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Topological entropy on points without physical-like behaviour

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Abstract

For C^1 transitive Anosov diffeomorphisms, we consider the sets of irregular points without physical-like behaviour and regular points without physical-like behaviour respectively and show that they all carry full topological entropy. Roughly speaking, physical-like measures do not affect the dynamical complexity of the regular set and the irregular set in the sense of topological entropy.

Keywords SRB-like, Physical-like or observable measure · Topological entropy · Uniformly hyperbolic systems

Mathematics Subject Classification 37D20 · 37D30 · 37C45 · 37A35 · 37B40

1 Introduction

The differentiable ergodic theory of dynamical systems is mainly developed in the $C^{1+\alpha}$ scenario. Relatively few results were obtained in the C^1 context. In this paper we focus our attention on C^1 dynamical systems, for which the Lebesgue measure is not necessarily invariant.

Among the most useful concepts in the ergodic theory, the physical probability measures play an important role. An invariant probability measure μ is called physical if for a Lebesgue-positive set of initial states x, the time-average of any continuous function φ along the orbit

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of x, up to time n, converges (when $n \to +\infty$) to the expected value of φ with respect to μ. Not any system, principally in the C^1 context, possesses physical measures. This problem can be easily dodged by substituting the definition of physical measure by a weaker concept: *physical-like* measure, also called SRB-like or observable measure (see Definition 2.4). Physical-like measures do always exist (see [11]).

For any $C^{1+\alpha}$ transitive Anosov diffeomorphism f, the classical Pesin theory ensures the existence of a unique physical measure μ_f, which is also of SRB (Sinai–Ruelle–Bowen) type. Besides, μ_f has a basin of statistical attraction with full Lebesgue measure and satisfies Pesin entropy formula. The *typical* points are those in the basin of statistical attraction of μ. Analogously, in [11] it was proved that for any C^0 system f, there exists a nonempty set $\mathcal{O}_f$ composed by all the observable or physical-like measures. Besides, the basin of statistical attraction of $\mathcal{O}_f$ has full Lebesgue measure, and if f is (for instance) a C^1 Anosov diffeomorphism, then any measure μ in $\mathcal{O}_f$ satisfies Pesin entropy formula (see [10]). In this general case, the *typical* points are those in the basin of statistical attraction of $\mathcal{O}_f$.

Along this paper, we will disregard the typical orbits, and look only at the orbits in the set of zero-Lebesgue measure that have non physical-like behaviour. Precisely, a point $x \in M$ *without physical-like behaviour*, is a point such that none of the limits when $n \to \infty$ of the convergent subsequences of its time-averages, is a physical-like measure. In other words x is anything but typical. We denote by Γ_f the set of such points. Thus

$$\Gamma_f = \{x : pw_f(x) \cap \mathcal{O}_f = \emptyset\}, \tag{1}$$

where $pw_f(x) :=$

$$\left\{ \mu \in \mathcal{P} : \lim_{j \to +\infty} \frac{1}{n_j} \sum_{i=1}^{n_j - 1} \delta_{f^i(x)} = \mu \text{ for some sequence } n_j \to +\infty \right\},$$

and $\mathcal{P}$ denotes the space of all the probability measures endowed with the weak* topology.

We will adopt also a topological point of view, and look at the increasing rate of the topological information quantity of f; namely, its topological entropy $h_{top}(f)$. In [8], Bowen defined the topological entropy $h_{top}(E)$ restricted to an arbitrary subset E of the space M. Among its properties: $h_{top}(E)$ increases with E, and $h_{top}(M) = h_{top}(f)$. We say that a set $E \subset M$ *has full topological entropy* if $h_{top}(E) = h_{top}(f)$. If so, the dynamics of f restricted to E produces the total increasing rate of topological information of the system. In other words, even if one disregards the orbits whose initial states are not in E, the information obtained from the sub-dynamics is, roughly speaking, the information of the whole system.

We also consider irregular and regular sets without physical-like behaviour. A point $x \in M$ is *irregular* if the sequence of time-averages along its orbit is not convergent, that is, $\# pw_f(x) > 1$. It is also called point with historic behaviour [23,26]. Otherwise, x is called *regular* (called quasi-regular in [13,18]). Let I_f be the set of irregular points and R_f be the set of regular points. In [2] it is proved that I_f carries full topological entropy for hyperbolic systems. This result is generalized to systems with specification-like properties [28,29]. The points without physical-like behaviour may be irregular or not. For any continuous map $f : M \mapsto M$, the set of irregular points without physical-like behaviour has zero Lebesgue measure by Theorem 2.5 below and also zero μ-measure for any f-invariant measure μ by Birkhoff ergodic theorem.

