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# STABILITY PROPERTIES OF STANDING WAVES FOR NLS EQUATIONS WITH THE $\delta'$-INTERACTION

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Abstract. We study the orbital stability of standing waves with discontinuous bump-like profile for the nonlinear Schrödinger model with the *repulsive* $\delta'$-interaction on the line. We consider the model with power non-linearity. In particular, it is showed that such standing waves are unstable in the energy space under some restrictions for parameters. The use of extension theory of symmetric operators by Krein-von Neumann is fundamental for estimating the Morse index of self-adjoint operators associated with our stability study. Moreover, for this purpose we use Sturm oscillation results and analytic perturbation theory. The Perron-Frobenius property for the repulsive $\delta'$-interaction is established.

The arguments presented in this investigation has prospects for the study of the stability of stationary waves solutions of other nonlinear evolution equations with point interactions.

**Mathematics Subject Classification (2000)**. Primary 34B24, 35B35, 35J10, 35Q51, 47A55; Secondary 34L40, 47A75, 47B25, 47E05.

**Key words**. Nonlinear Schrödinger equation, point interactions, orbital stability, bump solutions, self-adjoint extension, deficiency indices, Sturm-Liouville theory.

## 1. Introduction

In the last years the study of nonlinear dispersive models with point interactions has attracted a lot of attention of mathematicians and physicists. In particular, the prototype of framework for description of these phenomena have been on all the line and more recently on star graphs. Such models appear in nonlinear optics, Bose Einstein condensates, Wannier-Stark effect, and quantum graphs (or networks) (see [7, 22, 25, 26, 36, 56] and reference therein). The prototype equation for description of these models on the line is the nonlinear Schrödinger equation

$$iu_t(x,t) - \mathcal{A}u(x,t) + F(u(x,t)) = 0, \qquad x \neq 0, \tag{1.1}$$

where $(x,t) \in \mathbb{R}^+ \times \mathbb{R}$, $F(u)$ represents the nonlinearity, and $\mathcal{A}$ is a self-adjoint interaction operator with particular boundary conditions at $x = 0$. The most studied recently are the following two specific operators:

- Schrödinger operator $\mathcal{A} = \mathcal{A}_{\delta,-\alpha}$ with the $\delta$-interaction of intensity $-\alpha$ defined by

$$\mathcal{A}_{\delta,-\alpha}v(x) = -v''(x), \quad x \neq 0,$$
$$D(\mathcal{A}_{\delta,-\alpha}) = \big\{v \in H^1(\mathbb{R}) \cap H^2(\mathbb{R} - \{0\}) : v'(0+) - v'(0-) = -\alpha v(0)\big\}.$$

The operator $\mathcal{A}_{\delta,-\alpha}$ is formally defined by the expression $\mathcal{A}_{\delta,-\alpha} = -\frac{d^2}{dx^2} - \alpha\delta(x)$, where $\delta(x)$ is the Dirac delta distribution centered at $x = 0$. In this case, equation (1.1) is called the NLS-$\delta$ model.

- Schrödinger operator $\mathcal{A} = \mathcal{A}_{\delta',-\beta}$ with $\delta'$-interaction of intensity $-\beta$ defined by

$$\mathcal{A}_{\delta',-\beta}v(x) = -v''(x), \quad x \neq 0,$$
$$D_\beta := D(\mathcal{A}_{\delta',-\beta}) = \{H^2(\mathbb{R} - \{0\}) : v(0+) - v(0-) = -\beta v'(0), v'(0+) = v'(0-)\}.$$

We recall that $\mathcal{A}_{\delta',-\beta}$ is formally defined by the expression $\mathcal{A}_{\delta',-\beta} = -\frac{d^2}{dx^2} - \beta\langle\cdot,\delta'\rangle\delta'(x)$, and that the elements in $D_\beta$ do not need to be continuous, however they have a continuous derivative at $x = 0$. Thus, the function $v \in D_\beta$ such that $v'(0) = 0$ obviously belongs to $H^2(\mathbb{R})$. In particular, every even function belonging to $D_\beta$ is a $H^2(\mathbb{R})$-function.

The mathematical study of these two point interaction models with nonlinearities $F(u) = |u|^{p-1}u$, $p > 1$, and $F(u) = u\mathrm{Log}|u|^2$, has attracted a lot of attention, and currently it is a very active research area (see [2–6,12–16,19,20,23,28,31–35,37,38,40,44–47,50] and reference therein). Numerous analytical, numerical and experimental works deal with special solutions of (1.1). In particular, a big part of them consider so-called *standing wave solutions* which preserve the spatial shape and harmonically oscillate in time, namely, solutions of the form

$$u(x,t) = e^{i\omega t}\varphi(x).$$

For example, in the case of nonlinearity $F(u) = |u|^{p-1}u$, we induce that the profile $\varphi$ satisfies the equation

$$\mathcal{A}\varphi + \omega\varphi - |\varphi|^{p-1}\varphi = 0, \qquad \varphi \in D(\mathcal{A}). \tag{1.2}$$

In this paper we investigate the orbital stability of the standing waves of the NLS-$\delta'$ model with power nonlinearity

$$iu_t - \mathcal{A}_{\delta',-\beta}u + |u|^{p-1}u = 0 \tag{1.3}$$

in the case of $\beta < 0$ (*repulsive* $\delta'$-interaction). One of the main advantages of using delta-type potentials is the existence of an explicit expression for the profile $\varphi$ in (1.2). This allows one to prove very specific stability results, the proofs of which are considerably harder in the case of an effective linear potential term $V(x)$ in (1.1), i.e. for $\mathcal{A} = -\partial_x^2 - V(x)$ (see [21, 48, 53, 54, 59, 67] and reference therein). We recall that the general NLS model with external potential

$$iu_t + u_{xx} + V(x)u + F(u) = 0 \tag{1.4}$$

has been studied theoretically and experimentally in Bose-Einstein condensates (see [27, 64, 65] and reference therein). It model also represents a trapping (wave-guiding) structure for light beams, induced by an inhomogeneity of the local refractive index (see [7, 49, 52, 55, 60, 61, 68, 69] and reference therein). In particular, the $\delta$- and $\delta'$-interaction terms in (1.1) adequately represent narrow trap which is able to capture broad solitonic beams (see [69]).

For completeness we will briefly describe the main results on the stability of standing waves for the model (1.1) with the $\delta$- and $\delta'$-interaction. In [37] the authors showed that NLS-$\delta$ equation with power nonlinearity has a unique positive even solution (modulo rotation) for $\omega > \frac{\alpha^2}{4}$,

$$\varphi_{\omega,\alpha}(x) = \left[\frac{(p+1)\omega}{2}\operatorname{sech}^2\left(\frac{(p-1)\sqrt{\omega}}{2}|x| + \tanh^{-1}\left(\frac{\alpha}{2\sqrt{\omega}}\right)\right)\right]^{\frac{1}{p-1}}, \quad x \in \mathbb{R}.$$

For $\alpha < 0$ the continuous profile $\varphi_{\omega,\alpha}$ has exactly two bumps. In this case the standing wave $e^{i\omega t}\varphi_{\omega,\alpha}$ is unstable "almost for sure" in $H^1(\mathbb{R})$ for any $p > 1$ (see [31, 37, 63]). Mention that the classical variational argument (for instance, via the Nehari manifold analysis) is not applicable for the stability investigation.

Further, as far as we know, the NLS-$\delta'$ model has not been studied in the repulsive case, namely, for $\beta < 0$. From [3, Proposition 5.1] it follows that for $\beta < 0$ equation (1.2) has two types of solutions (odd and asymmetric, see Figures 1 (a)-(b) below)

$$\varphi^{odd}_{\omega,\beta}(x) = \text{sign}(x)\left[\frac{(p+1)\omega}{2}\,\text{sech}^2\left(\frac{(p-1)\sqrt{\omega}}{2}(|x|+y_0)\right)\right]^{\frac{1}{p-1}}, \quad x \neq 0; \quad \tfrac{4}{\beta^2} < \omega, \tag{1.5}$$

$$\varphi^{as}_{\omega,\beta}(x) = \begin{cases} \left[\frac{(p+1)\omega}{2}\,\text{sech}^2\left(\frac{(p-1)\sqrt{\omega}}{2}(x+y_1)\right)\right]^{\frac{1}{p-1}}, & x > 0; \\ -\left[\frac{(p+1)\omega}{2}\,\text{sech}^2\left(\frac{(p-1)\sqrt{\omega}}{2}(x-y_2)\right)\right]^{\frac{1}{p-1}}, & x < 0, \end{cases}, \quad \omega > \frac{4}{\beta^2}\frac{p+1}{p-1},$$

where $y_1$ and $y_2$ are negative constants depending on $\beta, p, \omega$, and satisfying specific relations, and $y_0 < 0$ in (1.5) is defined by

$$y_0 = \frac{1}{\sqrt{\omega}(p-1)}\text{Log}\Big(\frac{\beta\sqrt{\omega}+2}{\beta\sqrt{\omega}-2}\Big). \tag{1.6}$$

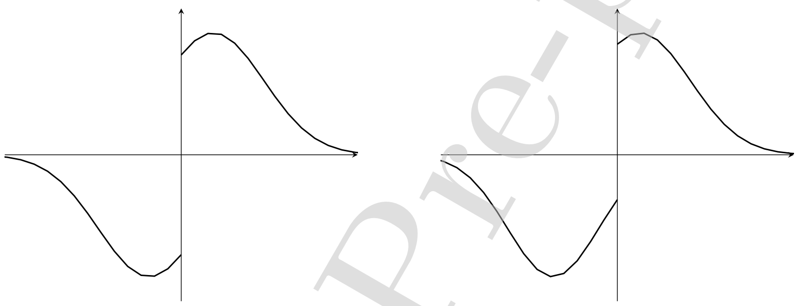

Figure 1(a). $\varphi^{odd}_{\omega,\beta}$ for $\beta < 0$ Figure 1(b). $\varphi^{as}_{\omega,\beta}$ for $\beta < 0$

For the case $\beta > 0$, the profiles in (1.5) are still solutions for (1.2) ($y_0, y_1, y_2$ are positive in this case) and they are of tail-type profile (see Figure 3 below). Their stability was studied in [3] by variational techniques and Grillakis, Shatah and Strauss stability approach (see [42]). In Section 5 below, via extension theory approach, we improved the stability results in [3].

The main purpose of this paper is to establish the stability properties of the odd discontinuous bump-like profile defined in (1.5) in the case $\beta < 0$. To our knowledge this problem is quite new. Our approach is based on the classical Sturm-Liouville theory (on the line and on the half-line) and the extension theory of symmetric operators by Krein-von Neumann, which provides the key ingredient for estimating the Morse index of specific self-adjoint Schrödinger operators associated with $\varphi^{odd}_{\omega,\beta}$'s profiles. The analytic perturbation theory and continuation arguments for analytic families of linear operators help us to obtain the precise values of the Morse indices. Moreover, we use in our analysis the fact that the mentioned self-adjoint Schrödinger operators satisfy the Perron-Frobenius property in the case of the repulsive $\delta'$-interaction (see Lemma 6.5).

Our main stability theorem for the odd bump-like $\varphi^{odd}_{\omega,\beta}$ is the following.

**Theorem 1.1.** *Let $\mathcal{A} = \mathcal{A}_{\delta',-\beta}$ in (1.2), $\beta < 0$, and $p > 1$. Let also $\varphi^{odd}_{\omega,\beta}$ be defined by* (1.5) *for $\omega > \frac{4}{\beta^2}$. If $\omega^* = \frac{4(p+1)}{\beta^2(p-1)}$, then the following assertions hold.*

1) *Let $p \in [5, +\infty)$. Then for all $\omega$, the standing wave $e^{i\omega t}\varphi^{odd}_{\omega,\beta}$ is linearly unstable (also orbitally unstable) in $H^1(\mathbb{R} - \{0\})$.*
2) *Let $p \in (1, 3]$. Then,*
   a) *for $p \in (1, 2]$ and $\omega < \omega^*$, the standing wave $e^{i\omega t}\varphi^{odd}_{\omega,\beta}$ is linearly unstable in $H^1(\mathbb{R} - \{0\})$;*
   b) *for $p \in (2, 3]$ and $\omega < \omega^*$, the standing wave $e^{i\omega t}\varphi^{odd}_{\omega,\beta}$ is linearly unstable (also orbitally unstable) in $H^1(\mathbb{R} - \{0\})$;*
   c) *for $\omega \geqq \omega^*$, the standing wave $e^{i\omega t}\varphi^{odd}_{\omega,\beta}$ is orbitally stable in $H^1_{\mathrm{odd}}(\mathbb{R} - \{0\})$.*

*Here $H^1_{\mathrm{odd}}(\mathbb{R} - \{0\})$ denotes the subspace of odd functions in $H^1(\mathbb{R} - \{0\})$.*

The case $p \in (3, 5)$ is studied in Remark 4.3-a) and it is based in numerical simulations. The nonlinear instability property of the standing wave $e^{i\omega t}\varphi^{\mathrm{odd}}_{\omega,\beta}$ established in Theorem 1.1, it is deduced of the spectral instability property of this profile (see Remark 2.4 below). The stability properties of the discontinuous non-symmetric bump-like profiles $\varphi^{as}_{\omega,\beta}$ in (1.5) will be the subject of an upcoming study of us.

Lastly, our method have allowed us to establish the first results of the orbital (in)stability of the Gaussian-type standing waves $u(x,t) = e^{i\omega t}\psi_{\omega,\gamma}$ with discontinuous bump-like profile

$$\psi_{\omega,\gamma}(x) = \mathrm{sign}(x)e^{\frac{\omega+1}{2}}e^{-\frac{1}{2}(|x|+\frac{2}{\gamma})^2}, \quad x \neq 0, \tag{1.7}$$

with $\omega \in \mathbb{R}$ and $\gamma < 0$, for the nonlinear Schrödinger equation with logarithmic nonlinearity and the $\delta'$-interaction

$$iu_t - \mathcal{A}_{\delta',-\gamma}u + u\mathrm{Log}|u|^2 = 0. \tag{1.8}$$

This study is currently being written.

**Notation**

Let $A$ be a densely defined closed symmetric operator on a Hilbert space $H$ with domain $D(A)$, and let $A^*$ be its adjoint. We denote *deficiency subspaces* of $A$ by $\mathcal{N}_+(A) := \mathrm{Ker}(A^* - i)$ and $\mathcal{N}_-(A) := \mathrm{Ker}(A^* + i)$. The *deficiency indices* of $A$ are denoted by $n_\pm(A) := \dim(\mathcal{N}_\pm(A))$. The number of negative eigenvalues counting multiplicities (Morse index) is denoted by $n(A)$. The spectrum (resp. point spectrum) of $A$ is denoted by $\sigma(\mathcal{A})$ (resp. $\sigma_p(\mathcal{A})$). The resolvent set of $A$ is denoted by $\rho(A)$. By $\dim(\mathrm{Ran}(A))$ we denote the dimension of the range of the operator A given by $\mathrm{Ran}(A) = \{Ax : x \in D(A)\}$.

Let $I$ be interval on the real line, by $\|\cdot\|_p$ we denote the norm in $L^p(I)$. In particular, $\|\cdot\|$ denotes the norm in $L^2(I)$, and $\langle\cdot,\cdot\rangle$ denotes the scalar product in $L^2(I)$. The Sobolev spaces are denoted by $H^k(I)$, $k \in \mathbb{N}$.

## 2. Stability analysis framework for NLS-$\delta'$ equation

Crucial role in the stability analysis is played by the symmetries of the NLS equation (1.1) with point interactions. The basic symmetry associated to the mentioned equation with the nonlinearity $F$ satisfying $F(e^{i\theta}u) = e^{i\theta}F(u)$ is phase-invariance (in particular, translation invariance does not hold due to the defect). Thus, it is reasonable to define orbital stability as follows.

**Definition 2.1.** The standing wave $u(x,t) = e^{i\omega t}\varphi(x)$ is said to be *orbitally stable* in a Hilbert space $X$ by the flow of equation (1.1) if for any $\varepsilon > 0$ there exists $\eta > 0$ with the following property: if $u_0 \in X$ satisfies $\|u_0 - \varphi\|_X < \eta$, then the solution $u(t)$ of (1.1) with $u(0) = u_0$ exists for any $t \in \mathbb{R}$ and

$$\sup_{t\in\mathbb{R}}\inf_{\theta\in\mathbb{R}}\|u(t)-e^{i\theta}\varphi\|_X<\varepsilon.$$

Otherwise, the standing wave $u(x,t)=e^{i\omega t}\varphi(x)$ is said to be *orbitally unstable* in $X$.

In our analysis for the $\delta'$-interaction models the energy space $X$ in Definition 2.1 will coincide with one of the spaces $H^1(\mathbb{R}-\{0\})$, $H^1_{\text{odd}}(\mathbb{R}-\{0\})$.

Investigation of the orbital stability by Grillakis, Shatah and Strauss approach (see [41,42]) requires well-posedness of the associated initial value problem to (1.3). A part of the following result was proved in [3],

**Theorem 2.2.** *Let $p>1$ and $\beta\neq 0$. Then equation* (1.3) *is locally well-posed in $H^1(\mathbb{R}-\{0\})$, namely, for any $u_0\in H^1(\mathbb{R}-\{0\})$ there exists $T=T(\|u_0\|_{H^1(\mathbb{R}-\{0\})})>0$ such that equation* (1.3) *has a unique solution $u\in C([-T,T];H^1(\mathbb{R}-\{0\}))$ satisfying $u(0)=u_0$. For each $T_0\in(0,T)$ the mapping data-solution*

$$\varphi\in B_\delta(u_0)\subset H^1(\mathbb{R}-\{0\})\to u\in C([-T_0,T_0];H^1(\mathbb{R}-\{0\})),\ \ u(0)=\varphi$$

*is continuous for some $\delta>0$ small. In particular, for $p>2$ this mapping is at least of class $C^2$. Moreover, if an initial data $u_0$ is odd, then the solution $u(t)$ is also odd.*

*The equation* (1.3) *has the following conservation laws (of energy and charge)*

$$E(u)=E(u_0),\qquad Q(u)=\tfrac{1}{2}\|u\|^2=\tfrac{1}{2}\|u_0\|^2,$$

*where the energy is defined by*

$$E(u)=\tfrac{1}{2}\|u'\|^2-\tfrac{1}{p+1}\|u\|_{p+1}^{p+1}-\tfrac{1}{2\beta}|u(0+)-u(0-)|^2.$$

*In particular, for $1<p<5$ the solutions to* (1.3) *are globally defined in time.*

*Proof.* The local well-posedness result in $H^1(\mathbb{R}-\{0\})$ follows from standard arguments of the Banach fixed point theorem and it was proved in [3]. The $C^2$-regularity of the mapping data-solution is not so standard. By convenience of the reader we give a sketch of the prove for the case $\beta<0$. Consider the mapping $J_{u_0}:C([-T,T];H^1(\mathbb{R}-\{0\}))\longrightarrow C([-T,T];H^1(\mathbb{R}-\{0\}))$ given by

$$J_{u_0}[u](t)=e^{-i\mathcal{A}_\beta}u_0+i\int_0^t e^{-i(t-s)\mathcal{A}_\beta}|u(s)|^{p-1}u(s)ds,$$

with $\mathcal{A}_\beta\equiv\mathcal{A}_{\delta',-\beta}$ and $e^{-it\mathcal{A}_\beta}$ being the unitary group associated to the linear equation

$$iu_t=\mathcal{A}_\beta\equiv-u_{xx}-\beta\langle\delta',\cdot\rangle\delta'(x)u. \tag{2.1}$$

One needs to show that the mapping $J_{u_0}$ is well-defined. Using the one-dimensional Gagliardo-Nirenberg inequality, the relation $|((|f|^{p-1}f)'|\leq C_0|f|^{p-1}|f'|$ and Hölder's inequality, we obtain for $u\in H^1(\mathbb{R}-\{0\})$

$$|||u|^{p-1}u||_{H^1(\mathbb{R}-\{0\})}\leq C_1||u||^p_{H^1(\mathbb{R}-\{0\})}. \tag{2.2}$$

Moreover, using (2.2) and the $L^2$-unitarity of $e^{-it\mathcal{A}_\beta}$, we get

$$||J_{u_0}[u](t)||_{H^1(\mathbb{R}-\{0\})}\leq C_2||u_0||_{H^1(\mathbb{R}-\{0\})}+C_3T\sup_{s\in[0,T]}||u(s)||^p_{H^1(\mathbb{R}-\{0\})},$$

where the positive constants $C_2,C_3$ do not depend on $u_0$. Therefore $J_{u_0}[u](t)\in H^1(\mathbb{R}-\{0\})$. The continuity and contraction property of $J_{u_0}$ are proved in a standard way. Therefore, we obtain the existence of a unique solution to the Cauchy problem associated to (1.3) on $H^1(\mathbb{R}-\{0\})$.

