

# Disintegrations of non-hyperbolic ergodic measures along the center foliation of DA maps

Ali Tahzibi and Jinhua Zhang\*

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## Abstract

We show that each non-hyperbolic ergodic measure of a partially hyperbolic diffeomorphism on  $\mathbb{T}^3$  which is homotopic to Anosov admits a full measure subset which intersects each center leaf in at most two points.

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## 1 Introduction

### 1.1 Background

One attempt of research in smooth ergodic theory is to verify up [to what extent](#) the known ergodic features of hyperbolic dynamics appear in beyond uniformly hyperbolic dynamics. Several [weak forms of hyperbolicity are proposed to relax](#) the notion of uniform hyperbolicity. Partial hyperbolicity had attracted [many](#) attentions. Classification of partially hyperbolic diffeomorphisms and proving ergodicity of volume measure are two main research topics. The following statement of Pugh and Shub: "a little hyperbolicity goes a long way toward [guaranteeing](#) ergodicity", has been a driving force for many ergodicity results in the partially hyperbolic setting.

In this paper we would like to address hyperbolicity in terms of global measure theoretical properties. Observe that for genuinely partially hyperbolic diffeomorphisms, a non-trivial center bundle is the obstruction to uniform hyperbolicity. In our quest for "a little hyperbolicity", we calculate the Lyapunov exponents along center bundle. If [all](#) the center Lyapunov exponents are non-zero almost everywhere, we are in the setting of so-called nonuniform hyperbolicity and several methods including Pesin's theory are available. So, vanishing center Lyapunov exponent is one of the main obstructions to understand ergodic properties of partially hyperbolic diffeomorphisms.

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A naive way of getting rid of center bundle is to study the quotient space of center foliation (whenever it exists). However, in general this quotient space is not even a Hausdorff topological space. By the way, one may consider disintegration of volume along the center foliation and it turns out that in some cases the Lebesgue measure disintegrates into Dirac measure (or finitely many Dirac) along the global center leaves! Shub and Wilkinson [SW] considered  $A \times Id : \mathbb{T}^3 \rightarrow \mathbb{T}^3$  where  $A$  is a linear Anosov diffeomorphism of  $\mathbb{T}^2$  and [by perturbations, they obtained an open set](#) of volume preserving partially hyperbolic dynamics with non-vanishing center Lyapunov exponent. Using compactness of center foliation and non vanishing center Lyapunov exponent, they proved that the center foliation is non-absolutely continuous, and in fact the volume disintegrates into atomic measures along center foliation [RW]. See also Pesin and Hirayama [PH] who proved atomic disintegration in higher dimensional center case where they assumed [that the sum of the center Lyapunov](#) exponents is non-zero.

Here we would like to address this phenomenon of Dirac disintegration in the setting of partially hyperbolic diffeomorphisms on 3-manifolds. Sometimes it is called *virtual hyperbolicity* (This term was coined by E. Lindenstrauss and K. Schmidt in [LiSch]): the existence of a full volume subset which intersects every center leaf in at most finitely many points (or orbits). This “little (virtual) hyperbolicity” may be useful to prove ergodic properties. For instance, the Dirac disintegration of [Lebesgue measure](#) along center foliation of derived from Anosov diffeomorphisms plays an important role to prove Bernoulli property in [PTV2], and the examples of Dirac disintegration of volume along center foliation of derived from Anosov diffeomorphisms enabled the authors of [PTV1] to construct examples of minimal yet measurable foliations.

In this paper, we will consider non-hyperbolic ergodic measures and study their disintegration along the center foliation.

There are many deep results in the active area: the classification of partially hyperbolic diffeomorphisms on 3-manifolds. In the spirit of these classification results, we highlight following three classes:

- fibered maps,
- perturbation of time-one map of Anosov flows,
- derived from Anosov diffeomorphisms.

By the results in the works of Avila, Viana and Wilkinson ([AVW1], [AVW2]), in the first two categories there is a “Dirac versus Lebesgue” dichotomy. R. Varão [V] gave an example to show that in the derived from Anosov case, the disintegration along center foliation may be neither atomic nor Lebesgue. It is important to note that in the first two categories, if the center Lyapunov exponent is non-zero then we are in the atomic disintegration case of dichotomy. However, there are examples with vanishing center Lyapunov exponent but Dirac disintegration along the center foliation (See examples in [1.3.2](#) and [1.3.3](#)).

Our main result is in the derived from Anosov setting where we prove that vanishing center exponent always yields atomic disintegration. In fact, this result holds for any invariant ergodic probability measure. This shows a main difference between our result and previous disintegration [results](#) where smoothness of measure is crucial. Besides its intrinsic interest, we use this result in some class of derived from Anosov examples to show that the so called invariance principle [Fur, L, AV] does not hold in this context. In this way we shed more

light on the atomic disintegration along the center foliation of all three partially hyperbolic categories above (See Section 1.3).

## 1.2 Statement of the results

Recall that a diffeomorphism  $f \in \text{Diff}^1(M)$  is *partially hyperbolic* if there exist a  $Df$ -invariant **continuous** splitting  $TM = E^s \oplus E^c \oplus E^u$  and  $N \in \mathbb{N}$  such that

$$\|Df^N|_{E^s(x)}\| < \min\{1, m(Df^N|_{E^c(x)})\} \leq \max\{1, \|Df^N|_{E^c(x)}\|\} < m(Df^N|_{E^u(x)}).$$

A diffeomorphism  $f \in \text{Diff}^1(\mathbb{T}^3)$  induces an isomorphism  $f_* : H_1(\mathbb{T}^3, \mathbb{Z}) \rightarrow H_1(\mathbb{T}^3, \mathbb{Z})$  which can be considered as a matrix in  $GL(3, \mathbb{Z})$ , and one says that  $f$  is *derived from Anosov* (or *homotopic to Anosov*) if  $f_*$  induces an Anosov automorphism on  $\mathbb{T}^3$ .

The existence of non-hyperbolic ergodic measures for robustly non-hyperbolic systems has been extensively investigated, see for instance [GIKN, KN, BBD, BZ]. Now, we investigate the disintegration of non-hyperbolic ergodic measures for derived from Anosov systems on  $\mathbb{T}^3$  which is one of the classical partially hyperbolic models and whose study originated from Mañé [M].