Theorem A *Let $f : M \mapsto M$ be a C^1 transitive Anosov diffeomorphism on a compact Riemannian manifold M. Then*

(1) the set $\Gamma_f \cap I_f$ of irregular points without physical-like behaviour has full topological entropy;
(2) the set $\Gamma_f \cap R_f$ of regular points without physical-like behaviour has full topological entropy.

To prove this theorem, we will use the following main tools: the topological and metric properties of asymptotically entropy-expansive maps ([5,14,17,19]), the formulae of the topological entropy of saturated sets according to [21], and Pesin entropy formula for physical-like measures of certain C^1 diffeomorphisms according to [10].

Organization of the paper. Section 2 is a review of definitions to make precise the statements of the theorems and their proofs. In Sect. 3 we give some key technique lemmas by an abstract framework and then end the proof of Theorem A.

2 Definitions

2.1 Physical-like or SRB-like measures

Let $f : M \to M$ be a continuous map on a compact manifold M, which does not necessarily preserve any smooth measures with respect to the Lebesgue measure. Let $\mathcal{P}$ denote the space of all the probability measures endowed with the weak* topology, and $\mathcal{P}_f \subset \mathcal{P}$ denote the space of f-invariant probability measures.

Definition 2.1 (*Empirical probabilities or time-averages and p-omega limit*).
For any point $x \in M$ and for any integer number $n \geq 1$, the *empirical probability or time-average measure* $\Upsilon_n(x)$ of the f-orbit of x up to time n, is defined by

$$\Upsilon_n(x) := \frac{1}{n}\sum_{j=0}^{n-1} \delta_{f^j(x)},$$

where δ_y is the Dirac probability measure supported at $y \in M$. Consider the sequence $\{\Upsilon_n\}_{n\in\mathbb{N}^+}$ of empiric probabilities in the space $\mathcal{P}$, and define the *p-omega-limit set* $p\omega_f(x) \subset \mathcal{P}$ as follows:

$$p\omega_f(x) := \Big\{\mu \in \mathcal{P} : \ \exists\, n_i \to +\infty \text{ such that } \lim_{i\to+\infty} \Upsilon_{n_i} = \mu\Big\}.$$

It is standard to check that $p\omega_f(x) \subset \mathcal{P}_f$. From [13] we know that $p\omega_f(x)$ is always nonempty, weak*-compact and connected.

Definition 2.2 (*Physical or SRB measures and their basins*)
We call a measure $\mu \in \mathcal{P}$ *physical or SRB* (Sinai–Ruelle–Bowen), if the set

$$G_\mu = \{x \in M : \ p\omega_f(x) = \{\mu\}\} \tag{2}$$

has positive Lebesgue measure. The set G_μ is called *basin of statistical attraction* of μ, or in brief, basin of μ (even if μ is not physical).

Remark 2.3 The above definition of physical or SRB measures is not adopted by all the authors. Some mathematicians require the measure μ to be ergodic to call it physical. Besides, some mathematicians when studying $C^{1+\alpha}$ systems do not define SRB as a synonym of physical measure, but take into account the property of absolute continuity on the unstable foliation. But, in the scenario of continuous systems, and even for C^1 systems, the unstable conditional measures can not be defined because the unstable foliation may not exist.

Definition 2.4 (*Physical-like measures and their* ε-basins, cf. [11])
Choose any metric dist* that induces the weak* topology on the space $\mathcal{P}$ of probability measures. A probability measure $\mu \in \mathcal{P}$ is called *physical-like* (or *SRB-like* or *observable*) if for any $\varepsilon > 0$ the set

$$G_\mu(\varepsilon) = \{x \in M: \ \text{dist}^*(p\omega_f(x), \mu) < \varepsilon\}, \tag{3}$$

has positive Lebesgue measure. The set $G_\mu(\varepsilon)$ is called *basin of* ε*-partial statistical attraction* of μ, or in brief, ε-basin of μ. We denote by $\mathcal{O}_f$ the set of physical-like measures for f. It is standard to check that every physical-like measure is f-invariant and that $\mathcal{O}_f$ does not depend on the choice of the metric in $\mathcal{P}$.

