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SOME POMU* COMMENTS ON LAGRANGIAN DUALITY, OPTIMALITY CONDITIONS AND CONVEXITY

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Abstract

In the classical literature of NLP, the Lagrangian duals (minimax and Wolfe's) are lengthily discussed under various assumptions of convexity and/or regularity. Their relations with optimality conditions (saddle point and stationary point) are also very much exploited. This paper exploits the characteristics of the optimal value function of the dual problems, trying to clarify the essential or non essential role of the usual assumptions, presenting some probably obvious results like the necessity and sufficiency of $v(\cdot)$ being convex and ℓ.s.c. in order to avoid duality gaps in the minimax dual, some characterizations of dual solutions, the role of weak duality theorems and the idea of $f(\cdot)$ uniqueness.

1. Introduction

The classical body of NLP literature lays heavily on optimality conditions and Lagrangian duality. For the case of minimax Lagrangian duality a number of interesting papers has clarified several aspects of its study, since the classical work of Geoffrion [2], passing by Tind and Wolsey [8] and reaching, for instance Flippo et alii [1].

The same drive has not been applied to Wolfe's dual, although its application on models with physical or economic interpretations is quite widespread, as much as stationary

* POMU stands for "probably obvious, maybe useful".

point conditions.

This paper intends to present explictly some results that being probably obvious are not clearly stated, but have the property of identifying which is the role of several usual assumptions, so it is hoped that its comments may be useful.

In section 2, we recall classical definitions, generalize the concept of "no duality gap", and obtain the obvious result that if there is no duality gap then the perturbation function is convex and lower semicontinuous. In section 3, we revisit the minimax dual trying to isolate the role of every usual assumption in the derivation of the related results. A similar task is undergone for Wolfe's dual, in section 4, showing the relations and dissimilarities between these results and stationary point conditions.

2. Basic presentation

We shall use an approach quite similar to Geoffrion's [2], but with enough differences to suggest some initial remarks.

Let us recall some classical definitions:

Definition 2.1. Given $X^o \subset R^n$, $X^o \neq \emptyset$,

$$f : X^o \to R$$

$$g : X^o \to R^m,$$

the *minization problem* (MP) is defined as:

"find, if there exists, $x^* \in X$, such that

$f(x^*) = \min\{f(x) \mid x \in X\}$,

where $X \triangleq \{x \in X^o \mid g(x) \leqq 0\}$."

This definition can be generalized to (MP_y), *minimization problem with RHS y*, for $y \in R^m$, when the feasible set (X) is replaced by $X_y = \{x \in X^o / g(x) \leqq y\}$. ∎

The usage of this more general definition, (MP_y), generates the need of defining:

Definition 2.2. The *feasible perturbations set Y and the perturbation function* $v(\cdot)$ are defined as:

(i) $Y = \{y \in R^m \mid \exists x \in X^o : g(x) \leq y\}$,

(ii) $v : Y \to R \cup \{-\infty\}$ by $v(y) = \inf\{f(x) \mid x \in X_y\}$. ■

It is convenient to note that $(X^o \not\equiv \emptyset \Rightarrow Y \not\equiv \emptyset)$ and $(Y \not\equiv \emptyset \Rightarrow Y \equiv Y + R^m_+$ and $\dot{Y} \not\equiv \emptyset)$.

Associated with (MP) and (MP_y) it is usual to introduce the Lagrangean function and two Lagrangean duals (minimax and Wolfe's) defined as:

Definition 2.3. As in (2.1), we define

(i) *the Lagrangean function* $L(x,u)$ as $L(\cdot,\cdot) : X^o \times R^m_+ \to R$, by $L(x,u) = f(x) + \langle u, g(x) \rangle$;