**Theorem A.** *Let  $f \in \text{Diff}^1(\mathbb{T}^3)$  be a partially hyperbolic diffeomorphism which is derived from Anosov. For any non-hyperbolic ergodic measure  $\nu$ , there exists a  $\nu$ -full measure set  $\Lambda_\nu$  which intersects each center leaf in at most two points.*

- Remark 1.1.**
1. *It has been shown in [VY] that if the linear Anosov diffeomorphism has 2-dimensional unstable bundle, then each ergodic measure with negative center Lyapunov exponent admits a full measure set which intersects each center leaf in at most one point.*
  2. *Note that the center foliation of  $f$  is orientable. If  $f$  preserves the orientation of the center foliation, the  $\nu$ -full measure set  $\Lambda_\nu$  can be chosen to intersect each center leaf in at most one point. If the  $\nu$ -full measure set  $\Lambda_\nu$  intersects almost every center leaf in two points, then the conditional measures of  $\nu$  along the center foliation are equidistributed on these two points.*
  3. *In terms of vanishing of transverse entropy (defined in [LY2]), F. Ledrappier and J. Xie [LX] showed that for  $C^2$ -diffeomorphisms, if the transverse entropy of an ergodic measure vanishes, then the conditional measure along a typical unstable manifold is actually carried by a single strong unstable manifold. Thus the disintegrations along the weak unstable manifolds are atomic. While we work for the case where the center Lyapunov exponent of a  $C^1$ -partially hyperbolic diffeomorphism vanishes, so our result is not implied by theirs.*

In [GS], the authors show that  $C^{1+\alpha}$  volume preserving partially hyperbolic diffeomorphisms  $f$  which are derived from Anosov on  $\mathbb{T}^3$  are ergodic. **Observe that this implies** that the Lebesgue measure is ergodic for  $f^k$  ( $k \in \mathbb{Z} \setminus \{0\}$ ), **as  $f^k$  is also derived from Anosov**. Thus, one has the following corollary.

**Corollary 1.2.** *Let  $f \in \text{Diff}_m^{1+\alpha}(\mathbb{T}^3)$  be a partially hyperbolic diffeomorphism which is derived from Anosov. If the center Lyapunov exponent of  $f$  vanishes, then there exists a Lebesgue-full measure set which intersects each center leaf in at most one point.*

### 1.3 Atomic disintegration and vanishing center exponent

In this section we first mention some byproducts of our main result for derived from Anosov diffeomorphisms. Then we give some examples of vanishing center Lyapunov exponent and atomic disintegration in the case of partially hyperbolic diffeomorphisms which are close to time one map of geodesic flow on negatively curved surface or have compact center foliations.

#### 1.3.1 Derived from Anosov diffeomorphisms

We will show that there exist derived from Anosov diffeomorphisms such that the invariance principle does not hold.

Recently, Crovisier and Poletti [CP] obtained the following invariance principle result. They consider partially hyperbolic and dynamically coherent diffeomorphisms  $f$  which act quasi-isometrically in the center: there exist  $K \geq 1$  and  $q > 0$  such that for every  $x, y \in M$  with  $y \in \mathcal{F}^c(x)$  and every  $n \in \mathbb{Z}$ , one has

$$K^{-1}d^c(x, y) - q \leq d^c(f^n(x), f^n(y)) \leq Kd^c(x, y) + q,$$

where  $d^c(\cdot, \cdot)$  denotes the distance along the center leaves.

The following result is a direct consequence of Theorem B in [CP] (see the paragraph before Theorem B in [CP]).

**Theorem 1.3** ([CP]). *Let  $f \in \text{Diff}^r(M)$  ( $r > 1$ ) be a partially hyperbolic and dynamically coherent diffeomorphism, and let  $m$  be an ergodic measure. Assume, in addition, that*

- $f$  acts quasi-isometrically in the center;
- $m$  has local  $cu \times s$ -product structure<sup>1</sup> and  $\lambda^c(f, m) = 0$ .

*Then there exists a family of local center measures  $\{m_x^c\}_{x \in \text{Supp}(m)}$  which are continuous,  $f$ -invariant,  $s$ -invariant,  $u$ -invariant and extend the center disintegration of  $m$  to the whole support.*

Now, we present an example showing that the above invariance principle does not hold in derived from Anosov setting. In particular we conclude that the quasi-isometric hypothesis in the above theorem is a necessary condition. Consider a family of examples in [PT] where the authors begin with an Anosov automorphism and make Baraviera-Bonatti [BB] type of perturbation carefully to obtain derived from Anosov diffeomorphisms with some nice properties. More precisely, they begin with a linear Anosov automorphism (chosen from a suitable family of automorphisms)  $A: \mathbb{T}^3 \rightarrow \mathbb{T}^3$  with eigenvalues  $\lambda^s < \lambda^c < 1 < \lambda^u$  where  $\lambda^c$  is very close to one (see Section 4 in [PT]), and after  $C^1$ -small perturbation they find a smooth path  $\{f_t\}_{t \in [0,1]} \subset \text{Diff}_m^\infty(\mathbb{T}^3)$  such that each  $f_t$  is partially hyperbolic,  $f_0 = A$  and  $f_1$  has positive center Lyapunov exponent. It is proved in [PT] that the disintegration of Lebesgue measure along center foliation of  $f_1$  is atomic. As the center bundle is one dimensional, there exists  $t_0 \in [0,1]$  such that  $f_{t_0}$  has vanishing Lyapunov exponent (notice that  $f_{t_0}$  is homotopic

<sup>1</sup>Roughly speaking,  $m$  has local  $cu \times s$ -product structure if  $m$  restricted to a small product neighborhood of  $z$  (that is a neighborhood can be identified as  $\mathcal{F}_{loc}^{cu}(z) \times \mathcal{F}_{loc}^s(z)$ ) is equivalent to the product measure  $m^{cu}(z) \times \hat{m}^s(z)$  where  $m^{cu}(z)$  is the conditional measure on  $\mathcal{F}_{loc}^{cu}(z)$  and  $\hat{m}^s(z)$  is the quotient measure on  $\mathcal{F}_{loc}^s(z)$ . One can refer to Definition 1.5 in [CP] and references therein for a precise definition.

to  $A$ ). By Corollary 1.2, the disintegration of  $m$  along center foliation of  $f_{t_0}$  is given by Dirac measures.

The family of diffeomorphisms  $\{f_t\}_{t \in [0,1]}$  is obtained by perturbing  $A$  locally (supported on a small ball) and the center-unstable bundle of  $A$  is preserved by the perturbations. Thus center-unstable foliation of  $f$  and  $A$  are the same and consequently all the leaves are planes and it is an absolutely continuous foliation (in fact smooth). This implies that volume measure  $m$  has local  $cu \times s$  product structure (see [CP, Proposition 6.1] for a proof).

Now we claim that the conclusions of the above theorem can not hold for such an example. Indeed, by theorem A the center disintegration is atomic. So take any such atom  $x \in \mathbb{T}^3$  and consider a small neighborhood of  $x$ . As  $f_{t_0}$  is not Anosov, by [GS, HaS] the diffeomorphism  $f_{t_0}$  is accessible<sup>2</sup> (here we have local non-joint integrability). We can take an  $su$ -path beginning from  $x$  and ending at some  $y \neq x$  in the center plaque of  $x$ . As  $m$  is fully supported, if there exists a continuous disintegration of  $m$  which is both  $s$ -invariant and  $u$ -invariant then we would conclude that  $y$  is also an atom of the disintegration of  $m$  along the center plaque of  $x$  which is a contradiction.

