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DISCONTINUOUS LOCAL SEMIFLOWS FOR KURZWEIL EQUATIONS LEADING TO LASALLE'S INVARIANCE PRINCIPLE FOR DIFFERENTIAL SYSTEMS WITH IMPULSES AT VARIABLE TIMES

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Resumo

Nós provaremos que um problema de valor inicial para uma classe de EDOs generalizadas, também conhecidas como equações de Kurzweil, gera um sistema semidinâmico local. Sob certas condições de perturbação, também mostramos que esta classe de EDOs generalizadas admite um semifluxo descontínuo, ao qual nos referiremos como um sistema semidinâmico impulsivo. Como consequência, obtemos o princípio de invariância de LaSalle para tal classe de EDOs generalizadas. Devido à importância do princípio de invariância de LaSalle em estudar a estabilidade de sistemas diferenciais, incluímos uma aplicação para sistemas diferenciais ordinários autônomos sob a ação impulsiva em tempos variáveis.

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S. AFONSO, E. BONOTTO, M. FEDERSON, AND Š. SCHWABIK

ABSTRACT. We consider an initial value problem for a class of generalized ODEs, also known as Kurzweil equations, and we prove the existence of a local semidynamical system there. Under certain perturbation conditions, we also show that this class of generalized ODEs admits a discontinuous semiflow which we shall refer to as an impulsive semidynamical system. As a consequence, we obtain LaSalle's invariance principle for such a class of generalized ODEs. Due to the importance of LaSalle's invariance principle in studying stability of differential systems, we include an application to autonomous ordinary differential systems with impulse action at variable times.

1. INTRODUCTION

In order to generalize certain results on continuous dependence of solutions of ordinary differential equations (ODEs) with respect to the initial data, J. Kurzweil introduced, in 1957, the notion of generalized ordinary differential equations for functions taking values in Euclidean and Banach spaces. We refer to these equations as generalized ODEs or *Kurzweil equations*. See references [15] and [17] for instance.

The correspondence between generalized ODEs and classic ODEs is very simple. It is known that the ordinary system

$$\dot{x} = f(x,t), \tag{1}$$

where $\dot{x} = dx/dt$, $\Omega \subset \mathbb{R}^n$ is open and $f : \Omega \times \mathbb{R} \to \mathbb{R}^n$, is equivalent to the "integral equation"

$$x(t) = x(t_0) + \int_{t_0}^{t} f(x(\tau),\tau)d\tau, \qquad t \geq t_0, \tag{2}$$

when the integral exists in some sense. It is also known that if the integral in (2) is considered in the sense of Riemann, Lebesgue (with the equivalent McShane definition) or Henstock-Kurzweil, for instance, then it can be approximated by a sum of the form

$$\sum_{i=1}^{m} f(x(\tau_i),\tau_i)(s_i - s_{i-1})$$

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where $t_0 = s_0 \leq s_1 \leq \ldots \leq s_m = t$ is a sufficiently fine partition of the interval $[t_0, t]$ and, for each $i = 1, 2, \ldots, m$, τ_i is "*close*" enough to the interval $[s_{i-1}, s_i]$.

Alternatively, if we define

$$F(x,s) = \int_{s_0}^{s} f(x,\sigma)d\sigma, \qquad (x,s) \in \Omega \times \mathbb{R},$$

then the integral in (2) can be approximated by

$$\sum_{i=1}^{m} \int_{s_{i-1}}^{s_i} f(x(\tau_i),\sigma)d\sigma = \sum_{i=1}^{m} \left[F(x(\tau_i),s_i) - F(x(\tau_i),s_{i-1})\right]. \tag{3}$$

In this case, the righthand side of (3) approximates the non-absolute Kurzweil integral which, when considered in (2), gives rise to a differential equation of type (1), but in a wider sense. Such differential equation is known as generalized ordinary differential equation or *Kurzweil equation* (see [1], [9] and [17]).

In the present paper, we consider a class of generalized ODEs and we prove the existence of a local semidynamical system. We also consider the case where the system of generalized ODEs is subject to some perturbations. In this case, we introduce the notion of an impulsive semidynamical system and we prove that one such system can be constructed for our class of generalized ODEs. With this result at hand we are able to present a version of LaSalle's invariance principle. In particular, a version of LaSalle's invariance principle for ordinary differential systems subject to impulse effects at variable times comes out naturally.

At this moment, we would like to make a comment on our treatment of differential systems with impulses at variable times which is in connection with the ideas and approach of S. K. Kaul [14] and K. Ciesielski [7] and [8] and differs from the approach of V. Lakshmikantham et al in [16]. In [16] and in some papers (see [16] Theorems 2.12.1 and 2.12.2, and also [19] and [20], for instance) the study of properties of differential systems with impulses is somehow reduced to the pre-assigned case by the imposition of additional hypotheses as the number of times the impulse surfaces are reached by the integral curve (usually exactly once), the assumption that the sequence of impulse surfaces is monotone increasing, etc. In the present paper, we an impulse operator, which can be the sum of several impulse operators, acting on a surface M (or in a collection of surfaces which can also be denoted by M) and transferring the solution to another surface N (or a collection of surfaces N).

We start our presentation by mentioning some basic facts of the Kurzweil integration theory and of the theory of generalized differential equations.

2. Generalized ODEs

A *tagged division* of a compact interval $[a,b] \subset \mathbb{R}$ is a finite collection

$$\{(\tau_i, [s_{i-1}, s_i]) : i = 1, 2, \ldots, k\},$$

where $a = s_0 \leq s_1 \leq \ldots \leq s_k = b$ is a division of $[a,b]$ and $\tau_i \in [s_{i-1}, s_i]$, $i = 1, 2, \ldots, k$.

A *gauge* on $[a,b]$ is any function $\delta : [a,b] \to (0,+\infty)$. Given a gauge δ on $[a,b]$, a tagged division $d = (\tau_i, [s_{i-1}, s_i])$ of $[a,b]$ is *δ-fine* if, for every i,

$$[s_{i-1}, s_i] \subset \{t \in [a,b] : |t-\tau_i| < \delta(\tau_i)\}.$$

Let X be a Banach space. In the sequel, we will use integration specified by the next definition.

Definition 2.1. *A function $U(\tau,t) : [a,b] \times [a,b] \to X$ is* Kurzweil integrable *over $[a,b]$, if there is a unique element $I \in X$ such that given $\varepsilon > 0$, there is a gauge δ of $[a,b]$ such that for every δ-fine tagged division $d = (\tau_i, [s_{i-1}, s_i])$ of $[a,b]$, we have*

$$\|S(U,d) - I\| < \varepsilon,$$

where $S(U,d) = \sum_i [U(\tau_i, s_i) - U(\tau_i, s_{i-1})]$. In this case, we write $I = \int_a^b DU(\tau,t)$ and use the convention $\int_a^b DU(\tau,t) = -\int_b^a DU(\tau,t)$, whenever $b < a$.

This type of integration belongs to Jaroslav Kurzweil and it was described extensively in Chapter I of [17] for the case $X = \mathbb{R}^n$ (see Definition 1.2n in [17]).

Checking the results concerning this integration in [17], it can be easily seen that many of the results presented there can be transferred without any changes to the case of X-valued functions $U(\tau,t) : [a,b] \times [a,b] \to X$. Let us mention a few of them. The integral has the usual properties of linearity, additivity with respect to adjacent intervals, etc.

An important result, which will be used latter, concerns the integrability on subintervals (see Theorem 1.10 in [17]).

Lemma 2.1. *Let $U(\tau,t) : [a,b] \times [a,b] \to X$ be integrable over $[a,b]$. Then $\int_c^d DU(\tau,t)$ exists, for each subinterval $[c,d] \subset [a,b]$.*

The next result is known as the Saks-Henstock Lemma. A proof of it can be found in [17], Lemma 1.13.

Lemma 2.2 (Saks-Henstock Lemma). *Let $U(\tau,t) : [a,b] \times [a,b] \to X$. If for every $\varepsilon > 0$, δ is a gauge of $[a,b]$ such that for every δ-fine tagged division $d = (\tau_i, [s_{i-1}, s_i])$ of $[a,b]$,*

$$\left\| \sum_i [U(\tau_i, s_i) - U(\tau_i, s_{i-1})] - \int_a^b DU(\tau,t) \right\| < \varepsilon$$

then for $a \le c_1 \le \eta_1 \le d_1 \le c_2 \le \eta_2 \le d_2 \le \ldots \le c_l \le \eta_l \le d_l \le b$, with $\eta_j \in [c_j, d_j] \subset [\eta_j - \delta(\eta_j), \eta_j + \delta(\eta_j)]$, $j = 1, 2, \ldots, l$, we have

$$\left\| \sum_j \left[U(\eta_j, d_j) - U(\eta_j, c_j) - \int_{c_j}^{d_j} DU(\tau,t) \right] \right\| < \varepsilon.$$

The following result, an important Hake-type theorem (see Theorem 1.14 in [17]), is based on Saks-Henstock Lemma (Lemma 2.2).

Lemma 2.3. *Let a function $U : [a, b] \times [a, b] \to X$ be given such that U is integrable over $[a, c]$ for every $c \in [a, b)$ and let the limit*

$$\lim_{c \to b-} \left[\int_a^c DU(\tau, t) - U(b, c) + U(b, b) \right] = I \in X$$

exist. Then the function U is integrable over $[a, b]$ and

$$\int_a^b DU(\tau, t) = I.$$

Similarly, if the function U is integrable over $[c, b]$ for every $c \in (a, b]$ and the limit

$$\lim_{c \to a+} \left[\int_c^b DU(\tau, t) + U(a, c) - U(a, a) \right] = I \in X$$

exists, then the function U is integrable over $[a, b]$ and

$$\int_a^b DU(\tau, t) = I.$$

This leads to the following result (see Theorem 1.16 in [17]).

Lemma 2.4. *Let $U : [a, b] \times [a, b] \to X$ be integrable over $[a, b]$ and $c \in [a, b]$. Then*

$$\lim_{s \to c} \left[\int_a^s DU(\tau, t) - U(c, s) + U(c, c) \right] = \int_a^c DU(\tau, t).$$

Lemma 2.4 above shows that the function defined by

$$s \in [a, b] \mapsto \int_a^s DU(\tau, t) \in X,$$

that is, the *indefinite integral of* U, may not be continuous in general. The indefinite integral is continuous at a point $c \in [a, b]$, if and only if, the function $U(c, \cdot) : [a, b] \to X$ is continuous at the point c. Notice that if $U : [a, b] \times [a, b] \to X$ is integrable over $[a, b]$, then by Lemma 2.1 the indefinite integral of the function U is well defined on the whole interval $[a, b]$.

Let $\Omega = \mathcal{O} \times [0, +\infty)$, where $\mathcal{O} \subset X$ is an open subset. Let us present the concept of a generalized ordinary differential equation with righthand side $G : \Omega \to X$.

Definition 2.2. *A function $x : [\alpha, \beta] \to X$ is called* a solution of the generalized ordinary differential equation

$$\frac{dx}{d\tau} = DG(x, t) \tag{4}$$

on the interval $[\alpha, \beta] \subset [0, +\infty)$, if $(x(t), t) \in \Omega$ for all $t \in [\alpha, \beta]$ and if the equality

$$x(v) - x(\gamma) = \int_\gamma^v DG(x(\tau), t)$$

holds for every γ, $v \in [\alpha, \beta]$.

Given an initial condition $(z_0, t_0) \in \Omega$ the following definition of a solution of the initial value problem for the equation (4) will be used.

Definition 2.3. *A function $x : [\alpha, \beta] \to X$ is a* solution of the generalized ordinary differential equation (4) with the initial condition $x(t_0) = z_0$ on the interval $[\alpha, \beta] \subset [0, +\infty)$ *if $t_0 \in [\alpha, \beta]$, $(x(t), t) \in \Omega$ for all $t \in [\alpha, \beta]$ and if the equality*

$$x(v) - z_0 = \int_{t_0}^{v} DG(x(\tau), t)$$

holds for every $v \in [\alpha, \beta]$.

Remark 2.1. *Let $U(\tau, t) = G(x(\tau), t)$. In the definition of $\int_a^b DG(x(\tau), t)$, there are only differences of the form*

$$U(\tau_i, s_i) - U(\tau_i, s_{i-1}) = G(x(\tau_i), s_i) - G(x(\tau_i), s_{i-1}).$$

Thus, adding to $G(x, t)$ a function varying only in x, the solutions of (4) do not change. In particular, subtracting $G(x, 0)$ from $G(x, t)$, we obtain a normalized representation G_1 *of G fulfilling $G_1(x, 0) = 0$ for every x.*

Now, we define a class of functions $G : \Omega \to X$ for which it is possible to obtain some information concerning the solutions of (4).

Definition 2.4. *Given a nondecreasing function $h : [0, +\infty) \to \mathbb{R}$, we say that a function $G : \Omega \to X$ belongs to the class $\mathcal{F}(\Omega, h)$ if $G(x, 0) = 0$ for all $x \in \mathcal{O}$,*

$$\|G(x, s_2) - G(x, s_1)\| \leq |h(s_2) - h(s_1)| \tag{5}$$

for all (x, s_2), $(x, s_1) \in \Omega$ and

$$\|G(x, s_2) - G(x, s_1) - G(y, s_2) + G(y, s_1)\| \leq \|x - y\| |h(s_2) - h(s_1)| \tag{6}$$

for all (x, s_2), (x, s_1), (y, s_2), $(y, s_1) \in \Omega$.

The next lemma will imply the fact that all solutions of the generalized differential equation (4), with G satisfying (5), are of locally bounded variation.

Lemma 2.5. *Suppose $G : \Omega \to X$ satisfies (5). If $[\alpha, \beta] \subset [0, +\infty)$ and $x : [\alpha, \beta] \to X$ is a solution of (4), then the inequality*

$$\|x(s_2) - x(s_1)\| \leq |h(s_2) - h(s_1)|$$

holds for every $s_1, s_2 \in [\alpha, \beta]$.