The fact that the solution preserves oddness follows from the particular form of the kernel $\mathcal{K}_\beta$ associated to $e^{-it\mathcal{A}_\beta}$ (in other words, the fundamental solution for (2.1)). In the case $\beta<0$, $\mathcal{K}_\beta$ is defined by (see [9])

$$\mathcal{K}_\beta(x,y;t)=\mathcal{K}(x-y;t)+\mathrm{sign}(xy)\mathcal{K}(|x|+|y|;t)-\frac{2}{\beta}\int_0^\infty \mathrm{sign}(xy)e^{\frac{2}{\beta}s}\mathcal{K}(s+|x|+|y|;t)ds,$$

where $\mathcal{K}(\cdot;t)$ is the fundamental solution to the classical linear Schrödinger equation $iu_t=-u_{xx}$ defined by

$$\mathcal{K}(x;t)=\frac{e^{-x^2/4it}}{(4i\pi t)^{1/2}},\quad t>0.$$

Next, we recall that the argument based on the contraction mapping principle above has the advantage that if $F(u,\overline{u})=|u|^{p-1}u$ has a specific regularity, then it is inherited by the mapping data-solution. Indeed, following the ideas in [14], we consider for $(v_0,v)\in B(u_0;\epsilon)\times C([-T,T];H^1(\mathbb{R}-\{0\}))$ the mapping

$$\Gamma(v_0,v)(t)=v(t)-J_{v_0}[v](t),\qquad t\in[-T,T].$$

Then $\Gamma(u_0,u)(t)=0$ for all $t\in[-T,T]$. For $p-1$ being an even integer, $F(u,\overline{u})$ is smooth, and therefore $\Gamma$ is smooth. Hence, using the arguments applied for obtaining the local well-posedness in $H^1(\mathbb{R}-\{0\})$ above, we can show that the operator $\partial_v\Gamma(u_0,u)$ is one-to-one and onto. Thus, by the Implicit Function Theorem there exists a smooth mapping $\Lambda: B(u_0;\delta)\to C([-T,T];H^1(\mathbb{R}-\{0\}))$ such that $\Gamma(v_0,\Lambda(v_0))=0$ for all $v_0\in B(u_0;\delta)$. This argument establishes the smoothness property of the mapping data-solution associated to equation (1.3) when $p-1$ is an even integer.

If $p-1$ is not an even integer and $p>2$, then $F(u,\overline{u})$ is $C^{[p]}$-function, and consequently the mapping data-solution is of class $C^{[p]}$ (see [58, Remark 5.7]). Therefore, for $p>2$ we conclude that the mapping data-solution is at least of class $C^2$. This finishes the proof. □

To formulate the stability criterium for the NLS-$\delta'$ equation in the framework of the Grillakis, Shatah and Strauss theory, we define the following two self-adjoint linear operators

$$\begin{cases} L^\beta_{1,\omega}=-\dfrac{d^2}{dx^2}+\omega-p|\varphi^{odd}_{\omega,\beta}|^{p-1}, \quad L^\beta_{2,\omega}=-\dfrac{d^2}{dx^2}+\omega-|\varphi^{odd}_{\omega,\beta}|^{p-1},\\ \mathrm{dom}(L^\beta_{j,\omega})=D_\beta,\ \ j\in\{1,2\}.\end{cases}\tag{2.3}$$

The operators $L^\beta_{1,\omega}$ and $L^\beta_{2,\omega}$ are associated with the action functional $S^\beta_\omega: H^1(\mathbb{R}-\{0\})\to\mathbb{R}$ defined by

$$S^\beta_\omega(\psi)=\tfrac12\|\psi'\|^2+\tfrac{\omega}{2}\|\psi\|^2-\tfrac{1}{p+1}\|\psi\|^{p+1}_{p+1}-\tfrac{1}{2\beta}|\psi(0+)-\psi(0-)|^2$$

in the sense of bilinear forms. Namely, for $\varphi_{\omega,\beta}=\varphi^{odd}_{\omega,\beta}$ we have

$$(S^\beta_\omega)''(\varphi_{\omega,\beta})(u,v)=\langle L^\beta_{1,\omega}u_1,v_1\rangle+\langle L^\beta_{2,\omega}u_2,v_2\rangle,\tag{2.4}$$

where $u=u_1+iu_2$ and $v=v_1+iv_2$. The functions $u_j,v_j$, $j\in\{1,2\}$, are real valued. With $(S^\beta_\omega)''(\varphi_{\omega,\beta})$ we associate the formally self-adjoint operator $\mathcal{H}_\omega\equiv(S^\beta_\omega)''(\varphi_{\omega,\beta})$,

$$\mathcal{H}_\omega=\begin{pmatrix} L^\beta_{1,\omega} & 0\\ 0 & L^\beta_{2,\omega}\end{pmatrix}.\tag{2.5}$$

Define the number $p(\omega_0)$ by

$$p(\omega_0)=\begin{cases}1, & \text{if } \partial_\omega\|\varphi_{\omega,\beta}\|^2>0 \text{ at } \omega=\omega_0,\\ 0, & \text{if } \partial_\omega\|\varphi_{\omega,\beta}\|^2<0 \text{ at } \omega=\omega_0,\end{cases}$$

By Stability/Instability Theorem in [42] we can state.

**Theorem 2.3.** *Suppose that* $Ker(L^{\beta}_{2,\omega}) = [\varphi_{\omega,\beta}]$, $Ker(L^{\beta}_{1,\omega}) = \{0\}$, *and the rest of the spectrum of* $L^{\beta}_{2,\omega}$ *and* $L^{\beta}_{1,\omega}$ *is bounded away from zero. Then the following assertion hold.*

1) *If* $n(\mathcal{H}_\omega) = p(\omega)$, *the standing wave* $e^{i\omega t}\varphi_{\omega,\beta}$ *is orbitally stable in* $H^1(\mathbb{R} - \{0\})$.
2) *If* $n(\mathcal{H}_\omega) - p(\omega)$ *is odd, the standing wave* $e^{i\omega t}\varphi_{\omega,\beta}$ *is orbitally unstable in* $H^1(\mathbb{R} - \{0\})$.

The same result holds in the space $H^1_{\text{odd}}(\mathbb{R} - \{0\})$.

*Remark* 2.4. We note that it is well known that the condition $n(\mathcal{H}_\omega) - p(\omega)$ odd in Theorem 2.3 implies the spectral instability of $e^{i\omega t}\varphi^{odd}_{\omega,\beta}$, namely, that the spectrum of the linear operator

$$\mathcal{G} = \begin{pmatrix} 0 & L^{\beta}_{2,\omega} \\ -L^{\beta}_{1,\omega} & 0 \end{pmatrix}, \tag{2.6}$$

associated to the linearization of the time-dependent NLS-$\delta'$ model around $\varphi^{odd}_{\omega,\beta}$, contains an eigenvalue with positive real part. To conclude orbital instability due to [42], it is sufficient to show estimate (6.2) in [42] for the semigroup $e^{t\mathcal{G}}$ generated by $\mathcal{G}$. In general, it is a nontrivial issue to be verified in the case of Schrödinger operators with point interactions (see Ohta [63] and Georgiev&Ohta [39]). However, we conjecture that for the operator $\mathcal{G}$ we have the spectral mapping theorem (that is, $\sigma(e^{\mathcal{G}}) = e^{\sigma(\mathcal{G})}$), which would imply estimate (6.2) in [42]. On the other hand, if we use Theorem 2.2, Remark 2 in Section 2 of [43] and the property that the mapping data-solution associated to equation (1.3) is of class $C^2$ around $\varphi^{odd}_{\omega,\beta}$ for $p > 2$, we can obtain that the spectral instability results imply nonlinear instability in Theorem 1.1 (see Angulo&Natali [17] and Angulo&Neves&Lopes [18] where this kind of strategy has been used for obtaining nonlinear instability results).

In the following sections we study spectral properties of $L^{\beta}_{2,\omega}$ and $L^{\beta}_{1,\omega}$ required by the above theorem.

## 3. Morse index of $\mathcal{H}_\omega$

The main result of this section is the following.

**Theorem 3.1.** *Let* $\beta < 0$ *and* $\omega > \frac{4}{\beta^2}$. *Let also* $\omega^* = \frac{4(p+1)}{\beta^2(p-1)}$ *and* $\mathcal{H}_\omega$ *be defined in (2.5). Then we have:*

1) *if* $\omega \geqq \omega^*$, *then* $n(\mathcal{H}_\omega) = 3$, *and* $n(\mathcal{H}_\omega|_{\text{odd}}) = 1$;
2) *if* $\omega < \omega^*$, *then* $n(\mathcal{H}_\omega) = 4$, *and* $n(\mathcal{H}_\omega|_{\text{odd}}) = 1$.

*Here* $\mathcal{H}_\omega|_{\text{odd}}$ *denotes the restriction of the operator* $\mathcal{H}_\omega$ *to the subspace of odd functions in* $D_\beta$.

The proof of Theorem 3.1 is given at the end of this section. First we will estimate the Morse index of the operators $L^{\beta}_{1,\omega}$ and $L^{\beta}_{2,\omega}$ using Sturm-Liouville theory, the Perron-Frobenius property, the extension theory of symmetric operators, and the theory of analytic perturbations.

### 3.1. Spectral analysis for $L^{\beta}_{1,\omega}$.

In this subsection we study the spectral properties of the operator $L^{\beta}_{1,\omega}$ defined in (2.3). For notational simplicity, throughout the paper we will write $\varphi_{\omega,\beta}$ instead of $\varphi^{odd}_{\omega,\beta}$.

Below we establish some properties of the profile $\varphi_{\omega,\beta}$ for $\beta < 0$. Initially, we have that $\varphi''_{\omega,\beta}(0+) = -\varphi''_{\omega,\beta}(0-)$, and $\varphi''_{\omega,\beta}(0+) = 0$ if and only if $\omega = \frac{4(p+1)}{\beta^2(p-1)}$. Indeed, from (1.5) we obtain

$$\begin{aligned}\varphi''_{\omega,\beta}(0+) = 0 \Leftrightarrow 0 &= \omega - \frac{(p+1)\omega}{2}\operatorname{sech}^2\Big(\frac{(p-1)\sqrt{\omega}}{2}y_0\Big)\\ &= \frac{(1-p)\omega}{2} + \frac{(p+1)\omega}{2}\tanh^2\Big(\frac{p-1}{2}\sqrt{\omega}y_0\Big).\end{aligned}$$

Next, using the definition of $y_0$ in (1.6) and the relation

$$\operatorname{arctanh}(x) = \frac{1}{2}\operatorname{Log}\Big(\frac{1+x}{1-x}\Big), \qquad |x| < 1,$$

we obtain $\tanh\Big(\frac{p-1}{2}\sqrt{\omega}y_0\Big) = \frac{2}{\beta\sqrt{\omega}}$, and thus

$$\varphi''_{\omega,\beta}(0+) = 0 \Leftrightarrow 0 = \frac{(1-p)\omega}{2} + \frac{(p+1)\omega}{2}\frac{4}{\omega\beta^2} \Leftrightarrow \omega = \frac{4(p+1)}{\beta^2(p-1)}.$$

Denoting $\omega^* = \frac{4(p+1)}{\beta^2(p-1)}$, from the above analysis we get

$$\varphi''_{\omega,\beta}(0+) > 0, \quad \text{for} \quad \omega < \omega^*, \quad \text{and} \quad \varphi''_{\omega,\beta}(0+) < 0, \quad \text{for} \quad \omega > \omega^*. \tag{3.1}$$

In particular, for $\omega = \omega^*$ we have the *crucial properties*

$$\varphi'_{\omega^*,\beta} \in D_\beta, \text{ and } \varphi'_{\omega^*,\beta} \in H^2(\mathbb{R}),$$

while $\varphi'_{\omega,\beta} \notin D_\beta$ for every $\omega \neq \omega^*$ (see Proposition 3.2 below).

Next, we consider the following domain (where the one-dimensional Laplacian operator on the positive half-line remains self-adjoint) $W_\theta$ for $\theta \in \mathbb{R}$

$$W_\theta = \{v \in H^2(0,+\infty) : v(0+) = \theta v'(0+)\}. \tag{3.2}$$

For any $\omega$ we have $\varphi_{\omega,\beta}|_{(0,+\infty)} \in W_{-\frac{\beta}{2}}$. Now, we determine $\theta_0 \in \mathbb{R}$ such that $\varphi'_{\omega,\beta} \in W_{\theta_0}$, for every $\omega \neq \omega^*$, i.e.,

$$\begin{aligned}\varphi'_{\omega,\beta}(0+) = \theta_0\varphi''_{\omega,\beta}(0+) \Leftrightarrow -\frac{\sqrt{\omega}}{\theta_0}\frac{2}{\beta\sqrt{\omega}} &= \frac{(1-p)\omega}{2} + \frac{(p+1)\omega}{2}\tanh^2\Big(\frac{p-1}{2}\sqrt{\omega}y_0\Big)\\ &= \frac{(1-p)\omega}{2} + \frac{(p+1)\omega}{2}\frac{4}{\omega\beta^2}.\end{aligned} \tag{3.3}$$

Thus, we obtain the relation

$$\theta_0 = \frac{-4\beta}{\omega\beta^2(1-p) + 4(p+1)}. \tag{3.4}$$

Therefore, from (3.1) it follows that $\theta_0 < 0$ for $\omega > \omega^*$, and $\theta_0 > 0$ for $\omega < \omega^*$.

3.1.1. **Kernel of $L^\beta_{1,\omega}$.** In this subsection we investigate the structure of the kernel of $L^\beta_{1,\omega}$.

**Proposition 3.2.** *Let $\beta < 0$, $\omega > \frac{4}{\beta^2}$, and $\varphi_{\omega,\beta} = \varphi^{odd}_{\omega,\beta}$. Let also $\omega^* = \frac{4(p+1)}{\beta^2(p-1)}$. Then the following assertions hold:*

1) *if $\omega \neq \omega^*$, then $Ker(L^\beta_{1,\omega}) = \{0\}$;*
2) *if $\omega = \omega^*$, then $Ker(L^\beta_{1,\omega^*}) = \big[\frac{d}{dx}\varphi_{\omega^*,\beta}\big]$.*

*Proof.* Let $v \in D_\beta$ such that $L^\beta_{1,\omega}v = 0$.

1) Suppose $\omega \neq \omega^*$. Denoting $\varphi_{\omega,\beta}(x) \equiv \text{sign}(x)\psi(x)$ (see (1.5)), we get from Sturm-Liouville oscillation theory on the half-line (see [24, Chapter II])
$$v(x) = \begin{cases} \mu\psi'(x), & x > 0; \\ -\nu\psi'(x), & x < 0, \end{cases} \tag{3.5}$$
with $\mu, \nu \in \mathbb{R}$. Since $v'(0+) = \mu\psi''(0+) = v'(0-) = -\nu\psi''(0-) = -\nu\psi''(0+)$, and $\psi''(0+) \neq 0$, then $\mu+\nu = 0$. From $v(0+) - v(0-) = -\beta v'(0)$ and $\psi'(0-) = -\psi'(0+)$ follows $-\beta\mu\psi''(0+) = \mu\psi'(0+) + \nu\psi'(0-) = \mu(\psi'(0+) - \psi'(0-)) = 2\mu\psi'(0+)$. Suppose $\mu \neq 0$, then $\psi'(0+) = -\frac{\beta}{2}\psi''(0+)$, which is false due to $\omega > \frac{4}{\beta^2}$. Indeed, from (1.5) we have
$$\psi'(0+) = -\frac{\beta}{2}\psi''(0+) \Leftrightarrow \frac{4}{\beta^2} = \frac{(1-p)\omega}{2} + \frac{(p+1)\omega}{2}\frac{4}{\omega\beta^2} \Leftrightarrow \omega = \frac{4}{\beta^2}.$$
Therefore, $\mu = \nu = 0$, and finally $v \equiv 0$.
2) Suppose $\omega = \omega^*$. From (3.5) and $\psi''(0+) = 0$ we obtain $(\mu - \nu)\psi'(0+) = 0$, consequently $\mu = \nu$. Therefore, $v = \mu\varphi'_{\omega^*,\beta}$.

□

*Remark* 3.3. Proposition 3.2 (see also Theorem 5.1 below) shows a very peculiar behavior of NLS models with singular interactions. Indeed, by the breakdown of translation symmetry for NLS models in (1.1) would expect that the kernel of the self-adjoint Schrödinger operator $L^{\beta}_{1,\omega}$ was always trivial for any admissible phase-parameter $\omega$ (such as in the case of a $\delta$-interaction ( [31])). We note that there are other settings for NLS models where this kernel behavior can happen. By instance, in the case of the NLS model in (1.4) with $F(u) = |u|^{p-1}u$, $p > 1$, and with an external real-valued, symmetric potential (even in $x$) $V$ satisfying: $V(x), xV'(x) \in L^\infty(\mathbb{R})$, $\lim_{|x|\to\infty} V(x) = 0$, and $-V$ having a non-degenerate maxima at $x = 0$ we have that the following linearized operator with domain $H^2(\mathbb{R})$
$$L_+ = -\frac{d^2}{dx^2} + E - V(x) + p|\psi_E|^{p-1}$$
has zero as a simple eigenvalue for exactly one value $E = E_*$ of a solution-curve $E \in I \to \psi_E \in H^2(\mathbb{R})$ of states for (1.4) (see Theorem 1 in Kirr&Kevrekidis&Pelinovsky [53] for more details).

3.1.2. **Morse index of $L^{\beta}_{1,\omega^*}$ for $\omega^* = \frac{4(p+1)}{\beta^2(p-1)}$.** In this subsection we obtain an estimate for the Morse index of $L^{\beta}_{1,\omega^*}$ defined on $D_\beta$. We start with the following result which is a consequence of the Cauchy uniqueness principle.