In all the known examples of volume preserving derived from Anosov diffeomorphisms on  $\mathbb{T}^3$  with Dirac disintegration (along center leaves), the center Lyapunov exponent is either zero or has opposite sign to the center exponent of their linear part. Thus it is natural to ask the following question.

**Question 1.** *Is there any example of volume preserving derived from Anosov diffeomorphism  $f$  such that the sign of its center Lyapunov exponent coincides with the sign of the center Lyapunov exponent of its linearization and the disintegration of Lebesgue measure along center foliation is Dirac?*

### 1.3.2 Perturbations of time-one maps of Anosov flows

Perturbations of time-1 maps of geodesic flows on the unit tangent bundle of negatively curved surfaces are classical examples of partially hyperbolic diffeomorphisms isotopic to identity. In this case a deep result of Avila-Viana-Wilkinson proves a dichotomy: either the center foliation is virtually hyperbolic or the dynamics is embedded into a flow.

**Theorem 1.4** (Main Theorem 1 in [AVW1]). *(Dirac - Lebesgue dichotomy) Let  $\phi^t : T^1S \rightarrow T^1S$  be the geodesic flow for a negatively curved closed surface  $S$  and let  $m$  be the  $\phi^t$ -invariant Liouville probability measure.*

*Then there is a  $C^1$ -open neighborhood  $\mathcal{U} \subset \text{Diff}_m^1(T^1S)$  of  $\phi^1$  such that for any smooth diffeomorphism  $f \in \mathcal{U}$ , one has*

1. *either there exist  $k \geq 1$  and a full  $m$ -measure set  $Z \subset T^1S$  that intersects every center leaf in exactly  $k$  orbits of  $f$ ,*
2. *or the center foliation is absolutely continuous; in this case,  $f$  is the time-one map of an  $m$ -preserving  $C^\infty$  flow.*

The study of Lyapunov exponents and the so called Invariance Principle is crucial in the proof of the above theorem. We point out that the above dichotomy does not coincide with hyperbolic versus non-hyperbolic behavior in the center direction. If  $m$  is non-uniformly

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<sup>2</sup>Recall that a partially hyperbolic diffeomorphism is *accessible* if any two points can be joined by a path which is a concatenation of finitely many curves lying either in a strong stable leaf or in a strong unstable leaf.

hyperbolic (the center Lyapunov exponent is non-zero), then we are in the first item of dichotomy. However in the presence of zero Lyapunov exponent we may still be in the first case.

We construct a family  $\{g_t\}_{t \in [-1,1]} \subset \text{Diff}_m^1(T^1S)$  of diffeomorphisms in a  $C^1$  small neighborhood of  $\phi^1$  such that  $g_0 = \phi^1$  and  $g_1$  (resp.  $g_{-1}$ ) has positive (resp. negative) center Lyapunov exponent. Indeed, by Baraviera-Bonatti [BB] local perturbation method consider two families of volume preserving diffeomorphisms  $(\xi_t)_{t \in [0,1]}$  and  $(\eta_t)_{t \in [-1,0]}$  such that

- $\eta_0 = Id$  and  $\lambda^c(g_0 \circ \eta_1) > 0$ ,
- $\xi_0 = Id$  and  $\lambda^c(g_0 \circ \xi_{-1}) < 0$ ,
- the supports of all  $\xi_t$  and  $\eta_t$  are in some small disjoint balls  $B_1$  and  $B_2$  respectively.

Now define

$$g_t = \begin{cases} g_0 \circ \eta_t & t \in [0, 1] \\ g_0 \circ \xi_t & t \in [-1, 0]. \end{cases}$$

Let  $\mathcal{C}$  be a periodic orbit of  $\phi^t$  which is disjoint from  $B_1$  and  $B_2$ . Then  $\mathcal{C}$  is normally hyperbolic for  $g_0$ , and thus admits continuation under small perturbations. By the conservative version of Franks lemma (see Proposition 7.4 in [BDP]), there exists  $g$  arbitrarily  $C^1$  close to  $g_0$  such that  $g$  admits a sink on the compact center leaf  $\mathcal{C}_g$  (continuation of  $\mathcal{C}$  for  $g$ ) and  $g$  coincides with  $g_0$  in the complement of a small neighborhood of  $\mathcal{C}$ . Finally we define  $f_t = g \circ g_0^{-1} \circ g_t$ . As the center bundle is one dimensional, the center Lyapunov exponent varies continuously. Since  $\lambda^c(f_1) > 0$  and  $\lambda^c(f_{-1}) < 0$ , by mean value theorem there exists  $t_*$  such that  $\lambda^c(f_{t_*}) = 0$ . Observe that  $f_{t_*}$  restricted to  $\mathcal{C}_g$  admits a sink and thus it can not be embedded into a flow as in Theorem 1.4.

### 1.3.3 Fibered partially hyperbolic diffeomorphisms (Circle center leaves)

An important class of partially hyperbolic diffeomorphisms are the so called fibered maps and they include skew products. A diffeomorphism  $f : M \rightarrow M$  is fibered partially hyperbolic if  $M$  admits an  $f$ -invariant structure  $\pi : M \rightarrow B$  of continuous fiber bundle with  $C^1$  fibers where the fibers are tangent to the center bundle of  $f$  (See [AVW2] for more details.) Here we assume the fibers are homeomorphic to circle. In this setting, the phenomena of Dirac center disintegration can appear in systems with non-vanishing center Lyapunov exponent as well as in systems with vanishing center Lyapunov exponent.

Shub and Wilkinson [SW] obtain a volume preserving skew-product system which has non-vanishing center Lyapunov exponent and has atomic center disintegration for the volume.

Katok's example shows that zero center Lyapunov exponent and Dirac disintegration can occur simultaneously. We recall Katok's example here. Let  $\{f_t\}_{t \in \mathbb{S}^1}$  be a smooth family of volume preserving Anosov diffeomorphisms of  $\mathbb{T}^2$  such that for any  $s \neq t$  one has that  $f_s$  and  $f_t$  are topologically but not smoothly conjugate. By a result of de la Llave [dL] the conjugacy between  $f_s$  and  $f_t$  can not be absolutely continuous. Now define  $F : \mathbb{T}^2 \times \mathbb{S}^1 \rightarrow \mathbb{T}^2 \times \mathbb{S}^1$  by  $F(x, t) = (f_t(x), t)$ . It is not difficult to see that  $\lambda^c(F, vol) = 0$  and the center holonomy between two transverse tori  $\mathbb{T}^2 \times \{s\}$  and  $\mathbb{T}^2 \times \{t\}$  is given by the conjugacy between  $f_t$  and  $f_s$  and hence not absolutely continuous. In fact considering the union of generic points of  $f_t$

for  $t \in \mathbb{S}^1$ , one obtains a full Lebesgue measure subset of  $\mathbb{T}^3$  which intersects any center leaf in at most one point.