For a proof of Lemma 2.5, see Lemma 3.10 in [17].

Let $\mathrm{var}_\alpha^\beta(x)$ denote the variation of a function $x : [\alpha, \beta] \to X$ in $[\alpha, \beta]$. Lemma 2.5 implies the following property of the solutions of (4).

Corollary 2.1. *Suppose $G : \Omega \to X$ satisfies* (5). *If $[\alpha, \beta] \subset [0, +\infty)$ and $x : [\alpha, \beta] \to X$ is a solution of* (4), *then x is of bounded variation on $[\alpha, \beta]$ and*

$$\operatorname{var}_\alpha^\beta x \le h(\beta) - h(\alpha) < +\infty.$$

In addition, every point in $[\alpha, \beta]$ at which the function h is continuous is a continuity point of the solution $x : [\alpha, \beta] \to X$.

Moreover, we have the following result (see Lemma 3.12 in [17]).

Lemma 2.6. *If $x : [\alpha, \beta] \to X$ is a solution of* (4) *and $G : \Omega \to X$ satisfies condition* (5), *then*

$$x(\sigma+) - x(\sigma) = \lim_{s \to \sigma+} x(s) - x(\sigma) = G(x(\sigma), \sigma+) - G(x(\sigma), \sigma)$$

for $\sigma \in [\alpha, \beta)$ and

$$x(\sigma) - x(\sigma-) = x(\sigma) - \lim_{s \to \sigma-} x(s) = G(x(\sigma), \sigma) - G(x(\sigma), \sigma-)$$

for $\sigma \in (\alpha, \beta]$, where

$$G(x, \sigma+) = \lim_{s \to \sigma+} G(x, s), \quad \text{for } \sigma \in [\alpha, \beta)$$

and

$$G(x, \sigma-) = \lim_{s \to \sigma-} G(x, s), \quad \text{for } \sigma \in (\alpha, \beta].$$

Note that in spite of (5) and of Lemma 2.5, all the onesided limits $G(x, \sigma+)$, $G(x, \sigma-)$, $x(\sigma+)$ and $x(\sigma-)$ exist in X, since h is a nondecreasing real function.

By a *step function* $f : [a, b] \to X$, we mean a function for which there is a finite division $a = \beta_0 < \beta_1 < \ldots < \beta_m = b$ such that in every open interval (β_{i-1}, β_i), $i = 1, \ldots, m$, the function f is equal to a constant $c_i \in X$.

Now we present a result on the existence of the integral involved in the definition of the solution of the generalized differential equation (4).

Proposition 2.1. *Let $G \in \mathcal{F}(\Omega, h)$. If $x : [\alpha, \beta] \to X$, with $[\alpha, \beta] \subset [0, +\infty)$, is the uniform limit of a sequence $(x_k)_{k \in \mathbb{N}}$ of step functions $x_k : [\alpha, \beta] \to X$ such that $(x(s), s)) \in \Omega$ and $(x_k(s), s) \in \Omega$, for every $k \in \mathbb{N}$ and for every $s \in [\alpha, \beta]$. Then the integral $\int_\alpha^\beta DG(x(\tau), t)$ exists and*

$$\int_\alpha^\beta DG(x(\tau), t) = \lim_{k \to \infty} \int_\alpha^\beta DG(x_k(\tau), t).$$

Proof. For each $k \in \mathbb{N}$, the integral $\int_\alpha^\beta DG(x_k(\tau), t)$ exists by Corollary 3.15, [17].

Given $\varepsilon > 0$, let $k_0 \in \mathbb{N}$ be such that for $k \ge k_0$, we have

$$\|x_k(s) - x(s)\| < \frac{\varepsilon}{2[h(\beta) - h(\alpha)]}, \quad s \in [\alpha, \beta],$$

and let δ be a gauge on $[a,b]$ such that for $k \geq k_0$, we have

$$\left\| \sum_{i=1}^{m} [G(x_k(\tau_i), t_i) - G(x_k(\tau_i), t_{i-1})] - \int_{\alpha}^{\beta} DG(x_k(\tau), t) \right\| < \frac{\varepsilon}{2}$$

for every δ-fine tagged division $D = \{\alpha = t_0 \leq \tau_1 \leq t_1 \leq \ldots \leq t_{m-1} \leq \tau_m \leq t_m = \beta\}$ of $[\alpha, \beta]$. Then for every $k \geq k_0$, we have

$$\left\| \sum_{i=1}^{m} [G(x(\tau_i), t_i) - G(x(\tau_i), t_{i-1})] - \int_{\alpha}^{\beta} DG(x_k(\tau), t) \right\| \leq$$

$$\leq \sum_{i=1}^{m} \| G(x(\tau_i), t_i) - G(x(\tau_i), t_{i-1}) - G(x_k(\tau_i), t_i) + G(x_k(\tau_i), t_{i-1}) \| +$$

$$+ \left\| \sum_{i=1}^{m} [G(x_k(\tau_i), t_i) - G(x_k(\tau_i), t_{i-1})] - \int_{\alpha}^{\beta} DG(x_k(\tau), t) \right\| \leq$$

$$\leq \sum_{i=1}^{m} [h(t_i) - h(t_{i-1})] \max_i \| x(\tau_i) - x_k(\tau_i) \| + \frac{\varepsilon}{2} =$$

$$= [h(\beta) - h(\alpha)] \max_i \| x(\tau_i) - x_k(\tau_i) \| + \frac{\varepsilon}{2} < \varepsilon$$

and the proof is complete. □

Next, we specialize Proposition 2.1 for a specific class of functions $x : [\alpha, \beta] \to X$.

A function $f : [a,b] \to X$ is called *regulated*, if at any point $t \in [a,b]$, it possesses onesided limits, that is the limit $\lim_{s \to t-} f(s) = f(t-) \in X$ exists for every $t \in (a,b]$ and the limit $\lim_{s \to t+} f(s) = f(t+) \in X$ exists for every $t \in [a,b)$. We write $f \in G([a,b], X)$ in this case. Therefore, if $f \in G([a,b], X)$, then for every $\varepsilon > 0$ and $t \in (a,b]$, there are a $\delta > 0$ and $f(t-) \in X$ such that

$$\| f(s) - f(t-) \| < \varepsilon, \quad \text{when} \quad t - \delta < s < t,$$

and, for every $\varepsilon > 0$ and $t \in [a,b)$, there are a $\delta > 0$ and $f(t+) \in X$ such that

$$\| f(s) - f(t+) \| < \varepsilon \quad \text{when} \quad t < s < t + \delta.$$

If we endow $G([a,b], X)$ with the usual supremum norm $\|f\|_\infty = \sup_{a \leq t \leq b} \|f(t)\|$, then $(G([a,b], X), \|\cdot\|_\infty)$ is a Banach space. For other properties of this space, the reader may want to consult [13]. For example, it is known that regulated functions are the uniform limit of step functions and this leads, by Proposition 2.1, to the next statement.

Lemma 2.7. *Let $G \in \mathcal{F}(\Omega, h)$ and $x : [\alpha, \beta] \to X$ be regulated (in particular, a function of bounded variation) on $[\alpha, \beta] \subset [0, +\infty)$ and $(x(s), s) \in \Omega$ for every $s \in [\alpha, \beta]$. Then the integral $\int_\alpha^\beta DG(x(\tau), t)$ exists and the function $s \mapsto \int_\alpha^s DG(x(\tau), t) \in X$ is of bounded variation in $[\alpha, \beta]$ (and therefore also regulated).*

The next result concerns the existence of a solution of (4) (see [10], Theorem 2.15).

Theorem 2.1 (Existence and uniqueness). *Let $G : \Omega \to X$ belong to the class $\mathcal{F}(\Omega, h)$, where the function h is continuous from the left. If for every $(\widetilde{x}, t_0) \in \Omega$ such that for $\widetilde{x}_+ = \widetilde{x} + G(\widetilde{x}, t_0+) - G(\widetilde{x}, t_0)$ we have $(\widetilde{x}_+, t_0) \in \Omega$, then there exists $\Delta > 0$ such that on the interval $[t_0, t_0 + \Delta]$ there exists a unique solution $x : [t_0, t_0 + \Delta] \to X$ of the generalized ordinary differential equation (4) for which $x(t_0) = \widetilde{x}$.*

Remark 2.2. *The assumption on the left continuity of the function h in Theorem 2.1 implies that the solutions of (4) are also left continuous (cf. Lemma 2.5). Given a solution x of (4), the limit $x(\sigma-)$ exists for every σ in the domain of x. This follows again by Lemma 2.5 and, by Lemma 2.6, we have the relation*

$$x(\sigma) = x(\sigma-) + G(x(\sigma), \sigma) - G(x(\sigma), \sigma-)$$

which describes the discontinuity of the given solution.

Remark 2.3. *We say that $x : [t_0,\ t_0 + b) \to X$ is the maximal solution of (4) with $x(t_0) = u \in \mathcal{O}$, if x is a solution of (4) on every interval $[t_0,\ t_0+\beta]$, $\beta < b$, and it cannot be continued to $[t_0,\ t_0 + b]$. We denote $b = \omega(u,\ G)$ in this case.*

For other properties of generalized differential equations and their applications, see [10], [11] and [12].

3. The compactness of the class $\mathcal{F}(\Omega, h)$

In this section, we will consider $X = \mathbb{R}^n$ and we are going to show that the class $\mathcal{F}(\Omega, h)$ is a compact space when h is a nondecreasing continuous function, where $\Omega = \mathcal{O} \times [0, +\infty)$, with $\mathcal{O} \subset \mathbb{R}^n$ an open set.

At first, we are going to endow the space $\mathcal{F}(\Omega, h)$ with a metric. Let $\{K_n\}_{n\geq 1}$ be a sequence of compact sets in Ω such that $K_n \subset int(K_{n+1})$ and $\Omega = \bigcup_{n=1}^{+\infty} K_n$. By $int(A)$ we mean the interior of a subset A of Ω. For each natural $n \geq 1$, we construct a pseudo-metric on $\mathcal{F}(\Omega,\ h)$ as follows: let

$$\|G_1 - G_2\|_n = \sup\{\|G_1(x,t) - G_2(x,t)\| : (x,t) \in K_n\},$$

where $\|\cdot\|$ is any metric in $\mathbb{R}^n$, and set

$$\rho_n(G_1, G_2) = \frac{\|G_1 - G_2\|_n}{1 + \|G_1 - G_2\|_n}.$$

The required metric is then given by

$$\rho(G_1, G_2) = \sum_{n=1}^{\infty} 2^{-n} \rho_n(G_1, G_2)$$

and $(\mathcal{F}(\Omega,\ h), \rho)$ is a metric space. Note that the metric ρ depends on the choice of the sequence K_n. However any other sequence of compact sets generates an equivalent metric.

The next result concerns the equicontinuity of the class $\mathcal{F}(\Omega,\ h)$.

Lemma 3.1. *Assume* $h : [0, +\infty) \to \mathbb{R}$ *is a nondecreasing continuous function. Then* $\mathcal{F}(\Omega, h)$ *is equicontinuous on compact subsets of* $\Omega = \mathcal{O} \times [0, +\infty)$*, where* $\mathcal{O} \subset \mathbb{R}^n$ *is open.*

Proof. Let $A \subset \mathcal{O}$ and $C \subset [0, +\infty)$ be compact subsets and let $(x, t) \in A \times C$ be an arbitrary point. Take an arbitrary $G \in \mathcal{F}(\Omega, h)$. Since $G(z, 0) = 0$ for every $z \in \mathcal{O}$, we have

$$\|G(x,t) - G(y,t)\| = \|G(x,t) - G(x,0) - G(y,t) + G(y,0)\|,$$

for each $y \in A$. Then, using condition (6), we have

$$\|G(x,t) - G(y,t)\| \leq \|x - y\||h(t) - h(0)| \leq \|x - y\|(|h(t)| + |h(0)|), \tag{7}$$

for $y \in A$.

On the other hand, by condition (5), we have

$$\|G(y,t) - G(y,s)\| \leq |h(t) - h(s)|, \tag{8}$$

for every $(y, s) \in A \times C$. Thus by (7) and (8), we obtain

$$\begin{aligned} \|G(x,t) - G(y,s)\| &\leq \|G(x,t) - G(y,t)\| + \|G(y,t) - G(y,s)\| \\ &\leq \|x - y\|(|h(t)| + |h(0)|) + |h(t) - h(s)|, \end{aligned}$$

for all $(y, s) \in A \times C$. Since h is a continuous function (therefore uniformly continuous on C), for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|h(t) - h(s)| < \frac{\varepsilon}{2}, \quad \text{whenever} \quad |t - s| < \delta, \; t, s \in C.$$

Moreover, by the compactness of C, there exists $M > 0$ such that $|h(s)| \leq M$ for all $s \in C$. Therefore,

$$\|G(x,t) - G(y,s)\| < \varepsilon$$

whenever $\|x - y\| < \frac{\varepsilon}{2}(M + |h(0)|)^{-1}$ and $|t - s| < \delta$, with $(y, s) \in A \times C$. This completes the proof. □

The compactness of the class $\mathcal{F}(\Omega, h)$ is presented next.