**Lemma 3.4.** *Let $\beta < 0$ and $\omega > \frac{4}{\omega\beta^2}$. Let also $L^{\beta}_{1,\omega}$ be defined by* (2.3)*. Suppose that $\lambda$ is a simple eigenvalue of $L^{\beta}_{1,\omega}$, then the associated eigenfunction is either even or odd.*

*Proof.* Let $v \in D_\beta - \{0\}$ such that $L^{\beta}_{1,\omega}v = \lambda v$. Define $\xi(x) = v(-x)$, for $x \neq 0$. Then $\xi \in D_\beta$ and $L^{\beta}_{1,\omega}\xi = \lambda\xi$. Thus, by simplicity of $\lambda$, there is $\mu \in \mathbb{R}$ such that $\xi(x) = \mu v(x)$ for every $x \neq 0$, hence $v(-x) = \mu v(x)$. Consider two cases.
1) Suppose $v'(0+) \neq 0$. Then since $-v'(0+) = -v'(0-) = \mu v'(0+)$, we have $\mu = -1$, and therefore $v$ is odd.
2) Suppose $v'(0+) = 0$. Then $v(0+) = v(0-)$ and $v \in H^2(\mathbb{R})$. Thus, $v(-x) = \mu v(x)$ on all $x \in \mathbb{R}$. Hence $v(0-) = \mu v(0+) = \mu v(0-)$, and by the Cauchy uniqueness principle, $v(0) \neq 0$. Therefore, $\mu = 1$, and consequently $v$ is even.

□

The following two statements follow from Sturm-Liouville oscillation results on the half-line (see [24]) and from the Perron-Frobenius property satisfied by the Schrödinger operators with a $\delta'$-interaction defined in (2.3) for $\beta < 0$ (see Lemma 6.5 below).

**Lemma 3.5.** *Let $\beta < 0$ and $\omega^* = \frac{4(p+1)}{\beta^2(p-1)}$. Consider the operator $L^{\beta}_{1,\omega^*}$ on the domain (with Neumann type condition at 0)*

$$S_0 = \{f \in H^2(0, +\infty) : f'(0+) = 0\}.$$

*Then the Morse index of such operator $L^{\beta}_{1,\omega^*}$ equals one.*

*Proof.* Define $\phi = \frac{d}{dx}\varphi_{\omega^*,\beta}|_{(0,+\infty)}$. Then $\phi \in S_0$ and $L^{\beta}_{1,\omega^*}\phi(x) = 0$ for all $x > 0$. Moreover, since $\phi$ has exactly one zero on $(0, +\infty)$, then $\lambda = 0$ is the second simple eigenvalue, and consequently there is a unique negative simple eigenvalue with positive associated eigenfunction. This finishes the proof. □

**Proposition 3.6.** *Let $\beta < 0$ and $\omega^* = \frac{4(p+1)}{\beta^2(p-1)}$. Then the Morse index of the operator $L^{\beta}_{1,\omega^*}$ defined on the domain $D_\beta$ satisfies $n(L^{\beta}_{1,\omega^*}) \geqq 2$. Moreover, there are at least two different negative eigenvalues $\lambda_{0,\omega^*} < \lambda_{1,\omega^*} < 0$, where $\lambda_{0,\omega^*}$ is the first simple eigenvalue with an associated positive and even eigenfunction.*

*Proof.* By item 2) in Proposition 3.2, the function $\varphi'_{\omega^*} = \frac{d}{dx}\varphi_{\omega^*,\beta}$ belongs to $D_\beta$, and $L^{\beta}_{1,\omega^*}\varphi'_{\omega^*} = 0$. Moreover, since $\varphi''_{\omega^*}(0) = 0$ then $\varphi'_{\omega^*} \in H^2(\mathbb{R})$.

Consider the operator $L^{\beta}_{1,\omega^*}$ defined on the domain $H^2(\mathbb{R})$, then $\varphi'_{\omega^*} \in \mathrm{Ker}(L^{\beta}_{1,\omega^*})$. Since $\varphi'_{\omega^*}$ has two different zeros, there exist exactly two negative simple eigenvalues $\lambda_0 < \lambda_1 < 0$, besides $\psi_{0,\omega^*} \in H^2(\mathbb{R})$ being a positive-even eigenfunction corresponding to $\lambda_0$. The eigenfunction $\psi_1 \in H^2(\mathbb{R})$ associated with $\lambda_1$ need to have exactly one zero at $x = a$. Next, by Lemma 3.4 (with $D_\beta$ substituted by $H^2(\mathbb{R})$), the function $\psi_1$ needs to be odd, and therefore $a = 0$.

Below we will analyze if the eigenfunctions $\psi'_{0,\omega^*}$ and/or $\psi_1$ belong to domain $D_\beta$. First, since $\psi_1(0) = 0$ and $\psi'_1(0) > 0$ (without loss of generality), then $\psi_1 \notin D_\beta$, and thus $\psi_1$ can not be an eigenfunction for $L^{\beta}_{1,\omega^*}$ defined on $D_\beta$. Further, by $\psi_{0,\omega^*}(0+) = \psi_{0,\omega^*}(0-)$ and $\psi'_{0,\omega^*}(0+) = \psi'_{0,\omega^*}(0-) = 0$, we can conclude that $\lambda_{0,\omega^*} := \lambda_0$ is a negative eigenvalue for $L^{\beta}_{1,\omega^*}$ acting on $D_\beta$ with $\psi_{0,\omega^*} \in H^2(\mathbb{R})$ being associated positive and even eigenfunction.

Consider the quadratic form $F_{\omega^*} : H^1(\mathbb{R} - \{0\}) \to \mathbb{R}$ associated with $L^{\beta}_{1,\omega^*}$ acting on $D_\beta$

$$F_{\omega^*}(v) = \int_{\mathbb{R}} (v')^2 + (\omega^* - p|\varphi_{\omega^*,\beta}|^{p-1})v^2 dx - \frac{1}{\beta}|v(0+) - v(0-)|^2.$$

Using that $F_{\omega^*}(\psi_{0,\omega^*}) = \langle L^{\beta}_{1,\omega^*}\psi_{0,\omega^*}, \psi_{0,\omega^*}\rangle = \lambda_{0,\omega^*}\|\psi_{0,\omega^*}\|^2 < 0$ and

$$F_{\omega^*}(\varphi_{\omega^*,\beta}) = \langle L^{\beta}_{1,\omega^*}\varphi_{\omega^*,\beta}, \varphi_{\omega^*,\beta}\rangle = (1-p)\int_{\mathbb{R}} |\varphi_{\omega^*,\beta}|^{p+1} dx < 0,$$

by orthogonality of $\psi_{0,\omega^*}$ and $\varphi_{\omega^*,\beta}$, we obtain that $F_{\omega^*}$ is negatively defined on a two-dimensional subspace $\mathcal{M} = [\psi_{0,\omega^*}, \varphi_{\omega^*,\beta}] \subset D_\beta$. Then the Morse index of $L^{\beta}_{1,\omega^*}$ acting on $D_\beta$ satisfies $n(L^{\beta}_{1,\omega^*}) \geqq 2$.

Finally, since $\psi_{0,\omega^*}$ is positive, we obtain from the Perron-Frobenius property in Lemma 6.5 that $\lambda_{0,\omega^*}$ is the smallest negative eigenvalue for $L^{\beta}_{1,\omega^*}$ defined on $D_\beta$, and therefore $\lambda_{0,\omega^*}$ is simple. This finishes the proof. □

*Remark* 3.7. By using Krein&von Neumann extension theory, we can show the estimate $n(L^{\beta}_{1,\omega^*}) \leqq 3$ (see Remark 3.14 below). Moreover, via perturbation analysis, we will establish in Lemma 3.17 that $n(L^{\beta}_{1,\omega^*}) = 2$. In Corollary 3.18 we show that eigenvalue $\lambda_{1,\omega^*}$ has an associated odd eigenfunction.

3.1.3. **Morse index of $L^{\beta}_{1,\omega}$ for $\omega > \omega^* = \frac{4(p+1)}{\beta^2(p-1)}$.** In this subsection we show that the Morse index of $L^{\beta}_{1,\omega}$ acting on $D_\beta$ for $\omega > \omega^*$ is exactly two. Our approach is based on the analysis of quadratic forms and the extension theory of symmetric operators. As we will see in Subsection 2.1.4, this result is basic for calculating (via a perturbation analysis) the Morse index for $L^{\beta}_{1,\omega}$ acting on $D_\beta$ in the case $\omega \leqq \omega^*$.

**Theorem 3.8.** *Let $\beta < 0$ and $\omega > \omega^* = \frac{4(p+1)}{\beta^2(p-1)}$. The Morse index of $L^{\beta}_{1,\omega}$ defined on $D_\beta$ equals two. Moreover, there are two different negative eigenvalues $\lambda_{1,\omega} > \lambda_{0,\omega}$ of this operator, and the associated eigenfunctions are even and odd, respectively.*

The proof of this theorem is not so immediate and therefore we will divide it into several lemmas. We start with the following observations. By Proposition 3.2 and (3.1), we get for $\varphi_\omega \equiv \varphi_{\omega,\beta}$, that $\varphi'_\omega \notin D_\beta$, and $\varphi''_\omega(0+) < 0$ for $\omega > \omega^*$. Thus, by (3.3)-(3.4), there is $\theta_0 < 0$ such that $\varphi'_\omega \in W_{\theta_0}$, where $W_{\theta_0}$ is defined by (3.2), and

$$L^{\beta}_{1,\omega}\varphi'_\omega(x) = 0, \quad \text{for } x > 0.$$

Therefore, since $\varphi'_\omega$ has exactly one zero in $(0, +\infty)$, it follows that there are an unique negative eigenvalue $\gamma_{0,\omega}$ and $\chi_{0,\omega} \in W_{\theta_0}$ such that

$$L^{\beta}_{1,\omega}\chi_{0,\omega} = \gamma_{0,\omega}\chi_{0,\omega}, \quad \text{on } (0, +\infty).$$

Moreover, $\chi_{0,\omega}$ can be chosen strictly positive on $[0, +\infty)$, and from (3.2) it follows that $\chi'_{0,\omega}(0+) < 0$. Next, from the Spectral Theorem we obtain for $f \in H^1(0, +\infty)$ such that $f \perp \chi_{0,\omega}$

$$Q_{\theta_0}(f) = \int_0^{+\infty} (f')^2 + (\omega - p\varphi^{p-1}_\omega)f^2 dx + \frac{1}{\theta_0}|f(0+)|^2 \geqq 0. \tag{3.6}$$

Here $Q_{\theta_0}$ denotes quadratic form associated with $L^{\beta}_{1,\omega}$ acting on $W_{\theta_0}$. Thus, the above analysis provides us the following result.

**Lemma 3.9.** *Let $\beta < 0$ and $\omega > \omega^* = \frac{4(p+1)}{\beta^2(p-1)}$. Then the Morse index of the operator $L^{\beta}_{1,\omega}$ defined on the domain $W_{-\frac{\beta}{2}}$ equals one. Moreover, for $\lambda_{1,\omega} < 0$, $\psi_{1,\omega} \in W_{-\frac{\beta}{2}}$ such that $L^{\beta}_{1,\omega}\psi_{1,\omega}(x) = \lambda_{1,\omega}\psi_{1,\omega}(x)$ for $x > 0$, we have $\psi'_{1,\omega}(0+) > 0$.*

*Proof.* Consider the quadratic form $Q_{-\frac{\beta}{2}}$ associated to $L^{\beta}_{1,\omega}$ defined on $W_{-\frac{\beta}{2}}$

$$Q_{-\frac{\beta}{2}}(f) = \int_0^{+\infty} (f')^2 + V_1(x)f^2 dx - \frac{2}{\beta}|f(0+)|^2$$

where $V_1(x) = \omega - p\varphi^{p-1}_\omega$. By (3.6), for $f \in H^1(0, +\infty)$ such that $f \perp \chi_{0,\omega}$ we have

$$Q_{-\frac{\beta}{2}}(f) = Q_{\theta_0}(f) - \Big(\frac{1}{\theta_0} + \frac{2}{\beta}\Big)|f(0+)|^2 \geqq 0,$$

since $\theta_0 < 0$ by (3.4). Thus, by the min-max principle, the Morse index of $L^{\beta}_{1,\omega}$ acting on $W_{-\frac{\beta}{2}}$, $n_1(L^{\beta}_{1,\omega})$, satisfies $n_1(L^{\beta}_{1,\omega}) \leqq 1$. Moreover, since $\varphi_\omega \in W_{-\frac{\beta}{2}}$ and $Q_{-\frac{\beta}{2}}(\varphi_\omega) < 0$, we get $n_1(L^{\beta}_{1,\omega}) \geqq 1$. Therefore, $n_1(L^{\beta}_{1,\omega}) = 1$, and consequently there are $\lambda_{1,\omega} < 0$, $\psi_{1,\omega} \in W_{-\frac{\beta}{2}}$

such that $L^{\beta}_{1,\omega}\psi_{1,\omega}(x) = \lambda_{1,\omega}\psi_{1,\omega}(x)$ for $x > 0$. Finally, since $\psi_{1,\omega}(0+) = -\frac{\beta}{2}\psi'_{1,\omega}(0+)$ and assuming that $\psi_{1,\omega}(0+) > 0$ (recall that $\psi_{1,\omega} > 0$ or $\psi_{1,\omega} < 0$ on $(0,+\infty)$), then $\psi'_{1,\omega}(0+) > 0$. □

**Lemma 3.10.** *Let $\beta < 0$ and $\omega > \omega^* = \frac{4(p+1)}{\beta^2(p-1)}$. Then the Morse index of the operator $L^{\beta}_{1,\omega}$ defined on $S_0 = \{v \in H^2(0,+\infty) : v'(0) = 0\}$ equals one. Moreover, for $\lambda_{0,\omega} < 0$, $\psi_{0,\omega} \in S_0$ such that $L^{\beta}_{1,\omega}\psi_{0,\omega}(x) = \lambda_{0,\omega}\psi_{0,\omega}(x)$ for $x > 0$, we have $\psi_{0,\omega} > 0$.*

*Proof.* Let $Q^*_0$ be the quadratic form associated with $L^{\beta}_{1,\omega}$ acting on $S_0$

$$Q^*_0(f) = \int_0^{+\infty} (f')^2 + V_1 f^2 dx,$$

where $V_1(x) = \omega - p\varphi_\omega^{p-1}$. Then, from (3.6) we obtain for $f \in H^1(0,+\infty)$ such that $f \perp \chi_{0,\omega}$, $Q^*_0(f) = Q_{\theta_0}(f) - \frac{1}{\theta_0}|f(0+)|^2 \geqq 0$, and therefore the Morse index of $L^{\beta}_{1,\omega}$, $n_0(L^{\beta}_{1,\omega})$, acting on $S_0$ satisfies $n_0(L^{\beta}_{1,\omega}) \leqq 1$. From Lemma 3.9 follows that

$$Q^*_0(\psi_{1,\omega}) = Q_{-\frac{\beta}{2}}(\psi_{1,\omega}) + \frac{2}{\beta}|\psi_{1,\omega}(0+)|^2 = \lambda_{1,\omega}\|\psi_{1,\omega}\|^2 + \frac{2}{\beta}|\psi_{1,\omega}(0+)|^2 < 0.$$

Therefore, $n_0(L^{\beta}_{1,\omega}) = 1$. This finishes the proof. □

**Lemma 3.11.** *Let $\lambda_{1,\omega}, \lambda_{0,\omega}$ be the negative eigenvalues for $L^{\beta}_{1,\omega}$ obtained in Lemmas 3.9-3.10, respectively, with associated positive eigenfunctions $\psi_{1,\omega}, \psi_{0,\omega}$, such that $\psi_{1,\omega} \in W_{-\frac{\beta}{2}}$ and $\psi_{0,\omega} \in S_0$. Then, $\lambda_{1,\omega} > \lambda_{0,\omega}$.*

*Proof.* From the proofs of Lemmas 3.9-3.10 it follows that without loss of generality we can assume $\psi_{1,\omega} > 0$ and $\psi_{0,\omega} > 0$. Integrating by parts, we obtain

$$\begin{aligned}\lambda_{1,\omega}\langle\psi_{1,\omega}, \psi_{0,\omega}\rangle &= \langle L^{\beta}_{1,\omega}\psi_{1,\omega}, \psi_{0,\omega}\rangle = \psi'_{1,\omega}(0+)\psi_{0,\omega}(0+) - \psi_{1,\omega}(0+)\psi'_{0,\omega}(0+) + \langle\psi_{1,\omega}, L^{\beta}_{1,\omega}\psi_{0,\omega}\rangle \\ &= \psi'_{1,\omega}(0+)\psi_{0,\omega}(0+) + \lambda_{0,\omega}\langle\psi_{1,\omega}, \psi_{0,\omega}\rangle.\end{aligned}$$

Thus, $(\lambda_{1,\omega} - \lambda_{0,\omega})\langle\psi_{1,\omega}, \psi_{0,\omega}\rangle = \psi'_{1,\beta}(0+)\psi_{0,\beta}(0+) > 0$. Therefore, $\lambda_{1,\omega} > \lambda_{0,\omega}$. □

Below we show the existence of at least two different negative eigenvalues of $L^{\beta}_{1,\omega}$ defined on $D_\beta$ by (2.3).

**Proposition 3.12.** *Let $\beta < 0$ and $\omega > \omega^* = \frac{4(p+1)}{\beta^2(p-1)}$. Consider the negative eigenvalues $\lambda_{1,\omega}, \lambda_{0,\omega}$ for $L^{\beta}_{1,\omega}$ determined in Lemmas 3.9-3.10. Then $\lambda_{1,\omega}, \lambda_{0,\omega}$ are simple eigenvalues for $L^{\beta}_{1,\omega}$ defined on the domain $D_\beta$. Moreover, the associated eigenfunctions are odd and even, respectively, and $\lambda_{0,\omega}$ is the first negative eigenvalue.*

*Proof.* We divide the proof into several steps.

1) Let $\lambda_{1,\omega} < 0$ and $\psi_{1,\omega} \in W_{-\frac{\beta}{2}}$ such that $L^{\beta}_{1,\omega}\psi_{1,\omega} = \lambda_{1,\omega}\psi_{1,\omega}$ on $(0,+\infty)$. Recall that $\psi'_{1,\omega}(0+) > 0$. Then the odd function

$$\Phi_{1,\omega}(x) = \begin{cases} \psi_{1,\omega}(x), & x \geqq 0, \\ -\psi_{1,\omega}(-x), & x < 0 \end{cases}$$

belongs to $H^2(\mathbb{R} - \{0\})$ and satisfies the relation

$$\Phi_{1,\omega}(0+) - \Phi_{1,\omega}(0-) = \psi_{1,\omega}(0+) + \psi_{1,\omega}(0+) = -\beta\psi'_{1,\omega}(0+) = -\beta\Phi'_{1,\omega}(0+) = -\beta\Phi'_{1,\omega}(0-).$$

Hence $\Phi_{1,\omega} \in D_\beta$. Moreover, $L^\beta_{1,\omega}\Phi_{1,\omega}(x) = \lambda_{1,\omega}\Phi_{1,\omega}(x)$ for $x < 0$. Therefore, $\lambda_{1,\omega}$ is an eigenvalue of $L^\beta_{1,\omega}$ defined on $D_\beta$ with the associated odd eigenfunction $\Phi_{1,\omega}$.