Note that Katok's example is not accessible. One can also construct an example of an accessible volume preserving partially hyperbolic diffeomorphism on a nilmanifold  $M^3$  which has vanishing center Lyapunov exponent and Dirac center disintegration. Take a one parameter family of volume preserving partially hyperbolic diffeomorphisms on  $M^3$  as in Section 1.3.2 and after a perturbation introduce a sink in a periodic center leaf of all diffeomorphisms in the path. There exists a parameter with vanishing center Lyapunov exponent and by rigidity result of [AVW2] (Theorems A and B) the conditional measures along center foliation are atomic. Finally, notice that all the partially hyperbolic diffeomorphisms on the nilmanifold  $M^3$  are accessible due to [HHU].

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## 2 Preliminaries

In this section, we collect the notions and results used in this paper.

### 2.1 Rohlin's disintegration theorem

Given a compact metric space  $X$ , a partition  $\mathcal{P}$  of  $X$  is called *measurable*, if there exists a sequence of finite (Borel) measurable partitions  $\mathcal{P}_1 < \mathcal{P}_2 < \dots < \mathcal{P}_n < \dots$  such that  $\mathcal{P} = \bigvee_{n \in \mathbb{N}} \mathcal{P}_n$ .

Given a probability measure  $\mu$  and a measurable partition  $\mathcal{P}$  on  $X$ , we denote by  $\hat{\mu}$  the quotient measure induced by  $\mu$  on the Lebesgue space  $X/\mathcal{P}$ .

**Theorem 2.1** ([Ro]). *Given a probability measure  $\mu$  and a measurable partition  $\mathcal{P}$  on  $X$ , there exists a (essentially) unique family of probability measures  $\{\mu_P\}_{P \in \mathcal{P}}$  such that*

- $\mu_P(P) = 1$  for  $\hat{\mu}$  a.e.  $P \in \mathcal{P}$ ;
- for any measurable set  $A \subset X$ , the map  $P \mapsto \mu_P(A)$  is measurable and

$$\mu(A) = \int \mu_P(A) d\hat{\mu}.$$

The family of probability measure  $\{\mu_P\}_{P \in \mathcal{P}}$  is called the *conditional measures or the disintegrations* of  $\mu$  with respect to  $\mathcal{P}$ .

### 2.2 Partially hyperbolic diffeomorphisms homotopic to Anosov

In this section, we recall some properties of partially hyperbolic diffeomorphisms on  $\mathbb{T}^3$  which are homotopic to Anosov.

**Theorem 2.2** ([Fr]). *Let  $f$  be a homeomorphism on  $\mathbb{T}^3$ , which is homotopic to a linear Anosov diffeomorphism  $A$ . Consider a lift  $F$  of  $f$  to  $\mathbb{T}^3$ . Then there exists a unique continuous surjective map  $\pi : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  such that*

- $\pi \circ F = A \circ \pi$ ;
- $\pi(x + n) = \pi(x) + n$ , for any  $x \in \mathbb{R}^3$  and  $n \in \mathbb{Z}^3$ .

**Remark 2.3.**  $\pi$  is homotopic to identity and  $\pi$  induces a continuous surjective map on  $\mathbb{T}^3$ , still denoted by  $\pi$ , such that  $\pi \circ f = A \circ \pi$ .

**Theorem 2.4** ([BI, Ha, U, Po]). *Let  $f$  be a partially hyperbolic diffeomorphism on  $\mathbb{T}^3$  which is homotopic to a linear Anosov diffeomorphism  $A$ . Let  $\pi$  be the semi-conjugacy between  $f$  and  $A$ . Then one has the following properties:*

- there exist unique foliations  $\mathcal{F}^{cs}$ ,  $\mathcal{F}^{cu}$  and  $\mathcal{F}^c$  tangent to the bundles  $E^s \oplus E^c$ ,  $E^c \oplus E^u$  and  $E^c$  respectively;
- the lift of  $\mathcal{F}^c$  to  $\mathbb{R}^3$  is quasi-isometric<sup>3</sup>;
- $A$  has simple spectrum;
- the semi-conjugacy sends the center foliation of  $f$  to the center foliation of  $A$ ;
- for a point  $x \in \mathbb{T}^3$ , the set  $\pi^{-1}(x)$  is a center closed segment (including a single point case).

## 2.3 Entropy along an expanding foliation

The entropy along unstable manifolds is defined in [LY1, LY2] and is generalized for an expanding foliation in [VY, Y, HHW].

Let  $g \in \text{Diff}^1(M)$ . A measurable partition  $\xi$  is *increasing* if  $g\xi < \xi$ . A measurable partition  $\xi$  is called  $\mu$ -*subordinated* to a foliation  $\mathcal{F}$  on  $M$  if for  $\mu$  a.e.  $x$ , the element  $\xi(x)$  is contained in the leaf  $\mathcal{F}(x)$  and there exists  $\delta_x > 0$  such that the  $\delta_x$  neighborhood  $\mathcal{F}_{\delta_x}(x)$  of  $x$  (with respect to the leaf topology) is contained in  $\xi(x)$ .

A foliation  $\mathcal{F}$  on  $M$  is called an *expanding foliation* of  $g \in \text{Diff}^1(M)$ , if

- $\mathcal{F}$  is  $g$ -invariant;
- each leaf of  $\mathcal{F}$  is  $C^1$  and  $g$  is uniformly expanding along the tangent bundle of  $\mathcal{F}$ .

**Lemma 2.5** (Proposition 3.1 in [LS] and Lemma 3.2 in [Y]). *Let  $g \in \text{Diff}^1(M)$ ,  $\mu$  be an invariant measure and  $\mathcal{F}$  be an expanding foliation  $g$ . Then there exists an increasing measurable partition which is  $\mu$ -subordinated to  $\mathcal{F}$ .*

Let  $\mu$  be an invariant measure of a partially hyperbolic diffeomorphism  $g$ , then the *unstable (metric) entropy* of  $\mu$  is defined as

$$h_\mu(g, \mathcal{F}^u) := H_\mu(\xi | g\xi),$$

where  $\xi$  is an increasing partition  $\mu$ -subordinated to the unstable foliation  $\mathcal{F}^u$ . It has been shown in [LY1] that the unstable entropy is independent of the choice of  $\xi$ . Analogously, one can define the stable metric entropy along a contracting foliation by considering  $g^{-1}$ .

The following results show that the disintegration of an ergodic measure along an expanding foliation is closely related to its metric entropy along this foliation.