Theorem 3.1. *Assume* $h : [0, +\infty) \to \mathbb{R}$ *is nondecreasing and continuous. Then the space* $\mathcal{F}(\Omega, h)$ *is compact.*

Proof. By Lemma 3.1, the family of functions in $\mathcal{F}(\Omega, h)$ is equicontinuous on compact subsets of $\Omega = \mathcal{O} \times [0, +\infty)$. Moreover, $\mathcal{F}(\Omega, h)$ is also uniformly bounded on compact sets. Indeed. Since $G(x, 0) = 0$ for all $G \in \mathcal{F}(\Omega, h)$ and $x \in \mathcal{O}$, we have

$$\|G(x,t)\| = \|G(x,t) - G(x,0)\| \leq |h(t) - h(0)| \leq |h(t)| + |h(0)|, \tag{9}$$

for every $t \in [0, +\infty)$, $x \in \mathcal{O}$ and $G \in \mathcal{F}(\Omega, h)$. Let $A \subset \mathcal{O}$ and $C \subset [0, +\infty)$ be compact sets and suppose $(x, t) \in A \times C$. Since h is a continuous function and C is a compact set, there exists $M > 0$ such that $|h(s)| \leq M$ for all $s \in C$. Then by (9) we have $\|G(x,t)\| \leq M + |h(0)|$ for all $(x, t) \in A \times C$ and all $G \in \mathcal{F}(\Omega, h)$. Therefore, by Ascoli's Theorem, for each sequence $\{G_n\}_{n \geq 1}$ in $\mathcal{F}(\Omega, h)$ there exists a subsequence $\{G_{n_k}\}_{k \geq 1}$ converging to a certain function G_0 uniformly on compact subsets. Since $\mathcal{F}(\Omega, h)$ is a closed set (see Definition 2.4), $G_0 \in \mathcal{F}(\Omega, h)$. This completes the proof. □

4. Existence of a local semidynamical system

We continue to consider the special case where $X = \mathbb{R}^n$.

Let $\mathcal{O} \subset \mathbb{R}^n$ be an open set. Consider the generalized ODE

$$\frac{dx}{d\tau} = DG(x, t), \tag{10}$$

where $G : \Omega \to \mathbb{R}^n$ belongs to $\mathcal{F}(\Omega, h)$ with $\Omega = \mathcal{O} \times [0, +\infty)$ and h is a nondecreasing continuous real function defined on $[0, +\infty)$.

By (5) and (6), G is continuous in both variables.

Let $\{G_k\}_{k\geq 1}$ be a sequence of elements of $\mathcal{F}(\Omega, h)$ and let $G \in \mathcal{F}(\Omega, h)$. We say that $\{G_k\}_{k\geq 1}$ converges to G in $\mathcal{F}(\Omega, h)$, and we write $G_k \stackrel{k\to+\infty}{\longrightarrow} G$, if and only if $G_k(x, t) \stackrel{k\to+\infty}{\longrightarrow} G(x, t)$ in $\mathbb{R}^n$ for each $(x, t) \in \Omega$, that is,

$$\|G_k(x, t) - G(x, t)\| \stackrel{k\to+\infty}{\longrightarrow} 0,$$

for every $(x, t) \in \Omega$, where $\|\cdot\|$ is a norm in $\mathbb{R}^n$. Moreover, given a sequence $\{v_k\}_{k\geq 1}$ in $\mathbb{R}^n$ and $v \in \mathbb{R}^n$, we write $(v_k, G_k) \stackrel{k\to+\infty}{\longrightarrow} (v, G)$ in $\mathbb{R}^n \times \mathcal{F}(\Omega, h)$, if and only if $\|v_k - v\| \stackrel{k\to+\infty}{\longrightarrow} 0$ and $\|G_k(x, s) - G(x, s)\| \stackrel{k\to+\infty}{\longrightarrow} 0$ for every $(x, s) \in \Omega$.

Now we introduce the notion of a local semidynamical system and prove that initial value problems for the generalized ODE (10) generate a local semidynamical system. See [1] for an analogous result in a different setting of functions G.

For each $(v, G) \in \mathcal{O} \times \mathcal{F}(\Omega, h)$, let $I_{(v,G)}$ be an interval of type $[0, b) \subset \mathbb{R}$, with $b \in \mathbb{R}_+$ and define

$$S = \{(t, v, G) \in \mathbb{R}_+ \times \mathcal{O} \times \mathcal{F}(\Omega, h) : t \in I_{(v,G)}\}.$$

Definition 4.1. *A mapping*

$$\pi : S \to \mathcal{O} \times \mathcal{F}(\Omega, h)$$

is called a local semidynamical system *on* $\mathcal{O} \times \mathcal{F}(\Omega, h)$, *if the following properties hold:*

i) $\pi(0, v, G) = (v, G)$, *for every* $(v, G) \in \mathcal{O} \times \mathcal{F}(\Omega, h)$;

ii) *Given* $(v, G) \in \mathcal{O} \times \mathcal{F}(\Omega, h)$, *if* $t \in I_{(v,G)}$ *and* $s \in I_{\pi(t,v,G)}$, *then* $t + s \in I_{(v,G)}$ *and* $\pi(s, \pi(t, v, G)) = \pi(t + s, v, G)$;

iii) *For each* $(v, G) \in \mathcal{O} \times \mathcal{F}(\Omega, h)$ *fixed,* $\pi(t, v, G)$ *is continuous at every* $t \in I_{(v,G)}$.

iv) $I_{(v,G)} = [0, b_{(v,G)})$ *is maximal in the following sense: either* $I_{(v,G)} = \mathbb{R}_+$ *or, if* $b_{(v,G)} \neq +\infty$, *then the positive orbit*

$$\{\pi(t, v, G) : t \in [0, b_{(v,G)})\} \subset \mathcal{O} \times \mathcal{F}(\Omega, h)$$

cannot be continued to a larger interval $[0, b_{(v,G)} + c)$, $c > 0$;

v) *If* $(v_k, G_k) \overset{k\to+\infty}{\longrightarrow} (v, G)$, *where* (v, G) *and* $(v_k, G_k) \in \mathcal{O} \times \mathcal{F}(\Omega, h)$, $k = 1, 2, ...,$ *then*

$$I_{(v,G)} \subset \liminf I_{(v_k,G_k)}.$$

Remark 4.1. *Note that, if the domain of* π *is* $\mathbb{R}_+ \times \mathcal{O} \times \mathcal{F}(\Omega, h)$, *then conditions iv) and v) are satisfied trivially. When this is the case, we call* π *a* global semidynamical system.

Now, let $G \in \mathcal{F}(\Omega,\ h)$ be given. For each $t \geq 0$, we define the translate G_t of G by

$$G_t(x,\ s) = G(x,\ t+s) - G(x,\ t), \tag{11}$$

where $(x,\ s) \in \Omega$. Then the following properties can be easily checked:

i) $G_0 = G$ (normalization of G);

ii) $G_{t+\tau} = (G_t)_\tau$ for all $t, \tau \geq 0$ (semigroup property);

iii) the mapping $(t, G) \mapsto G_t$ is continuous.

Now we define a subset of $\mathcal{F}(\Omega, h)$ with the important property that it contains the translates G_t of all its elements G.

Definition 4.2. *Given a nondecreasing continuous function* $h : [0, +\infty) \to \mathbb{R}$, *we say that a function* $G : \Omega \to X$ *belongs to the class* $\mathcal{F}^*(\Omega, h)$, *if* G *belongs to the class* $\mathcal{F}(\Omega, h)$ *and the function* h *satisfies*

$$|h(t_1+s) - h(t_2+s)| \leq |h(t_1) - h(t_2)|, \quad t_1, t_2, s \in [0,\ +\infty).$$

Remark 4.2. *It follows from Theorem 3.1 that the class* $\mathcal{F}^*(\Omega, h)$ *is compact.*

The following statement is easy to check.

Lemma 4.1. *Let* $G \in \mathcal{F}^*(\Omega,\ h)$. *Then the translates* G_t *of* G *belong to* $\mathcal{F}^*(\Omega,\ h)$ *for each* $t \geq 0$.

Since we are assuming that $G \in \mathcal{F}^*(\Omega, h)$, with h nondecreasing and continuous, it is clear from (11) that for each $t \geq 0$, G_t is continuous.

Our aim now is to construct a local semidynamical system for an initial value problem concerning the generalized ODE (10). At first, we state the main result of this section, namely Theorem 4.1, which generalizes [1], Theorem 6.3 and [9], Theorem 4.1. Then we present several auxiliary results and, finally, we give a proof of Theorem 4.1.

Theorem 4.1. *Assume that for each* $u \in \mathcal{O}$ *and each* $G \in \mathcal{F}^*(\Omega,\ h)$, $x(t,\ u,\ G)$ *is the unique maximal solution of the initial value problem*

$$\frac{dx}{d\tau} = DG(x,\ t), \qquad x(0) = u. \tag{12}$$

Let $[0,\ \omega(u,\ G))$, $\omega(u,\ G) > 0$, *be the maximal interval of definition of* $x(\cdot\,, u, G)$. *Define* $\pi : S \to \mathcal{O} \times \mathcal{F}^*(\Omega,\ h)$ *by*

$$\pi(t,\ u,\ G) = (x(t,\ u,\ G),\ G_t), \tag{13}$$

where $S = \{(t, u, G) \in \mathbb{R}_+ \times \mathcal{O} \times \mathcal{F}^*(\Omega, h) : t \in I_{(u,G)}\}$. *Then* π *is a local semidynamical system on* $\mathcal{O} \times \mathcal{F}^*(\Omega,\ h)$.

Note that the maximal interval $I_{(u,G)}$ of the semidynamical system given by (13) coincides with $[0, \omega(u, G))$ necessarily, since the second component G_t of the flow is defined for all $t \in [0, +\infty)$.

In order to prove Theorem 4.1, we need to prove that the conditions in Definition 4.1 hold. Therefore we need some auxiliary lemmas. The first result we present, namely Lemma 4.2, says that the function π defined by (13) in Theorem 4.1 satisfies the identity and semigroup properties.

Lemma 4.2. *The mapping π defined in Theorem 4.1 satisfies the following conditions:*

a) $\pi(0, u, G) = (u, G)$ *for each* $(u, G) \in \mathcal{O} \times \mathcal{F}^*(\Omega, h)$*;*
b) *If* $t \in I_{(u,G)}$ *and* $s \in I_{\pi(t,u,G)}$*, then* $t+s \in I_{(u,G)}$ *and* $\pi(s, \pi(t, u, G)) = \pi(t+s, u, G)$ *for all* $(u, G) \in \mathcal{O} \times \mathcal{F}^*(\Omega, h)$*.*

Proof.

a) By definition, is clear that
$$\pi(0, u, G) = (x(0, u, G), G_0) = (u, G),$$
for all $(u, G) \in \mathcal{O} \times \mathcal{F}^*(\Omega, h)$.

b) In order to prove this item we borrow some ideas from [18]. Let $t \in I_{(u,G)}$, $s \in I_{\pi(t,u,G)}$ and $(u, G) \in \mathcal{O} \times \mathcal{F}^*(\Omega, h)$. Denote
$$x(\tau) = x(\tau, u, G),$$
$$\psi(\tau) = x(\tau, x(t), G_t)$$
and put
$$\xi(\tau) = x(\tau + t),$$
where x is the maximal solution of (12) and ψ is a solution of the generalized ODE
$$\frac{d\psi}{d\tau} = D[G_t(\psi, s)], \tag{14}$$
with initial condition
$$\psi(0) = x(t) = x(t, u, G). \tag{15}$$
We assert that ξ is a solution of problem (14)-(15). Indeed. We have
$$\xi(\sigma) - \xi(0) = x(\sigma + t) - x(t) = \int_t^{\sigma+t} DG(x(\tau), s).$$
By the change of variable $\phi(s) = s + t$, it follows by a substitution theorem (see [17], Theorem 1.18) that
$$\int_t^{t+\sigma} DG(x(\tau), s) = \int_{\phi(0)}^{\phi(\sigma)} DG(x(\tau), s) = \int_0^{\sigma} DG(x(\phi(\varsigma)), \phi(\mu)) =$$
$$= \int_0^{\sigma} DG(x(\varsigma + t), \mu + t).$$

Thus,

$$\xi(\sigma)-\xi(0)=\int_0^\sigma DG(x(\tau+t),s+t)=\int_0^\sigma DG_t(\xi(\tau),s).$$

Moreover $\xi(0)=x(t)=x(t,\,u,\,G)$. Hence, by the uniqueness of the solution of (12) (see Theorem 2.1), we get

$$\psi(\sigma)=\xi(\sigma)=x(\sigma+t),\qquad \text{for all}\quad \sigma\in I_{\pi(t,u,G)}=[0,\omega(u,\,G)). \tag{16}$$

Therefore,

$$\begin{aligned}\pi(s,\,\pi(t,\,u,\,G)) &= \pi(s,\,x(t,\,u,\,G),\,G_t)=\pi(s,\,x(t),\,G_t)=\\ &= (x(s,\,x(t),\,G_t),\,(G_t)_s)=(\xi(s),\,(G_t)_s)=\\ &= (\xi(s),\,G_{s+t})=(x(s+t),\,G_{t+s})=\\ &= (x(s+t,\,u,\,G),\,G_{t+s})=\pi(s+t,\,u,\,G)\end{aligned}$$

and the proof is complete. □

The next result is not difficult to prove. It says that the motion $\pi(\cdot,u,G)$ of (u,G) is continuous on the interval $I_{(u,G)}$.

Lemma 4.3. *Let π be the mapping defined in Theorem 4.1. For each fixed $(u,G)\in\mathcal{O}\times\mathcal{F}^*(\Omega,h)$, $\pi(t,\,u,\,G)$ is continuous at every $t\in I_{(u,G)}$.*

Lemma 4.4. *Let $x(t,\,u,\,G)$ be the unique maximal solution of (12) defined on $[0,\,\omega(u,\,G))$. Suppose $\omega=\omega(u,\,G)<+\infty$. If $x(t,\,u,\,G)\to z$ as $t\to\omega^-$, then $z\notin\mathcal{O}$.*

Proof. Let $\omega=\omega(u,\,G)<+\infty$. Then a pair (z,ω), with $z\in\mathcal{O}$, cannot be a limit point of the solution $(x(t,\,u,\,G),\,t)$ as $t\to\omega^-$. Indeed. Suppose the contrary. Since $x(t,\,u,\,G)\to z$ as $t\to\omega^-$, we can define $x(\omega)=z$. By the existence theorem (Theorem 2.1), we can extend the solution to an interval strictly greater than $[0,\,\omega)$ and this is a contradiction. Thus we finished the proof. □

The next proposition concerns the continuous dependence of a solution of a generalized ODE on the initial data. A similar statement was proved in [17] for the case when $X=\mathbb{R}^n$. We postpone its proof for an analogous statement in the more general case of a Banach space to the Appendix of our paper.