2) Let $\lambda_{0,\omega} < 0$ and $\psi_{0,\omega} \in S_0$ such that $L^\beta_{1,\omega}\psi_{0,\omega} = \lambda_{0,\omega}\psi_{0,\omega}$ on $(0,+\infty)$. Recall that $\psi'_{0,\omega}(0+) = 0$ and $\psi_{0,\omega}(0+) > 0$. Then the even function

$$\Phi_{0,\omega}(x) = \begin{cases} \psi_{0,\omega}(x), & x \geqq 0, \\ \psi_{0,\omega}(-x), & x < 0 \end{cases}$$

belongs to $H^2(\mathbb{R})$ and satisfies the relations $\Phi'_{0,\omega}(0) = 0$, and

$$\Phi_{0,\omega}(0+) - \Phi_{0,\omega}(0-) = 0 = -\beta\Phi'_{0,\omega}(0).$$

Moreover, $L^\beta_{1,\omega}\Phi_{0,\omega}(x) = \lambda_{0,\omega}\Phi_{0,\omega}(x)$ for $x < 0$. Therefore, $\lambda_{0,\omega}$ is the eigenvalue for $L^\beta_{1,\omega}$ defined on $D_\beta$ with the associated even positive eigenfunction $\Phi_{0,\omega} \in H^2(\mathbb{R})$. Therefore, $\lambda_{0,\omega}$ is the first negative eigenvalue of $L^\beta_{1,\omega}$ acting on $D_\beta$ by the Perron-Frobenius property established in Lemma 6.5.

3) Let us show that $\lambda_{1,\omega}$ is a simple eigenvalue for $L^\beta_{1,\omega}$ acting on $D_\beta$. Indeed, take $f \in D_\beta$ such that $L^\beta_{1,\omega}f = \lambda_{1,\omega}f$. In what follows we use the decomposition $f = f_{\text{even}} + f_{\text{odd}}$.

Since $f'(0+) = f'(0-)$, it follows that $f'_{\text{even}}(0+) = 0$, and therefore $f_0 \equiv f_{\text{even}}|_{(0,+\infty)} \in S_0 = \{v \in H^2(0,+\infty) : v'(0) = 0\}$. Noting that $L^\beta_{1,\omega}$ maps even (odd) functions into even (odd) functions, we obtain $L^\beta_{1,\omega}f_{\text{even}} = \lambda_{1,\omega}f_{\text{even}}$ and $L^\beta_{1,\omega}f_{\text{odd}} = \lambda_{1,\omega}f_{\text{odd}}$ on all the line. Therefore, if $f_0 \neq 0$, then $\lambda_{1,\omega} < 0$ is an eigenvalue for $L^\beta_{1,\omega}$ defined on $S_0$, and consequently, by Lemma 3.10, we get $\lambda_{1,\omega} = \lambda_{0,\omega}$ which contradicts with Lemma 3.11. Therefore, $f_{\text{even}} \equiv 0$ and $f \equiv f_{\text{odd}}$.

The last equality induces that $f(0-) = -f(0+)$. Then, by definition of $D_\beta$, we get $f(0+) = -\frac{\beta}{2}f'(0+)$, and therefore $f|_{(0,+\infty)} \in W_{\frac{-\beta}{2}}$, and it is the eigenfunction for $L^\beta_{1,\omega}$ acting on $W_{\frac{-\beta}{2}}$. Therefore, by Lemma 3.9, we obtain $f(x) = \theta\psi_{1,\omega}(x)$ for $x > 0$. Thus, from the definition of $\Phi_{1,\omega}$ follows $f = \theta\Phi_{1,\omega}$ on the line. This finishes the proof. □

In the following Proposition we estimate the Morse index of $L^\beta_{1,\omega}$ from above.

**Proposition 3.13.** *Let $\beta < 0$ and $\omega > \omega^* = \frac{4(p+1)}{\beta^2(p-1)}$. Then the Morse index of $L^\beta_{1,\omega}$ defined on the domain $D_\beta$ satisfies the estimate $n(L^\beta_{1,\omega}) \leqq 3$.*

*Proof.* Our strategy of proof is based in the Krein&von Neumann extension theory of symmetric operators. Let $\mu$ be the unique positive zero of $\varphi'_{\omega,\beta}$. From Proposition 6.4 follows that the symmetric operator $L$ defined by

$$L = -\frac{d^2}{dx^2} + \omega - p|\varphi_{\omega,\beta}|^{p-1}, \quad D(L) = \{v \in H^2(\mathbb{R}) : v'(0) = 0, v(\pm\mu) = 0\},$$

has deficiency index $n_\pm(L) = 3$.

Moreover, the operator $(L^\beta_{1,\omega}, D_\beta)$ belongs to the 9-parameter family of self-adjoint extensions of $L$. Let us show that $L$ is non-negative for $\beta < 0$. Indeed, it is easy to verify that for $v \in D(L)$ the following identity holds

$$Lv = \frac{-1}{\varphi'_{\omega,\beta}}\frac{d}{dx}\left[(\varphi'_{\omega,\beta})^2\frac{d}{dx}\left(\frac{v}{\varphi'_{\omega,\beta}}\right)\right], \quad x \neq 0, \pm\mu. \tag{3.7}$$

Integration by parts yields

$$\langle Lv, v\rangle = \int_{-\infty}^{-\mu-} (\varphi'_{\omega,\beta})^2 \left(\frac{d}{dx}\left(\frac{v}{\varphi'_{\omega,\beta}}\right)\right)^2 dx + \int_{-\mu+}^{0-} (\varphi'_{\omega,\beta})^2 \left(\frac{d}{dx}\left(\frac{v}{\varphi'_{\omega,\beta}}\right)\right)^2 dx$$
$$+ \int_{0+}^{\mu-} (\varphi'_{\omega,\beta})^2 \left(\frac{d}{dx}\left(\frac{v}{\varphi'_{\omega,\beta}}\right)\right)^2 dx + \int_{\mu+}^{+\infty} (\varphi'_{\omega,\beta})^2 \left(\frac{d}{dx}\left(\frac{v}{\varphi'_{\omega,\beta}}\right)\right)^2 dx$$
$$- \left[v'v - v^2\frac{\varphi''_{\omega,\beta}}{\varphi'_{\omega,\beta}}\right]_{-\infty}^{-\mu-} - \left[v'v - v^2\frac{\varphi''_{\omega,\beta}}{\varphi'_{\omega,\beta}}\right]_{-\mu+}^{0-} - \left[v'v - v^2\frac{\varphi''_{\omega,\beta}}{\varphi'_{\omega,\beta}}\right]_{0+}^{\mu-} - \left[v'v - v^2\frac{\varphi''_{\omega,\beta}}{\varphi'_{\omega,\beta}}\right]_{\mu+}^{+\infty}. \tag{3.8}$$

Noting that $v(\pm\mu) = 0$, and $\pm\mu$ are the first-order zeroes for $\varphi'_{\omega,\beta}$ (indeed, $\varphi''_{\omega,\beta}(\pm\mu) \neq 0$), we have, for instance, for the eighth term in (3.8)

$$- \left[v'v - v^2\frac{\varphi''_{\omega,\beta}}{\varphi'_{\omega,\beta}}\right]_{\mu+}^{\infty} = -\varphi''_{\omega,\beta}(\mu) \lim_{x\to\mu^+} \frac{v^2(x)}{\varphi'_{\omega,\beta}(x)} = -2\varphi''_{\omega,\beta}(\mu) \lim_{x\to\mu^+} \frac{v(x)v'(x)}{\varphi''_{\omega,\beta}(x)} = 0.$$

Analogously the fifth term in (3.8) is zero. Next, since $\varphi'_{\omega,\beta}(0+) = \varphi'_{\omega,\beta}(0-)$ and $\varphi''_{\omega,\beta}(0-) = -\varphi''_{\omega,\beta}(0+) > 0$ for $\omega > \omega^*$, we obtain

$$- \left[v'v - v^2\frac{\varphi''_{\omega,\beta}}{\varphi'_{\omega,\beta}}\right]_{-\mu+}^{0-} - \left[v'v - v^2\frac{\varphi''_{\omega,\beta}}{\varphi'_{\omega,\beta}}\right]_{0+}^{\mu-} = -v^2\frac{\varphi''_{\omega,\beta}}{\varphi'_{\omega,\beta}}\Big|_{x=0-}^{x=0+} = v^2(0)\frac{\varphi''_{\omega,\beta}(0-) - \varphi''_{\omega,\beta}(0+)}{\varphi'_{\omega,\beta}(0+)} \geqq 0. \tag{3.9}$$

Therefore, we get $L \geqq 0$ on $D(L)$, and consequently the family of self-adjoint extension $L^{\beta}_{1,\omega}$ has discrete spectrum in $(-\infty, 0)$ that consists of at most of $n_{\pm}(L) = 3$ eigenvalues counting multiplicities (see Proposition 6.3). In particular, the Morse index of $L^{\beta}_{1,\omega}$ acting on $D_\beta$ satisfies $n(L^{\beta}_{1,\omega}) \leqq 3$. This finishes the proof. □

*Remark* 3.14. 1) Observe that, when we deal with deficiency indices, the operator $L$ is assumed to act on complex-valued functions which however does not affect the analysis of negative spectrum of $L^{\beta}_{1,\omega}$ acting on real-valued functions.

2) The strategy used in the proof of Proposition 3.13 does not work for the case $\omega < \omega^*$. Indeed, $\varphi''_{\omega,\beta}(0+) > 0$ and $\varphi''_{\omega,\beta}(0-) < 0$ for $\omega < \omega^*$ (see (3.1)), hence the term in (3.9) turns to be negative. However, the extension theory still can be applied for estimating the Morse index. Indeed, by adding the condition $v(0) = 0$ in the definition of $D(L)$ we obtain the "rough" estimate $n(L^{\beta}_{1,\omega}) \leqq 4$.

3) The proof of Proposition 3.13 can be adapted to obtain the estimate $n(L^{\beta}_{1,\omega^*}) \leqq 3$ for $\omega^* = \frac{4(p+1)}{\beta^2(p-1)}$. Indeed, considering the profile $\varphi_{\omega^*,\beta}$ in equality (3.9), we obtain

$$v^2(0)\frac{\varphi''_{\omega^*,\beta}(0-) - \varphi''_{\omega^*,\beta}(0+)}{\varphi'_{\omega^*,\beta}(0+)} = 0,$$

by $\varphi''_{\omega^*,\beta}(0-) = \varphi''_{\omega^*,\beta}(0+) = 0$.

Now we are ready to prove Theorem 3.8.

*Proof.* [**Theorem 3.8**] From Propositions 3.12 and 3.13 we have $2 \leqq n(L^{\beta}_{1,\omega}) \leqq 3$, and there are at least two simple negative eigenvalues $\lambda_{0,\omega} < \lambda_{1,\omega} < 0$.

Suppose that $n(L^{\beta}_{1,\omega}) = 3$. Since $\lambda_{1,\beta}, \lambda_{0,\beta}$ are simple, there exists one more simple eigenvalue $\lambda < 0$ such that $\lambda_{1,\beta} \neq \lambda$ and $\lambda_{2,\beta} \neq \lambda$. Thus, there is $\phi \in D_\beta$ satisfying $L^{\beta}_{1,\omega}\phi = \lambda\phi$. By Lemma 3.1, the function $\phi$ is either even or odd.

1) Assume that $\phi$ is even. Then $\phi|_{(0,+\infty)} \in S_0$, and, by Lemma 3.10, we obtain $\lambda_{2,\beta} = \lambda$, which is a contradiction.
2) Assume that $\phi$ is odd. Then $\phi(0-) = -\phi(0+)$, and, by the definition of $D_\beta$, we have $\phi(0+) = -\frac{\beta}{2}\phi'(0+)$. Therefore, $\phi|_{(0,+\infty)} \in W_{-\frac{\beta}{2}}$, and it is the eigenfunction for $L^{\beta}_{1,\omega}$ acting on $W_{-\frac{\beta}{2}}$. Then, by Lemma 3.9, we have $\lambda_{3,\beta} = \lambda$, which is a contradiction.

Finally, $n(L^{\beta}_{1,\omega}) = 2$, and the Theorem is proved. □

3.1.4. **Morse index of $L^{\beta}_{1,\omega}$ for $\omega \leqq \omega^* = \frac{4(p+1)}{\beta^2(p-1)}$.** Below we use the analytic perturbation theory and a classical continuation argument based on the Riesz-projection to describe the Morse index of the family of self-adjoint operators $L^{\beta}_{1,\omega}$ as $\frac{4}{\beta^2} < \omega \leqq \omega^*$.

The following Lemma states the analyticity of the family of operators $L^{\beta}_{1,\omega}$ as a function of $\omega$.

**Lemma 3.15.** *Let $\beta < 0$. Then, as a function of $\omega$, $(L^{\beta}_{1,\omega})$ is a real-analytic family of self-adjoint operators of type (B) in the sense of Kato.*

*Proof.* The linear operator $L^{\beta}_{1,\omega}$ defined on $D_\beta$ by (2.3) is the self-adjoint operator on $L^2(\mathbb{R})$ associated with the following bilinear-symmetric form defined for $v, w \in H^1(\mathbb{R} - \{0\})$ by

$$B^{\beta}_{1,\omega}(v,w) = \langle v_x, w_x\rangle + \langle(\omega - p|\varphi_{\omega,\beta}|^{p-1})v, w\rangle - \frac{1}{\beta}(v(0+) - v(0-))(w(0+) - w(0-)). \quad (3.10)$$

Thus from [57, Theorem VII-4.2], $(L^{\beta}_{1,\omega})$ will be a real-analytic family of self-adjoint operators of type (B) in the sense of Kato as long as the family of bilinear-symmetric forms $(B^{\beta}_{1,\omega})$ is real-analytic of type (B) on $H^1(\mathbb{R} - \{0\}) \times H^1(\mathbb{R} - \{0\})$, namely,

a) $D(B^{\beta}_{1,\omega}) = H^1(\mathbb{R} - \{0\}) \times H^1(\mathbb{R} - \{0\})$, for all $\omega$,
b) the family $(B^{\beta}_{1,\omega})$ is bounded from below and closed,
c) the mapping $\omega \to B^{\beta}_{1,\omega}(v,v)$ is analytic for every $v \in H^1(\mathbb{R} - \{0\})$.

The conditions *a*) and *b*) above follows immediately from the bounded property of $\varphi_{\omega,\beta}$. Moreover, noting that the mapping

$$\omega \in (\tfrac{4}{\beta^2}, +\infty) \to \omega - p|\varphi_{\omega,\beta}|^{p-1} = \omega - \frac{p(p+1)\omega}{2}\,\mathrm{sech}^2\left(\frac{(p-1)\sqrt{\omega}}{2}(|x| + y_0(\omega))\right)$$

is real-analytic, we obtain the condition *c*) above. This finishes the proof. □

Further, we obtain the following result which is a consequence of the Kato-Rellich Theorem (see [66]).

**Lemma 3.16.** *There exist $\delta_0 > 0$ and two analytic functions $\Pi : (\omega^* - \delta_0, \omega^* + \delta_0) \to \mathbb{R}$ and $\Omega : (\omega^* - \delta_0, \omega^* + \delta_0) \to L^2(\mathbb{R})$ such that*

1) *$\Pi(\omega^*) = 0$ and $\Omega(\omega^*) = \frac{d}{dx}\varphi_{\omega^*,\beta}$.*
2) *For all $\omega \in (\omega^* - \delta_0, \omega^* + \delta_0)$, $\Pi(\omega)$ is a simple isolated eigenvalue of $L^{\beta}_{1,\omega}$, and $\Omega(\omega)$ is an associated eigenvector for $\Pi(\omega)$.*

3) $\delta_0$ *can be chosen small enough to ensure that for* $\omega \in (\omega^* - \delta_0, \omega^* + \delta_0)$ *the spectrum of* $L^\beta_{1,\omega}$ *is positive, except at most the first* $n(L^\beta_{1,\omega^*}) + 1$ *eigenvalues.*

*Proof.* We divide the proof into several steps.

a) There is $M > 0$ such that $\sigma(L^\beta_{1,\omega}) \cap (-\infty, -M] = \emptyset$ for $\omega \in [\omega^* - a, \omega^* + a]$ and $a > 0$ small enough.

b) Using item 2) of Lemma 3.2 and Proposition 3.6, we define $\sigma_0 = \{0\} \cup \mathcal{N}$ with $\mathcal{N}$ being a finite set of negative eigenvalues of $L^\beta_{1,\omega^*}$. Recall that this set consists of at most three negative eigenvalues due to $n(L^\beta_{1,\omega^*}) \leqq 3$ (see Remark 3.14). Thus, we can separate the spectrum $\sigma(L^\beta_{1,\omega^*})$ of $L^\beta_{1,\omega^*}$ into two parts $\sigma_0, \sigma_1$ by a closed curve $\Gamma \subset \rho(L^\beta_{1,\omega^*})$ such that $\sigma_0$ belongs to the inner domain of $\Gamma$ and $\sigma_1$ to the outer domain of $\Gamma$. Indeed, such curve could be chosen a circle passing through the points $-M$ and $\theta_{\omega^*} - \epsilon$, where $\epsilon < \theta_{\omega^*}/2$ and $\theta_{\omega^*} = \inf\{\lambda : \lambda \in \sigma(L^\beta_{1,\omega^*}), \ \ \lambda > 0\}$. Note that $\sigma_1 \subset [\theta_{\omega^*}, +\infty)$.

c) Observe that $L^\beta_{1,\omega}$ converges to $L^\beta_{1,\omega^*}$ as $\omega \to \omega^*$ in the generalized sense. Indeed, denoting $W_\omega = \omega - p|\varphi_{\omega,\beta}|^{p-1}$, we obtain

$$\begin{aligned} \widehat{\delta}(L^\beta_{1,\omega}, L^\beta_{1,\omega^*}) &= \widehat{\delta}(L^\beta_{1,\omega^*} + (W_\omega - W_{\omega^*}), L^\beta_{1,\omega^*}) \\ &\leqq \|W_\omega - W_{\omega^*}\| \to 0, \qquad \text{as } \omega \to \omega^*, \end{aligned}$$

where $\widehat{\delta}$ is the gap metric (see [57, Chapter IV]). By [57, Theorem 3.16, Chapter IV] and Lemma 3.15, we have $\Gamma \subset \rho(L^\beta_{1,\omega})$ for $\omega \in [\omega^* - \delta_1, \omega^* + \delta_1]$ and $\delta_1 > 0$ small enough. Moreover, $\sigma(L^\beta_{1,\omega})$ is likewise separated by $\Gamma$ into two parts so that the part of $\sigma(L^\beta_{1,\omega})$ inside $\Gamma$ consists of a finite number of eigenvalues with total multiplicity (algebraic) $n(L^\beta_{1,\omega^*}) + 1$ (recall that zero is not an eigenvalue of $L^\beta_{1,\omega}$ for $\omega \neq \omega^*$).

d) For $\epsilon$ small enough define $\Gamma_0 = \{z \in \mathbb{C} : |z| = \epsilon\}$ such that $\Gamma_0 \cap \mathcal{N} = \emptyset$, $\Gamma_0 \subset \mathrm{int}(\Gamma)$, therefore from the non-degeneracy of 0 for $L^\beta_{1,\omega^*}$, we obtain that there exists $\delta_2 < \delta_1$ such that for $\omega \in (\omega^* - \delta_2, \omega^* + \delta_2) - \{\omega^*\}$ we get $\sigma(L^\beta_{1,\omega}) \cap \mathrm{int}(\Gamma_0) = \{\lambda_\omega\}$, where $\lambda_\omega$ is a *non-zero simple eigenvalue* of $L^\beta_{1,\omega}$, and $\lambda_\omega \to 0$ as $\omega \to \omega^*$.

e) Considering the contour $\Gamma_0$ above and applying Kato-Rellich Theorem (see [66, Theorem XII.8]), we get the existence of $0 < \delta_0 < \delta_2$ and two analytic functions $\Pi, \Omega$ defined in the neighborhood of $\omega^*$, $(\omega^* - \delta_0, \omega^* + \delta_0)$, such that $1), 2)$ and $3)$ hold.