<sup>3</sup>A foliation  $\mathcal{F}$  with  $C^1$ -leaves on  $\mathbb{R}^3$  is quasi-isometric, if there exists  $a, b > 0$  such that  $d_{\mathcal{F}}(x, y) < a d(x, y) + b$  for any  $x \in \mathbb{R}^3$  and  $y \in \mathcal{F}(x)$ .

**Proposition 2.6** (Proposition 2.5 in [VY]). *Let  $g \in \text{Diff}^1(M)$ ,  $\nu$  be an ergodic measure and  $\mathcal{F}$  be an expanding foliation. Then the followings are equivalent:*

- $h_\nu(g, \mathcal{F}) = 0$ .
- *there exists a  $\nu$ -full measure subset intersecting each center leaf in at most one point.*

**Proposition 2.7** (Proposition 2.7 in [VY]). *Let  $g \in \text{Diff}^1(M)$ ,  $\nu$  be an ergodic measure and  $\mathcal{F}$  be an expanding foliation. Then the followings are equivalent:*

- $h_\nu(g, \mathcal{F}) > 0$ .
- *any  $\nu$ -full measure subset intersects almost every center leaf in an uncountable set.*

In Section 3, we will apply the results of this section to  $g = A$  and  $\mathcal{F} = W^c$ , where  $A$  is a linear Anosov with expanding center foliation  $W^c$ .

### 3 Entropy and disintegrations along the center foliation: Proof of Theorem A

In this section, we first show that the projection of a non-hyperbolic ergodic measure has zero entropy along the center foliation, then we study the disintegration along the center foliation.

**Proposition 3.1.** *Let  $f \in \text{Diff}^1(\mathbb{T}^3)$  be a partially hyperbolic diffeomorphism homotopic to a linear Anosov diffeomorphism  $A$ . Let  $\pi : \mathbb{T}^3 \rightarrow \mathbb{T}^3$  be the semi-conjugacy between  $f$  and  $A$  given by Theorem 2.2. For any non-hyperbolic ergodic measure  $\nu$ , one has  $h_{\hat{\nu}}(A, W^c) = 0$  where  $\hat{\nu} := \pi_* \nu$ .*

Ledrappier and Young [LY2] considered the entropy along Pesin unstable laminations. And by Ruelle's inequality, positive entropy gives positive Lyapunov exponents. However, here we need to deal with the entropy along a foliation with a vanishing Lyapunov exponent. To overcome this difficulty, we use the semi-conjugacy to the hyperbolic automorphism.

*Proof of Proposition 3.1.* Up to replacing  $f$  by  $f^{-1}$ , one can assume that  $A$  is expanding along the center, as the result is symmetric in considering  $f^{-1}$ . Let us denote by  $\mathcal{F}^c$  and  $W^c$  the center foliations of  $f$  and  $A$  respectively. Let  $\nu$  be an ergodic measure with vanishing center Lyapunov exponent, then  $\hat{\nu} := \pi_* \nu$  is an ergodic measure of  $A$ . The proof proceeds by contradiction. Assume, on the contrary, that  $h_{\hat{\nu}}(A, W^c) > 0$ .

**Claim 3.2.**  *$(f, \nu)$  is isomorphic to  $(A, \hat{\nu})$  via  $\pi$ .*

*Proof of Claim 3.2.* By Theorem 2.4, the set  $\pi^{-1}(x)$  is a center segment (could be trivial) for any  $x \in \mathbb{T}^3$ ; in particular, on each leaf of  $W^c$ , there are at most countably many points whose pre-images under  $\pi$  are non-trivial center segments. Consider the set

$$\mathcal{S} = \{x \in \mathbb{T}^3 \mid \pi^{-1}(x) \text{ is a non-trivial center segment}\}.$$

Notice that  $f^{-1}(\pi^{-1}(x)) = \pi^{-1}(A^{-1}(x))$  for any  $x \in \mathbb{T}^3$ . Thus, the set  $\mathcal{S}$  is  $A$ -invariant. By the ergodicity of  $\hat{\nu}$ , one has  $\hat{\nu}(\mathcal{S}) = 0$  or 1. As  $h_{\hat{\nu}}(A, W^c) > 0$ , by Proposition 2.7, for any  $\hat{\nu}$ -full

measure subset  $K$ , one has that  $K \cap W^c(x)$  is an uncountable set for  $\hat{\nu}$  a.e.  $x \in \mathbb{T}^3$ . Since on each leaf of  $W^c$  there are at most countably many points whose pre-images under  $\pi$  are non-trivial center segments, thus  $\hat{\nu}(\mathcal{S}) = 0$  proving that  $\pi$  is an isomorphism between  $(f, \nu)$  and  $(A, \hat{\nu})$ .  $\square$

Now, consider a measurable partition  $\hat{\xi}^c$  which is increasing and  $\hat{\nu}$ -subordinated to the foliation  $W^c$  with  $\text{Diam}(\hat{\xi}^c) < 1$ . Let  $\{\hat{\nu}_x^c\}$  be the conditional measures of  $\hat{\nu}$  with respect to the measurable partition  $\hat{\xi}^c$ . By Shannon-McMillan-Breiman theorem for unstable entropy [LY2, Lemma 9.3.1], one has

$$\lim_{n \rightarrow \infty} -\frac{\log \hat{\nu}_x^c(\bigvee_{i=0}^{n-1} A^{-i} \hat{\xi}^c(x))}{n} = h_{\hat{\nu}}(A, W^c), \text{ for } \hat{\nu} \text{ a.e. } x \in \mathbb{T}^3.$$

As  $\pi : \mathbb{T}^3 \rightarrow \mathbb{T}^3$  is continuous, the partition  $\xi^c := \pi^{-1}(\hat{\xi}^c)$  is measurable. Since  $\pi$  is homotopic to identity and the center foliation is quasi-isometric, there exists  $\eta_0 > 0$  such that  $\text{Diam}(\xi^c) < \eta_0$ <sup>4</sup>. By Theorem 2.4,  $\pi$  sends the center leaves of  $f$  to the center leaves of  $A$  and  $\pi^{-1}(x)$  is a center segment for any  $x \in \mathbb{T}^3$ , and thus the partition  $\xi^c$  is  $\nu$ -subordinated to the center foliation  $W^c$ . Notice that  $\xi^c = \pi^{-1}(\hat{\xi}^c) < \pi^{-1}A^{-1}(\hat{\xi}^c) = f^{-1}\pi^{-1}(\hat{\xi}^c) = f^{-1}(\xi^c)$ . To summarize, the partition  $\xi^c$  is an increasing measurable partition which is  $\nu$ -subordinated to  $\mathcal{F}^c$ . We denote by  $\{\nu_x^c\}$  the conditional measures of  $\nu$  with respect to the measurable partition  $\xi^c$ .