Theorem 2.5 *(Characterization of physical-like measures* [11])
Let $f : M \to M$ be a continuous map on a compact manifold M. Then, the set $\mathcal{O}_f$ of physical-like measures is nonempty, weak compact, and contains the limits of the convergent subsequences of the empiric probabilities for Lebesgue almost all the initial states $x \in M$. Besides, no proper subset of $\mathcal{O}_f$ has the latter three properties simultaneously.*

2.2 Topological definitions

In this subsection we list some other concepts that we will use along the proofs. Indeed, we will not formally use the mathematical conditions that impose those definitions, but only some already known relations among them. So here, we just cite the bibliography where the definitions can be found.

Topological entropy of a subset $E \subset M$. We adopt Bowen's definition of the topological entropy $h_{top}(E)$ of an arbitrary subset $E \subset M$, for any compact metric space M and any continuous map f on M (see [8]).

Entropy-expansive and asymptotically entropy-expansive maps. We refer to [5,14,17,19] for the definitions of expansive, entropy-expansive and asymptotically entropy-expansive maps. From those definitions, trivially every expansive homeomorphism is entropy-expansive, and every entropy expansive map (not necessarily an homeomorphism) is asymptotically entropy-expansive.

Specification and g**-almost product properties** We adopt the definition of the specification property of the map f, as for instance in [4,6,7,13,24,28]). We note that the original definition of specification, due to Bowen [6], was stronger than the specification property that we adopt here.

We recall the definition of the blowup functions $g : \mathbb{N} \mapsto \mathbb{N}$, and of the g-almost product property of the map f, in [21]). Every continuous map f that has the specification property, also has the g-almost product property for some blow up function g ([21, Proposition 2.1]). In other words, the g-almost product property is weaker than the specification property.

2.3 Saturated sets and saturation property of the entropy

Let $f : M \mapsto M$ be a continuous map on a compact metric space M. We reformulate the definition of the saturated sets in [21], as follows:

Definition 2.6 (*Saturated sets*) Let $K \subseteq \mathcal{P}_f$. We call the (maybe empty) following set $G_K \subset M$, *the saturated set of* K:

$$G_K = \{x \in M : \ pw_f(x) = K\}. \tag{4}$$

Note that $G_\mu = G_{\{\mu\}}$ for any invariant measure μ from (2) and (4). Remark that if $G_K \neq \emptyset$, then K must be nonempty, compact and connected since from [13] $pw_f(x)$ is always nonempty, compact and connected for any point x. For convenience, we introduce the following definition inspired in the results of [21]:

Definition 2.7 (*Saturation property of the entropy*) We say that the continuous system $f : M \mapsto M$ has the *saturation property of the entropy*, if for any nonempty, weak* compact and connected set $K \subseteq \mathcal{P}_f$, the following equality holds:

$$h_{top}(G_K) = \inf\{h_\mu(f) : \ \mu \in K\},$$

where $h_\mu(f)$ is the metric entropy of f with respect to the probability measure μ.

Say a system has the singleton saturation property if for any invariant (not necessarily ergodic) measure μ,

$$h_{top}(G_\mu) = h_\mu(f). \tag{5}$$

This property holds for any system with the g-almost product structure [21]. In [8, Theorem 3], Bowen proved that equality (5) holds for any ergodic measure μ, for any continuous map f on a compact metric space M (without any other assumptions).

Define an interval $[\mu, \nu] := \{t\mu + (1-t)\nu, \ t \in [0, 1]\}$ for $\mu, \nu \in \mathcal{P}_f$. Say f has the interval saturation property when for any intervals $[\mu, \nu]$ (trivial or not) we have $h_{top}(G_{[\mu,\nu]}) = \min\{h_\mu(f), h_\nu(f)\}$. Obviously saturation property of the entropy is stronger than interval saturation property and the later is stronger than singleton saturation property. Interval saturation property is satisfied for many systems with specification-like properties: strong nonuniform specification [30], system with saturation property of the entropy such as asymptotically entropy-expansive system with the g-almost product structure [21].

2.4 Φ-irregualr, regular and level sets

We denote by $C^0(M, \mathbb{R})$ the space of continuous real functions $\varphi : M \mapsto \mathbb{R}$. Given $\varphi \in C^0(M, \mathbb{R})$, consider the set I_f^φ defined by

$$I_f^\varphi := \left\{ x \in M : \ \frac{1}{n} \sum_{j=0}^{n-1} \varphi\big(f^j(x)\big) \text{ is not convergent} \right\}.$$

The points in I_f^φ are called $\varphi-$irregular. Then, due to the weak* topology, we have

$$I_f = \bigcup_{\varphi \in C^0(M,\mathbb{R})} I_f^\varphi. \tag{6}$$

The set of strongly regular points (called regular in [18]), denoted by SR_f, which means that

$$SR_f = \bigcup_{\mu \in \mathcal{P}_f,\, \mu\, ergodic} (G_\mu \cap S_\mu),$$

where S_μ denotes the support of μ. This set has full measure for any invariant measure by Birkhoff ergodic theorem and ergodic decomposition theorem.