□

Below we analyze how the simple perturbed eigenvalue $\Pi(\omega)$ moves depending on the relative position of $\omega$ and $\omega^*$.

**Lemma 3.17.** *Let* $\beta < 0$. *Then*

1) *there exists* $0 < \delta < \delta_0$ *such that* $\Pi(\omega) > 0$ *for any* $\omega \in (\omega^*, \omega^* + \delta)$, *and* $\Pi(\omega) < 0$ *for any* $\omega \in (\omega^* - \delta, \omega^*)$;

2) *for* $\omega = \omega^*$ *we have* $n(L^\beta_{1,\omega^*}) = 2$, *and consequently* $n(L^\beta_{1,\omega}) = 3$ *for* $\omega \in (\omega^* - \delta, \omega^*)$.

*Proof.* 1) As the proof of this part is a bit tedious we divide it into several steps.

a) From Taylor's theorem we have the following expansions

$$\Pi(\omega) = \gamma(\omega - \omega^*) + O(|\omega - \omega^*|^2), \quad \text{and} \quad \Omega(\omega) = \frac{d}{dx}\varphi_{\omega^*,\beta} + \phi_0(\omega - \omega^*) + O(|\omega - \omega^*|^2), \quad (3.11)$$

where $\gamma = \Pi'(\omega^*) \in \mathbb{R}$ and $\phi_0 = \Omega'(\omega^*) \in L^2(\mathbb{R})$. The desired result will follow if we show that $\gamma > 0$.

Observe that there exists $\chi_0 \in H^1(\mathbb{R} - \{0\})$ such that for $\omega$ close to $\omega^*$ we have

$$\varphi_{\omega,\beta} = \varphi_{\omega^*,\beta} + (\omega - \omega^*)\chi_0 + O(|\omega - \omega^*|^2). \tag{3.12}$$

Denote $\varphi_\omega = \varphi_{\omega,\beta}$. To find $\gamma$ we compute $\langle L^\beta_{1,\omega}\Omega(\omega), \frac{d}{dx}\varphi_{\omega^*}\rangle$ in two different ways. Since $L^\beta_{1,\omega}\Omega(\omega) = \Pi(\omega)\Omega(\omega)$, it follows from (3.11)

$$\langle L^\beta_{1,\omega}\Omega(\omega), \varphi_{\omega^*}\rangle = \gamma(\omega - \omega^*)\Big\|\frac{d}{dx}\varphi_{\omega^*}\Big\|^2 + O(|\omega - \omega^*|^2). \tag{3.13}$$

From Proposition 3.2 it follows that $\frac{d}{dx}\varphi_{\omega^*} \in \mathrm{Ker}(L^\beta_{1,\omega^*})$ for all $\beta$, hence

$$\begin{aligned} L^\beta_{1,\omega}\Big(\frac{d}{dx}\varphi_{\omega^*}\Big) &= L^\beta_{1,\omega^*}\Big(\frac{d}{dx}\varphi_{\omega^*}\Big) + (\omega - \omega^*)\frac{d}{dx}\varphi_{\omega^*} + p(|\varphi_{\omega^*}|^{p-1} - |\varphi_\omega|^{p-1})\frac{d}{dx}\varphi_{\omega^*} \\ &= (\omega - \omega^*)\frac{d}{dx}\varphi_{\omega^*} + p(|\varphi_{\omega^*}|^{p-1} - |\varphi_\omega|^{p-1})\frac{d}{dx}\varphi_{\omega^*}. \end{aligned} \tag{3.14}$$

Using the relation

$$|\varphi_\omega|^{p-1} = |\varphi_{\omega^*}|^{p-1} + (p-1)\,\mathrm{sign}(x)|\varphi_{\omega^*}|^{p-2}\chi_0(x)(\omega - \omega^*) + O(|\omega - \omega^*|^2),$$

self-adjointness of $L^\beta_{1,\omega}$, (3.14) and (3.11), we obtain

$$\begin{aligned} &\left\langle L^\beta_{1,\omega}\Omega(\omega), \frac{d}{dx}\varphi_{\omega^*}\right\rangle \\ &= \left\langle \Omega(\omega), (\omega - \omega^*)\frac{d}{dx}\varphi_{\omega^*} - p(p-1)\,\mathrm{sign}(x)|\varphi_{\omega^*}|^{p-2}\chi_0\frac{d}{dx}\phi_{\omega^*}(\omega - \omega^*)\right\rangle + O(|\omega - \omega^*|^2) \\ &= (\omega - \omega^*)\|\frac{d}{dx}\varphi_{\omega^*}\|^2 - p(p-1)\left\langle \frac{d}{dx}\varphi_{\omega^*}, \mathrm{sign}(x)|\varphi_{\omega^*}|^{p-2}\chi_0\frac{d}{dx}\varphi_{\omega^*}(\omega - \omega^*)\right\rangle + O(|\omega - \omega^*|^2) \\ &= (\omega - \omega^*)\|\frac{d}{dx}\varphi_{\omega^*}\|^2 - p(p-1)\left\langle \chi_0, |\varphi_{\omega^*}|^{p-3}\varphi_{\omega^*}(\frac{d}{dx}\varphi_{\omega^*})^2\right\rangle(\omega - \omega^*) + O(|\omega - \omega^*|^2). \end{aligned} \tag{3.15}$$

b) The next step in is to study the expression $\langle \chi_0, |\varphi_{\omega^*}|^{p-3}\varphi_{\omega^*}(\frac{d}{dx}\varphi_{\omega^*})^2\rangle$ in (3.15). Observe that for every $\psi \in H^1(\mathbb{R} - \{0\})$ the quadratic form $B^\beta_{1,\omega^*}$ defined by (3.10) satisfies

$$B^\beta_{1,\omega^*}(\chi_0, \psi) = -\langle \varphi_{\omega^*}, \psi\rangle + O(|\omega - \omega^*|). \tag{3.16}$$

Indeed, since $L^\beta_{2,\omega}\varphi_\omega = 0$, we induce that for any $\psi \in H^1(\mathbb{R} - \{0\})$

$$0 = \langle L^\beta_{2,\omega}\varphi_\omega, \psi\rangle = B^\beta_{2,\omega}(\varphi_\omega, \psi), \quad \omega > 4/\beta^2,$$

where $B^\beta_{2,\omega}$ is the quadratic form associated with $L^\beta_{2,\omega}$ defined by (2.3). Let us analyze the integral term of $B^\beta_{2,\omega}(\varphi_\omega, \psi)$. From (3.12) we obtain

$$I := \int_{\mathbb{R}} \varphi'_\omega \psi' dx = \int_{\mathbb{R}} (\varphi'_{\omega^*}\psi' + (\omega - \omega^*)\chi'_0\psi')dx + O(|\omega - \omega^*|^2),$$

and

$$\begin{aligned} II &:= \int_{\mathbb{R}} (\omega - |\varphi_\omega(x)|^{p-1})\varphi_\omega\psi dx = \int_{\mathbb{R}} \left[(\omega - |\varphi_{\omega^*}|^{p-1})\varphi_\omega\psi + (|\varphi_{\omega^*}|^{p-1} - |\varphi_\omega|^{p-1})\varphi_\omega\psi\right] dx \\ &= \int_{\mathbb{R}} \left[(\omega - |\varphi_{\omega^*}|^{p-1})(\varphi_{\omega^*} + (\omega - \omega^*)\chi_0)\psi - (p-1)(\omega - \omega^*)|\varphi_{\omega^*}|^{p-1}\chi_0\psi\right] dx + O(|\omega - \omega^*|^2). \end{aligned}$$

Therefore, $I + II = A + B$ where

$$\begin{aligned}A =& \int_{\mathbb{R}} \left[\varphi'_{\omega^*}\psi' + (\omega - |\varphi_{\omega^*}|^{p-1})\varphi_{\omega^*}\psi\right] dx\\ =&\langle L^{\beta}_{2,\omega^*}\varphi_{\omega^*}, \psi\rangle + \frac{1}{\beta}[\varphi_{\omega^*}(0+) - \varphi_{\omega^*}(0-)][\psi(0+) - \psi(0-)] + (\omega - \omega^*) \int_{\mathbb{R}} \varphi_{\omega^*}\psi dx\\ =&\frac{1}{\beta}[\varphi_{\omega^*}(0+) - \varphi_{\omega^*}(0-)][\psi(0+) - \psi(0-)] + (\omega - \omega^*) \int_{\mathbb{R}} \varphi_{\omega^*}\psi dx,\end{aligned}$$

and

$$\begin{aligned}B =& \int_{\mathbb{R}} \left[(\omega - \omega^*)\chi'_0\psi' + (\omega - \omega^*)\chi_0\psi(\omega - p|\varphi_{\omega^*}|^{p-1})\right] dx\\ =&(\omega - \omega^*) \int_{\mathbb{R}} \left[\chi'_0\psi' + \chi_0\psi(\omega^* - p|\varphi_{\omega^*}|^{p-1})\right] dx + O(|\omega - \omega^*|^2).\end{aligned}$$

Now, on the other hand, we obtain

$$\begin{aligned}I + II =&B^{\beta}_{2,\omega}(\varphi_\omega, \psi) + \frac{1}{\beta}[\varphi_\omega(0+) - \varphi_\omega(0-)][\psi(0+) - \psi(0-)]\\ =&\frac{1}{\beta}[\varphi_\omega(0+) - \varphi_\omega(0-)][\psi(0+) - \psi(0-)]\\ =&\frac{1}{\beta}[\varphi_{\omega^*}(0+) - \varphi_{\omega^*}(0-)][\psi(0+) - \psi(0-)]\\ &+\frac{1}{\beta}(\omega - \omega^*)[\chi_0(0+) - \chi_0(0-)][\psi(0+) - \psi(0-)] + O(|\omega - \omega^*|^2),\end{aligned}$$

thus, we obtain for every $\psi \in H^1(\mathbb{R} - \{0\})$,

$$\begin{aligned}B^{\beta}_{1,\omega^*}(\chi_0, \psi) =& \int_{\mathbb{R}} \left[\chi'_0\psi' + (\omega^* - p|\varphi_{\omega^*}|^{p-1}\chi_0\psi\right] dx - \frac{1}{\beta}[\chi_0(0+) - \chi_0(0-)][\psi(0+) - \psi(0-)]\\ =& - \langle\varphi_{\omega^*}, \psi\rangle + O(|\omega - \omega^*|).\end{aligned} \tag{3.17}$$

Finally, (3.16) is proven.

Using the above analysis, we conclude that there is $g_0 \in H^1(\mathbb{R} - \{0\})$ with $\|g_0\| + \|g'_0\| = O(|\omega - \omega^*|)$ and such that $B^{\beta}_{1,\omega^*}(\chi_0, \psi) = \langle -\varphi_{\omega^*} + g_0, \psi\rangle$. Therefore, $\chi_0 \in D(L^{\beta}_{1,\omega^*})$ and $L^{\beta}_{1,\omega^*}\chi_0 = -\varphi_{\omega^*} + g_0$.

Next we show that for $\psi \in H^1(\mathbb{R} - \{0\})$

$$\begin{aligned}B^{\beta}_{1,\omega^*}(\omega^*\varphi_{\omega^*} - |\varphi_{\omega^*}|^{p-1}\varphi_{\omega^*}, \psi) = p(p-1)\langle\varphi_{\omega^*}|\varphi_{\omega^*}|^{p-3}(\varphi'_{\omega^*})^2, \psi\rangle\\ - \frac{p-1}{\beta}|\varphi_{\omega^*}(0+)|^{p-1}[\varphi_{\omega^*}(0+) - \varphi_{\omega^*}(0-)][\psi(0+) - \psi(0-)].\end{aligned} \tag{3.18}$$

Indeed, using that $(|\varphi_{\omega^*}|^{p-1}\varphi_{\omega^*})'(x) = p|\varphi_{\omega^*}(x)|^{p-1}\varphi'_{\omega^*}(x)$ for every $x \neq 0$, we obtain

$$\begin{aligned}J_1 =& \int_{\mathbb{R}} (|\varphi_{\omega^*}|^{p-1}\varphi_{\omega^*})'\psi' dx = p|\varphi_{\omega^*}(0+)|^{p-1}\varphi'_{\omega^*}(0+)[\psi(0-) - \psi(0+)] - p\int_{\mathbb{R}} (|\varphi_{\omega^*}|^{p-1}\varphi'_{\omega^*})'\psi dx\\ =& -p\int_{\mathbb{R}} \left[(|\varphi_{\omega^*}|^{p-1}\varphi''_{\omega^*} + (p-1)|\varphi_{\omega^*}|^{p-3}\varphi_{\omega^*}(\varphi'_{\omega^*})^2)\psi\right] dx\\ &+ \frac{p}{\beta}|\varphi_{\omega^*}(0+)|^{p-1}[\varphi_{\omega^*}(0+) - \varphi_{\omega^*}(0-)][\psi(0+) - \psi(0-)].\end{aligned}$$

Therefore, since $\varphi''_{\omega^*}(x) = \omega^*\varphi_{\omega^*}(x) - |\varphi_{\omega^*}(x)|^{p-1}\varphi_{\omega^*}(x)$ for $x \neq 0$, we obtain

$$
\begin{aligned}
&J_1 + \int_{\mathbb{R}} (\omega^* - p|\varphi_{\omega^*}|^{p-1})|\varphi_{\omega^*}|^{p-1}\varphi_{\omega^*}\psi dx \\
&= -\int_{\mathbb{R}} \left[p(p-1)|\varphi_{\omega^*}|^{p-3}\varphi_{\omega^*}(\varphi'_{\omega^*})^2\psi + (p-1)\omega^*|\varphi_{\omega^*}|^{p-1}\varphi_{\omega^*}\psi\right] dx \\
&\quad + \frac{p}{\beta}|\varphi_{\omega^*}(0+)|^{p-1}[\varphi_{\omega^*}(0+) - \varphi_{\omega^*}(0-)][\psi(0+) - \psi(0-)].
\end{aligned}
$$

Hence

$$
\begin{aligned}
B^{\beta}_{1,\omega^*}(|\varphi_{\omega^*}|^{p-1}\varphi_{\omega^*}, \psi) &= -\int_{\mathbb{R}} \left[p(p-1)|\varphi_{\omega^*}|^{p-3}\varphi_{\omega^*}(\varphi'_{\omega^*})^2\psi + (p-1)\omega^*|\varphi_{\omega^*}|^{p-1}\varphi_{\omega^*}\psi\right] dx \\
&\quad + \frac{p-1}{\beta}|\varphi_{\omega^*}(0+)|^{p-1}[\varphi_{\omega^*}(0+) - \varphi_{\omega^*}(0-)][\psi(0+) - \psi(0-)].
\end{aligned}
\tag{3.19}
$$

Thus, combining $B^{\beta}_{1,\omega^*}(\varphi_{\omega^*}, \psi) = \langle L^{\beta}_{1,\omega^*}\varphi_{\omega^*}, \psi\rangle = -\langle (p-1)|\varphi_{\omega^*}|^{p-1}\varphi_{\omega^*}, \psi\rangle$ and (3.19), we arrive at (3.18). Therefore, by (3.17) and (3.18),

$$
\begin{aligned}
&-p(p-1)\left\langle \chi_0, |\varphi_{\omega^*}|^{p-3}\varphi_{\omega^*}(\frac{d}{dx}\varphi_{\omega^*})^2 \right\rangle = -B^{\beta}_{1,\omega^*}(\omega^*\varphi_{\omega^*} - |\varphi_{\omega^*}|^{p-1}\varphi_{\omega^*}, \chi_0) \\
&-\frac{p-1}{\beta}|\varphi_{\omega^*}(0+)|^{p-1}[\varphi_{\omega^*}(0+) - \varphi_{\omega^*}(0-)][\chi_0(0+) - \chi_0(0-)] \\
&= \langle \varphi_{\omega^*}, \omega^*\varphi_{\omega^*} - |\varphi_{\omega^*}|^{p-1}\varphi_{\omega^*}\rangle \\
&-\frac{p-1}{\beta}|\varphi_{\omega^*}(0+)|^{p-1}(\varphi_{\omega^*}(0+) - \varphi_{\omega^*}(0-))(\chi_0(0+) - \chi_0(0-)) + O(|\omega - \omega^*|).
\end{aligned}
\tag{3.20}
$$

c) Now, from (3.15), (3.20) and using again $\varphi''_{\omega^*}(x) = \omega^*\varphi_{\omega^*}(x) - |\varphi_{\omega^*}(x)|^{p-1}\varphi_{\omega^*}(x)$, $x \neq 0$, we arrive to

$$
\begin{aligned}
&\langle L^{\beta}_{1,\omega}\Omega(\omega), \frac{d}{dx}\varphi_{\omega^*}\rangle = (\omega - \omega^*)\|\frac{d}{dx}\varphi_{\omega^*}\|^2 \\
&- p(p-1)\langle \chi_0, |\varphi_{\omega^*}|^{p-3}\varphi_{\omega^*}(\frac{d}{dx}\varphi_{\omega^*})^2\rangle(\omega - \omega^*) + O(|\omega - \omega^*|^2) \\
&= (\omega - \omega^*)\|\frac{d}{dx}\varphi_{\omega^*}\|^2 + (\omega - \omega^*)\langle \varphi_{\omega^*}, \omega^*\varphi_{\omega^*} - |\varphi_{\omega^*}|^{p-1}\varphi_{\omega^*}\rangle \\
&- \frac{p-1}{\beta}|\varphi_{\omega^*}(0+)|^{p-1}(\varphi_{\omega^*}(0+) - \varphi_{\omega^*}(0-))(\chi_0(0+) - \chi_0(0-))(\omega - \omega^*) + O(|\omega - \omega^*|^2) \\
&= \beta[\varphi'_{\omega^*}(0+)]^2(\omega - \omega^*) \\
&- \frac{p-1}{\beta}|\varphi_{\omega^*}(0+)|^{p-1}[\varphi_{\omega^*}(0+) - \varphi_{\omega^*}(0-)][\chi_0(0+) - \chi_0(0-)](\omega - \omega^*) + O(|\omega - \omega^*|^2).
\end{aligned}
\tag{3.21}
$$

d) Define $f(\omega) = \varphi_\omega(0+)$. Then from (1.5) we induce

$$
f(\omega) = \left(\frac{p+1}{2}\right)^{\frac{1}{p-1}} \left[\frac{\beta^2\omega - 4}{\beta^2}\right]^{\frac{1}{p-1}}.
$$

Thus, by (3.12) and Taylor's theorem, we obtain

$$\begin{aligned}(\chi_0(0+)-\chi_0(0-))(\omega-\omega^*) &= 2\varphi_\omega(0+)-2\varphi_{\omega^*}(0+)+O(|\omega-\omega^*|^2)\\ &= 2f'(\omega^*)(\omega-\omega^*)+O(|\omega-\omega^*|^2).\end{aligned}$$

Therefore, from (3.21) we have

$$\begin{aligned}\langle L^\beta_{1,\omega}\Omega(\omega), \frac{d}{dx}\varphi_{\omega^*}\rangle &= \beta[\varphi'_{\omega^*}(0+)]^2(\omega-\omega^*)\\ &\quad - \frac{4(p-1)}{\beta}|\varphi_{\omega^*}(0+)|^{p-1}\varphi_{\omega^*}(0+)f'(\omega^*)(\omega-\omega^*)+O(|\omega-\omega^*|^2).\end{aligned} \tag{3.22}$$

Combining (3.13), (3.22) and $-\beta\varphi'_{\omega^*}(0+)=2\varphi_{\omega^*}(0+)$, we obtain

$$\begin{aligned}\gamma\Big\|\frac{d}{dx}\varphi_{\omega^*}\Big\|^2 &= \beta[\varphi'_{\omega^*}(0+)]^2-\frac{4(p-1)}{\beta}|\varphi_{\omega^*}(0+)|^{p-1}\varphi_{\omega^*}(0+)f'(\omega^*)+O(|\omega-\omega^*|)\\ &= \frac{4\varphi_{\omega^*}(0+)}{\beta}\Big[\varphi_{\omega^*}(0+)-(p-1)|\varphi_{\omega^*}(0+)|^{p-1}f'(\omega^*)\Big]+O(|\omega-\omega^*|).\end{aligned}$$

Therefore, from the definition of $f$ we get

$$\varphi_{\omega^*}(0+)-(p-1)|\varphi_{\omega^*}(0+)|^{p-1}f'(\omega^*)=\frac{1-p}{2}f(\omega^*),$$

and consequently the relation

$$\gamma\Big\|\frac{d}{dx}\varphi_{\omega^*}\Big\|^2=\frac{4(1-p)\varphi_{\omega^*}(0+)}{2\beta}f(\omega^*)+O(|\omega-\omega^*|) \tag{3.23}$$

implies that $\gamma>0$ for $|\omega-\omega^*|$ small enough. This finishes the proof of item 1).