Since  $\pi$  is an isomorphism between  $(f, \nu)$  and  $(A, \hat{\nu})$ , one has

$$h_{\hat{\nu}}(A, W^c) = H_{\nu}(f^{-1}(\xi^c)|\xi^c) = \lim_{n \rightarrow \infty} -\frac{\log \nu_x^c(\bigvee_{i=0}^{n-1} f^{-i} \xi^c(x))}{n}, \text{ for } \nu \text{ a.e. } x \in \mathbb{T}^3.$$

Now fix  $\varepsilon \in (0, \min\{h_{\hat{\nu}}(A, W^c)/20, 1/8\})$  small, and let us denote

$$\Lambda_{N,\varepsilon} = \{x \in \mathbb{T}^3 \mid \nu_x^c(f^{-n}(\xi^c(f^n(x)))) \leq e^{-n(h_{\hat{\nu}}(A, W^c) - \varepsilon)}, \text{ for any } n \geq N\},$$

$$K_{N,\varepsilon} = \{x \in \mathbb{T}^3 \mid e^{-|n|\varepsilon} \leq \log \|Df^n|_{E^c(x)}\| \leq e^{|n|\varepsilon}, \text{ for any } |n| \geq N\}.$$

Then  $\nu(\Lambda_{N,\varepsilon} \cap K_{N,\varepsilon})$  tends to 1 when  $N$  tends to infinity for any  $\varepsilon > 0$ .

For  $\delta > 0$ , we define the set

$$L_{\delta} = \{x \in \mathbb{T}^3 \mid \mathcal{F}_{\delta}^c(x) \subset \xi^c(x)\}.$$

As  $\xi^c$  is  $\nu$ -subordinated to the center foliation  $\mathcal{F}^c$ ,  $\nu(L_{\delta})$  tends to 1 when  $\delta$  tends to 0. Choose  $\delta > 0$  small such that  $\nu(L_{\delta}) > 3/4$ , and up to shrinking  $\delta$ , one can assume that

$$e^{-\varepsilon} < \|Df|_{E^c(x)}\| / \|Df|_{E^c(y)}\| < e^{\varepsilon} \text{ for any } x, y \in \mathbb{T}^3 \text{ with } d(x, y) < \delta.$$

We consider the set

$$R_{N,\varepsilon} = \{x \in \mathbb{T}^3 \mid \nu(L_{\delta}) - \varepsilon < \frac{1}{n} \sum_{i=0}^{n-1} \chi_{L_{\delta}}(f^i(x)) < \nu(L_{\delta}) + \varepsilon, \text{ for any } n \geq N\}.$$

By Birkhoff ergodic theorem,  $\nu(R_{N,\varepsilon})$  tends to 1 when  $N$  tends to infinity. Let  $\tau_{n,\varepsilon} = [8n\varepsilon] + 1$  for  $n \in \mathbb{N}$ .

<sup>4</sup>The diameter of a partition  $\eta$  is defined as  $\text{Diam}(\eta) := \sup_{C \in \eta} \text{Diam}(C)$ .

**Claim 3.3.** For any  $n > N$  and any  $x \in R_{N,\varepsilon}$ , there exists  $j \in \{n, \dots, n + \tau_{n,\varepsilon} - 1\}$  such that  $f^j(x) \in L_\delta$ .

*Proof of Claim 3.3.* By the definition of  $R_{N,\varepsilon}$ , for  $n > N$ , one has

$$v(L_\delta) - \varepsilon < \frac{1}{n} \sum_{i=0}^{n-1} \chi_{L_\delta}(f^i(x)) < v(L_\delta) + \varepsilon,$$

$$v(L_\delta) - \varepsilon < \frac{1}{n + \tau_{n,\varepsilon}} \sum_{i=0}^{n + \tau_{n,\varepsilon} - 1} \chi_{L_\delta}(f^i(x)) < v(L_\delta) + \varepsilon.$$

Thus for  $n > N$ , as  $\varepsilon < 1/8$ , one has

$$\sum_{i=n}^{n + \tau_{n,\varepsilon} - 1} \chi_{L_\delta}(f^i(x)) > (n + \tau_{n,\varepsilon})(v(L_\delta) - \varepsilon) - n(v(L_\delta) + \varepsilon) > \tau_{n,\varepsilon}(v(L_\delta) - \varepsilon) - 2n\varepsilon > \tau_{n,\varepsilon}/2 - 2n\varepsilon > 0.$$

This ends the proof of Claim 3.3.  $\square$

Fix  $N$  large enough such that there exists a subset  $\tilde{\Lambda} \subset \Lambda_{N,\varepsilon} \cap K_{N,\varepsilon} \cap R_{N,\varepsilon} \cap L_\delta$  satisfying that

- $v(\tilde{\Lambda}) > 1/2$ ;
- $v_x^c(\Lambda_{N,\varepsilon} \cap K_{N,\varepsilon} \cap R_{N,\varepsilon} \cap L_\delta) > 1/2$  for any  $x \in \tilde{\Lambda}$ .

Recall that  $\xi^c < f^{-1}(\xi^c)$ . For each  $x \in \tilde{\Lambda}$  and  $n > N$ , the elements of the partition  $f^{-n}\xi^c|_{\xi^c(x)}$  on  $\xi^c(x)$  which contain points in  $\Lambda_{N,\varepsilon} \cap K_{N,\varepsilon} \cap R_{N,\delta} \cap L_\delta$  have  $v_x^c$ -measure at most  $e^{-n(h_\nu(A,W^c) - \varepsilon)}$ , and therefore there are at least  $l_n := \lfloor e^{n(h_\nu(A,W^c) - \varepsilon)} / 2 \rfloor$  elements of  $f^{-n}\xi^c|_{\xi^c(x)}$  intersecting  $\Lambda_{N,\varepsilon} \cap K_{N,\varepsilon} \cap R_{N,\delta} \cap L_\delta$ . Let  $x_1, \dots, x_{l_n} \in \Lambda_{N,\varepsilon} \cap K_{N,\varepsilon} \cap R_{N,\delta} \cap L_\delta \cap \xi^c(x)$  satisfy that

$$f^{-n}\xi^c(x_i) \cap f^{-n}\xi^c(x_j) = \emptyset \quad \text{for } i \neq j.$$

By the definition of  $L_\delta$ , one has  $\mathcal{F}_\delta^c(x_i) \subset \xi^c(x)$ . Now, we decompose the set  $\{x_1, \dots, x_{l_n}\}$  according to their return time to  $L_\delta$ . For each  $j \in \{0, \dots, \tau_{n,\varepsilon} - 1\}$ , let  $Z_j = \{x_i | f^{j+n}(x_i) \in L_\delta\}$ . By Claim 3.3, it holds  $\cup_{j=0}^{\tau_{n,\varepsilon} - 1} Z_j = \{x_1, \dots, x_{l_n}\}$ , then there exists  $j_0 \in \{0, \dots, \tau_{n,\varepsilon} - 1\}$  such that  $\#Z_{j_0} = \max \#Z_j \geq \frac{l_n}{\tau_{n,\varepsilon}}$ . By the definition of  $L_\delta$ , one has

$$\mathcal{F}_\delta^c(f^{j_0+n}(x_i)) \subset \xi^c(f^{j_0+n}(x_i)) \quad \text{for } x_i \in Z_{j_0}.$$

Let us denote  $\gamma_n = C^{-N} \cdot e^{-3n\varepsilon} \cdot \delta$ , where  $C = \sup_{x \in \mathbb{T}^3} \|Df|_{E^c(x)}\| > 1$ .