Proposition 4.1. *Assume that $\Omega=\mathcal{O}\times[c,\,d]$ and $G_k:\Omega\to\mathbb{R}^n$ belongs to the class $\mathcal{F}(\Omega,\,h)$, for $k=0,1,2,\ldots$, where $[c,d]\subset[0,\,+\infty)$. Suppose*

$$\lim_{k\to+\infty}G_k(x,\,t)=G_0(x,\,t),$$

for $(x,\,t)\in\mathcal{O}\times[c,\,d]$. Let $[\alpha,\,\beta]\subset[c,\,d]$ and $x_k:[\alpha,\,\beta]\to\mathbb{R}^n$, $k=1,2,\ldots$, be solutions of the generalized ODE

$$\frac{dx}{d\tau}=DG_k(x,\,t),$$

on $[\alpha, \beta]$ *such that*

$$\lim_{k\to+\infty} x_k(s) = x_0(s), \quad s \in [\alpha, \beta],$$

and $(x(s), s) \in \Omega$ *for* $s \in [\alpha, \beta]$. *Then* $x_0 : [\alpha, \beta] \to \mathbb{R}^n$ *satisfies:*

i) $\|x_0(s_2) - x_0(s_1)\| \leq h(s_2) - h(s_1)$, *if* $s_1 \leq s_2$, $s_1, s_2 \in [\alpha, \beta]$;

ii) $\lim_{k\to+\infty} x_k(s) = x_0(s)$ *uniformly on* $[\alpha, \beta]$;

iii) x_0 *is a solution of the generalized ODE,* $\frac{dx}{d\tau} = DG_0(x, t)$, *on* $[\alpha, \beta]$.

The next proposition concerns the continuous dependence of a solution of a generalized ODE with respect to parameters which are presented in the form of sequences. A proof using the known Helly-type selection principle for functions which take values in $\mathbb{R}^n$ can be found in [17], Theorem 8.6.

Proposition 4.2. *Suppose* $\Omega = \mathcal{O} \times [c, d]$ *and* $G_k : \Omega \to \mathbb{R}^n$ *belongs to the class* $\mathcal{F}(\Omega, h)$, *for* $k = 0, 1, 2, \ldots$, *where* $[c, d] \subset [0, +\infty)$. *Suppose*

$$\lim_{k\to+\infty} G_k(x, t) = G_0(x, t), \quad (x, t) \in \mathcal{O} \times [c, d].$$

Let $[\alpha, \beta] \subset [c, d]$ *and* $x_0 : [\alpha, \beta] \to \mathbb{R}^n$ *be the unique solution of*

$$\frac{dx}{d\tau} = DG_0(x, t), \quad x(\alpha) = y_0,$$

$y_0 \in \mathcal{O}$, *on* $[\alpha, \beta]$. *Assume further that there is a sequence* $\{y_k\}_{k\geq 1} \in \mathcal{O}$, $k = 1, 2, \ldots$ *satisfying*

$$\lim_{k\to+\infty} y_k = y_0.$$

Then there exists a positive integer k_1 *such that, for all* $k > k_1$, *there exists a solution* x_k *of the generalized differential equation*

$$\frac{dx}{d\tau} = DG_k(x, t)$$

on $[\alpha, \beta]$ *with* $x_k(\alpha) = y_k$ *and* $\lim_{k\to+\infty} x_k(s) = x_0(s)$, $s \in [\alpha, \beta]$.

The next theorem is crucial in the proof of Theorem 4.1.

Theorem 4.2. *Let* $x(t, u, G)$ *be the unique solution of* (12) *defined on the maximal interval* $[0, \omega(u, G))$, *with* $\omega(u, G) > 0$. *Then* $\omega(u, G)$ *is lower semicontinuous on* $\mathcal{O} \times \mathcal{F}^*(\Omega, h)$.

Proof. This proof follows the ideas of the proof presented by Z. Artstein for Theorem A.8 in [1].

Let $(y_0, G_0) \in \mathcal{O} \times \mathcal{F}^*(\Omega, h)$ and $(y_k, G_k) \in \mathcal{O} \times \mathcal{F}^*(\Omega, h)$, $k = 1, 2, \ldots$, be such that $(y_k, G_k) \stackrel{k\to+\infty}{\longrightarrow} (y_0, G_0)$. Let $t_k \stackrel{k\to+\infty}{\longrightarrow} t_0$ and consider $x(s) = x(s, y_0, G_0)$ as being the unique solution of the system

$$\frac{dx}{d\tau} = DG_0(x, s), \quad x(0) = y_0$$

on the maximal interval $[0, \omega(y_0, G_0))$ with $\omega(y_0, G_0) > 0$. By Proposition 4.2, there exists a positive integer k_1 such that for each $k \geq k_1$, there is a solution $x_k(s, y_k, G_k)$ of the generalized differential equation

$$\frac{dx}{d\tau} = DG_k(x, s), \quad x(0, y_k, G_k) = y_k$$

on $[0, \gamma]$, $0 < \gamma < \omega(y_0, G_0)$, with $\lim_{k \to +\infty} x_k(s, y_k, G_k) = x(s, y_0, G_0)$, for all $s \in [0, \gamma]$. Note that γ is independent of $k \geq k_1$ (see Proposition 4.2).

Define the set $A \subset [0, +\infty)$ by

$$A = \{b \geq 0 : \text{ for } k \geq k_1 \text{ the functions } x_k(s, y_k, G_k) \text{ are defined on } [0, b] \text{ and are equicontinuous on } [0, b]\}.$$

Note that the functions $x_k(\cdot, y_k, G_k)$, $k \geq k_1$, are equicontinuous on $[0, \gamma]$. Indeed, since by Lemma 2.5,

$$\|x_k(s_2, y_k, G_k) - x_k(s_1, y_k, G_k)\| \leq |h(s_2) - h(s_1)|, \quad s_1, s_2 \in [0, \gamma],$$

and since h is independent of k, the equicontinuity of $x_k(s, y_k, G_k)$ follows easily. Therefore $A \neq \emptyset$.

Let $\beta = \sup A$. We shall show that $[0, \beta)$ is the maximal positive interval of definition of $x(\cdot, y_0, G_0)$. This will imply the lower semicontinuity of ω.

Let $0 \leq b < \beta$. By Lemma 2.5, we have

$$\begin{aligned} \|x_k(s, y_k, G_k)\| &\leq \|y_k\| + \|x_k(s, y_k, G_k) - y_k\| \leq \\ &\leq \|y_k\| + [h(s) - h(0)] \leq \|y_k\| + [h(b) - h(0)], \end{aligned}$$

for each $s \in [0, b]$ and, since $y_k \overset{k \to +\infty}{\longrightarrow} y_0$, the sequence of functions $x_k(\cdot, y_k, G_k)$ is, for $k > k_2$, k_2 sufficiently larger than k_1, an equibounded sequence. Thus we have an infinite pointwise precompact family $\{x_k(s, y_k, G_k)\}$ of uniformly bounded variation. This implies, by a Helly's type choice principle (see [2]), that the sequence $x_k(\cdot, y_k, G_k)$, for $k > k_2$, is precompact in $C([0, b], \mathbb{R}^n)$. Using Proposition 4.1, it can be seen that every limit point of this sequence is a solution of the system

$$\frac{dx}{d\tau} = DG_0(x, s), \quad x(0) = y_0$$

on $[0, b]$. The uniqueness of solutions of this equation implies that there is only one limit point of the sequence $\{x_k(s, y_k, G_k)\}$ for $k > k_2$ and, therefore, the whole sequence converges uniformly to the solution $x(s, y_0, G_0)$ on $[0, b]$.

Suppose $x(\beta) = x(\beta, y_0, G_0)$ is defined. Then $x(\beta) \in \mathcal{O}$. Thus, by Theorem 2.1, there is a $\Delta_\beta > 0$ such that $x(s, y_0, G_0)$ is defined for $s \in [\beta, \beta + \Delta_\beta]$. By Proposition 4.2, for sufficiently large k the solutions $x_k(s, y_k, G_k)$ are also defined on the interval $[0, \beta + \Delta_\beta]$ and are equicontinuous there. But this contradicts the fact that $\beta = \sup A$. Hence $x(\beta, y_0, G_0)$ is not defined and $\beta = \omega(y_0, G_0)$. □

Now we are able to prove Theorem 4.1.

Proof. (of Theorem 4.1) By Lemma 4.2, we obtain items i) and ii). Item iii) follows from Lemma 4.3, item iv) follows from Lemma 4.4 and item v) follows from Theorem 4.2. □

5. Existence of an impulsive semidynamical system

Our aim is this section is to define a semidynamical system subject to instantaneous perturbations.

5.1. An impulsive initial value problem.

Consider the initial value problem

$$\frac{dx}{d\tau} = DG(x,\, s), \qquad x(0) = u, \tag{17}$$

for G in $\mathcal{F}^*(\Omega,\, h)$, where $\Omega = \mathcal{O} \times [0,\, +\infty)$ with $\mathcal{O} \subset \mathbb{R}^n$ an open set, u in $\mathcal{O}$ and $x(t,\, u,\, G)$ is the unique solution of (17) (see Theorem 2.1).

Now we will describe the impulse effects acting on a generalized ordinary differential equation. The moments of time of such impulses are not pre-assigned, but vary on time.

Let M be a closed subset of $\mathbb{R}^n$. We assume that M satisfies the following condition: if for any $G \in \mathcal{F}^*(\Omega,\, h)$ and for any $u \in \mathcal{O}$, the solution of (17) is such that $x(t_0, u, G) \in \mathrm{M}$ for some $t_0 > 0$, then there exists an $\varepsilon > 0$ such that

$$x(t,u,G) \notin \mathrm{M} \quad \text{for} \quad t \in (t_0 - \varepsilon, t_0) \cup (t_0, t_0 + \varepsilon).$$

This last condition means that the points of M are isolated in each trajectory of the system (17).

Now, define a function $\varphi : \mathcal{O} \times \mathcal{F}^*(\Omega, h) \to (0, +\infty]$ by

$$\varphi(u,G) = \begin{cases} s, & \text{if } x(s,u,G) \in \mathrm{M} \text{ and } x(t,u,G) \notin \mathrm{M} \text{ for } 0 < t < s, \\ +\infty, & \text{if } x(t,u,G) \notin \mathrm{M} \text{ for all } t > 0. \end{cases} \tag{18}$$

This means that $\varphi(u,G)$ is the least positive time at which the trajectory of u meets M.

Given system (17), we consider the following impulsive initial value problem associated to (17)

$$\begin{cases} \dfrac{dx}{d\tau} = DG(x,s), \\ \mathrm{I} : \mathrm{M} \to \mathrm{N}, \\ x(0) = u, \end{cases} \tag{19}$$

where I is a continuous function, $\mathrm{N} = \mathrm{I}(\mathrm{M})$, $\mathrm{I}(\mathrm{M} \cap \mathcal{O}) \subset \mathcal{O} \setminus \mathrm{M}$ and M satisfies the condition that the points of M are isolated in each trajectory of the system (17). The solution of (19), which we denote by $\widetilde{x}(t,u,G)$, is described in the following lines.

If $\varphi(u,G) = +\infty$, then $\widetilde{x}(t,u,G) = x(t,u,G)$, for all $t \geq 0$, where $x(t,u,G)$ is solution of (17). However if $\varphi(u,G) = s_0$, we define $\widetilde{x}(t,u,G)$ on $[0, s_0]$ by

$$\widetilde{x}(t,u,G) = \begin{cases} x(t,u,G), & 0 \leq t < s_0, \\ u_1^+, & t = s_0, \end{cases}$$

where $u_1^+ = \mathrm{I}(u_1)$ and $u_1 = x(s_0, u, G)$. Denote u by u_0^+.

Since $s_0 < +\infty$, the process now continues from u_1^+ on. Thus, if $\varphi(u_1^+, G) = +\infty$, then we define $\widetilde{x}(t, u, G) = x(t - s_0, u_1^+, G)$, $s_0 \leq t < +\infty$, where $x(\cdot, u_1^+, G)$ is the solution of the system $\dfrac{dx}{d\tau} = DG(x, s)$, $x(0) = u_1^+$. When $\varphi(u_1^+, G) = s_1$, we define $\widetilde{x}(t, u, G)$ on $[s_0,\ s_0 + s_1]$ by

$$\widetilde{x}(t, u, G) = \begin{cases} x(t - s_0, u_1^+, G), & s_0 \leq t < s_0 + s_1 \\ u_2^+, & t = s_0 + s_1, \end{cases}$$

where $u_2^+ = \mathrm{I}(u_2)$ and $u_2 = x(s_1, u_1^+, G)$.

Now we suppose that $\widetilde{x}(t, u, G)$ is defined on the interval $[t_{n-1}, t_n]$ and that $\widetilde{x}(t_n, u, G) = u_n^+$, where $t_n = \sum_{i=0}^{n-1} s_i$ with $n \geq 1$. If $\varphi(u_n^+, G) = +\infty$, then $\widetilde{x}(t, u, G) = x(t - t_n, u_n^+, G)$, $t \geq t_n$. But if $\varphi(u_n^+, G) = s_n$, then

$$\widetilde{x}(t, u, G) = \begin{cases} x(t - t_n, u_n^+, G), & t_n \leq t < t_{n+1} \\ u_{n+1}^+, & t = t_{n+1}, \end{cases}$$

where $u_{n+1}^+ = \mathrm{I}(u_{n+1})$ and $u_{n+1} = x(s_n, u_n^+, G)$. Notice that $\widetilde{x}(t, u, G)$ is defined on each interval $[t_n,\ t_{n+1}]$, where $t_0 = 0$ and $t_{n+1} = \sum_{i=0}^{n} s_i$, $n = 0, 1, 2, \ldots$. Thus $\widetilde{x}(t, u, G)$ is defined on $[0,\ t_{n+1}]$.

The process above ends after a finite number of steps, whenever $\varphi(u_n^+, G) = +\infty$ for some n. Or it continues indefinitely, if $\varphi(u_n^+, G) < +\infty$, $n = 0,\ 1,\ 2, \ldots$, and thus $\widetilde{x}(t, u, G)$ is defined on the interval $[0,\ T(u, G))$, where $T(u, G) = \sum_{i=0}^{\infty} s_i$.