2) Let $\omega=\omega^*$. By Propositions 3.12 and 3.13, $2\leqq n(L^\beta_{1,\omega^*})\leqq 3$. Suppose that $n(L^\beta_{1,\omega^*})=3$. From the analyticity of the mapping $\omega\to L^\beta_{1,\omega}$ we get that $n(L^\beta_{1,\omega})\neq 2$ for $\omega$ close enough to $\omega^*$ such that $\omega>\omega^*$, which is the contradiction with the statement of Theorem 3.8. Thus, $n(L^\beta_{1,\omega^*})=2$, and therefore, by item 1), for $\omega<\omega^*$ we have $n(L^\beta_{1,\omega})=3$.

□

**Corollary 3.18.** *Let $\beta<0$ and $\omega^*=\frac{4(p+1)}{\beta^2(p-1)}$. Then the second negative eigenvalue $\lambda_{1,\omega^*}$ of $L^\beta_{1,\omega^*}$ is simple with an associated odd eigenfunction.*

*Proof.* By Propositions 3.6 and 3.17, the second negative eigenvalue $\lambda_{1,\omega^*}$ for $L^\beta_{1,\omega^*}$ is simple, and, by Lemma 3.4, the associated eigenfunction $\psi_{1,\omega^*}$ is either even or odd. Suppose that $\psi_{1,\omega^*}$ is even, then $\psi\equiv\psi_{1,\omega^*}|_{(0,+\infty)}\in S_0$ and $L^\beta_{1,\omega^*}\psi(x)=\lambda_{1,\omega^*}\psi(x)$ for $x>0$. Therefore, from Proposition 3.6 it follows that the Morse index of $L^\beta_{1,\omega^*}$ acting on $S_0$ is two. This contradicts with Lemma 3.5. □

Now we are the in position to investigate the Morse index of $L^\beta_{1,\omega}$ for any $\omega>\frac{4}{\beta^2}$. We use the classical continuation argument based on the Riesz-projection.

**Theorem 3.19.** *Let $\beta<0$ and $\frac{4}{\beta^2}<\omega<\omega^*$. Then the Morse index of $L^\beta_{1,\omega}$ defined on the domain $D_\beta$ equals three. Moreover, $\Pi(\omega)$ is the third negative simple eigenvalue.*

*Proof.* Let $\omega<\omega^*$. We define $\widetilde{\omega}$ by,

$$\widetilde{\omega}=\inf\left\{r : r\in(\tfrac{4}{\beta^2},\omega^*)\ s.t.\ L^\beta_{1,\omega}\ \text{has three negative eigenvalues for all}\ \omega\in(r,\omega^*)\right\}.$$

Lemma 3.17 implies that $\widetilde{\omega}$ is well defined, and $\widetilde{\omega} \in [\frac{4}{\beta^2}, \omega^*)$. We claim that $\widetilde{\omega} = \frac{4}{\beta^2}$. Suppose that $\widetilde{\omega} > \frac{4}{\beta^2}$. Let $N = n(L^\beta_{1,\widetilde{\omega}})$, and $\Gamma$ be a closed curve such that $0 \in \Gamma \subset \rho(L^\beta_{1,\widetilde{\omega}})$, and all the negative eigenvalues of $L^\beta_{1,\widetilde{\omega}}$ belong to the inner domain of $\Gamma$. The existence of such $\Gamma$ can be deduced from the lower semi-boundedness of the quadratic form associated to $L^\beta_{1,\widetilde{\omega}}$. Next, using Lemma 3.15 and steps a) and b) of the proof of Lemma 3.16, we deduce that there is $\epsilon > 0$ such that for $\omega \in [\widetilde{\omega} - \epsilon, \widetilde{\omega} + \epsilon]$ we have $\Gamma \subset \rho(L^\beta_{1,\omega})$, and the mapping $\omega \to (L^\beta_{1,\omega} - \xi)^{-1}$ is analytic for $\xi \in \Gamma$. Therefore, the existence of an analytic family of Riesz-projections $\omega \to P(\omega)$ given by

$$P(\omega) = -\frac{1}{2\pi i} \oint_\Gamma (L^\beta_{1,\omega} - \xi)^{-1} d\xi$$

implies that $\dim(\operatorname{Ran} P(\omega)) = \dim(\operatorname{Ran} P(\widetilde{\omega})) = N$ for all $\omega \in [\widetilde{\omega} - \epsilon, \widetilde{\omega} + \epsilon]$. Further, there is $r_0 \in (\widetilde{\omega}, \widetilde{\omega} + \epsilon)$, and $L^\beta_{1,\omega}$ has exactly three negative eigenvalues for all $\omega \in (r_0, \omega^*)$. Therefore, $L^\beta_{1,\widetilde{\omega}+\epsilon}$ has three negative eigenvalues and $N = 3$, hence $L^\beta_{1,\omega}$ has three negative eigenvalues for $\omega \in (\widetilde{\omega} - \epsilon, \omega^*)$, which contradicts with the definition of $\widetilde{\omega}$. Thus, $\widetilde{\omega} = \frac{4}{\beta^2}$. This finishes the proof. □

**Proposition 3.20.** *Let $\beta < 0$. The function $\Omega(\omega)$ defined in Lemma 3.16 and associated to the third eigenvalue of $L^\beta_{1,\omega}$ can be extended to $(\frac{4}{\beta^2}, +\infty)$. Moreover, $\Omega(\omega)$ is an even function for $\omega > \frac{4}{\beta^2}$.*

*Proof.* By Lemma 3.15 and Theorem XII.7 in [66], the set $G_0 = \{(\omega, \lambda) | \omega > \frac{4}{\beta^2}, \lambda \in \rho(L^\beta_{1,\omega})\}$ is open, and $(\omega, \lambda) \in G_0 \to (L^\beta_{1,\omega} - \lambda)^{-1}$ is an analytic function in both variables. Thus, we can repeat the arguments of Lemma 3.16 and Lemma 3.17 at each point $\omega$ and on each neighborhood of $\omega$ to see that the functions $\Omega(\omega)$ and $\Pi(\omega)$ are analytic for every $\omega \in (\frac{4}{\beta^2}, +\infty)$.

Below we consider the case of $\omega > \omega^*$ (the case $\frac{4}{\beta^2} < \omega < \omega^*$ is similar). We know from Lemma 3.4 and Lemma 3.16 that the eigenvectors $\Omega(\omega)$ are even or odd, and $\Omega(\omega^*) = \frac{d}{dx}\varphi_{\omega^*,\beta}$ is even. Therefore, from the equality $\lim\limits_{\omega \to \omega^*+} \langle \Omega(\omega), \Omega(\omega^*) \rangle = \|\Omega(\omega^*)\|^2 \neq 0$ one has $\langle \Omega(\omega), \Omega(\omega^*) \rangle \neq 0$ for $\omega$ close to $\omega^*$ and $\omega > \omega^*$. Thus, $\Omega(\omega)$ is even for $\omega \in [\omega^*, \omega^* + \delta)$.

Let $\eta$ be defined by

$$\eta = \sup\{r : r > \omega^*, \Omega(\omega) \text{ is even for any } \omega \in [\omega^*, r)\}.$$

Suppose that $\eta < \infty$. If $\Omega(\eta)$ is even, then by continuity there exists $\delta_0 > 0$ such that $\Omega(\omega)$ is even for $\omega \in (\eta - \delta, \eta + \delta)$. Thus, from the definition of $\eta$ we obtain that $\Omega(\omega)$ is even for $\omega \in [\omega^*, \eta + \delta)$, which is a contradiction. Therefore, from Lemma 3.4 it follows that $\Omega(\eta)$ is odd. Since $\Omega(\eta)$ is the limit of even functions, $\Omega(\eta)$ is even. Hence $\Omega(\eta) \equiv 0$, which is a contradiction since $\Omega(\eta)$ is an eigenvector. Therefore, $\eta = +\infty$. □

The following result completes the study of the parity of the eigenfunctions to $L^\beta_{1,\omega}$ in the case $\omega < \omega^*$.

**Proposition 3.21.** *Let $\beta < 0$ and $\omega < \omega^*$. Then the associated eigenfunctions for the three negative simple eigenvalues of $L^\beta_{1,\omega}$ are even, odd and even, respectively.*

*Proof.* From the Perron-Frobenius property of $L^\beta_{1,\omega}$ established in Lemma 6.5, we obtain that the eigenfunction associated to the first negative eigenvalue is positive and even. Moreover, by Theorem 3.19 and Proposition 3.20, the eigenfunction associated to the third negative eigenvalue is also even.

By Corollary 3.18 and Kato-Rellich theorem, there are $\delta_3 > 0$ small and two analytic functions $\Pi_1 : (\omega^* - \delta_3, \omega^* + \delta_3) \to \mathbb{R}$ and $\Omega_1 : (\omega^* - \delta_3, \omega^* + \delta_3) \to L^2(\mathbb{R})$ such that $\Pi_1(\omega^*) = \lambda_{1,\omega^*}$ and $\Omega_1(\omega^*) = \psi_{1,\omega^*}$, where $\psi_{1,\omega^*}$ is an odd eigenfunction associated to $\lambda_{1,\omega^*}$. Following the ideas in the proof of Theorem 3.19, we obtain that $\Pi_1$ and $\Omega_1$ are holomorphic for every $\omega > \frac{4}{\beta^2}$. Moreover, $\Pi_1(\omega)$ represents the second simple negative eigenvalue of $L^\beta_{1,\omega}$ for $\omega < \omega^*$. Thus, by Lemma 3.4, the eigenfunction $\Omega_1(\omega)$ is even or odd. Then, by the equality $\lim\limits_{\omega\to\omega^*-} \langle \Omega_1(\omega), \Omega_1(\omega^*)\rangle = \|\Omega_1(\omega^*)\|^2 \neq 0$, one gets that $\langle \Omega_1(\omega), \Omega_1(\omega^*)\rangle \neq 0$ for $\omega$ close to $\omega^*$. Thus, $\Omega_1(\omega)$ is odd. □

3.1.5. **Spectral analysis for $L^\beta_{2,\omega}$.** In this subsection we describe the spectral properties of the self-adjoint operator $L^\beta_{2,\omega}$ defined by (2.3). Our principal result is the following.

**Theorem 3.22.** *Let $\beta < 0$, $\omega > \frac{4}{\beta^2}$, and $L^\beta_{2,\omega}$ be defined by (2.3). Then $Ker(L^\beta_{2,\omega}) = [\varphi_{\omega,\beta}]$, and the Morse index of $L^\beta_{2,\omega}$ is exactly one. In particular, the eigenfunction associated to the negative eigenvalue is even and positive.*

*Proof.* Following the ideas in the proof of Proposition 3.2, we obtain $\mathrm{Ker}(L^\beta_{2,\omega}) = [\varphi_{\omega,\beta}]$.

To determine the Morse index, we divide the analysis into several steps.

1) Let us show $n(L^\beta_{2,\omega}) \geqq 1$. Consider the quadratic form $F$ associated to $L^\beta_{2,\omega}$

$$F(u) = \|u'\|^2 + \omega\|u\|^2 - \langle |\varphi_{\omega,\beta}|^{p-1}u, u\rangle - \frac{1}{\beta}|u(0+) - u(0-)|^2, \quad u \in H^1(\mathbb{R} - \{0\}).$$

For $u = |\varphi_{\omega,\beta}| \in H^1(\mathbb{R})$, we obtain by $|\varphi_{\omega,\beta}(0+)| = |\varphi_{\omega,\beta}(0-)|$, formula (1.2), and integration by parts,

$$\begin{aligned} F(|\varphi_{\omega,\beta}|) &= (\varphi_{\omega,\beta}(0-) - \varphi_{\omega,\beta}(0+))\varphi'_{\omega,\beta}(0+) + \int_{-\infty}^{0-} \varphi_{\omega,\beta}(-\varphi''_{\omega,\beta} + \omega\varphi_{\omega,\beta} - |\varphi_{\omega,\beta}|^{p-1}\varphi_{\omega,\beta})dx \\ &+ \int_{0+}^{+\infty} \varphi_{\omega,\beta}(-\varphi''_{\omega,\beta} + \omega\varphi_{\omega,\beta} - |\varphi_{\omega,\beta}|^{p-1}\varphi_{\omega,\beta})dx = \beta|\varphi'_{\omega,\beta}(0+)|^2 < 0. \end{aligned}$$

Thus, the mini-max principle yields $n(L^\beta_{2,\omega}) \geqq 1$.

2) Let us show $n(L^\beta_{2,\omega}) \leqq 2$. Consider the symmetric operator $L_{\min}$ defined by

$$L_{\min} = -\frac{d^2}{dx^2} + \omega - |\varphi_{\omega,\beta}|^{p-1}, \quad D(L_{\min}) = \{v \in H^2(\mathbb{R}) : v(0) = v'(0) = 0\}.$$

The deficiency numbers of $L_{\min}$ are $n_\pm(L_{\min}) = 2$ (see [11, Chapter I.4]). Moreover, the following von Neumann decomposition holds

$$D(L^*_{\min}) = H^2(\mathbb{R} - \{0\}) = D(L_{\min}) \oplus [v^1_i, v^2_i] \oplus [v^1_{-i}, v^2_{-i}],$$

where

$$v^1_{\pm i} = \begin{cases} e^{i\sqrt{\pm i}x}, & x > 0; \\ 0, & x < 0. \end{cases}, \qquad v^2_{\pm i} = \begin{cases} 0, & x > 0; \\ e^{-i\sqrt{\pm i}x}, & x < 0. \end{cases}, \quad \Im(\sqrt{\pm i}) > 0.$$

Thus, all the self-adjoint extensions of $L_{\min}$ are given by a 4-parameter family of self-adjoint operators. In particular, $(L^\beta_{2,\omega}, D_\beta)$ belongs to this family.

Let us show that $L_{\min}$ is non-negative for $\beta<0$. Indeed, it is easy to verify that for $\beta<0$ and $v\in D(L_{\min})$ the following identity holds

$$L_{\min}v=\frac{-1}{\varphi_{\omega,\beta}}\frac{d}{dx}\left[\varphi^2_{\omega,\beta}\frac{d}{dx}\left(\frac{v}{\varphi_{\omega,\beta}}\right)\right],\quad x\neq 0. \tag{3.24}$$

Using (3.24) and integrating by parts, we get

$$\langle L_{\min}v,v\rangle=\int\limits_{-\infty}^{+\infty}\varphi^2_{\omega,\beta}\left(\frac{d}{dx}\left(\frac{v}{\varphi_{\omega,\beta}}\right)\right)^2dx+\left[v'v-v^2\frac{\varphi'_{\omega,\beta}}{\varphi_{\omega,\beta}}\right]_{0-}^{0+}. \tag{3.25}$$

The integral terms in (3.25) are non-negative and equal zero if and only if $v\equiv 0$. Due to the conditions $v(0)=v'(0)=0$, non-integral term vanishes, and we get $L_{\min}\geqq 0$ on $D(L_{\min})$. Thus, the Morse index of $L^\beta_{2,\omega}$ on $D_\beta$ satisfies $n(L^\beta_{2,\omega})\leqq 2$ (see Proposition 6.3).

3) Operator $L^\beta_{2,\omega}$ defined on the domain $W_{-\frac{\beta}{2}}$ has Morse index equal to zero. Indeed, since $L^\beta_{2,\omega}\varphi_{\omega,\beta}(x)=0$ for all $x\neq 0$, $\varphi_{\omega,\beta}|_{(0,+\infty)}\in W_{-\frac{\beta}{2}}$, and $\varphi_{\omega,\beta}|_{(0,+\infty)}>0$, we obtain from the classical oscillation theory on the half-line that $L^\beta_{2,\omega}$ defined on $W_{-\frac{\beta}{2}}$ has not negative eigenvalues.