Let

$$R_f^e = \bigcup_{\mu\in\mathcal{P}_f,\text{ ergodic}} G_\mu, \quad R_f^{ne} = \bigcup_{\mu\in\mathcal{P}_f,\text{ not ergodic}} G_\mu.$$

The points in R_f^e and R_f^{ne} are called ergodic regular points and non-ergodic regular points respectively.

Let $\phi : M \to \mathbb{R}$ be a continuous function and denote the interval $[\inf_{\mu\in\mathcal{P}_f}\int\phi d\mu, \sup_{\mu\in\mathcal{P}_f}\int\phi d\mu]$ by L_ϕ and its interior by $Int(L_\phi)$. For $a\in\mathbb{R}$, define level set

$$R_\phi(a) := \left\{x : \lim_{n\to+\infty}\frac{1}{n}\sum_{j=0}^{n-1}\varphi\big(f^j(x)\big) = a\right\}.$$

By weak* topology, a necessary condition for $R_\phi(a)\neq\emptyset$ is $a\in L_\phi$. Its topological entropy has a variational principle characterized by $\sup_{\mu\in\mathcal{P}_f,\int\phi d\mu=a} h_\mu(f)$ in [1] for hyperbolic systems and [21,27] for systems with specification-like properties.

2.5 Dominated splitting

Definition 2.8 (*Dominated splitting*) Let $f : M\to M$ be a C^1 diffeomorphism on a compact Riemannian manifold M. Let $TM = E\oplus F$ be a Df-invariant and continuous splitting such that $dim(E)\cdot dim(F)\neq 0$. It is called a *dominated splitting* if there exists $\sigma > 1$ such that

$$\|Df|_{E(x)}\|\|Df^{-1}_{f(x)}|_{F(f(x))}\| \le \sigma^{-1} \quad \forall x\in M.$$

Remark 2.9 The continuity of the splitting in the latter definition is redundant (see [3, p. 288]). The classical definition of dominated splitting is $TM = E\oplus F$ such that there exists $C > 0$ and $\sigma > 1$:

$$\|Df^n|_{E(x)}\|\cdot\|Df^{-n}|_{F(f^n(x))}\| \le C\sigma^{-n}, \forall x\in M,\ n\ge 1.$$

It is equivalent to Definition 2.8 (see [12]).

2.6 Pesin entropy formula

Definition 2.10 (*Pesin entropy formula*) Let $f : M\mapsto M$ be a C^1 diffeomorphism on a compact Riemannian manifold M and let $\mu\in\mathcal{P}_f$. We say that μ *satisfies Pesin Entropy Formula* if

$$h_\mu(f) = \int\sum_{\chi_i(x)\ge 0}\chi_i(x)d\mu, \tag{7}$$

where $h_\mu(f)$ is the metric entropy of μ and $\chi_1(x)\ge\chi_2(x)\cdots\ge\chi_{\dim(M)}(x)$ denote the Lyapunov exponents of μ-a.e. $x\in M$. We denote

$$PE_f := \big\{\mu\in\mathcal{P}_f :\ \mu \text{ satisfies Pesin Entropy Formula (7)}\big\}.$$

Remark 2.11 A convex subset K of $\mathcal{P}_f$ is called a face when almost every ergodic component of measures in K also belongs to K. Note that PE_f is a face, and either $PE_f = \mathcal{P}_f$ or the interior of PE_f in $\mathcal{P}_f$ is empty, due to the affinity property of metric entropy function and Ruelle's inequality [22].

2.7 Equilibrium states

Let us return to the continuous setting and state the basic definitions and properties of the thermodynamic formalism (see for instance [16]). Let f be a continuous map on a compact manifold M. Fix a continuous real function $\psi : M \mapsto \mathbb{R}$, which is called the *potential*. Consider the following real number $p_f(\psi)$:

$$p_f(\psi) := \sup_{\mu \in \mathcal{P}_f} \left(h_\mu(f) - \int \psi \, d\mu \right).$$

The number $p_f(\psi)$ is called the *pressure* with respect to the potential ψ.