5.2. **An impulsive semidynamical system.** In this subsection, we are going to show that problem (19) admits a discontinuous semiflow which we will call an impulsive semidynamical system.

Impulsive systems where the motion is defined for all $t \geq 0$ are the most important and interesting ones. Moreover, in many cases, the systems defined in $[0, w)$, $w < \infty$, can be extended, via isomorphisms, to $[0, +\infty)$ (see [8]). Thus we may restrict ourselves to such systems. We will therefore assume that the solutions of equations (17) and (19) are defined in the whole interval $[0, +\infty)$.

We recall that a local semidynamical system π corresponding to problem (17) and defined in $\mathbb{R}_+ \times \mathcal{O} \times \mathcal{F}^*(\Omega,\ h)$ is given by

$$\pi(t,\ u,\ G) = (x(t,\ u,\ G),\ G_t).$$

Hence π is a global semidynamical system (see Remark 4.1). We denote such system by $(\mathcal{O} \times \mathcal{F}^*(\Omega,\ h), \pi)$ and, from now on, we drop the term "global" and we refer to such system simply as a semidynamical system.

For every $(u, G) \in \mathcal{O} \times \mathcal{F}^*(\Omega,\ h)$, the continuous function $\pi_{(u,G)} : \mathbb{R}_+ \to \mathcal{O} \times \mathcal{F}^*(\Omega,\ h)$ defined by $\pi_{(u,G)}(t) = \pi(t, u, G)$ is called the *motion* of (u, G). Given $(u, G) \in \mathcal{O} \times \mathcal{F}^*(\Omega,\ h)$, the *positive orbit* of (u, G) is given by

$$\pi^+(u, G) = \{\pi(t, u, G) : t \geq 0\}.$$

Definition 5.1. *An impulsive semidynamical system on $\mathcal{O} \times \mathcal{F}^*(\Omega,\, h)$ is a mapping*

$$\widetilde{\pi} : \mathbb{R}_+ \times \mathcal{O} \times \mathcal{F}^*(\Omega,\, h) \to \mathcal{O} \times \mathcal{F}^*(\Omega,\, h)$$

such that

a) $\widetilde{\pi}(0, u, G) = (u, G)$ *for each* $(u, G) \in \mathcal{O} \times \mathcal{F}^*(\Omega,\, h)$;
b) $\widetilde{\pi}(s, \widetilde{\pi}(t, u, G)) = \widetilde{\pi}(t + s, u, G)$, *with* $(u, G) \in \mathcal{O} \times \mathcal{F}^*(\Omega,\, h)$ *and* $t, s \in [0, +\infty)$;
c) *for each* $(u, G) \in \mathcal{O} \times \mathcal{F}^*(\Omega,\, h)$, *the mapping* $\widetilde{\pi}(\cdot, u, G)$ *is continuous from the right at every point in* $[0, +\infty)$ *and the left limits* $\widetilde{\pi}(t-, u, G)$ *exist for all* $t > 0$.

Given $(u, G) \in \mathcal{O} \times \mathcal{F}^*(\Omega,\, h)$, the *positive impulsive orbit* of (u, G) is defined by the set

$$\widetilde{\pi}^+(u, G) = \{\widetilde{\pi}(t, u, G) : t \geq 0\}.$$

Let $(\mathcal{O} \times \mathcal{F}^*(\Omega,\, h), \pi)$ be a semidynamical system corresponding to problem (17). Then its motion is given by

$$\pi(t, u, G) = (x(t, u, G), G_t),$$

where $x(t, u, G)$ is the unique solution of (17) defined on the whole interval $[0, +\infty)$.

Now, define the mapping

$$\widetilde{\pi} : \mathbb{R}_+ \times \mathcal{O} \times \mathcal{F}^*(\Omega,\, h) \to \mathcal{O} \times \mathcal{F}^*(\Omega,\, h)$$

by

$$\widetilde{\pi}(t, u, G) = \pi(t - t_n, u_n^+, G), \qquad \text{for } t_n \leq t < t_{n+1} \text{ and } n = 0, 1, 2, \ldots, \tag{20}$$

where $u = u_0^+$, $t_0 = 0$ and $t_n = \sum_{i=0}^{n-1} s_i$ with $n \geq 1$. Recall that $s_n = \varphi(u_n^+, G)$, $n = 0, 1, 2, \ldots$.

Note that

$$\widetilde{\pi}(t, u, G) = (\widetilde{x}(t, u, G), G_{t-t_n}),$$

for $t_n \leq t < t_{n+1}$, $n = 0, 1, 2, \ldots$, where $\widetilde{x}(t, u, G)$ is solution of (19).

Theorem 5.1. $\widetilde{\pi}$ *given by* (20) *is an impulsive semidynamical system associated to* (19). *We denote such system by* $(\mathcal{O} \times \mathcal{F}^*(\Omega,\, h), \widetilde{\pi})$.

Proof. The proof of Proposition 2.1 in [4] can be applied to prove conditions a) and b) from Definition 5.1 with obvious modifications. Since $\widetilde{x}(t, u, G)$ and G_t are continuous from the right at every point $t \in [0, +\infty)$ and the left limits $\widetilde{x}(t-, u, G)$ and G_{t-} exist for all $t > 0$, condition c) from Definition 5.1 follows. Hence $(\mathcal{O} \times \mathcal{F}^*(\Omega,\, h), \widetilde{\pi})$ defines an impulsive semidynamical system corresponding to problem (19). □

For details about the theory of impulsive semidynamical systems in the classic ordinary case, the reader may to consult [3]-[8] and also [14].

In the next section, we will present a version of LaSalle's invariance principle for generalized ODEs. In order to state such result, we strongly used the existence of an impulsive semidynamical system $(\mathcal{O} \times \mathcal{F}^*(\Omega,\, h), \widetilde{\pi})$ (Theorem 5.1). Later, we will apply LaSalle's invariance principle for generalized ODEs and the correspondence between impulsive ODEs and generalized ODEs to get a LaSalle's invariance principle for impulsive autonomous ODEs without the need to construct a local discontinuous semiflow.

6. LaSalle's invariance principle

We shall consider that the funcion φ defined in (18) is continuous on $(\mathcal{O}\backslash \mathrm{M})\times\mathcal{F}^*(\Omega,\, h)$. In [7], the reader may find conditions that the impulsive set M must fulfill so that the function φ is continuous.

We start by introducing the concept of a limit set for an impulsive semidynamical system in the frame of generalized systems. Let $(\mathcal{O}\times\mathcal{F}^*(\Omega,\, h),\, \widetilde{\pi})$ be an impulsive semidynamical system as presented in Theorem 5.1. The set of limit points of $\widetilde{\pi}(t,\, u,\, G)$, when $t\to+\infty$, is given by

$$\Omega^+(u,\, G)=\{(u^*,\, G^*)\in\mathcal{O}\times\mathcal{F}^*(\Omega,\, h):\, \widetilde{\pi}(\lambda_n,\, u,\, G)\stackrel{n\to+\infty}{\longrightarrow}(u^*,\, G^*)$$
$$\text{for some sequence of positive real numbers}\quad \lambda_n\stackrel{n\to+\infty}{\longrightarrow}+\infty\}.$$

We call $\Omega^+(u,\, G)$ the *positive limit set* of $\widetilde{\pi}(t,\, u,\, G)$.

A subset Γ of $\mathcal{O}\times\mathcal{F}^*(\Omega,\, h)$ is said to be *positively* $\widetilde{\pi}$*−invariant*, if for any $(v_0,\, G_0)\in\Gamma$, we have $\widetilde{\pi}(t,\, v_0,\, G_0)\in\Gamma$ for every $t\in[0,\, +\infty)$.

In the following lines, we will prove that the limit set $\Omega^+(u,\, G)$ is positively $\widetilde{\pi}$-invariant. But at first, let us present an auxiliary lemma which is a version of Lemma 2.3 from [14] for the impulsive system $(\mathcal{O}\times\mathcal{F}^*(\Omega,\, h),\, \widetilde{\pi})$. The proof follows analogously and we include it here for the sake of self-containedness of this paper.

Lemma 6.1. *Let* $(\mathcal{O}\times\mathcal{F}^*(\Omega,\, h),\, \widetilde{\pi})$ *be the impulsive semidynamical system corresponding to* (19). *Suppose* $u\in\mathcal{O}\setminus\mathrm{M}$ *and* $\{v_n\}_{n\geq 1}$ *is a sequence in* $\mathcal{O}$ *which converges to the initial value* u *of* (19). *Let* $\{G_n\}_{n\geq 1}$ *be a sequence in* $\mathcal{F}^*(\Omega,\, h)$ *such that* $G_n\stackrel{n\to+\infty}{\longrightarrow}G$. *Then, for any* $t\geq 0$, *there exists a sequence of real numbers* $\{\varepsilon_n\}_{n\geq 1}$, *with* $\varepsilon_n\stackrel{n\to+\infty}{\longrightarrow}0$, *such that*

$$\widetilde{\pi}(t+\varepsilon_n, v_n, G_n)\stackrel{n\to+\infty}{\longrightarrow}\widetilde{\pi}(t, u, G).$$

Proof. For each $n\in\mathbb{N}$, $n\geq 1$, let $x(t, v_n, G_n)$ be the solution of problem

$$\begin{cases}\dfrac{dx}{d\tau}=DG_n(x, s),\\ x(0)=v_n,\end{cases}\tag{21}$$

defined for all $t\geq 0$. By Proposition 4.2, we have

$$x(t, v_n, G_n)\stackrel{n\to+\infty}{\longrightarrow}x(t,\, u,\, G),$$

where $x(t, u, G)$ is the solution of (17).

Since $G_n\stackrel{n\to+\infty}{\longrightarrow}G$, the sequence of translates $(G_n)_t$ of G_n also converges to the translate G_t of G. Thus

$$\pi(t, v_n, G_n)\stackrel{n\to+\infty}{\longrightarrow}\pi(t, u, G),$$

for each $t\geq 0$.

If $\varphi(u, G) = +\infty$ the result follows. Suppose $\varphi(u, G) < +\infty$.

In the sequel, we use some ideas borrowed from [14], Lemma 2.3, to prove the result.

At first, suppose $0 \le t < s_0$, $s_0 = \varphi(u, G)$. By the continuity of φ on $(\mathcal{O} \setminus \mathrm{M}) \times \mathcal{F}^*(\Omega, h)$, given $\varepsilon > 0$, $\varepsilon < s_0 - t$, there exists a natural number n_0 such that $-\varepsilon < \varphi(v_n, G_n) - \varphi(u, G)$ for all $n \ge n_0$. Thus for $n \ge n_0$, we have $t < s_0 - \varepsilon < \varphi(v_n, G_n)$ and then

$$\widetilde{\pi}(t, v_n, G_n) = \pi(t, v_n, G_n) \overset{n\to+\infty}{\longrightarrow} \pi(t, u, G) = \widetilde{\pi}(t, u, G).$$

In this case, take $\varepsilon_n = 0$ for each natural number $n = 1, 2, \ldots$. Thus $\widetilde{\pi}(t + \varepsilon_n, v_n, G_n) \overset{n\to+\infty}{\longrightarrow} \widetilde{\pi}(t, u, G)$.

Now, suppose $t = s_0$. Let $\varepsilon_n = \varphi(v_n, G_n) - \varphi(u, G)$. Then,

$$\widetilde{\pi}(t + \varepsilon_n, v_n, G_n) = \widetilde{\pi}(\varphi(v_n, G_n), v_n, G_n) = \pi(0, \mathrm{I}((v_n)_1), G_n),$$

where $(v_n)_1 = x(\varphi(v_n, G_n), v_n, G_n)$, $n = 1, 2, \ldots$.

Since $\mathrm{I}((v_n)_1) \overset{n\to+\infty}{\longrightarrow} \mathrm{I}(u_1)$, we have

$$\widetilde{\pi}(t + \varepsilon_n, v_n, G_n) = \pi(0, \mathrm{I}((v_n)_1), G_n) \overset{n\to+\infty}{\longrightarrow} \pi(0, u_1^+, G) = \widetilde{\pi}(t, u, G).$$

But if $t > \varphi(u, G)$, then $t = \sum_{i=0}^{m-1} s_i + t'$ for some $m \in \mathbb{N}^*$ and $0 \le t' < s_m$.

Let $t_n = \sum_{i=0}^{m-1} \varphi((v_n)_i^+, G_n)$, where $(v_n)_0^+ = v_n$, $(v_n)_i = x(\varphi((v_n)_{i-1}^+, G_n), (v_n)_{i-1}^+, G_n)$ and $\mathrm{I}((v_n)_i) = (v_n)_i^+$for $1 \le i \le m - 1$. Then

$$\widetilde{\pi}(t_n, v_n, G_n) = ((v_n)_m^+, G_n) \overset{n\to+\infty}{\longrightarrow} (u_m^+, G).$$

Define $\varepsilon_n = t_n + t' - t$, $n = 1, 2, \ldots$. Since $u_m^+ \notin \mathrm{M}$ (because $\mathrm{I}(\mathrm{M}) \cap \mathrm{M} = \emptyset$ and $t' < s_m = \varphi(u_m^+, G)$), it follows by the previous case that

$$\widetilde{\pi}(t + \varepsilon_n, v_n, G_n) = \widetilde{\pi}(t', \widetilde{\pi}(t_n, v_n, G_n)) \overset{n\to+\infty}{\longrightarrow} \widetilde{\pi}(t', u_m^+, G) = \widetilde{\pi}(t, u, G)$$

and this completes the proof. □

By applying Lemma 6.1, the proof of Proposition 6.1 follows straightforwardly.

Proposition 6.1. *Suppose $\Omega^+(u, G) \cap (\mathrm{M} \times \mathcal{F}^*(\Omega, h)) = \emptyset$. Then $\Omega^+(u, G)$ is positively $\widetilde{\pi}$−invariant.*

The next result gives us a sufficient condition under which $\Omega^+(u, G)$ is a non-empty set.

