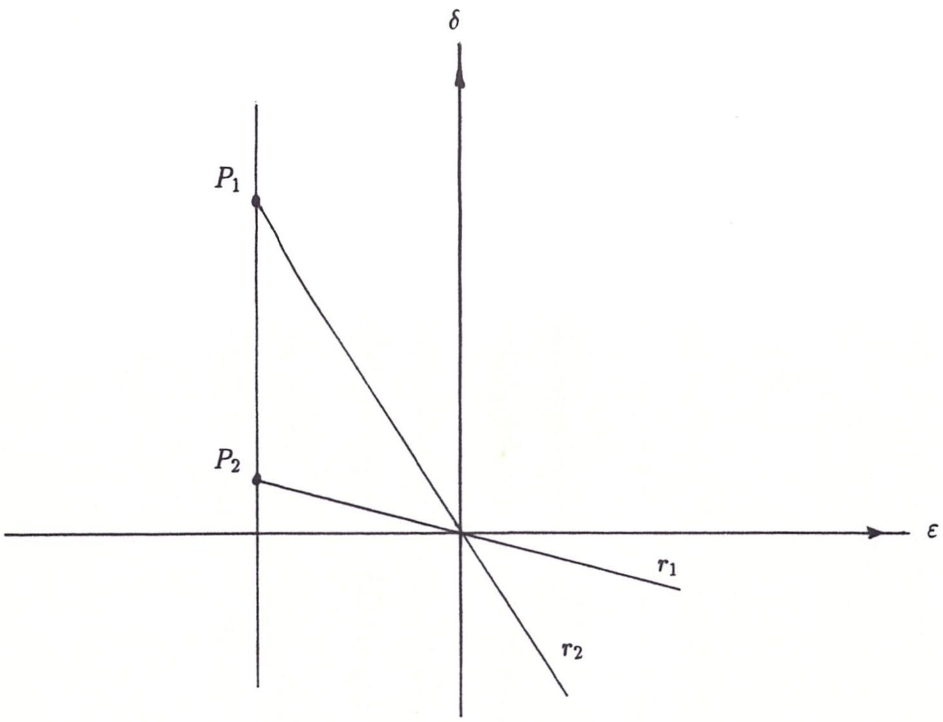

FIGURE 4.1

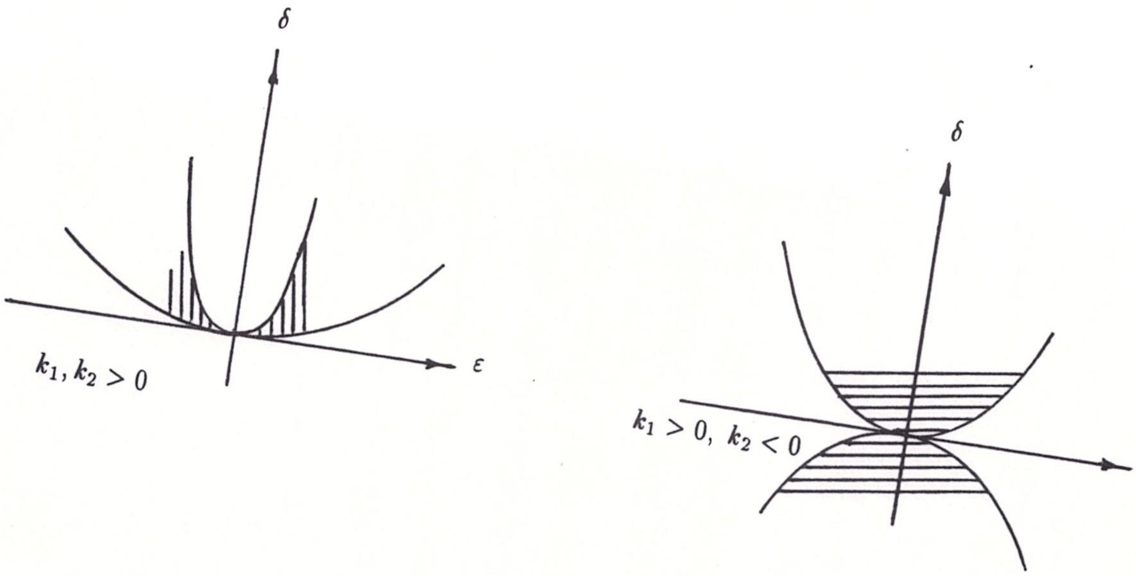

FIGURE 4.2

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