Definition 2.12 The (maybe empty) set $ES_f(\psi)$ of f-invariant probability measures, is defined by

$$ES_f := \left\{ \mu \in \mathcal{P}_f : h_\mu(f) - \int \psi \, d\mu = p_f(\psi) \right\}.$$

The measures μ in ES_f are called *equilibrium states* of f with respect to the potential ψ. So $ES_f(\psi)$ is the *set of equilibrium states*.

Remark 2.13 Due to the affinity property of the entropy function, $ES_f(\psi)$ is a face and either is the whole space $\mathcal{P}_f$, or it has empty interior in $\mathcal{P}_f$. If f is asymptotically entropy-expansive, then it is known that the entropy function is upper semi-continuous. Thus, $ES_f(\psi)$ is besides nonempty and weak*-compact (see for instance [16, Theorem 4.2.3]).

3 Abstract framework and proof of theorem A

3.1 Points with *L*-behaviour

Let $f : M \to M$ be a continuous map on a compact metric space M. Let $L \subseteq \mathcal{P}_f$. Recall

$$G_L := \{x \in M | pw_f(x) = L\}.$$

Define

$$H_L := \{x \in M | pw_f(x) \subseteq L\}.$$

Here we call H_L the set of points with L-behaviour. Note that $G_L \subseteq H_L$. Moreover for $L \subseteq L'$ we have $H_L \subseteq H_{L'}$. For $K \subseteq \mathcal{P}_f$ the set of points without K-behaviour is then $H_{\mathcal{P}_f \setminus K}$.

For any real number $t \geq 0$, define the (maybe empty) set

$$Q(t) := \{x : \exists \mu \in pw_f(x) \ s.t. \ h_\mu(f) \leq t\}.$$

From [8, Theorem 2]:

$$h_{top}(Q(t)) \leq t. \tag{8}$$

Observe that for any $L \subseteq \mathcal{P}_f$ one has $h_{top}(H_L) \leq \sup_{\mu \in L} h_\mu(f)$ by (8). Recall Bowen's result of [8, Theorem 3] that for any ergodic μ and measurable set Λ with $\mu(\Lambda) = 1$,

$$h_{top}(\Lambda) \geq h_\mu(f). \tag{9}$$

Thus by (8) and (9) one gets that for any ergodic μ, $h_{top}(G_\mu \cap S_\mu) = h_{top}(G_\mu) = h_\mu(f)$.

Lemma 3.1 *For any $L \subseteq \mathcal{P}_f$ one has*

(1)

$$\sup_{\mu \in L} h_\mu(f) \geq h_{top}(H_L) \geq h_{top}(H_L \cap R_f) \geq h_{top}(H_L \cap SR_f) = \sup_{\mu \in L,\, \mu\, ergodic} h_\mu(f);$$

(2) When L is a face, then the inequalities in item (1) are all equalities;
(3) When f has singleton saturation property, we also have

$$\sup_{\mu \in L} h_\mu(f) = h_{top}(H_L) = h_{top}(H_L \cap R_f),$$

and

$$h_{top}(H_L \cap R_f^{ne}) \geq \sup_{\mu \in L,\ non\text{-}ergodic} h_\mu(f);$$

(4) When f has interval saturation property,
if $\phi : M \to \mathbb{R}$ is a continuous function with $\inf_{\mu \in \mathcal{P}_f} \int \phi d\mu < \sup_{\mu \in \mathcal{P}_f} \int \phi d\mu$, then

$$h_{top}(I_f^\phi \cap H_L) \geq \sup_{[\mu,\nu] \subseteq L,\, \int \phi d\mu \neq \int \phi d\nu} \min\{h_\mu(f), h_\nu(f)\}.$$

Proof (1) Item (1) is from (8) and (9).
(2) When L is a face, $\sup_{\mu \in L} h_\mu(f) = \sup_{\mu \in L,\, \mu\, ergodic} h_\mu(f)$ so that item (2) is obtained by item (1).
(3) For any $\mu \in L$, $G_\mu \subseteq H_L \cap R_f$ and if further μ is non-ergodic, then $G_\mu \subseteq H_L \cap R_f^{ne}$. Thus by using singleton saturation property one has item (3).
(4) Note that for any non trivial intervals $[\mu, \nu]$, $G_{[\mu,\nu]} \subseteq I_f$ and if further $\int \phi d\mu \neq \int \phi d\nu$, then $G_{[\mu,\nu]} \subseteq I_f^\phi$. If $[\mu, \nu] \subseteq L$, then $G_{[\mu,\nu]} \subseteq H_L$ so that item (4) follows from interval saturation property. □

3.2 Entropy of measures outside a (weak) face

Here we introduce a possibly weak version of face called weak face. A convex subset $K \subseteq \mathcal{P}_f$ is called a weak face, if for any $\mu \in K$ with $\mu = \lambda\nu + (1-\lambda)\omega$ for some $\lambda \in (0, 1)$, one has $\nu, \omega \in K$. We say K is proper if $K \neq \mathcal{P}_f$. Say f satisfies the entropy-dense property if for any $\mu \in \mathcal{P}_f$, for any neighborhood G of μ in $\mathcal{P}_f$, and for any $\eta > 0$, there exists an ergodic measure $\nu \in \mathcal{P}_f$ such that $h_\nu > h_\mu - \eta$. From [20] entropy-dense holds for any system with $g-$almost product structure.