(ii) the *minimax dual* (D_y), *associated to* (MP_y), for $y \in Y$,
"find, if there exists, $u^* \in U$, such that
$F(u^*) - \langle u^*, y \rangle = \max\{F(u) - \langle u, y \rangle \mid u \in U\}$.
(a) $U = \{u \in R^m_+ \mid \inf\{L(x,u) \mid x \in X^o\} \in R\}$,
(b) $F : U \to R$, by $F(u) = \inf\{L(x,u) \mid x \in X^o\}$;"

(iii) the *Wolfe's dual* (WD_y), *associated to* (MP_y), for $y \in Y$,
"find, if there exists, $(x^*, u^*) \in DU$, such that
$L(x^*, u^*) - \langle u^*, y \rangle = \max\{L(x,u) - \langle u, y \rangle \mid (x,u) \in DU\}$,
where $DU = \{(x,u) \in X^o \times R^m_+ \mid \nabla_x L(x,u) = \nabla f(x) + \sum_{i=1}^m u_i \nabla g_i(x) = 0\}$";*

(iv) the *optimal value functions for the minimax problem* $(w(\cdot))$ and *for the Wolfe's dual* $(wd(\cdot))$, as

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* When discussing Wolfe's dual or any other concept using (DU), we will be assuming $f(\cdot)$ and $g(\cdot)$ continuously differentiable on the open set X^o.

(a) $w : Y \to R \cup \{+\infty, -\infty\}$ by $w(y) = \sup\{F(u) - < u, y > | \, u \in U\}$;

(b) $wd : Y \to R \cup \{+\infty, -\infty\}$ by $wd(y) = \sup\{L(x,u) - < u, y > | \, (x,u) \in DU\}$. ■

It is interesting to note that at this point we can present a strong structural result for the optimal value functions for the two dual problems here analysed:

Fact 2.4) If the dual problem is feasible and bounded, then its optimal value function is convex and lower semicontinuous.

Proof: These are classical results on convex functions and on ℓ.s.c. functions (see for instance C.1.6 and 4.1.13 in Mangasarian [5]). ■

The condition of bounded dual is applied for each $\bar{y} \in Y$, i.e., optimal value of the dual of $(MP_y) \neq +\infty$. For the minimax dual, it will be shown that as a weak duality theorem holds then the dual is bounded for every $y \in Y$. Otherwise, we could state the result using domain of finiteness, which is convex. *From now on we will assume that the dual function is bounded by above in* Y.

Fact 2.5) If the dual problem has a solution for $y = \bar{y}$, then its optimal value function is subdifferentiable at $\bar{y}$ and a subgradient is obtainable from the dual optimal solution.

Proof: Specific proofs will be presented for each of the two dual problems in their corresponding sections. ■

This result becomes much more interesting if compled with the following definition:

Definition 2.6) It is said that there is *no duality gap* for (MP) and its dual (Wolfe's or minimax) if for every $\bar{y} \in Y$, *there is no duality gap at* $\bar{y}$, i.e., the optimal values of $(MP_{\bar{y}})$ and its corresponding dual are equal.

This definition is not the classical one, that would correspond to no duality gap at $\bar{y} = 0$. It is our contention, that such a cardinal result (in the sense that it is based upon a particular value of the RHS) is not the best approach to obtain structural results, that should depend upon assumptions just on X^o, $f(\cdot)$ and $g(\cdot)$, except for some "frontier conditions", like $\bar{y}$ belongs or not to the interior of Y.

This "non-cardinal" definition of "no duality gap" leads to a trivial, but interesting fact:

Fact 2.7) If there is no duality gap for (MP) and some of its Lagrangean dual problems (minimax or Wolfe's), then the perturbation function is convex and lower semicontinous, assuming that there is $\bar{y}$ such that $v(\bar{y}) \in R$.

Proof: It follows directly from the definitions and (2.4). ■

This result may be the key for understanding the reason for convexity assumptions on Lagrangean duality theory and the related topic of optimality conditions, classically atributed to Kuhn and Tucker.