4) Since $\beta<0$, we have from Lemma 6.5 (Perron-Frobenious property) that the first negative eigenvalue for $L^\beta_{2,\omega}$ on $D_\beta$, $\lambda_0$, it is simple with an associated positive and even eigenfunction $\phi_0$ (after replacing $\phi_0$ by $-\phi_0$ if necessary). Thus, $\phi_0\in H^2(\mathbb{R})$ (since $\phi_0'(0)=0$). Next, we suppose that $n(L^\beta_{2,\omega})=2$, and $\lambda_1$ is the second negative simple eigenvalue with $\lambda_0<\lambda_1<0$. Let $\phi_1\in D_\beta$ be such that $L^\beta_{2,\omega}\phi_1=\lambda_1\phi_1$. Then from Lemma 3.4 it follows that $\phi_1$ is either even or odd. Suppose that $\phi_1$ is even, then $\phi_1\in H^2(\mathbb{R})$, and it has at least two zeros. Thus, considering $L^\beta_{2,\omega}$ defined on $H^2(\mathbb{R})$, we obtain that there is a simple eigenvalue $\lambda\in(\lambda_0,\lambda_1)$ being the second one, and with an eigenfunction $f_\lambda\in H^2(\mathbb{R})$ having exactly one zero. Then since $f_\lambda$ is either even or odd, we obtain that $f_\lambda$ is odd. Further, consider the quadratic form $F_{-\frac{\beta}{2}}:H^1(0,+\infty)\to\mathbb{R}$ associated to $L^\beta_{2,\omega}$ acting on $W_{-\frac{\beta}{2}}$,

$$F_{-\frac{\beta}{2}}(g)=\int_0^{+\infty}(g')^2+V_2g^2dx-\frac{2}{\beta}|g(0+)|^2,$$

where $V_2(x)=\omega-|\varphi_{\omega,\beta}|^{p-1}$. Therefore, since $f_\lambda(0)=0$,

$$F_{-\frac{\beta}{2}}(f_\lambda)=-f_\lambda(0+)f_\lambda'(0+)+\int_0^{+\infty}f_\lambda(-f_\lambda''+V_2f_\lambda)dx=\lambda\int_0^{+\infty}f_\lambda^2dx<0.$$

This contradicts with item 3) above. Therefore, $\phi_1$ is odd. Then, $\phi_1|_{(0,+\infty)}\in W_{-\frac{\beta}{2}}$, and for every $x>0$ we have $L^\beta_{2,\omega}\phi_1(x)=\lambda_1\phi_1(x)$, which is again a contradiction with item 3), and therefore $n(L^\beta_{2,\omega})=1$. This finishes the proof of the Theorem. □

*Proof.* [**Theorem 3.1**]

1) Let $\omega\geqq\omega^*$. From Theorems 3.8, 3.22, and Lemma 3.17 we have $n(L^\beta_{1,\omega})=n(L^\beta_{1,\omega^*})=2$ and $n(L^\beta_{2,\omega})=1$. Thus, $n(\mathcal{H}_\omega)=3$. Further, from Propositions 3.6, 3.12, Corollary 3.18, and Theorem 3.22 we obtain $n(L^\beta_{1,\omega}|_{\rm odd})=n(L^\beta_{1,\omega^*}|_{\rm odd})=1$ and $n(L^\beta_{2,\omega}|_{\rm odd})=0$. Thus, $n(\mathcal{H}_\omega|_{\rm odd})=1$.

2) Let $\omega < \omega^*$. From Proposition 3.21 we obtain $n(L^\beta_{1,\omega}) = 3$, and therefore from Theorem 3.22 it follows that $n(\mathcal{H}_\omega) = 4$. Moreover, since $n(L^\beta_{1,\omega}|_{\rm odd}) = 1$, we obtain $n(\mathcal{H}_\omega|_{\rm odd}) = 1$.

□

## 4. Slope analysis

In this subsection we calculate the sign of $\partial_\omega\|\varphi_{\omega,\beta}\|^2$. The main result is the following.

**Theorem 4.1.** *Let $\beta < 0$ and $\omega > \frac{4}{\beta^2}$. Then the following assertions hold.*

1) *If $p \in (1, 3]$, then $\partial_\omega\|\varphi_{\omega,\beta}\|^2 > 0$.*
2) *If $p \in (3, 5)$, then there is $\omega_0 \equiv \omega_0(p) > \frac{4}{\beta^2}$ such that*
$$\frac{p-5}{2(p-1)}\int_{B(p,\omega_0)}^{+\infty} \operatorname{sech}^{\frac{4}{p-1}}(x)dx = \frac{1}{\beta\sqrt{\omega_0}}\Big[1-\frac{4}{\beta^2\omega_0}\Big]^{\frac{3-p}{p-1}}$$
*where $B(p,\omega_0) = \frac{(p-1)\sqrt{\omega_0}}{2}y_0$, and $y_0$ is defined in (1.6). Moreover, $\partial_\omega\|\varphi_{\omega,\beta}\|^2 < 0$ for $\omega \in (\frac{4}{\beta^2}, \omega_0)$, and $\partial_\omega\|\varphi_{\omega,\beta}\|^2 > 0$ for $\omega \in (\omega_0, +\infty)$.*
3) *If $p \in [5, +\infty)$, then $\partial_\omega\|\varphi_{\omega,\beta}\|^2 < 0$.*

*Remark* 4.2. Let $\omega^*(p) = \frac{4(p+1)}{\beta^2(p-1)}$, then from numerical simulations (see Remark 4.3 below) we obtain for $p \in (3, 5)$ specific relations between $\omega^*(p)$ and $\omega_0(p)$.

*Proof.* By (1.5),
$$\|\varphi_{\omega,\beta}\|^2 = C_p\omega^{\frac{5-p}{2(p-1)}}\int_{B(p,\omega)}^{+\infty}\operatorname{sech}^{\frac{4}{p-1}}(x)dx := C_p\omega^{\frac{5-p}{2(p-1)}}H(\omega),$$
where $B(p,\omega) = \frac{(p-1)\sqrt{\omega}}{2}y_0$, and $C_p$ is a positive constant depending only of $p$. Therefore,
$$\partial_\omega\|\varphi_{\omega,\beta}\|^2 = \frac{C_p}{2}\omega^{\frac{7-3p}{2(p-1)}}\Big[\frac{5-p}{p-1}H(\omega) + \frac{2}{\beta\sqrt{\omega}}\Big[1-\frac{4}{\beta^2\omega}\Big]^{\frac{3-p}{p-1}}\Big] := \frac{C_p}{2}\omega^{\frac{7-3p}{2(p-1)}}g(\omega). \tag{4.1}$$
From (4.1) we get immediately that $\partial_\omega\|\varphi_{\omega,\beta}\|^2 < 0$ for $p \geqq 5$. Next, we analyze the behavior of the function $g(\omega)$ for $p \in (1, 5)$. We have
$$\begin{aligned} g'(\omega) &= \frac{2}{\beta\omega^{3/2}}\frac{3-p}{p-1}\Big[1-\frac{4}{\beta^2\omega}\Big]^{\frac{4-2p}{p-1}}, \\ g''(\omega) &= \frac{3-p}{(p-1)^2}\frac{1}{\beta\omega^{5/2}}\Big[1-\frac{4}{\beta^2\omega}\Big]^{\frac{5-3p}{p-1}}\Big[\frac{4(5-p)}{\beta^2\omega}-3(p-1)\Big]. \end{aligned} \tag{4.2}$$
Thus, from (4.2) it follows that $\frac{4(5-p)}{\beta^2\omega} - 3(p-1) < 0$ for all $p \geqq 2$. Further, for $p < 5$ we obtain
$$a_0 := \lim_{\omega\to+\infty} g(\omega) = \frac{5-p}{2(p-1)}\int_0^\infty \operatorname{sech}^{\frac{4}{p-1}}(x)dx > 0, \tag{4.3}$$
and
$$\lim_{\omega\to\frac{4}{\beta^2}} g(\omega) = \begin{cases} 2a_0, & p \in (1, 3]; \\ -\infty, & p \in (3, 5). \end{cases} \tag{4.4}$$
Thus, if we consider $p \in [2, 3]$, then $g'(\omega) \leqq 0$ and $g''(\omega) \geqq 0$. Thus, from (4.3)-(4.4) it follows that $g(\omega) > 0$ for all $\omega > \frac{4}{\beta^2}$. Next, for $p \in (3, 5)$ we have $g'(\omega) > 0$ and $g''(\omega) < 0$. Therefore, from (4.3)-(4.4) we obtain the existence of a unique $\omega_0 > \frac{4}{\beta^2}$ such that $g(\omega_0) = 0$.

Thus, for $\omega \in (\frac{4}{\beta^2}, \omega_0)$ we have $g(\omega) < 0$, and for $\omega \in (\omega_0, +\infty)$ we have $g(\omega) > 0$. Finally, for $p \in (1, 2)$, the analysis based on (4.2)-(4.3)-(4.4) implies that $g(\omega) > 0$ for all $\omega > \frac{4}{\beta^2}$. This finishes the proof. □

*Proof.* [**Proof of Theorem 1.1**]

We divide the analysis into several steps. Let $\omega^* = \frac{4(p+1)}{\beta^2(p-1)}$.

- Case $\omega \neq \omega^*$: we have from Lemma 3.2 that $\mathrm{Ker}(L^\beta_{1,\omega}) = \{0\}$ and $\mathrm{Ker}(L^\beta_{2,\omega}) = [\varphi_{\omega,\beta}]$.
  1) Let $p \in [5, +\infty)$. Since $\partial_\omega \|\varphi_{\omega,\beta}\|^2 < 0$, then we have $p(\omega) = 0$ and consequently:
     a) if $\omega < \omega^*$, then $n(\mathcal{H}_\omega|_{\mathrm{odd}}) = 1$, and therefore $e^{i\omega t}\varphi_{\omega,\beta}$ is orbitally unstable in $H^1_{\mathrm{odd}}(\mathbb{R} - \{0\})$ (and so is in $H^1(\mathbb{R} - \{0\})$);
     b) if $\omega > \omega^*$, then $n(\mathcal{H}_\omega) = 3$, and therefore $e^{i\omega t}\varphi_{\omega,\beta}$ is orbitally unstable in $H^1(\mathbb{R} - \{0\})$;
  2) Let $p \in (1, 3]$. Since $\partial_\omega \|\varphi_{\omega,\beta}\|^2 > 0$, then we have $p(\omega) = 1$ and consequently:
     a) if $\omega < \omega^*$, then $n(\mathcal{H}_\omega) - p(\omega) = 4 - 1 = 3$, and therefore $e^{i\omega t}\varphi_{\omega,\beta}$ is orbitally unstable in $H^1(\mathbb{R} - \{0\})$;
     b) if $\omega > \omega^*$, then $n(\mathcal{H}_\omega|_{\mathrm{odd}}) = 1 = p(\omega)$, and therefore $e^{i\omega t}\varphi_{\omega,\beta}$ is orbitally stable in $H^1_{\mathrm{odd}}(\mathbb{R} - \{0\})$.
- Case $\omega = \omega^*$: we have from Lemma 3.2 that $\mathrm{Ker}(L^\beta_{1,\omega^*}) = [\frac{d}{dx}\varphi_{\omega^*,\beta}]$ and $\mathrm{Ker}(L^\beta_{2,\omega^*}) = [\varphi_{\omega^*,\beta}]$. Therefore, $\mathrm{Ker}(L^\beta_{1,\omega^*}|_{\mathrm{odd}}) = \{0\}$.
  1) Let $p \in [5, +\infty)$. From Theorem 4.1 we have $p(\omega) = 0$. Next, from Lemma 3.17, Corollary 3.18 and Theorem 3.22 we obtain $n(L^\beta_{1,\omega^*}|_{\mathrm{odd}}) = 1$ and $n(L^\beta_{2,\omega^*}|_{\mathrm{odd}}) = \{0\}$. Therefore, $n(\mathcal{H}_{\omega^*}|_{\mathrm{odd}}) = 1$ and so $e^{i\omega t}\varphi_{\omega^*,\beta}$ is orbitally unstable in $H^1_{\mathrm{odd}}(\mathbb{R} - \{0\})$ (and so is in $H^1(\mathbb{R} - \{0\})$);
  2) Let $p \in (1, 3]$. From Theorem 4.1 we have $p(\omega) = 1$. Thus, since $n(\mathcal{H}_{\omega^*}|_{\mathrm{odd}}) = 1$, we obtain $e^{i\omega t}\varphi_{\omega^*,\beta}$ is orbitally stable in $H^1_{\mathrm{odd}}(\mathbb{R} - \{0\})$

This finishes the proof of the stability theorem. □

*Remark* 4.3. a) In the case $p \in (3, 5)$, we can write the mapping $J(\omega) = \frac{1}{2}g(\omega)$ defined in (4.1) for $\omega = \omega^*(p) = \frac{4(p+1)}{\beta^2(p-1)}$ as

$$G(p) \equiv J(\omega^*(p)) = \frac{5-p}{2(p-1)} \int_{-\sqrt{\frac{p-1}{p+1}}}^{1} (1-t^2)^{\frac{3-p}{p-1}} dt - \sqrt{\frac{p-1}{p+1}} \Big(\frac{2}{p+1}\Big)^{\frac{3-p}{p-1}}.$$

Then, by considering the Gamma function $\Gamma(\cdot)$ and the Gauss hypergeometric function ${}_2F_1(\cdot,\cdot,\cdot,\cdot)$ (see [1]) we obtain that

$$G(p) = \frac{5-p}{2(p-1)} \Big[ \frac{\sqrt{\pi}\,\Gamma(\frac{2}{p-1})}{2\Gamma(\frac{p+3}{2(p-1)})} + \sqrt{\frac{p-1}{p+1}}\, {}_2F_1\Big(\frac{1}{2}, \frac{p-3}{p-1}, \frac{3}{2}, \frac{p-1}{p+1}\Big) \Big] - \sqrt{\frac{p-1}{p+1}} \Big(\frac{2}{p+1}\Big)^{\frac{3-p}{p-1}}. \quad (4.5)$$

Thus, by using the Mathematical software we obtain the graph for the mapping $G(p)$, $p \in (1, 5.6)$, in Figure 2 below. Now, from a more accurate analysis we have $G(p) = 0$ if and only if $p = 3.15743 \equiv p_0$. Thus, for $p \in (3, p_0)$ we have $J(\omega^*(p)) > 0$ and therefore from the increasing property of $g$ follows that $\omega^*(p) > \omega_0(p)$. For $p \in (p_0, 5)$ we have $J(\omega^*(p)) < 0$ and so $\omega^*(p) < \omega_0(p)$. Lastly, since $G(p_0) = J(\omega^*(p_0)) = 0$ then $\omega^*(p_0) = \omega_0(p_0)$.

Then, from Theorems 2.3-3.1-4.1 we can conclude the following stability results:

i) Let $p \in (3, p_0)$. Then, for $\omega_0 \equiv \omega_0(p)$:

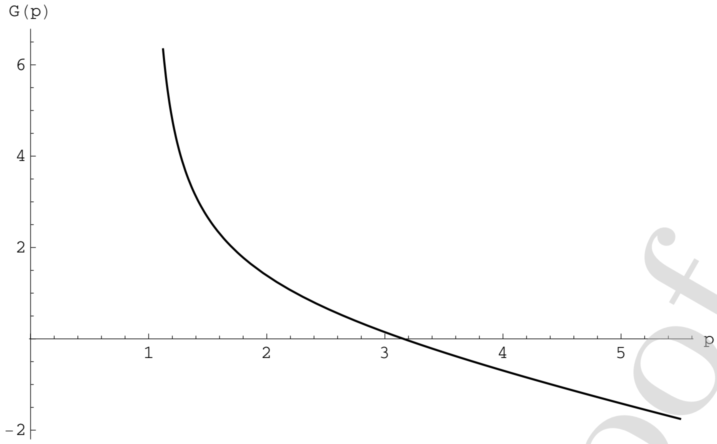


Figure 2. Graph of $G$ in (4.5)

- if $\omega \in (\frac{4}{\beta^2}, \omega_0)$, then $\partial_\omega \|\varphi_{\omega,\beta}\|^2 < 0$. Thus, $p(\omega) = 0$, and by $n(\mathcal{H}_\omega|_{\text{odd}}) = 1$, we obtain $e^{i\omega t}\varphi_{\omega,\beta}$ is orbitally unstable in $H^1_{\text{odd}}(\mathbb{R} - \{0\})$ (and so is in $H^1(\mathbb{R} - \{0\})$);
- if $\omega \in (\omega_0, \omega^*(p))$, then $\partial_\omega \|\varphi_{\omega,\beta}\|^2 > 0$. Thus, $p(\omega) = 1$, and consequently $n(\mathcal{H}_\omega) - p(\omega) = 4 - 1 = 3$ which implies that $e^{i\omega t}\varphi_{\omega,\beta}$ is orbitally unstable in $H^1(\mathbb{R} - \{0\})$;
- if $\omega > \omega^*(p)$, then $\partial_\omega \|\varphi_{\omega,\beta}\|^2 > 0$. Thus $p(\omega) = 1$, and by $n(\mathcal{H}_\omega|_{\text{odd}}) = 1$, we obtain that $e^{i\omega t}\varphi_{\omega,\beta}$ is orbitally stable in $H^1_{\text{odd}}(\mathbb{R} - \{0\})$.

ii) Let $p \in (p_0, 5)$. Then, for $\omega_0 \equiv \omega_0(p)$:
- if $\omega \in (\frac{4}{\beta^2}, \omega^*(p))$, then $\partial_\omega \|\varphi_{\omega,\beta}\|^2 < 0$. Thus, $p(\omega) = 0$, and by $n(\mathcal{H}_\omega|_{\text{odd}}) = 1$, we obtain $e^{i\omega t}\varphi_{\omega,\beta}$ is orbitally unstable in $H^1_{\text{odd}}(\mathbb{R} - \{0\})$ (and so is in $H^1(\mathbb{R} - \{0\})$);
- if $\omega \in (\omega^*(p), \omega_0)$, then $\partial_\omega \|\varphi_{\omega,\beta}\|^2 < 0$. Thus, $p(\omega) = 0$, and consequently $n(\mathcal{H}_\omega) - p(\omega) = 3$ which implies that $e^{i\omega t}\varphi_{\omega,\beta}$ is orbitally unstable in $H^1(\mathbb{R} - \{0\})$;
- if $\omega > \omega_0$, then $\partial_\omega \|\varphi_{\omega,\beta}\|^2 > 0$. Thus $p(\omega) = 1$, and by $n(\mathcal{H}_\omega|_{\text{odd}}) = 1$, we obtain that $e^{i\omega t}\varphi_{\omega,\beta}$ is orbitally stable in $H^1_{\text{odd}}(\mathbb{R} - \{0\})$.

iii) Let $p = p_0$. Then, $\omega^*(p_0) = \omega_0(p_0)$ and so we obtain:
- if $\omega \in (\frac{4}{\beta^2}, \omega^*(p_0))$, then $\partial_\omega \|\varphi_{\omega,\beta}\|^2 < 0$. Thus, $p(\omega) = 0$, and by $n(\mathcal{H}_\omega|_{\text{odd}}) = 1$, we obtain $e^{i\omega t}\varphi_{\omega,\beta}$ is orbitally unstable in $H^1_{\text{odd}}(\mathbb{R} - \{0\})$ (and so is in $H^1(\mathbb{R} - \{0\})$);
- if $\omega > \omega^*(p_0)$, then $\partial_\omega \|\varphi_{\omega,\beta}\|^2 > 0$. Thus, $p(\omega) = 1$, and by $n(\mathcal{H}_\omega|_{\text{odd}}) = 1$, we obtain that $e^{i\omega t}\varphi_{\omega,\beta}$ is orbitally stable in $H^1_{\text{odd}}(\mathbb{R} - \{0\})$.

iv) For $p \in (3, p_0)$ and $\omega = \omega^* > \omega_0(p)$, we obtain that $e^{i\omega^* t}\varphi_{\omega^*,\beta}$ is orbitally stable in $H^1_{\text{odd}}(\mathbb{R} - \{0\})$. In fact, this is deduced from $\text{Ker}(L^\beta_{1,\omega^*}|_{\text{odd}}) = \{0\}$, $\text{Ker}(L^\beta_{2,\omega^*}|_{\text{odd}}) = [\varphi_{\omega^*,\beta}]$, $n(\mathcal{H}_{\omega^*}|_{\text{odd}}) = 1$ and $\partial_\omega \|\varphi_{\omega,\beta}\|^2|_{\omega=\omega^*} > 0$.

v) For $p \in (p_0, 5)$ and $\omega = \omega^* < \omega_0(p)$, we obtain that $e^{i\omega^* t}\varphi_{\omega^*,\beta}$ is orbitally unstable in $H^1(\mathbb{R} - \{0\})$. In fact, this is deduced from $n(\mathcal{H}_{\omega^*}|_{\text{odd}}) = 1$ and $\partial_\omega \|\varphi_{\omega,\beta}\|^2|_{\omega=\omega^*} < 0$.

b) If $n(\mathcal{H}_\omega) - p(\omega)$ is even, the criterium in Theorem 2.3 does not provide any information about the stability of $e^{i\omega t}\varphi_{\omega,\beta}$ in all $H^1(\mathbb{R} - \{0\})$. For instance, in the cases $p \in (1, 3]$

and $\omega > \omega^*$, $p \in (3, p_0)$ and $\omega > \omega^*(p)$, $p \in (p_0, 5)$ and $\omega > \omega_0(p)$, and $p = p_0$ and $\omega > \omega^*(p_0)$.

c) For $p \in (3,5)$ and due to the ideas in [63], we conjecture that in the case $\omega = \omega_0(p)$ (namely, $\partial_\omega \|\varphi_{\omega,\beta}\|^2|_{\omega=\omega_0} = 0$), the standing wave $e^{i\omega t}\varphi_{\omega,\beta}$ is orbitally unstable.