Lemma 3.2 *Let $f : M \to M$ be a continuous map on a compact metric space M. Assume K is a proper weak face of $\mathcal{P}_f$ then*

(1) one has

$$\sup_{\mu\ not\ ergodic \notin K} h_\mu(f) = h_{top}(f), \tag{10}$$

(2) If $\phi : M \to \mathbb{R}$ is a continuous function with $\inf_{\mu \in \mathcal{P}_f} \int \phi d\mu < \sup_{\mu \in \mathcal{P}_f} \int \phi d\mu$, then

$$\sup_{[\mu,\nu] \subseteq \mathcal{P}_f \setminus K,\, \int \phi d\mu \neq \int \phi d\nu} \min\{h_\mu(f), h_\nu(f)\} = h_{top}(f), \tag{11}$$

and for any $a \in Int(L_\phi(a))$,

$$\sup_{\mu \text{ not ergodic} \in \mathcal{P}_f \setminus K, \int \phi d\mu = a} h_\mu(f) = \sup_{\mu \in \mathcal{P}_f, \int \phi d\mu = a} h_\mu(f). \tag{12}$$

(3) If further K is closed and that ergodic measures are entropy-dense, then

$$\sup_{\mu \text{ ergodic} \notin K} h_\mu(f) = h_{top}(f), \tag{13}$$

Remark 3.3 (1) There are other contexts (K is no more assumed to be a weak face, even convex) where we can check equation (13): Assume $\sup_{\mu \in K} h_\mu(f) < h_{top}(f)$ or assume K contains finitely (resp., countably) ergodic measures and for any $h < h_{top}(f)$ there are infinitely (resp., uncountably) many ergodic measures with entropy larger than h, then equation (13) holds true;

(2) Let us explain why it is required that $a \in Int(L_\phi(a))$. Here we give an example for which equation (12) is false when a is an endpoint of L_ϕ. Let $\sigma : \Sigma_2 \to \Sigma_2$ be the full shift of two symbols and let $\Lambda \subsetneq \Sigma_2$ be a minimal subset with positive entropy and only one invariant measure supported on Λ (which is ensured by [15]). Take a periodic orbit $Orb(p)$ contained in $\Sigma_2 \setminus \Lambda$ and define a continuous function $\phi : \Sigma_2 \to \mathbb{R}$ satisfying that $\phi|_\Lambda = 0$, $\phi|_{Orb(p)} = 1$ and $0 < \phi(x) < 1$ for other points of x. $a = 0$ is an endpoint of L_ϕ but no non-ergodic invariant measure μ with $\int \phi d\mu = 0$ so that for any K,

$$\sup_{\mu \text{ not ergodic} \in \mathcal{P}_f \setminus K, \int \phi d\mu = 0} h_\mu(f) = 0 < h_{top}(\Lambda) = \sup_{\mu \in \mathcal{P}_f, \int \phi d\mu = 0} h_\mu(f).$$

Proof (1) Equation (10) follows from the affinity of the entropy. More precisely, Let $\nu_1 \in K, \nu_2 \notin K$ and fix $\epsilon > 0$. By variational principle, one can take ergodic ν such that $h_\nu(f) > h_{top}(f) - \epsilon$ and then one can take $\theta \in (0, 1)$ close to 1 such that $\theta h_\nu(f) > h_{top}(f) - \epsilon$. If $\nu \in K$, then take $\mu = \theta\nu + (1-\theta)\nu_2$ and otherwise take $\mu = \theta\nu + (1-\theta)\nu_1$. In any case μ should be not in K, not ergodic and $h_\mu(f) \geq \theta h_\nu(f) > h_{top}(f) - \epsilon$.