In order to conclude this section, let us present the following definitions:

Definition 2.8) The pair $(\bar{x}, \bar{u}) \in X^o \times R^m$ obbeys the *sadlle point optimality conditions for* $(MP)_y$ if and only if:

(i) $L(\bar{x}, \bar{u}) = \min\{L(x, \bar{u}) \mid x \in X^o\}$;

(ii) $\bar{u} \geq 0$;

(iii) $g(\bar{x}) \leq y$;

(iv) $< \bar{u}, g(\bar{x}) - y >= 0$. ■

Definition 2.9) The pair $(\bar{x}, \bar{u}) \in X^o \times R^m$ obbeys the *stationary point optimality conditions for* $(MP)_y$ if and only if:

(i) $(\overline{x}, \overline{u}) \in DU = \{(x,u) \in X^o \times R^m_+ \ / \ \nabla_x L(x,u) = 0\}$;

(ii) $g(\overline{x}) \leq y$;

(iii) $< \overline{u}, g(\overline{x}) - y >= 0$. ■

3. Minimax duality and saddle point optimality conditions.

The definition of the minimax dual is such that a weak duality theorem is automatically valid:

Fact 3.1) (Weak duality theorem.) **If $\overline{x}$ is feasible in (MP_y) and $\overline{u}$ is feasible in (D_y), then the objective function of the minimization problem evaluated at $\overline{x}$ is not less than the dual objective function evaluated at $\overline{u}$.**

Proof: It follows directly from

$$F(\overline{u}) - < \overline{u}, y > \leq f(\overline{x}) + < \overline{u}, g(\overline{x}) - y > \leq f(\overline{x}).$$ ■

This result is basic for the following verification:

Fact 3.2) $(\overline{x}, \overline{u})$ obbeys the saddle point optimality condition for $(MP)_y$ if and only if

(i) $\overline{x}$ solves (MP_y),

(ii) $\overline{u}$ solves (Dy),

(iii) $w(y) = v(y)$.

Proof: Using the weak duality theorem it is a simple matter of identifying primal feasibility, dual feasibility and complementary slackness. ■

This result leads naturally to the question of existence for solutions for (Dy). In order to characterize this existence it suffices to state:

Fact 3.3) The dual problem (Dy) has a solution at $y = \overline{y}$ if and only if $w(\cdot)$ is subdifferentiable at $y = \overline{y}$. Moreover, $\overline{u}$ is a solution of $(D\overline{y})$ if and only if $(-\overline{u})$ is a subgradient of $w(\cdot)$ at $y = \overline{y}$.

Proof: Obviously, it suffices to prove the last statement.

(a) $\overline{u}$ solves $D\overline{y} \Rightarrow (-\overline{u})$ is a subgradient of $w(\cdot)$ at $y = \overline{y}$.

$w(y) = \sup_{u \geq 0}\{\inf_{x \in X^\circ} f(x) + < u, g(x) - y >\} \geq$ (definition of $w(\cdot)$)

$\geq \inf_{x \in X^\circ} \{f(x) + < \overline{u}, g(x) - y >\} =$ ($\overline{u}$ solves $D\overline{y} \Rightarrow \overline{u} \in U$)

$= w(\overline{y}) + < -\overline{u}, y - \overline{y} >$ $(w(\overline{y}) = \inf_{x \in X^\circ}\{f(x) + < \overline{u}, g(x) - \overline{y} >\})$.

(b) $(-\overline{u})$ is a subgradient of $w(\cdot)$ at $\overline{y} \Rightarrow \overline{u}$ solves $D\overline{y}$.