Proposition 6.2. *Let $(\mathcal{O} \times \mathcal{F}^*(\Omega, h), \widetilde{\pi})$ be the impulsive semidynamical system corresponding to (19). If $\widetilde{x}(t, u, G)$ remains in a compact subset C of $\mathcal{O}$ for all $t \in [0, +\infty)$, then $\Omega^+(u, G)$ is non-empty.*

Proof. Let $\{\lambda_n\}_{n\ge1}$ be a sequence of positive real numbers such that $\lambda_n \overset{n\to+\infty}{\longrightarrow} +\infty$. For each natural number n, let $p(n) \in \mathbb{N}^*$ be such that $t_{p(n)} \le \lambda_n < t_{p(n)+1}$, where $t_{p(n)} = \sum_{i=0}^{p(n)-1} s_i$. Then

$$\widetilde{\pi}(\lambda_n, u, G) = \pi(\lambda_n - t_{p(n)}, u_{p(n)}^+, G) = (x(\lambda_n - t_{p(n)}, u_{p(n)}^+, G), G_{\lambda_n - t_{p(n)}}).$$

By the compactness of $\mathcal{C}$, the sequence $\{\widetilde{x}(\lambda_n, u, G)\}_{n\geq 1}$ admits a convergent subsequence, say,

$$\widetilde{x}(\lambda_{n_k}, u, G) = x(\lambda_{n_k} - t_{p(n_k)}, u^+_{p(n_k)}, G) \overset{k\to+\infty}{\longrightarrow} u^* \in \mathcal{C}.$$

Also, since $\mathcal{F}^*(\Omega, h)$ is compact (see Remark 4.2), the sequence $\{G_{\lambda_{n_k}-t_{p(n_k)}}\}_{k\geq 1}$ admits a convergent subsequence. Then we can assume that $G_{\lambda_{n_k}-t_{p(n_k)}} \overset{k\to+\infty}{\longrightarrow} G^*$ in $\mathcal{F}^*(\Omega, h)$. Thus,

$$\widetilde{\pi}(\lambda_{n_k}, u, G) \overset{k\to+\infty}{\longrightarrow} (u^*, G^*)$$

and since $\lambda_{n_k} \overset{k\to+\infty}{\longrightarrow} +\infty$, we have $(u^*, G^*) \in \Omega^+(u, G)$. □

Now, we present the concept of a Lyapunov function, defined in $\mathcal{O}\times\mathcal{F}^*(\Omega, h)$, with respect to the impulsive semidynamical system $(\mathcal{O} \times \mathcal{F}^*(\Omega, h), \widetilde{\pi})$.

Definition 6.1. *A nonnegative function $V : \mathcal{O} \times \mathcal{F}^*(\Omega, h) \to \mathbb{R}_+$ satisfying the conditions*

i) *V is continuous on $\mathcal{O} \times \mathcal{F}^*(\Omega, h)$,*

ii) *$\dot{V}(u, G) \leq 0$ for $(u, G) \in \mathcal{O} \times \mathcal{F}^*(\Omega, h)$, where*

$$\dot{V}(u, G) = \limsup_{h\to 0^+} \frac{V(\widetilde{\pi}(h, u, G)) - V(u, G)}{h},$$

is called a Lyapunov function *associated to the impulsive semidynamical system $(\mathcal{O}\times\mathcal{F}^*(\Omega, h), \widetilde{\pi})$.*

Item ii) in Definition 6.1 implies that $V(\widetilde{\pi}(t, u, G)) \leq V(u, G)$ for every $t \geq 0$.

The next result is a version of LaSalle's invariance principle. Its proof follows some ideas of [6], Theorem 3.1.

Theorem 6.1 (LaSalle's Invariance Principle). *Let $(\mathcal{O}\times\mathcal{F}^*(\Omega, h), \widetilde{\pi})$ be the impulsive semidynamical system corresponding to system (19). Suppose $\widetilde{x}(t, u, G)$ remains in a compact subset C of $\mathcal{O}$ for all $t \in [0, +\infty)$. Let $V : \mathcal{O} \times \mathcal{F}^*(\Omega, h) \to \mathbb{R}_+$ be a Lyapunov function as defined in Definition 6.1. Define $E = \{z \in \mathcal{O} \times \mathcal{F}^*(\Omega, h) : \dot{V}(z) = 0\}$. Let W be the largest set in E which is positively $\widetilde{\pi}$-invariant. If $\Omega^+(u, G) \cap (\mathrm{M} \times \mathcal{F}^*(\Omega, h)) = \emptyset$, then $\Omega^+(u, G)$ is contained in W.*

Proof. By Proposition 6.2 the positive limit set $\Omega^+(u, G)$ is nonempty. Let $(u^*, G^*) \in \Omega^+(u, G)$. We have two cases to consider: when $\Omega^+(u, G)$ is a singleton and otherwise.

Suppose $\Omega^+(u, G)$ is a singleton, that is, $\Omega^+(u, G) = \{(u^*, G^*)\}$. By Proposition 6.1, the set $\Omega^+(u, G)$ is positively $\widetilde{\pi}-$invariant. Then $\widetilde{\pi}(t, u^*, G^*) = (u^*, G^*)$ for all $t \geq 0$. Hence $\dot{V}(u^*, G^*) = 0$ and $\Omega^+(u, G) \subset E$. Since W is the largest set in E which is positively $\widetilde{\pi}-$invariant, we have $\Omega^+(u, G) \subset W$.

Now, suppose $\Omega^+(u, G)$ is not a singleton. Let $(u_1, G_1), (u_2, G_2) \in \Omega^+(u, G)$. Then there are sequences $\{\lambda_n\}_{n\geq 1}$ and $\{\kappa_n\}_{n\geq 1}$ of positive real numbers $\lambda_n \overset{n\to+\infty}{\longrightarrow} +\infty$ and $\kappa_n \overset{n\to+\infty}{\longrightarrow} +\infty$ such that

$$\widetilde{\pi}(\lambda_n, u, G) \overset{n\to+\infty}{\longrightarrow} (u_1, G_1) \quad \text{and} \quad \widetilde{\pi}(\kappa_n, u, G) \overset{n\to+\infty}{\longrightarrow} (u_2, G_2).$$

We can choose subsequences such that $\lambda_{n_k} \leq \kappa_{n_k}$, $k = 1, 2, \ldots$. Then

$$V(\tilde{\pi}(\kappa_{n_k}, u, G)) \leq V(\tilde{\pi}(\lambda_{n_k}, u, G)). \tag{22}$$

Since V is continuous, when $k \to +\infty$ in (22), we have $V(u_2, G_2) \leq V(u_1, G_1)$. On the other hand, we can choose subsequences $\{\kappa_{n_m}\}$ and $\{\lambda_{n_m}\}$ such that $\kappa_{n_m} \leq \lambda_{n_m}$, $m = 1, 2, \ldots$, and then $V(u_1, G_1) \leq V(u_2, G_2)$. Hence $V(u_1, G_1) = V(u_2, G_2)$, that is, $V(u^*, G^*)$ is equal to a constant for every $(u^*, G^*) \in \Omega^+(u, G)$. Consequently $\dot{V}(u^*, G^*) = 0$ for every $(u^*, G^*) \in \Omega^+(u, G)$, since $\Omega^+(u, G)$ is positively $\tilde{\pi}$−invariant. Therefore $\Omega^+(u, G) \subset W$ and we finished the proof. □

7. An application

The aim of this section is to present a version of LaSalle's invariance principle for autonomous ordinary differential equations with impulses at variable times. Clearly the principle also applies in the case where there is absence of impulses.

Let us consider the following initial value problem

$$\begin{cases} \dot{x} = f(x), \\ \mathrm{I} : \mathrm{M} \to \mathrm{N}, \\ x(0) = x_0, \end{cases} \tag{23}$$

where $\dot{x} = \frac{dx}{dt}$, $f : \mathcal{O} \to \mathbb{R}^n$, $\mathcal{O}$ is an open set of $\mathbb{R}^n$, M is a closed subset of $\mathbb{R}^n$ and $\mathrm{I} : \mathrm{M} \to \mathrm{N}$ is a continuous mapping called the impulse operator such that $\mathrm{I}(\mathrm{M}) \cap \mathrm{M} = \emptyset$. The reader may find the fundamental theory related to (23) in [16], for instance.

We denote the solution of (23) by $x(t, x_0, f)$.

We shall also consider that M satisfies the following condition: if $x(t_0, x_0, f) \in \mathrm{M}$ for some $t_0 > 0$, where $x(t) = x(t, x_0, f)$ is solution of (23), then there exists an $\varepsilon > 0$ such that

$$x(t, x_0, f) \notin \mathrm{M}, \quad \text{for} \quad t \in (t_0 - \varepsilon, t_0) \cup (t_0, t_0 + \varepsilon).$$

This means that the solution x of (23) touches M only at isolated points.

Let us denote by $\mathcal{A}$ the set of all functions $f : \mathcal{O} \to \mathbb{R}^n$ which satisfy the following conditions:

(A) there is a positive constant $K > 0$ such that for all $x \in \mathcal{O}$,

$$\|f(x)\| \leq K;$$

(B) there is a positive constant $L > 0$ such that for all $x, y \in \mathcal{O}$,

$$\|f(x) - f(y)\| \leq L\|x - y\|.$$

We define a function $\phi : \mathcal{O} \times \mathcal{A} \to (0, +\infty]$ associated to system (23) by

$$\phi(u, f) = \begin{cases} s, & \text{if } x(s, u, f) \in \mathrm{M} \text{ and } x(t, u, f) \notin \mathrm{M} \text{ for } 0 < t < s, \\ +\infty, & \text{if } x(t, u, f) \notin \mathrm{M} \text{ for all } t > 0. \end{cases}$$

Lemma 7.1. *Assume $f \in \mathcal{A}$. For each $x \in \mathcal{O}$ and each $t \geq 0$, define*

$$F(x,t) = f(x)t.$$

Then $F \in \mathcal{F}^*(\Omega, h)$, where $\Omega = \mathcal{O} \times [0, +\infty)$ and $h(t) = (K+L)t$.

Proof. At first, note that h is increasing, continuous and

$$|h(t+s_2) - h(t+s_1)| = |(K+L)(s_2 - s_1)| = |h(s_2) - h(s_1)|,$$

for every $s_1, s_2, t \in [0, +\infty)$.

For each $x \in \mathcal{O}$, we also have $F(x,0) = 0$. Moreover,

$$\begin{aligned} \|F(x,s_2) - F(x,s_1)\| &= \|f(x)\| \, |s_2 - s_1| \leq K|s_2 - s_1| \leq \\ &\leq (K+L)|s_2 - s_1| = |h(s_2) - h(s_1)|, \end{aligned}$$

for all (x, s_2), $(x, s_1) \in \Omega$, and

$$\begin{aligned} \|F(x,s_2) - F(x,s_1) - F(y,s_2) + F(y,s_1)\| &= \|(f(x) - f(y))(s_2 - s_1)\| \leq \\ &\leq L\,\|x-y\|\,|s_2 - s_1| \leq (K+L)\,\|x-y\|\,|s_2 - s_1| \\ &= \|x-y\|\,|h(s_2) - h(s_1)|, \end{aligned}$$

for all (x, s_2), (x, s_1), (y, s_2), $(y, s_1) \in \Omega$. Therefore $F \in \mathcal{F}^*(\Omega, h)$. □

Let $x(t, x_0, f)$ be a solution of (23) defined on $[0, +\infty)$. The set of all *limit points* of $x(t, x_0, f)$, when $t \to +\infty$, is defined by

$$\begin{aligned} \omega(x_0, f) = \{x^* \in \mathcal{O} : x(\lambda_n, x_0, f) \overset{n \to +\infty}{\longrightarrow} x^*, \text{ for some sequence} \\ \text{of positive real numbers } \{\lambda_n\}_{n \geq 1} \text{ such that } \lambda_n \overset{n \to +\infty}{\longrightarrow} +\infty\}. \end{aligned}$$

We call $\omega(x_0, f)$ the *ω-limit set* of the solution $x(t, x_0, f)$.

A subset A of $\mathcal{O}$ is said to be *positively invariant* with respect to system (23), if for each $x_0 \in A$, $x(t, x_0, f) \in A$ for every $t \in [0, +\infty)$.

The following result follows similarly as Lemma 6.1. It will help us prove that, under certain conditions, the ω-limit set of $x(t, x_0, f)$ is positively invariant.

Lemma 7.2. *Suppose $x_0 \in \mathcal{O} \setminus \mathrm{M}$ and $\{x_n\}_{n \geq 1}$ is a sequence in $\mathcal{O}$ which converges to the point x_0. Let $x(t) = x(t, x_n, f)$, $n = 0, 1, 2, \ldots$, be the solution of*

$$\begin{cases} \dot{x} = f(x), \\ \mathrm{I} : \mathrm{M} \to \mathrm{N}, \\ x(0) = x_n, \end{cases}$$

defined on the interval $[0, +\infty)$. Then, for any $t \geq 0$, there exists a sequence of real numbers $\{\varepsilon_n\}_{n \geq 1}$, with $\varepsilon_n \overset{n \to +\infty}{\longrightarrow} 0$, such that

$$x(t + \varepsilon_n, x_n, f) \overset{n \to +\infty}{\longrightarrow} x(t, x_0, f).$$

Lemma 7.3. *Given $x_0 \in \mathcal{O}$, suppose $\omega(x_0, f) \cap \mathrm{M} = \emptyset$. Then the ω-limit set $\omega(x_0, f)$ is positively invariant with respect to system* (23).