(2) Consider a continuous function with $\inf_{\mu \in \mathcal{P}_f} \int \phi d\mu < \sup_{\mu \in \mathcal{P}_f} \int \phi d\mu$. Fix $\epsilon > 0$. From Eq. (10) we have got that there exists $\nu \notin K$ such that $h_\nu(f) > h_{top}(f) - \epsilon$. Take $\eta \in \mathcal{P}_f$ such that $\int \phi d\nu \neq \int \phi d\eta$, then one can take $\mu = s\nu + (1-s)\eta$ for s close enough to 1 such that $h_\mu(f) \geq s h_\nu(f) > h_{top}(f) - \epsilon$ and then $\mu \notin K$ since K is a weak face. Note that $\int \phi d\nu \neq \int \phi d\mu$, $\mu, \nu \notin K$ and $\min\{h_\mu(f), h_\nu(f)\} > h_{top}(f) - \epsilon$ so that one has equation (11).

Now we start to show Eq. (12). Fix $a \in Int(L_\phi(a))$. Since above μ, ν satisfies that $\int \phi d\nu \neq \int \phi d\mu$, $\mu, \nu \notin K$, then there is $\omega \in \{\mu, \nu\}$ such that $\omega \notin K$ and $\int \phi d\omega \neq a$. Without loss of generality, we may assume $\int \phi d\omega < a$. Take $\eta \in \mathcal{P}_f$ such that $\int \phi d\eta > a$ and then there exists a unique number $t \in (0, 1)$ such that $\int \phi d\nu' = a$, where $\nu' = t\omega + (1-t)\eta$. Note that ν' is not ergodic and $\nu' \notin K$ since K is a weak face. Write $t_a := \sup_{\mu \in \mathcal{P}_f, \int \phi d\mu = a} h_\mu(f)$. Take $\mu' \in \mathcal{P}_f$ with $\int \phi d\mu' = a$ such that $h_{\mu'}(f) > t_a - \epsilon$. Then it is enough to end the proof of Eq. (12) by taking $\lambda = s\mu' + (1-s)\nu'$ for s close enough to 1.

(3) Fix $\epsilon > 0$. By equation (10) we can take $\mu \notin K$ such that $h_\mu(f) > h_{top}(f) - \epsilon$. Then take ergodic measure ν close to μ with $h_\nu(f) > h_\mu(f) - \epsilon > h_{top}(f) - 2\epsilon$. As K is closed we may choose $\nu \notin K$. This finishes the proof of item (3). □

3.3 Points without *K*-behaviour

One can combine Lemmas 3.1 and 3.2 together to get following:

Theorem 3.4 *Let $f : M \mapsto M$ be a continuous map on a compact metric space M and let $K \subseteq \mathcal{P}_f$ is a proper weak face. If f has interval saturation property, then*

(1) the set of non-ergodic regular points without K-behaviour has full topological entropy;
(2) For any $\phi \in C^0(M, \mathbb{R})$ with $\inf_{\mu \in \mathcal{P}_f} \int \phi d\mu < \sup_{\mu \in \mathcal{P}_f} \int \phi d\mu$ and $a \in Int(L_\phi)$, the set of non-ergodic regular points without K-behavior has full topological entropy inside the level set $R_\phi(a)$.
(3) For any $\phi \in C^0(M, \mathbb{R})$ with $\inf_{\mu \in \mathcal{P}_f} \int \phi d\mu < \sup_{\mu \in \mathcal{P}_f} \int \phi d\mu$, the set of $\phi-$irregular points without K-behaviour has full topological entropy;
(4) If further K is closed and f has entropy-dense property, then the set of strongly regular points without K-behaviour has full topological entropy.

Proof (1) Take $L = \mathcal{P}_f \backslash K$ in the second part of item (3) of Lemma 3.1 and then by (10) one gets item (1).
(2) From [21, Proposition 7.1] singleton saturation property implies that

$$h_{top}(R_\phi(a)) = \sup_{\mu \in \mathcal{P}_f, \int \phi d\mu = a} h_\mu(f).$$

Thus one can take $L = \{\mu : \int \phi d\mu = a\} \cap \mathcal{P}_f \backslash K$ in the second part of item (3) of Lemma 3.1 and then by (12) one gets item (2).
(3) Take $L = \mathcal{P}_f \backslash K$ in item (4) of Lemma 3.1 and then by (11) one gets item (3).
(4) Take $L = \mathcal{P}_f \backslash K$ in item (1) of Lemma 3.1 and then by equation (13) one gets item (4). □

This result is the key argument of present paper and is suitable to all following systems (since saturation property of the entropy and entropy-dense property hold for asymptotically entropy-expansive system with the g-almost product structure [20,21]):

Example 3.5 (1) Any C^1 transitive Anosov diffeomorphism satisfies Theorem 3.4, since it is expansive and has specification property which is stronger than g-almost product property by [21].
(2) Any mixing subshift of finite type satisfies Theorem 3.4, since it is known that it is expansive and has specification.
(3) Any $\beta-$shift satisfies Theorem 3.4, since it is expansive and has g-almost product property by [21].