As $(x \in X^\circ \Rightarrow g(x) \in Y)$ and $w(\cdot)$ is subdifferentiable at $\overline{y}$,

$$w(g(x)) \geq w(\overline{y}) + < -\overline{u}, g(x) - \overline{y} >, \qquad (b1)$$

$$\overline{u} \geq 0 \qquad \text{(as } w(\cdot) \text{ is non-increasing).} \qquad (b2)$$

Analogously, $\forall x \in X^\circ$,

$$f(x) \geq v(g(x)); \qquad (b3)$$

and by the weak duality theorem:

$$v(g(x)) \geq w(g(x)). \qquad (b4)$$

From (b1), (b3), (b4), it follows that

$$\forall x \in X^\circ, \quad f(x) + < \overline{u}, g(x) - \overline{y} > \geq w(\overline{y}),$$

then

$$\inf_{x \in X^\circ} \{(f(x) + < \overline{u}, g(x) - \overline{y} >\} \geq w(\overline{y}). \qquad (b5)$$

Using the facts (b5) and (b2), we can conclude that $\overline{u} \in U$, so

$$w(\overline{y}) \geq \inf\{f(x) + < \overline{u}, g(x) - \overline{y} > \mid x \in X^\circ\}. \qquad (b6)$$

From (b2), (b5) and (b6), the result follows. ■

It is interesting to note that $w(\cdot)$ is convex, so the subdifferentiability is guaranteed in the interior of Y (it was already indicated $\dot{Y} \not\equiv \emptyset$) (the proof of this result is classical).

A stronger result is obtained assuming subdifferentiability of the perturbation function $v(\cdot)$:

Fact 3.4) The dual problem (Dy) has a solution at $y = \overline{y}$ and there is no duality gap at this point if and only if $v(\cdot)$ is subdifferentiable at $\overline{y}$.

Proof:

(a) $\overline{u}$ solves $(D\overline{y})$ and $w(\overline{y}) = v(\overline{y}) \Longrightarrow v(\cdot)$ is subdifferentiable at $y = \overline{y}$.

We can assert that $\forall y \in Y$

$v(y) \geqq w(y)$, by the weak duality theorem

$\geqq w(\overline{y}) + < -\overline{u}, y - \overline{y} >$, by (3.3)

$\geqq v(\overline{y}) + < -\overline{u}, y - \overline{y} >$, as $v(\overline{y}) = w(\overline{y})$.

■

(b) $-\overline{u}$ is a subgradient of $v(\cdot)$ at $y = \overline{y} \Longrightarrow \begin{cases} \overline{u} \text{ solves } (D\overline{y}) \\ w(\overline{y}) = v(\overline{y}) \end{cases}$

In a way completely analogous to (3.3)b, we obtain:

$\overline{u} \in U$ and $F(\overline{u}) \geqq v(\overline{y}) + < \overline{u}, \overline{y} >$.

So

$w(\overline{y}) \geqq F(\overline{u}) - < \overline{u}, \overline{y} > \geqq v(\overline{y}) \geqq w(\overline{y})$,

where the last inequality is derived from the weak duality theorem.

Clearly, these last inequalities imply

$w(\overline{y}) = F(\overline{u}) - < \overline{u}, \overline{y} > = v(\overline{y})$. ■

One important corollary of this result is that

Fact 3.5) If $v(\cdot)$ is convex then there is no duality gap in the interior of the feasible perturbation set.

Proof: It follows from the subdifferentiability of a convex function in an open set and that for $\hat{y} \in \partial Y$, $\forall \varepsilon > 0$, $(\hat{y} + \varepsilon \underline{1}) \in \dot{Y}$, so $\forall \varepsilon > 0$,

$$v(\hat{y}) \geq v(\hat{y} + \varepsilon \underline{1}) \geq v(\overline{y}) + < -\overline{u}, \hat{y} + \varepsilon \underline{1} - \overline{y} >$$

$$\Rightarrow v(\hat{y}) \geq \sup_{\varepsilon > 0} v(\overline{y}) - < \overline{u}, \hat{y} - \overline{y} > + \varepsilon < -\overline{u}, \underline{1} > = v(\overline{y}) + < -\overline{u}, \hat{y} - \overline{y} >,$$

where the last inequality follows from $\overline{u} \geq 0$ (where $\overline{u}$ is the subgradient of $v(\cdot)$ at $\overline{y} \in \dot{Y}$). ■

With these results we can state:

Fact 3.6) There is no duality gap if and only if $v(\cdot)$ is convex and lower semicontinuous.