Proof. Let $x^* \in \omega(x_0, f)$, then there exists a sequence $\{\lambda_n\}_{n\geq 1} \subset \mathbb{R}_+$, $\lambda_n \stackrel{n\to+\infty}{\longrightarrow} +\infty$, such that

$$x(\lambda_n, x_0, f) \stackrel{n\to+\infty}{\longrightarrow} x^* \in \mathcal{O}.$$

By Lemma 7.2, there exists a sequence of real numbers $\{\varepsilon_n\}_{n\geq 1}$, with $\varepsilon_n \stackrel{n\to+\infty}{\longrightarrow} 0$, such that

$$x(t+\varepsilon_n, x(\lambda_n, x_0, f), f) \stackrel{n\to+\infty}{\longrightarrow} x(t, x^*, f) \in \mathcal{O},$$

for any $t \geq 0$. Since $x(t+\varepsilon_n, x(\lambda_n, x_0, f), f) = x(\lambda_n + \varepsilon_n + t, x_0, f)$ (because the system is autonomous) and $t + \varepsilon_n + \lambda_n \stackrel{n\to+\infty}{\longrightarrow} +\infty$, the result follows. □

Now we define a Lyapunov functional $U : \mathcal{O} \times \mathcal{A} \to \mathbb{R}_+$ with respect to equation (23).

Definition 7.1. *A nonnegative function $U : \mathcal{O} \times \mathcal{A} \to \mathbb{R}_+$ is said to be a* Lyapunov function *associated to system* (23)*, if the following properties hold:*

(i) *U is continuous on $\mathcal{O} \times \mathcal{A}$,*
(ii) *$\dot{U}(x_0, f) \leq 0$, for $(x_0, f) \in \mathcal{O} \times \mathcal{A}$, where*

$$\dot{U}(x_0, f) = \limsup_{\eta\to 0^+} \frac{U(x(\eta, x_0, f), f) - U(x_0, f)}{\eta}.$$

Now we are able to present LaSalle's Invariance Principle for an impulsive autonomous ordinary system.

Theorem 7.1 (LaSalle's Invariance Principle). *Suppose $x(t) = x(t, x_0, f)$ stays in a compact subset of $\mathcal{O}$ for all $t \in [0, +\infty)$, where $x(t) = x(t, x_0, f)$ is the solution of* (23) *with $f \in \mathcal{A}$. Suppose U is a Lyapunov function as defined in Definition 7.1. Define $H_f = \{\widetilde{x} \in \mathcal{O} : \dot{U}(\widetilde{x}, f) = 0\}$ and let $\mathcal{N}$ be the largest set in H_f which is positively invariant. If $\omega(x_0, f) \cap \mathrm{M} = \emptyset$, then $\omega(x_0, f) \subset \mathcal{N}$.*

Proof. Let $z(t)$ be a solution of the non-impulsive system

$$\begin{cases} \dot{x} = f(x), \\ x(0) = x_0. \end{cases}$$

Clearly z satisfies

$$z(t) = x_0 + \int_0^t f(z(s))ds, \quad t \geq 0.$$

By defining

$$F(x, t) = f(x)t, \qquad (x, t) \in \mathcal{O} \times [0, +\infty),$$

then $z(t)$ is also a solution of the generalized differential equation

$$\begin{cases} \dfrac{dx}{d\tau} = DF(x, t), \\ x(0) = x_0. \end{cases} \tag{24}$$

This fact is easy to check (see the Introduction of this paper).

By Lemma 7.1, $F \in \mathcal{F}^*(\Omega, h)$, where $h(t) = (K+L)t$ and $\Omega = \mathcal{O} \times [0, +\infty)$.

By Theorem 4.1, the mapping $\pi : [0, +\infty) \times \mathcal{O} \times \overline{\mathcal{F}}(\Omega, h) \to \mathcal{O} \times \mathcal{F}^*(\Omega,\, h)$ given by

$$\pi(t,\, x_0,\, F) = (z(t,\, x_0,\, F),\, F_t), \tag{25}$$

is a semidynamical system on $\mathcal{O} \times \mathcal{F}^*(\Omega,\, h)$ associated to system (24). By Theorem 5.1, $(\mathcal{O} \times \mathcal{F}^*(\Omega,\, h), \widetilde{\pi})$ is the impulsive semidynamical system associated to the system

$$\begin{cases} \dfrac{dx}{d\tau} = DF(x,\, t) \\ \mathrm{I} : \mathrm{M} \to \mathrm{N} \\ x(0) = x_0 \end{cases} \tag{26}$$

where the associated impulsive semiflow is given by

$$\widetilde{\pi}(t,\, x_0,\, F) = (x(t,\, x_0,\, F),\, F_{t-t_n}),$$

with $x(t) = \widetilde{z}(t)$ being the unique maximal solution of (23), for $t_n \leq t < t_{n+1}$, $n = 0, 1, 2, \ldots$ with $t_0 = 0$ and $t_{n+1} = \sum_{i=0}^{n} s_i$, $n = 0, 1, 2, \ldots$ (recall that $s_i = \varphi((x_0)_i^+, F)$).

Define $V : \mathcal{O} \times \mathcal{F}^*(\Omega,\, h) \to \mathbb{R}_+$ by

$$V(x(t), F) = U(x(t), f), \quad t \geq 0.$$

Note that

$$F_h(x, t) = F(x, t+h) - F(x, h) = f(x)t = F(x, t),$$

for all $x \in \mathcal{O}$ and for all $t, h \geq 0$. Then, $V(x(t), F_h) = V(x(t), F) = U(x(t), f)$ for all $h \geq 0$. Therefore,

$$\begin{aligned} \dot{V}(x(t),\, F) &= \limsup_{h \to 0+} \frac{V(x(h,\, x(t),\, F),\, F_h) - V(x(t),\, F)}{h} \\ &= \limsup_{h \to 0+} \frac{U(x(h,\, x(t),\, f), f) - U(x(t),\, f)}{h} \\ &= \dot{U}(x(t), f). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} V(\mathrm{I}(u_1), F) &= V(u_1^+, F) = V(x(\varphi(u, F), u, F), F) = \\ &= V(x(\varphi(u, F), u, F), F_{\varphi(u,F)}) = \\ &= V(\pi(\varphi(u, F), u, F)), \end{aligned}$$

for each $(u, F) \in \mathcal{O} \times \mathcal{F}^*(\Omega,\, h)$.

Now, set

$$E = \{\nu \in \mathcal{O} \times \mathcal{F}^*(\Omega,\, h) : \dot{V}(\nu) = 0\}$$

and let $W \subset E$ be the largest set in E which is positively $\widetilde{\pi}$−invariant. Since $\omega(x_0, f) \cap \mathrm{M} = \emptyset$, then $\Omega^+(x_0,\, F) \cap (\mathrm{M} \times \mathcal{F}^*(\Omega,\, h)) = \emptyset$. Thus, by Theorem 6.1, $\Omega^+(x_0,\, F) \subset W$.

We claim that $\omega(x_0, f) \subset \mathcal{N}$. Indeed. Let $x^* \in \omega(x_0, f)$. Then there exists a sequence of positive real numbers $\{\lambda_n\}_{n\geq 1}$, $\lambda_n \stackrel{n\to+\infty}{\longrightarrow} +\infty$, such that

$$x(\lambda_n) = x(\lambda_n, x_0, f) \stackrel{n\to+\infty}{\longrightarrow} x^*.$$

Note that $V(x(\lambda_n), F_{\lambda_n}) = U(x(\lambda_n), f)$. By the compactness of $\mathcal{F}^*(\Omega, h)$, we can assume that there exists $F^* \in \mathcal{F}^*(\Omega, h)$ such that $F_{\lambda_n} \stackrel{n\to+\infty}{\longrightarrow} F^*$. Then

$$\limsup_{n\to+\infty} V(x(\lambda_n), F_{\lambda_n}) = \limsup_{n\to+\infty} U(x(\lambda_n), f),$$

that is,

$$\dot{U}(x^*, f) = \dot{V}(x^*, F^*).$$

Since $(x^*, F^*) \in \Omega^+(x_0, F)$, it follows from Theorem 6.1 that $\dot{V}(x^*, F^*) = 0$. Then

$$\dot{U}(x^*, f) = \dot{V}(x^*, F^*) = 0$$

and hence $x^* \in H_f$. Consequently $\omega(y_0, f) \subset H_f$.

Since $\omega(y_0, f)$ is positively invariant and $\mathcal{N}$ is the largest set in H_f which is positively invariant, it follows that $\omega(y_0, f) \subset \mathcal{N}$ and this completes the proof. □

8. Appendix

The main goal of this concluding part of our paper is to prove Proposition 8.1 which is Proposition 4.1 in a setting of arbitrary Banach-space valued functions.

Let $(X, \|\cdot\|)$ be a Banach space. We denote by $BV([0, +\infty), X)$ the space of functions $f : [0, +\infty) \to X$ which are locally of bounded variation, that is, for each compact interval $[a, b] \subset [0, +\infty)$, the restriction of f to $[a, b]$, $f|_{[a,b]}$, is of bounded variation. In $BV([a, b], X)$, we consider the usual norm given by $\|f\|_{BV([a,b])} = \|f(a)\| + \text{var}_b^a f$, where $\text{var}_b^a f$ stands for the variation of f in the interval $[a, b]$.

We assume that $\Omega = \mathcal{O} \times [c, d]$, where $\mathcal{O} \subset X$ is open. Moreover, we assume that the function h is nondecreasing and continuous from the left on $[0, +\infty)$ and the sequence

$$0 < t_1 < t_2 < \ldots < t_k < \ldots,$$

with $t_k \stackrel{k\to+\infty}{\longrightarrow} +\infty$, represents the points of discontinuity of h. The next results hold in the case where there is absence of impulses.

Lemma 8.1. *Assume that each $G_k : \mathcal{O} \times [c, d] \to X$, $k = 0, 1, 2, \ldots$, belongs to the class $\mathcal{F}(\Omega, h)$, where h is nondecreasing and left continuous. Let $G_k \stackrel{k\to+\infty}{\longrightarrow} G_0$ in $\mathcal{F}(\Omega, h)$. Let $\psi_k \in G([c, d], X)$, $k = 1, 2, \ldots$ be such that*

$$\|\psi_k - \psi_0\|_\infty = \sup_{c\leq t\leq d} |\psi_k(t) - \psi_0(t)| \stackrel{k\to+\infty}{\longrightarrow} 0,$$

where $[c, d] \subset [0, +\infty)$. Then

$$\left\| \int_c^d DG_k(\psi_k(\tau), s) - \int_c^d DG_0(\psi_0(\tau), s) \right\| \stackrel{k\to+\infty}{\longrightarrow} 0.$$

Proof. First of all, note that $\psi_0 \in G([c, d], X)$, because ψ_0 is, by assumption, the uniform limit of regulated functions on $[c, d]$.

By Lemma 2.7, all the integrals $\int_c^d DG_k(\psi_k(\tau), s)$, $k = 0, 1, 2, \dots$ exist.

Assume that $\varepsilon > 0$ is given. Then the regulated function $\psi_0 \in G([c, d], X)$ can be uniformly approximated by a step function, that is there is a step function $y : [c, d] \to X$ such that

$$\|y - \psi_0\|_\infty = \sup_{c \le t \le d} \|y(t) - \psi_0(t)\| < \varepsilon.$$

Since $\|\psi_k - \psi_0\|_\infty \overset{k \to +\infty}{\longrightarrow} 0$, there exists a positive integer N_0, such that

$$\|\psi_k - \psi_0\|_\infty < \varepsilon,$$

for all $k > N_0$.

Assume that $k > N_0$. Then we have

$$\left\| \int_c^d DG_k(\psi_k(\tau), s) - \int_c^d DG_0(\psi_0(\tau), s) \right\| \le \left\| \int_c^d D[G_k(\psi_k(\tau), s) - G_k(\psi_0(\tau), s)] \right\| + \\ + \left\| \int_c^d D[G_k(\psi_0(\tau), s) - G_k(y(\tau), s)] \right\| + \left\| \int_c^d D[G_k(y(\tau), s) - G_0(y(\tau), s)] \right\| + \\ + \left\| \int_c^d D[G_0(y(\tau), s) - G_0(\psi_0(\tau), s)] \right\|. \tag{27}$$

Let us consider the first summand on the righthand side of the inequality in (27).

Let δ be a gauge defined in $[c, d]$ corresponding to $\varepsilon > 0$ in the definition of the integral $\int_c^d D[G_k(\psi_k(\tau), s) - G_k(\psi_0, s)]$ and let $(\tau_i, [s_{i-1}, s_i])_{1 \le i \le p}$ be a δ-fine tagged division of $[c, d]$. Then we have

$$\left\| \int_c^d D[G_k(\psi_k(\tau), s) - G_k(\psi_0(\tau), s)] \right\| \le \left\| \int_c^d D[G_k(\psi_k(\tau), s) - G_k(\psi_0(\tau), s)] - \right. \\ \left. - \sum_{i=1}^p [(G_k(\psi_k(\tau_i), s_i) - G_k(\psi_k(\tau_i), s_{i-1})) - (G_k(\psi_0(\tau_i), s_i) - G_k(\psi_0(\tau_i), s_{i-1}))] \right\| + \\ \left\| \sum_{i=1}^p [(G_k(\psi_k(\tau_i), s_i) - G_k(\psi_k(\tau_i), s_{i-1})) - (G_k(\psi_0(\tau_i), s_i) - G_k(\psi_0(\tau_i), s_{i-1}))] \right\| \le \\ < \varepsilon + \sum_{i=1}^p \|G_k(\psi_k(\tau_i), s_i) - G_k(\psi_k(\tau_i), s_{i-1}) - G_k(\psi_0(\tau_i), s_i) + G_k(\psi_0(\tau_i), s_{i-1})\| \le \\ \overset{(6)}{\le} \varepsilon + \sum_{i=1}^p |\psi_k(\tau_i) - \psi_0(\tau_i)| \, [h(s_i) - h(s_{i-1})] \le \\ \le \varepsilon + \|\psi_k - \psi_0\|_\infty \sum_{i=1}^p [h(s_i) - h(s_{i-1})] \le \\ \le \varepsilon + \varepsilon[h(d) - h(c)] = \varepsilon(1 + [h(d) - h(c)]).$$

For the second and fourth summands on the righthand side of (27), we can show analogously that

$$\left\| \int_c^d D[G_k(\psi_0(\tau),\, s) - G_k(y(\tau),\, s)] \right\| < \varepsilon(1 + [h(d) - h(c)])$$

and

$$\left\| \int_c^d D[G_0(y(\tau),\, s) - G_0(\psi_0(\tau),\, s)] \right\| < \varepsilon(1 + [h(d) - h(c)]).$$

Thus

$$\left\| \int_c^d DG_k(\psi_k(\tau),\, s) - \int_c^d DG_0(\psi_0(\tau),\, s) \right\| <$$

$$< 3\varepsilon(1 + [h(d) - h(c)]) + \left\| \int_c^d D[G_k(y(\tau),\, s) - G_0(y(\tau),\, s)] \right\|.$$

Let us now consider the integral $\int_c^d D[G_k(y(\tau),\, s) - G_0(y(\tau),\, s)]$.