**Claim 3.4.** For any  $n > N$  and any  $x_i \in Z_{j_0}$ , one has  $\mathcal{F}_{\gamma_{n+j_0}}^c(x_i) \subset f^{-j_0-n}(\xi^c(f^{j_0+n}(x_i)))$ .

*Proof of Claim 3.4.* We shall inductively show that  $f^j(\mathcal{F}_{\gamma_{n+j_0}}^c(x_i)) \subset \mathcal{F}_\delta^c(f^j(x_i))$  for  $j \leq j_0 + n$ .

For any  $j \leq N$ , one has that

$$f^j(\mathcal{F}_{\gamma_{n+j_0}}^c(x_i)) \subset \mathcal{F}_{C^j \gamma_{n+j_0}}^c(f^j(x_i)) \quad \text{and} \quad C^j \cdot \gamma_{n+j_0} \leq e^{-3(n+j_0)\varepsilon} \delta < \delta.$$

Assume that  $f^j(\mathcal{F}_{\gamma_{n+j_0}}^c(x_i)) \subset \mathcal{F}_\delta^c(f^j(x_i))$  already holds for some  $N \leq k \leq j_0 + n - 1$  and any  $j \leq k$ . Then for any point  $y \in \mathcal{F}_{\gamma_{n+j_0}}^c(x_i)$ , one has

$$d^c(f^{k+1}(y), f^{k+1}(x_i)) = \|Df^{k+1}|_{E^c(y_{k+1})}\| \cdot d^c(y, x_i) \leq \|Df^{k+1}|_{E^c(y_{k+1})}\| \cdot \gamma_{n+j_0},$$

where  $y_{k+1} \in \mathcal{F}_{\gamma_{n+j_0}}^c(x_i)$  and  $d^c(\cdot, \cdot)$  denotes the distance along the center leaf. Since  $f^j(y_{k+1}) \in \mathcal{F}_\delta^c(f^j(x_i))$  for  $0 \leq j \leq k$ , by the choice of  $\delta$ , one has that

$$\|Df^{k+1}|_{E^c(y_{k+1})}\| = \prod_{j=0}^k \|Df|_{E^c(f^j(y_{k+1}))}\| \leq e^{(k+1)\varepsilon} \prod_{j=0}^k \|Df|_{E^c(f^j(x_i))}\| \leq e^{2(k+1)\varepsilon}.$$

Thus  $d^c(f^{k+1}(y), f^{k+1}(x_i)) \leq e^{2(k+1)\varepsilon} \cdot \gamma_{n+j_0} < \delta$ , proving that  $f^{k+1}(\mathcal{F}_{\gamma_{n+j_0}}^c(x_i)) \subset \mathcal{F}_\delta^c(f^{k+1}(x_i))$ .  $\square$

As  $f^{\xi^c} < \xi^c$ , by Claim 3.4, for any  $x_i \in Z_{j_0}$ , one has

$$\mathcal{F}_{\gamma_{n+j_0}}^c(x_i) \subset f^{-j_0-n}(\xi^c(f^{j_0+n}(x_i))) \subset f^{-n}(\xi^c(f^n(x_i))).$$

By the choice of  $x_i$ , the element  $\xi^c(x)$  contains at least  $\#Z_{j_0}$ -pairwise disjoint center segments of length  $\gamma_{n+j_0}$ , and thus

$$\eta_0 \geq \text{Diam}(\xi^c(x)) \geq \#Z_{j_0} \cdot \gamma_{n+j_0} \geq l_n \cdot \gamma_{n+j_0} / \tau_{n,\varepsilon} \geq [e^{n(h_{\hat{\nu}}(A, W^c) - \varepsilon)} / 2] C^{-N} \cdot e^{-3(n+[8n\varepsilon]+1)\varepsilon} \cdot \delta / ([8n\varepsilon]+1).$$

By the choice of  $\varepsilon$ , the right side tends to infinity when  $n$  tends to infinity, which gives the contradiction.  $\square$

Now, we are ready to give the proof of our main result.

*Proof of Theorem A.* Let  $\pi : \mathbb{T}^3 \rightarrow \mathbb{T}^3$  be the semi-conjugacy given by Theorem 2.2. By Theorem 2.4, the set  $\pi^{-1}(\pi(x))$  is center segment (could be trivial) for any  $x \in \mathbb{T}^3$ . Since  $\pi : \mathbb{T}^3 \rightarrow \mathbb{T}^3$  is homotopic to identity and the center foliation is quasi-isometric, there exists a constant  $K > 0$  such that for any  $x \in \mathbb{T}^3$ , the length of  $\pi^{-1}(\pi(x))$  is bounded from above by  $K$ . For any  $x \in \mathbb{T}^3$ , let us denote  $\Gamma_x = \pi^{-1}(\pi(x)) \subset \mathcal{F}^c(x)$ , then  $f(\Gamma_x) = \Gamma_{f(x)}$ .

Let  $\nu$  be a non-hyperbolic ergodic measure, then  $\hat{\nu} := \pi_*(\nu)$  is an ergodic measure of  $A$ . By Proposition 3.1,  $h_{\hat{\nu}}(A, W^c) = 0$ . By Proposition 2.6, there exists a  $\hat{\nu}$ -full measure subset  $\hat{\Lambda}$  intersecting each center leaf in at most one point. Let  $\Lambda = \pi^{-1}(\hat{\Lambda})$  which is a  $\nu$ -full measure subset. For any  $x \in \Lambda$ , one has that  $\Lambda \cap \mathcal{F}^c(x) = \Gamma_x$  which is a center segment (could be trivial). As  $\pi$  is a continuous map and the partition  $\mathcal{G}$  of  $\mathbb{T}^3$  whose elements consist of a single point is measurable, the partition  $\{\Gamma_x\} = \pi^{-1}(\mathcal{G})$  ( $\nu$  almost everywhere) is measurable.

If there exists a  $\nu$ -positive measure set in which the center segment  $\Gamma_x$  is trivial, then one can conclude by the ergodicity of  $\nu$ .