Proof: The only if part is a repetition of fact 2.7. So we just have to show that

$v(\cdot)$ is convex and lower semicontinuous $\Rightarrow \forall y \in Y, \quad v(y) = w(y)$.

For $y \in \dot{Y}$, the result follows from (3.5).

For $\hat{y} \in \partial Y$, we consider $y^n = (\hat{y} + \frac{1}{n}\underline{1}) \in \dot{Y}$, and we have:

$w(\hat{y})$	$\geq \lim_{n \to \infty} \inf \; w(y^n)$,	as $y^n \geq \hat{y}$ and so $w(y^n) \leq w(\hat{y})$,
	$= \lim_{n \to \infty} \inf \; v(y^n)$,	using (3.5) as $y^n \in \dot{Y}$,
	$\geq v(\hat{y})$,	as $v(\cdot)$ is lower semicontinuous in Y.
	$\geq w(\hat{y})$,	by the weak duality theorem.

So $\quad w(\hat{y}) = v(\hat{y})$, what concludes the proof. ■

This result, coupled with (3.2), allow us to state

Fact 3.7) The saddle point optimality conditions are necessary for all $y \in Y$ if and only if $v(\cdot)$ is convex and subdifferentiable.

Proof: The result is trivial from the previous facts, except that it must be shown that a subdifferentiable funcion in a set is lower semicontinuous in this set and this is trivial. ■

The interest of these results is that we do not only show in an easy manner the only if part of (3.7), but also that is quite clear that there is a part due to "no duality gap" (3.6) and a part due to the existence of a dual solution (3.3). We also note that Geoffrion [2] proved a similar result to (3.6), under the assumption that $v(\cdot)$ is convex and, in this note, we show that convexity is intimately tied with the absence of a duality gap, in the extended sense of (2.6), i.e., for all feasible perturbations (RHS's).

4. Wolfe's duality and stationary point conditions

The definition of the dual due to Wolfe, does not imply automatically the validity of a weak duality theorem. In order to have this result, we must introduce some assumptions, like convexity of all functions ($f(\cdot)$ and $g(\cdot)$). This comment implies that for Wolfe's dual we can state a somehow weaker version of (3.2):

Fact 4.1) If there holds a weak duality theorem for MP_y and WD_y, then:

$$(\overline{x},\overline{u}) \text{ obbeys the stationary point conditions for } (MP_y) \iff \begin{cases} \text{(i) } \overline{x} \text{ solves } (MP_y); \\ \text{(ii) } (\overline{x},\overline{u}) \text{ solves } (WD_y); \\ \text{(iii) } wd(y) = v(y) \end{cases}$$

Proof: Ommited, as it is identical to the one of (3.2). ■

It must be noted that the assumption of the validity of a weak duality theorem implies that dual feasibility $((\hat{x},\hat{u}) \in DU)$ is a quite strong property, i.e.,

Fact 4.2) If there holds a weak duality theorem, then

$$(\hat{x},\hat{u}) \in DU \Rightarrow \begin{cases} \text{(i) } \hat{x} \text{ solves } (MP_{g(\hat{x})}); \\ \text{(ii) } (\hat{x},\hat{u}) \text{ solves } (WD_{g(\hat{x})}); \\ \text{(iii) } wd(g(\hat{x})) = v(g(\hat{x})) = f(\hat{x}). \end{cases}$$

Proof: Ommited, as it is a trivial corollary of (4.1). ■

It is interesting to note that if a weak duality theorem holds, then

$\{((x^1,u^1),(x^2,u^2)) \in DU \times DU$ and $g(x^1) = g(x^2)\} \Rightarrow \{f(x^1) = f(x^2)\}$.