Since $y : [c,d] \to X$ is a step function, there is a finite number $p \in \mathbb{N}$ of points $c = r_0 < r_1 < r_2 < \cdots < r_{p-1} < r_p = d$ such that for $\tau \in (r_{j-1}, r_j)$, $j = 1, 2, \ldots, p$, we have $y(\tau) = c_j \in X$ (y assumes a constant value c_j in each open interval (r_{j-1}, r_j), $j = 1, 2, \ldots, p$). In this case, an explicit formula for the integral $\int_c^d DG_k(y(\tau),\, s)$, for every $k = 0, 1, 2, \ldots$, can be given, namely,

$$\int_c^d DG_k(y(\tau),\, s) = \sum_{j=1}^{p} \int_{r_{j-1}}^{r_j} DG_k(y(\tau),\, s)$$

and (using the Hake-type theorem given in Lemma 2.3 and a properly chosen partition of $[r_{j-1}, r_j]$)

$$\int_{r_{j-1}}^{r_j} DG_k(y(\tau), t) = G_k(c_j, r_j-) - G_k(c_j, r_{j-1}+) + G_k(y(r_{j-1}), r_{j-1}+) -$$

$$- G_k(y(r_{j-1}), r_{j-1}) - G_k(y(r_j), r_j-) + G_k(y(r_j), r_j).$$

Looking at the difference for the sums on the righthand side for G_k and G_0 in the last equality, we get

$$\lim_{k\to\infty} \int_{r_{j-1}}^{r_j} D[G_k(y(\tau), t) - G_0(y(\tau), t)] = 0,$$

because since $G_k \in \mathcal{F}(\Omega, h)$, we have

$$\|G_k(x, t_2) - G_k(x, t_1)\| \leq |h(t_2) - h(t_1)|$$

for every $(x, t_1), (x, t_2) \in \Omega$ and this leads to the conclusion that $\lim_{\rho\to 0+} G_k(x, t+\rho) = G_k(x, t+)$ and $\lim_{\rho\to 0+} G_k(x, t-\rho) = G_k(x, t-)$ for every $(x, t) \in \Omega$ uniformly with respect to $k = 0, 1, \ldots$.

Hence by the assumption that $G_k \overset{k\to+\infty}{\longrightarrow} G_0$, we obtain

$$\begin{aligned}\lim_{k\to+\infty} G_k(x,t+) &= \lim_{k\to+\infty}\lim_{\rho\to 0+} G_k(x,t+\rho) = \\ &= \lim_{\rho\to 0+}\lim_{k\to+\infty} G_k(x,t+\rho) = \\ &= \lim_{\rho\to 0+} G_0(x,t+\rho) = G_0(x,t+)\end{aligned}$$

provided $(x,t)\in\Omega$.

Since $\varepsilon > 0$ can be taken arbitrarily small, we obtain the conclusion of the lemma. □

Corollary 8.1. *Let $G_k \overset{k\to+\infty}{\longrightarrow} G_0$ in $\mathcal{F}(\Omega, h)$. Let $\psi_k \in BV([c, d], X)$, $k = 0,1,2,\ldots$, be such that $\|\psi_k - \psi_0\|_{BV([c,d])} \overset{k\to+\infty}{\longrightarrow} 0$, where $[c, d] \subset [0, +\infty)$. Then*

$$\left\| \int_c^d DG_k(\psi_k(\tau), s) - \int_c^d DG_0(\psi_0(\tau), s)\right\| \overset{k\to+\infty}{\longrightarrow} 0.$$

Proof. Taking into account that $BV([c,d],X) \subset G([c,d],X)$ and that for every $t\in[c,d]$, we have

$$\begin{aligned}\|\psi_k(t)-\psi_0(t)\| &\le \|\psi_k(c)-\psi_0(c)\| + \|\psi_k(t)-\psi_0(t)-(\psi_k(c)-\psi_0(c))\| \le \\ &\le \|\psi_k(c)-\psi_0(c)\| + \mathrm{var}_c^t(\psi_k-\psi_0) \le \\ &\le \|\psi_k(c)-\psi_0(c)\| + \mathrm{var}_c^d(\psi_k-\psi_0) = \|\psi_k-\psi_0\|_{BV([c,d])},\end{aligned}$$

we can easily see that $\|\psi_k-\psi_0\|_\infty \overset{k\to+\infty}{\longrightarrow} 0$ and the result follows from Lemma 8.1. □

Proposition 8.1. *Assume that $G_k : \mathcal{O}\times[c, d] \to X$ belongs to the class $\mathcal{F}(\Omega, h)$, for $k = 0,1,2,\ldots$, $[c,d]\subset[0, +\infty)$ and that*

$$\lim_{k\to+\infty} G_k(x, t) = G_0(x, t),$$

for $(x, t)\in\mathcal{O}\times[c, d]$. Let $[\alpha, \beta]\subset[c, d]$ and $x_k : [\alpha, \beta]\to X$, $k = 1,2,\ldots$, be solutions of the generalized ODE

$$\frac{dx}{d\tau} = DG_k(x, t),$$

on $[\alpha,\beta]$ such that

$$\lim_{k\to+\infty} x_k(s) = x_0(s), \qquad s\in[\alpha, \beta], \tag{28}$$

and $(x(s), s)\in\Omega$ for $s\in[\alpha, \beta]$. Then $x_0 : [\alpha, \beta]\to X$ satisfies:

i) $\|x_0(s_2)-x_0(s_1)\| \le h(s_2)-h(s_1)$, *if $s_1\le s_2$, $s_1, s_2\in[\alpha, \beta]$;*

ii) $\lim_{k\to+\infty} x_k(s) = x_0(s)$ *uniformly on $[\alpha, \beta]$;*

iii) *x_0 is a solution of the generalized ODE $\frac{dx}{d\tau} = DG_0(x, t)$, on $[\alpha,\beta]$.*

Proof. Assume that $\alpha \le s_1 \le s_2 \le \beta$. Then for any $k \in \mathbb{N}$, we have

$$\|x_0(s_2) - x_0(s_1)\| \le \|x_0(s_2) - x_k(s_2)\| + \|x_k(s_2) - x_k(s_1)\| + \|x_k(s_1) - x_0(s_1)\|.$$

Take an arbitrary $\varepsilon > 0$. By (28), there is an $\ell \in \mathbb{N}$ such that, for $k > \ell$, we have

$$\|x_k(s_2) - x_0(s_2)\| < \frac{\varepsilon}{2} \quad \text{and} \quad \|x_k(s_1) - x_0(s_1)\| < \frac{\varepsilon}{2}.$$

Using Lemma 2.5, we have

$$\|x_k(s_2) - x_k(s_1)\| \le h(s_2) - h(s_1), \qquad k = 1, 2, 3, \ldots,$$

and

$$\|x_0(s_2) - x_0(s_1)\| < \varepsilon + h(s_2) - h(s_1).$$

Since $\varepsilon > 0$ can be taken arbitrarily small, we obtain

$$\|x_0(s_2) - x_0(s_1)\| \le h(s_2) - h(s_1)$$

and this implies *i*).

For proving *ii*), let us assume that $[\alpha, \beta]$ does not contain points of discontinuity of the function h, that is, let us suppose $h : [\alpha, \beta] \to \mathbb{R}$ is continuous. By Heine-Borel Theorem, h is uniformly continuous on $[\alpha, \beta]$. This means that for every $\varepsilon > 0$, there is a $\delta > 0$ such that $|h(s) - h(t)| < \varepsilon$, whenever $|s - t| < \delta$.

Let $\varepsilon > 0$ be arbitrary and let $\delta > 0$ correspond to it in the definition of uniform continuity. Then intervals of the form $(t - \delta, t + \delta)$, $t \in [\alpha, \beta]$, cover $[\alpha, \beta]$. Since $[\alpha, \beta]$ is compact, there is a finite set $r_1, \ldots, r_\ell$ such that $[\alpha, \beta]$ is covered by the finite number of intervals $(r_j - \delta, r_j + \delta)$, $j = 1, \ldots, \ell$.

Take $k^* \in \mathbb{N}$ such that (by (28)), for $k > k^*$, we have

$$\|x_k(r_j) - x_0(r_j)\| < \varepsilon,$$

for all $j = 1, \ldots, \ell$. Let $s \in [\alpha, \beta]$ be given. Then there exists $j \in \{1, \ldots, \ell\}$ such that $s \in (r_j - \delta, r_j + \delta)$. Then, for $k > k^*$, we have

$$\begin{aligned} \|x_k(s) - x_0(s)\| &\le \|x_k(s) - x_k(r_j)\| + \|x_k(r_j) - x_0(r_j)\| + \|x_0(r_j) - x_0(s)\| \le \\ &\le |h(s) - h(r_j)| + \varepsilon + |h(s) - h(r_j)| \le 3\varepsilon. \end{aligned}$$

Since this can be done for any $s \in [\alpha, \beta]$, we obtain *ii*) in this case.

Recall that h is nondecreasing, continuous from the left on $[0, +\infty)$ and $0 < t_1 < t_2 < \ldots < t_k < \ldots$, with $t_k \overset{k\to+\infty}{\longrightarrow} +\infty$, is the sequence of points of discontinuity of h. Thus if $[\alpha, \beta] \cap \left(\cup_{k=1}^{+\infty}\{t_k\}\right) \neq \emptyset$, since $t_k \to +\infty$, there exists a finite number of points t_j's in $[\alpha, \beta]$, that is, $\alpha \le t_m < t_{m+1} < \ldots < t_{m+p} \le \beta$.

Assume $\varepsilon > 0$ is given. By (28), there is a $k^* \in \mathbb{N}$ such that, for any $j \in \{m, m+1, \ldots, m+p\}$, we get

$$\|x_k(t_j) - x_0(t_j)\| < \varepsilon,$$

whenever $k > k^*$.

Consider any of the intervals $[\alpha, t_m], [t_m, t_{m+1}], \ldots, [t_{m+p-1}, t_{m+1}], [t_{m+p}, \beta]$. Denote it by $[a, b]$ and define

$$h^*(s) = \begin{cases} h(s), & s \in (a, b] \\ h(a+), & s = a. \end{cases}$$

Then, by the assumptions on h, the function $h^* : [a, b] \to \mathbb{R}$ is nondecreasing and continuous. For all $k = 0, 1, \ldots$, put $x_k^*(a) = x_k(a+)$ and $x_k^*(s) = x_k(s)$, $s \in (a, b]$.

It is easy to see that, by (28), we have

$$\lim_{k\to+\infty} x_k^*(s) = x_0^*(s), \quad s \in (a, b],$$

$$\lim_{k\to+\infty} x_k^*(a) = \lim_{k\to+\infty} x_k(a+) = x_0(a+) = x_0^*(a)$$

and

$$\|x_k^*(s_2) - x_k^*(s_1)\| \leq |h^*(s_2) - h^*(s_1)|,$$

for $a \leq s_1 \leq s_2 \leq b$. Using the previous result, $x_k^* \overset{k\to+\infty}{\longrightarrow} x_0^*$ uniformly on $[a, b]$. Hence, for every $\varepsilon > 0$, there exists $k_* \in \mathbb{N}$ such that

$$\|x_k(s) - x_0(s)\| = \|x_k^*(s) - x_0^*(s)\| < \varepsilon,$$

whenever $s \in (a, b]$, $\|x_k(a) - x_0(a)\| < \varepsilon$, $k > k_*$ and $x_k \overset{k\to+\infty}{\longrightarrow} x_0$ uniformly on $[a, b]$. Since this can be done for every of the finite number of intervals of the type $[a, b]$, we obtain *ii*) and its general form as stated in the conclusion of the theorem.

Now, let us prove *iii*).

By the definition of a solution of the generalized ODE $\frac{dx}{d\tau} = DG_k(x, t)$, $k = 1, 2, 3, \ldots$, we have

$$x_k(s_2) - x_k(s_1) = \int_{s_1}^{s_2} DG_k(x_k(\tau), t) \tag{29}$$

for every $s_1, s_2 \in [\alpha, \beta]$. By Corollary 8.1, we have

$$\left\| \int_{s_1}^{s_2} DG_k(x_k(\tau), s) - \int_{s_1}^{s_2} DG_0(x_0(\tau), s) \right\| \overset{k\to+\infty}{\longrightarrow} 0,$$

for any $s_1, s_2 \in [\alpha, \beta]$. Using (29), we have

$$\left\| x_0(s_2) - x_0(s_1) - \int_{s_1}^{s_2} DG_0(x_0(\tau), t) \right\| \leq$$

$$= \|x_k(s_2) - x_0(s_2)\| + \|x_k(s_1) - x_0(s_1)\| + \left\| \int_{s_1}^{s_2} DG_k(x_k(\tau), s) - \int_{s_1}^{s_2} DG_0(x_0(\tau), s) \right\|.$$

Then, when $k \to +\infty$, we obtain

$$x_0(s_2) - x_0(s_1) = \int_{s_1}^{s_2} DG_0(x_0(\tau), t)$$

for every $s_1, s_2 \in [\alpha, \beta]$. Therefore x is a solution of to the generalized ODE $\frac{dx}{d\tau} = DG_0(x, t)$ on $[\alpha, \beta]$ and we finished the proof. □

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