Remark 3.6 In [30] the authors have proved under another property of specification, called nonuniform specification (see [30] for more details), that for all neighborhood V of $[\mu, \nu]$ one has $h_{top}(H_V) \geq \min\{h_\mu(f), h_\nu(f)\}$. Note that nonuniform specification does not a priori follow from the g-almost product property. Following the above scheme of proof one gets easily that: *Assume K is a proper closed weak face of $\mathcal{P}_f$ and f satisfies the nonuniform specification. Then the set of irregular points without K-behaviour has full topological entropy.*

3.4 Proof of theorem A

One can apply the abstract Theorem 3.4 for K being the face of measures satisfying the Pesin entropy formula or the set of equilibrium states for some given potential. Theorem A can be deduced from Theorem 3.4 and following lemma.

Lemma 3.7 *Let $f : M \mapsto M$ be a C^1 diffeomorphism on a compact Riemannian manifold M with a dominated splitting $TM = E \oplus F$. Assume that the Lyapunov exponents are non positive along E and non negative along F. Then f is asymptotically entropy-expansive. If moreover f*

satisfies the g-almost product property, and if not all the invariant measures satisfy Pesin Entropy Formula, then f has saturation and entropy-dense property, PE_f is a nonempty proper closed face and $\mathcal{O}_f \subseteq PE_f = ES_f(\psi)$ where $\psi(x) = \log|\det Df|_{F(x)}|$ and $ES_f(\psi)$ denote its equilibrium states.

Remark 3.8 (1) In general it is unknown whether PE_f is closed even if one assumes that the metric entropy function is upper-continuous, since we do not know whether $\int \sum_{\chi_i(x)\geq 0} \chi_i(x) d\mu$ is continuous w.r.t. μ and Ruelle's inequality and upper-continuity of metric entropy may be not enough. Moreover, in general it is also unknown whether PE_f coincides with $ES_f(\psi)$ for some continuous function ψ.

(2) The results of Theorem A for 'without physical-like behaviour' can be replaced by ones for 'without physical-like and smooth behaviour', since in fact we proved such results for 'without Pesin behaviour' and by [25] every smooth measure also satisfies Pesin's entropy formula. It is known that any C^1 volume-preserving Anosov diffeomorphism is transitive Anosov so that Theorem A applies but it is still unknown whether the volume measure is ergodic or not. In this example for any invariant subset Λ with positive Lebesgue measure, $Leb|_\Lambda$ is a smooth measure where $Leb|_\Lambda(A) := \frac{Leb(A\cap\Lambda)}{Leb(\Lambda)}$ for any Borel measurable $A \subseteq M$ and Leb denotes the volume measure on M.

(3) The assumptions of Lemma 3.7 is little weaker than transitive Anosov diffeomorphisms which may be suitable for some partially hyperbolic systems.

Proof From [20] entropy-dense holds for any system with g-almost product structure. The system f is asymptotically entropy-expansive by [9, Theorem 7.6].

As proved in [21, Theorem 3.1], for any continuous map f on a compact metric space, if f is asymptotically entropy-expansive, then f satisfies the uniform separation property. Besides, in [21, Theorem1.1] it is proved that if a continuous map f satisfies the uniform separation and the g-almost product properties, then f has the saturation property of the entropy. Thus f has saturation property.

That $\mathcal{O}_f \subseteq PE_f$ is [10, Corollary 2]. Taking into account that $\mathcal{O}_f \neq \emptyset$ (see Theorem 2.5), $PE_f \neq \emptyset$.

By Remark 2.11 PE_f is a face and it should be closed since f is asymptotically entropy-expansive and the function ψ is continuous. More precisely, asymptotically entropy-expansive implies that the entropy function $\mu \mapsto h_\mu(f)$ is upper semi-continuous (see [31, Theorem 8.2] for the expansive case, and [5] for the entropy-expansive case; with a standard adaptation the proofs are extended to the asymptotically entropy-expansive case). Joining the upper-continuity of the entropy function, continuity of $\int \psi d\mu$ w.r.t μ with Ruelle's inequality [22], it is deduced that PE_f is weak*-compact. It is proper since by assumption $PE_f \neq \mathcal{P}_f$. □

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