Although it is quite tempting to state that if for every dual feasible point $(\hat{x},\hat{u})$, $(\hat{x},\hat{u})$ solves $(WD)_{g(\hat{x})}$, $\hat{x}$ solves $(MP_{g(\hat{x})})$ and $w(g(\hat{x})) = v(g(\hat{x}))$ then a weak duality theorem holds, this is not the case as it can be seen taking $X^o = (-1,0) \cup (0,1)$, $g(x) = x$, $f(x) = (1-x)$, for $x \in (-1,0)$ and $f(x) = x^2$, for $x \in (0,1)$.

Unfortunately we do not have, for Wolfe's dual, a clear cut characterization of existence of optimal solutions of the dual as we had in (3.3). We can assert the somehow modest results:

Fact 4.3) If $(\overline{x},\overline{u})$ solves $(WD_{\overline{y}})$ then $(-\overline{u})$ is a subgradient of $wd(\cdot)$ at $y = \overline{y}$.

Proof: As $(\overline{x},\overline{u}) \in DU$ and $wd(\overline{y}) = L(\overline{x},\overline{u}) - <\overline{u},\overline{y}>$,

$L(\overline{x},\overline{u}) - <\overline{u},y> = wd(\overline{y}) - <\overline{u},y-\overline{y}> \leq wd(y)$. ■

Fact 4.4) If $(-\overline{u})$ is a subgradient of $wd(\cdot)$ at $y = \overline{y}$,

if there exists $\overline{x} \in X^o$ such that $(\overline{x},\overline{u}) \in DU$ and

if there holds a weak duality theorem,

then $(\overline{x},\overline{u})$ solves $(WD_{\overline{y}})$.

Note: As $wd(\cdot)$ is non-increasing, i.e., $y^1 \geq y^2 \Rightarrow wd(y^1) \leq wd(y^2)$), any subgradient of $wd(\cdot)$ has no positive components.

Proof: As $(-\overline{u})$ is a subgradient of $wd(\cdot)$ at $y = \overline{y}$,

$$wd(g(\overline{x})) \geq wd(\overline{y}) - < \overline{u}, g(\overline{x}) - \overline{y} >.$$

As $(\overline{x}, \overline{u}) \in DU$, by (4.2) and (4.3)

$$wd(\overline{y}) \geq wd(g(\overline{x})) - < \overline{u}, \overline{y} - g(\overline{x}) > = f(\overline{x}) + < \overline{u}, g(\overline{x}) > - < \overline{u}, \overline{y} >.$$

So $wd(\overline{y}) = f(\overline{x}) + < \overline{u}, g(\overline{x}) - \overline{y} >$. ∎

The statement "correspondent" to (3.3) that says "if $wd(\cdot)$ is subdifferentiable at $y = \overline{y}$ then $(WD_{\overline{y}})$ has a solution" is false as it can be seen setting $X^o = (0,1)$, $f(x) = -x^3$ and $g(x) = x$. This is also an example of the non automatic validity of a weak duality theorem.

Similar modifications can be applied to (3.4) leading to:

Fact 4.5) If there holds a weak duality theorem and $(\overline{x}, \overline{u})$ solves $(WD_{\overline{y}})$ and there is no duality gap at $y = \overline{y}$ then $(-\overline{u})$ is a subgradient of $v(\cdot)$ at $y = \overline{y}$.

Proof: Omitted, as it is a simple variation of (3.4.a). ∎

Fact 4.6) If there holds a weak duality theorem, if $(-\overline{u})$ is subgradient of $v(\cdot)$ at $y = \overline{y}$ and there exists $\overline{x} \in X^o$ such that $(\overline{x}, \overline{u}) \in DU$ then $(\overline{x}, \overline{u})$ solves $(WD_{\overline{y}})$ and $v(\overline{y}) = wd(\overline{y})$.