It remains the case where  $\Gamma_x$  is a non-trivial center segment for  $\nu$  a.e.  $x \in \mathbb{T}^3$ . Let  $\{\nu_x^c\}$  be the conditional measures of  $\nu$  with respect to the measurable partition  $\{\Gamma_x\}$ . As  $f(\Gamma_x) = \Gamma_{f(x)}$  and  $\nu$  is  $f$ -invariant, by the uniqueness of the conditional measures, one has  $f_* \nu_x^c = \nu_{f(x)}^c$ . By the ergodicity of  $\nu$ , one has that

- either  $\nu_x^c$  has atoms for  $\nu$ . a.e.  $x \in \mathbb{T}^3$ ; or

- $\nu_x^c$  has no atoms for  $\nu$ . a.e.  $x \in \mathbb{T}^3$ .

We claim that the later case cannot happen.

**Claim 3.5.** *For  $\nu$ . a.e.  $x \in \mathbb{T}^3$ , the conditional measure  $\nu_x^c$  has atoms.*

*Proof of Claim 3.5.* Assume, on the contrary, that  $\nu_x^c$  has no atoms for  $\nu$ . a.e.  $x \in \mathbb{T}^3$ . Notice that the center foliation  $\mathcal{F}^c$  is orientable and we fix an orientation. Up to replacing  $f$  by  $f^2$  and  $\nu$  by its ergodic components under  $f^2$ , one can assume that  $f$  preserves the orientation of  $\mathcal{F}^c$ . We denote by  $\Gamma_x = [\alpha_x, \omega_x]^c$  such that the orientation from  $\alpha_x$  directed to  $\omega_x$  gives the same orientation on  $\Gamma_x$  as we fixed.

For  $\nu$ . a.e.  $x \in \mathbb{T}^3$ , one can consider the function  $\phi_x^c : [\alpha_x, \omega_x]^c \rightarrow [0, 1]$  defined by  $\phi_x^c(z) = \nu_x^c([\alpha_x, z]^c)$ . As the conditional measures have no atoms, the function  $\phi_x^c$  is a continuous and non-decreasing function. Let  $\beta_x \in [\alpha_x, \omega_x]^c$  be the point such that  $\phi_x^c(\beta_x) = \frac{1}{2}$  and the length of the center segment  $[\alpha_x, \beta_x]^c$  is the smallest one. By the fact that  $f(\Gamma_x) = \Gamma_{f(x)}$  and  $f_*(\nu_x^c) = \nu_{f(x)}^c$ , one has  $f(\beta_x) = \beta_{f(x)}$ . Now, we can consider the family of measures  $\{2 \cdot \nu_x^c|_{[\alpha_x, \beta_x]^c}\}$  which gives an invariant probability measure  $\mu$  of  $f$ . By definition, the measure  $\mu$  is absolutely continuous with respect to the  $\nu$  and  $\mu \neq \nu$  which contradicts the ergodicity of  $\nu$ .  $\square$

By Claim 3.5, there exists an integer  $k_0 \in \mathbb{N}$  such that

$$\nu(\{x \in \mathbb{T}^3 \mid \text{there exists } z \in \Gamma_x \text{ with } \nu_x^c(z) \geq 1/k_0\}) > 0.$$

As  $f_*\nu_x^c = \nu_{f(x)}^c$  and  $f(\Gamma_x) = \Gamma_{f(x)}$ , the set  $\{x \in \mathbb{T}^3 \mid \text{there exists } z \in \Gamma_x \text{ with } \nu_x^c(z) \geq 1/k_0\}$  is  $f$ -invariant, and thus by the ergodicity of  $\nu$ , one has

$$\nu(\{x \in \mathbb{T}^3 \mid \text{there exists } z \in \Gamma_x \text{ with } \nu_x^c(z) \geq 1/k_0\}) = 1.$$

Notice that the center foliation  $\mathcal{F}^c$  is orientable and now we fix an orientation of  $\mathcal{F}^c$ . For each  $\Gamma_x$ , let  $x^- \in \Gamma_x$  and  $x^+ \in \Gamma_x$  be the leftest and rightest points in  $\Gamma_x$  with  $\nu_x^c$ -measure no less than  $1/k_0$  (one could have  $x^- = x^+$ ). If  $f$  preserves the orientation of the center foliation, then  $f(x^-) = (f(x))^-$  and  $f(x^+) = (f(x))^+$  due to the fact that  $f_*\nu_x^c = \nu_{f(x)}^c$ . Consider the probability measure  $\mu = \int \delta_{x^-} d\nu$ , then  $\mu$  is absolutely continuous with respect to  $\nu$  and the probability measure  $\mu$  is  $f$ -invariant. By the ergodicity of  $\nu$ , one has that  $\nu = \mu$ . Thus the disintegration of  $\nu$  along the center foliation has exactly one point. If  $f$  reverses the orientation of the center foliation, then  $f(x^-) = (f(x))^+$  and  $f(x^+) = (f(x))^-$  due to the fact that  $f_*\nu_x^c = \nu_{f(x)}^c$ . Consider the probability measure  $\mu = \frac{1}{2} \int (\delta_{x^-} + \delta_{x^+}) d\nu$ , then  $\mu$  is absolutely continuous with respect to  $\nu$  and the probability measure  $\mu$  is  $f$ -invariant. By the ergodicity of  $\nu$ , one has that  $\nu = \mu$ . Thus the disintegration of  $\nu$  along the center foliation has at most two points, and the conditional measures along the center foliation are equi-distributed on the atoms.  $\square$

**Remark 3.6.** 1. *For a non-hyperbolic ergodic measure  $\nu$ , if its disintegration along the center foliation has two atoms, then  $f$  must reverse the orientation of the center foliation and  $\nu = \frac{1}{2}(\nu_1 + \nu_2)$  where  $\nu_1, \nu_2$  are two different non-hyperbolic  $f^2$ -ergodic measures, and the disintegrations of  $\nu_1$  and  $\nu_2$  along the center foliation have only one atom. By our proof,  $\pi_*(\nu_1) = \pi_*(\nu_2)$  and thus  $\pi_*(\nu)$  is  $A^2$ -ergodic.*

2. Consider a linear Anosov diffeomorphism  $A$  on  $\mathbb{T}^3$  which reverses the center orientation. Consider a fixed point  $p$  of  $A$ , then one can deform  $A$  at  $p$  (following [M]) to get a partially hyperbolic diffeomorphism  $g$  such that  $p$  is still a fixed point of  $g$  and  $g$  admits a non-hyperbolic periodic point which is on the center leaf of  $p$  and whose period is exactly 2. This gives a non-hyperbolic ergodic measure of  $g$  whose disintegration along the center has exactly two atoms.

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*Ali Tahzibi*

Instituto de Ciências Matemáticas e de Computação,  
 Universidade de São Paulo (USP)  
 13566-590, São Carlos-SP, Brasil  
 tahzibi@icmc.usp.br

*Jinhua Zhang*

School of Mathematical Sciences  
 Beihang University  
 Beijing, 100191, P. R. China  
 jinhua\_zhang@buaa.edu.cn  
 zjh200889@gmail.com