Proof: Omitted, as it is a simple variation of (3.4.b). ∎

It is important to note that a result similar to (3.5) is not valid. This can be shown using $X^o = (0,1)$, $f(x) = -x$ and $g(x) = x$ at $\overline{y} = 2$, for instance. This is a case of

Y convex, $v(\cdot)$ and $wd(\cdot)$ convex, $wd(\cdot) \leq v(\cdot)$ where the subgradient of $wd(\cdot)$ at $\bar{y} = 2$ is (-1), the subgradient of $v(\cdot)$ at $\bar{y} = 2$ is 0, $DU = \{(x, 1) \mid x \in (0, 1)\}$ and for all $(x, u) \in DU$, $L(x, u) = 0$. Clearly this is also an example that (3.6) does not hold for Wolfe's dual.

The interesting point is that although we can not formulate an equivalent result for (3.7), we can derive a quite trivial condition for the stationary point conditions being sufficient. It is important to remember that these conditions are necessary under some regularity conditions (see, for instance, Mangasarian [5]), like all the binding constraints at the point of minimium being pseudo concave.

Fact 4.7) Let

$Y^* = \{y \in Y \mid MP_y$ has a solution and for at least one of the solutions of (MP_y) the stationary point conditions are necessary$\}$,

then the stationary point conditions are sufficient for $y \in Y^*$, if and only if for every $y \in Y^*$, the stationary points are $f(\cdot)$ unique, i.e., $\{(\bar{x}, \bar{u})$ and $(\hat{x}, \hat{u})$ obbey the stationary point conditions $\Rightarrow f(\bar{x}) = f(\hat{x})\}$.

Proof: Ommitted, as it is trivial. ∎

Corollary 4.8) If $g : X^o \to R^m$ is pseudo-concave, then for every (MP_y) that has an optimal solution, the stationary point conditions are sufficient if and only if the stationary points are $f(\cdot)$ unique.

Proof: Trivial from (4.7). ∎

One interesting point of this result is the verification that the sufficiency of the stationary point conditions have no obligatoly connection with Wolfe's duality, in the sense that the example $f(x) = -x^3$, $g(x) = x$, $X^o = (0,1)$ has duality gaps, there is no weak duality, but for $Y^* = \{y \in Y \mid y < 1\}$ the stationary point conditions are sufficient. Similar results may be obtained under monotonicity assumptions, but the main value of (4.7) is to state a clear and easy condition for the sufficiency of the stationary point conditions, with no ties even to a weak duality theorem, as it happens in the majority of the published results.

5. Conclusion

These results, whose formulation and demonstrations are quite simple, try to shed some light on the role of convexity in Lagrangian duality and to explore the consequent limitations and/or power of this tool. For instance, the result that shows that for the minimax duality a necessary and sufficient condition for no duality gap is convexity and lower semicontinuity reinforces the restriction of the scope in which classical Bender's decomposition is valid. The discussion on Wolfe's dual enforces the fact that, unlike minimax duality and saddle point conditions, there are no relations between Wolfe's dual and stationary point conditions unless a weak duality theorem holds. The specific role of subdifferentiability of the optimal value functions $(v(\cdot), w(\cdot)$ and $wd(\cdot))$ is also presented, trying to separate different results that may lead to strong duality theorems.

An interesting avenue of research is the specification of conditions where a reverse duality theorem holds for Wolfe's dual. All the known results use the fact that under some assumptions if $(\overline{x}, \overline{u})$ solves (DW) then the stationary point conditions are obbeyed and verifying $f(\cdot)$ uniqueness, the point $\overline{x}$ solves (MP). It appears that a line similar to the one presented here may lead to interesting formulations.

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RELATÓRIOS TÉCNICOS

DEPARTAMENTO DE CIÊNCIA DA COMPUTAÇÃO

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V.W. Setzer
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