## 5. Extension theory and tail stability properties

In this section, we will show that the approach for studying the stability of the bump-like profiles $\varphi_{\omega,\beta}$ can also be applied for the tail-type standing waves in the case $\beta > 0$ (attractive $\delta'$-interaction) in the model (1.3). We note that an stability analysis for this case was elaborated in [3]. Our proof does not use variational tools. Further, we improved Proposition 6.3 and Proposition 6.11 in [3]. In particular, we obtain an stability property of the standing wave for the case $\omega = \frac{4(p+1)}{\beta^2(p-1)}$.

Next, we establish the spectral properties of the operators $L^\beta_{j,\omega}$, $j \in \{1,2\}$, defined by (2.3) for $\beta > 0$ with $\varphi_{\omega,\beta} = \varphi^{odd}_{\omega,\beta}$ (see Figure 3 below) having tail-like profile. Our study give a complete picture of the spectrum of these self-adjoint operators.

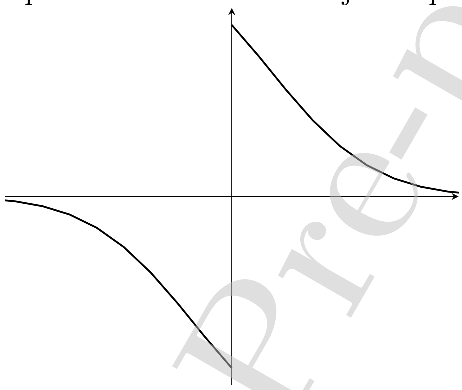

Figure 3. $\varphi^{odd}_{\omega,\beta}$ for $\beta > 0$

**Theorem 5.1.** *Let $\beta > 0$ and $\omega > \frac{4}{\beta^2}$. Let also $L^\beta_{j,\omega}$, $j \in \{1,2\}$, be defined by (2.3). Then the following assertions hold.*

1) *If $\omega^* = \frac{4(p+1)}{\beta^2(p-1)}$, then $Ker(L^\beta_{1,\omega^*}) = \left[\frac{d}{dx}\varphi_{\omega^*,\beta}\right]$, and $n(L^\beta_{1,\omega^*}) = 1$. Moreover, the eigenfunction associated to the negative eigenvalue is odd.*
2) *If $\omega \neq \omega^*$, then $Ker(L^\beta_{1,\omega}) = \{0\}$.*
3) *If $\omega > \omega^*$, then $n(L^\beta_{1,\omega}) = 2$. Moreover, the eigenfunctions associated to the negative eigenvalues are odd and even, respectively.*
4) *If $\omega < \omega^*$, then $n(L^\beta_{1,\omega}) = 1$. Moreover, the eigenfunction associated to the negative eigenvalue is odd.*
5) *$Ker(L^\beta_{2,\omega}) = [\varphi_{\omega,\beta}]$ and $L^\beta_{2,\omega} \geqq 0$.*

*Proof.* 1) For $\omega^* = \frac{4(p+1)}{\beta^2(p-1)}$ we have $\varphi''_{\omega^*}(0+) = 0$. Thus, repeating the arguments from the proof of Proposition 3.2, we obtain $\mathrm{Ker}(L^\beta_{1,\omega^*}) = \left[\frac{d}{dx}\varphi_{\omega^*,\beta}\right]$. Further, from $\langle L^\beta_{1,\omega^*}\varphi_{\omega^*,\beta}, \varphi_{\omega^*,\beta}\rangle < 0$ follows $n(L^\beta_{1,\omega^*}) \geqq 1$. Using the extension theory, we deduce $n(L^\beta_{1,\omega^*}) \leqq 1$. Indeed, the symmetric operator $L_0$

$$L_0 = -\frac{d^2}{dx^2} + \omega - p|\varphi_{\omega^*,\beta}|^{p-1}, \quad D(L_0) = \{v \in H^2(\mathbb{R}) : v'(0) = 0\},$$

has deficiency numbers $n_{\pm}(L_0) = 1$ (see [11]). By the von Neumann decomposition (see Theorem 6.1), we have

$$D(L_0^*) = \{v \in H^2(\mathbb{R} - \{0\}) : v'(0+) = v'(0-)\} = D(L_0) \oplus [v_i] \oplus [v_{-i}],$$

where $v_{\pm i}$ are defined in the proof of Proposition 3.13. All self-adjoint extensions of $L_0$ are given by the one-parameter family $(L^{\beta}_{1,\omega^*}, D_\beta)$, $\beta \in \mathbb{R}$. Next, we show that $L_0 \geqq 0$ on $D(L_0)$ for $\beta > 0$. Indeed, for $v \in D(L_0)$ we obtain (see (3.7) and (3.8))

$$\langle L_0 v, v\rangle = \int\limits_{-\infty}^{+\infty} (\varphi'_{\omega^*,\beta})^2 \left(\frac{d}{dx}\left(\frac{v}{\varphi'_{\omega^*,\beta}}\right)\right)^2 dx - \left[v'v - v^2 \frac{\varphi''_{\omega^*,\beta}}{\varphi'_{\omega^*,\beta}}\right]_{-\infty}^{0-} - \left[v'v - v^2 \frac{\varphi''_{\omega^*,\beta}}{\varphi'_{\omega^*,\beta}}\right]_{0+}^{+\infty}. \tag{5.1}$$

The non-integral term in (5.1) admits the form

$$v^2(0)\frac{\varphi''_{\omega^*,\beta}(0-) - \varphi''_{\omega^*,\beta}(0+)}{\varphi'_{\omega^*,\beta}(0+)} = 0,$$

since $\varphi'_{\omega^*,\beta}(0+) = \varphi'_{\omega^*,\beta}(0-) < 0$ and $\varphi''_{\omega^*,\beta}(0+) = -\varphi''_{\omega^*,\beta}(0-) = 0$. Therefore, $\langle L_0 v, v\rangle \geqq 0$. Thus, from Proposition 6.3 it follows $n(L^{\beta}_{1,\omega^*}) \leqq 1$.

Next, let $\lambda_{\omega^*} < 0$ and $\psi_{\omega^*} \in D_\beta$ be such that $L^{\beta}_{1,\omega^*}\psi_{\omega^*} = \lambda_{\omega^*}\psi_{\omega^*}$. Let us show that $\psi_{\omega^*}$ is odd. By Lemma 3.4 for the case $\beta > 0$, we deduce that $\psi_{\omega^*}$ is either even or odd. Suppose that it is even, then $\psi_{\omega^*} \in H^2(\mathbb{R})$, and $\psi_{\omega^*}(0+) = 0$.

Consider the operator $L^{\beta}_{1,\omega^*}$ defined on $S_0 = \{v \in H^2(\mathbb{R}_+) : v'(0+) = 0\}$. Since $\phi = \varphi'_{\omega^*,\beta}|_{(0,+\infty)}$ satisfies $L^{\beta}_{1,\omega^*}\phi = 0$, and $\phi < 0$ with $\phi'(0+) = 0$, it follows $L^{\beta}_{1,\omega^*} \geqq 0$ on $S_0$. From the other hand, $\psi_{\omega^*}|_{(0,+\infty)} \in S_0$, and $n(L^{\beta}_{1,\omega^*}) \geqq 1$ on $S_0$, which is the contradiction with the positivity of $L^{\beta}_{1,\omega^*}$ Therefore, $\psi_{\omega^*}$ is odd.

2) The proof is similar to the one of Proposition 3.2.
3) and 4) Combining the analytic perturbation theory arguments, item 1), and Lemma 3.17 applied to $\varphi_{\omega^*,\beta}$, for $\beta > 0$, we obtain from relation (3.23) that $\gamma < 0$ in decomposition (3.11) (due to $\varphi_{\omega^*,\beta}(0+) < 0$). Thus, from (3.11) we obtain that the second simple eigenvalue $\Pi_1(\omega)$ is negative for $\omega > \omega^*$, and $\Pi_1(\omega)$ is positive for $\omega < \omega^*$. Moreover, the associated eigenfunction $\Omega_1(\omega)$ is even, and the eigenfunction associated to the first negative eigenvalue is an odd function for all $\omega \neq \omega^*$.
5) Repeating the arguments from the proof of Proposition 3.2, we obtain $\mathrm{Ker}(L^{\beta}_{2,\omega}) = [\varphi_{\omega,\beta}]$. Further, by (3.24), (3.25) and inequality $\varphi_{\omega,\beta}(0+)\varphi_{\omega,\beta}(0-) < 0$,

$$\langle L^{\beta}_{2,\omega} v, v\rangle = A + \left[v'v - v^2\frac{\varphi'_{\omega,\beta}}{\varphi_{\omega,\beta}}\right]_{0-}^{0+} = A - \frac{|\varphi_{\omega,\beta}(0+)v(0-) - \varphi_{\omega,\beta}(0-)v(0+)|^2}{\beta\varphi_{\omega,\beta}(0+)\varphi_{\omega,\beta}(0-)} \geqq 0,$$

where $v \in D_\beta$, and $A \geqq 0$ is the integral term.

□

Recall the operator $\mathcal{H}_\omega$ defined by (2.5). Thus, from Theorem 5.1 we obtain the following:
1) if $\omega > \omega^*$ then $n(\mathcal{H}_\omega) = 2$, and $n(\mathcal{H}_\omega|_{\mathrm{odd}}) = 1$;
2) if $\omega < \omega^*$, then $n(\mathcal{H}_\omega) = 1$.
3) if $\omega = \omega^*$, then $n(\mathcal{H}_{\omega^*}|_{\mathrm{odd}}) = 1$ and $\mathrm{Ker}(L^{\beta}_{1,\omega^*}|_{odd}) = \{0\}$

Finally, we can establish the stability result for the NLS-$\delta'$ equation in the case $\beta > 0$ (see Proposition 6.11 and Theorem 6.13 in [3]).

**Theorem 5.2.** *Let $\beta > 0$, $\omega > \frac{4}{\beta^2}$, and $\omega^* = \frac{4(p+1)}{\beta^2(p-1)}$. Then*

1) *if $\omega < \omega^*$ and $p > 1$, the standing wave $e^{i\omega t}\varphi_{\omega,\beta}$ is orbitally stable in $H^1(\mathbb{R} - \{0\})$;*
2) *Let $\omega > \omega^*$. For $p \in (1, 2]$, the standing wave $e^{i\omega t}\varphi_{\omega,\beta}$ is linearly unstable in $H^1(\mathbb{R} - \{0\})$. For $p > 2$, the standing wave $e^{i\omega t}\varphi_{\omega,\beta}$ is nonlinearly unstable in $H^1(\mathbb{R} - \{0\})$.*
3) *if $\omega = \omega^*$ and $p > 1$, the standing wave $e^{i\omega t}\varphi_{\omega^*,\beta}$ is orbitally stable in $H^1_{odd}(\mathbb{R} - \{0\})$.*

*Proof.* a) By Theorem 5.1, for $\omega \neq \omega^*$ we get $\mathrm{Ker}(L^{\beta}_{2,\omega}) = [\varphi_{\omega,\beta}]$, $L^{\beta}_{2,\omega} \geqq 0$, and $\mathrm{Ker}(L^{\beta}_{1,\omega}) = \{0\}$. Thus, we obtain the following:

1) for every $p > 1$ and $\omega < \omega^*$ from Proposition 6.5 in [3] we have $\partial_\omega \|\varphi_{\omega,\beta}\|^2 > 0$. Thus, since $n(\mathcal{H}_\omega) = 1$, the standing wave $e^{i\omega t}\varphi_{\omega,\beta}$ is orbitally stable in $H^1(\mathbb{R} - \{0\})$,
2) let $\omega > \omega^*$. In this case we have $n(\mathcal{H}_\omega) = 2$. If $p \in (1, 5]$, by Proposition 6.5 in [3], we deduce $\partial_\omega \|\varphi_{\omega,\beta}\|^2 > 0$. Thus, since

$$n(\mathcal{H}_\omega) - p(\omega) = 2 - 1 = 1,$$

by Theorem 2.3 and Remark 2.4 we obtain that the standing wave $e^{i\omega t}\varphi_{\omega,\beta}$ is linearly unstable in $H^1(\mathbb{R} - \{0\})$ for $p \in (1, 2]$ and orbitally unstable in $H^1(\mathbb{R} - \{0\})$ for $p \in (2, 5]$.
Let $p > 5$. The sign of $\partial_\omega \|\varphi_{\omega,\beta}\|^2$ was established in Proposition 6.5 of [3]. If $\partial_\omega \|\varphi_{\omega,\beta}\|^2 > 0$, the instability of $e^{i\omega t}\varphi_{\omega,\beta}$ follows immediately. Suppose now that $\partial_\omega \|\varphi_{\omega,\beta}\|^2 < 0$. Then, since $n(\mathcal{H}_\omega|_{\mathrm{odd}}) = 1$, we conclude that $e^{i\omega t}\varphi_{\omega,\beta}$ is orbitally unstable in $H^1_{\mathrm{odd}}(\mathbb{R} - \{0\})$ and, a fortiori, it is orbitally unstable in $H^1(\mathbb{R} - \{0\})$.

b) By Theorem 5.1, for $\omega = \omega^*$ we get $\mathrm{Ker}(L^{\beta}_{2,\omega^*}) = [\varphi_{\omega^*,\beta}]$, $L^{\beta}_{2,\omega^*} \geqq 0$, and $\mathrm{Ker}(L^{\beta}_{1,\omega^*}|_{\mathrm{odd}}) = \{0\}$. Thus, we obtain the following:

3) let $\omega = \omega^*$. From the proof of Proposition 6.5 in [3] (see (6.22)-(6.23)-(6.32)) we deduce $\partial_\omega \|\varphi_{\omega,\beta}\|^2|_{\omega=\omega^*} > 0$. Thus, since $n(\mathcal{H}_{\omega^*}|_{odd}) = 1$, the standing wave $e^{i\omega^* t}\varphi_{\omega^*,\beta}$ is orbitally stable in $H^1_{\mathrm{odd}}(\mathbb{R} - \{0\})$.

□

## 6. Appendix

In this Appendix, we prove some non-standard results used in the body of the manuscript. In particular, we prove a Perron-Frobenius property for $\delta'$-interactions. We start by convenience of the reader establishing some results of the extension theory.

### 6.1. Extension theory.

Let $A$ be a densely defined symmetric operator on a Hilbert space $H$ with adjoint $A^*$. Consider the deficiency subspaces $\mathcal{N}_\pm(A) = \mathrm{Ker}(A^* \mp i)$ of $A$ and the deficiency numbers $n_\pm(A)$. Then, we have the following two results that have been used in this work (see [62]).

**Theorem 6.1.** *(von Neumann decomposition) Let $A$ be as above, then*

$$D(A^*) = D(A) \oplus \mathcal{N}_+(A) \oplus \mathcal{N}_-(A). \tag{6.1}$$

*Therefore, for $u \in D(A^*)$ such that $u = f + f_i + f_{-i}$, with $f \in D(A)$, $f_{\pm i} \in \mathcal{N}_\pm(A)$,*

$$A^*u = Af + if_i - if_{-i}.$$

*Remark* 6.2. The direct sum in (6.1) is not necessarily orthogonal.

The second result reads as follows.

**Proposition 6.3.** *Let $A$ be a densely defined lower semi-bounded symmetric operator (that is, $A \geq mI$) with finite deficiency indices $n_\pm(A) = k < \infty$ in the Hilbert space $H$, and let $\widetilde{A}$ be a self-adjoint extension of $A$. Then the spectrum of $\widetilde{A}$ in $(-\infty, m)$ is discrete and consists of at most $k$ eigenvalues counting multiplicities.*

The result below was used in the proof of Proposition 3.13.

**Proposition 6.4.** *Let $\omega > \frac{4}{\beta^2}$ and $\beta \in \mathbb{R}$. Then, the operator $L^\beta_{1,\omega}$ in (2.3) belongs to the family of self-adjoint extensions of the following symmetric operator $L$*

$$L = -\frac{d^2}{dx^2} + \omega - p|\varphi_{\omega,\beta}|^{p-1}, \quad D(L) = \{v \in H^2(\mathbb{R}) : v'(0) = 0, v(\pm\mu) = 0\},$$

*where $\mu$ denotes the unique positive zero of even function $\varphi'_{\omega,\beta}$.*

*Proof.* It is not difficult to see that the following symmetric operator $\widetilde{L}$,

$$\widetilde{L} = -\frac{d^2}{dx^2}, \quad D(\widetilde{L}) = \{v \in H^2(\mathbb{R}) : v'(0) = 0, v(\pm\mu) = 0\},$$

has deficiency numbers $n_\pm(\widetilde{L}) = 3$. Indeed, by classical arguments from the theory of ODE's, the deficiency subspaces of $\widetilde{L}$ are given by

$$\mathcal{N}_-(\widetilde{L}) = \operatorname{Ker}(\widetilde{L}^* + i) = \left[v_i, e^{i\sqrt{i}|x-\mu|}, e^{i\sqrt{i}|x+\mu|}\right]$$

and $\mathcal{N}_+(\widetilde{L}) = \operatorname{Ker}(\widetilde{L}^* - i) = \left[v_{-i}, e^{i\sqrt{-i}|x-\mu|}, e^{i\sqrt{-i}|x+\mu|}\right]$, where

$$v_{\pm i} = \begin{cases} e^{i\sqrt{\pm i}x}, & x > 0, \\ -e^{i\sqrt{\pm i}x}, & x < 0, \end{cases}, \quad \Im(\sqrt{\pm i}) > 0.$$

Thus, since $\varphi_{\omega,\beta} \in L^\infty(\mathbb{R})$ follows from extension theory that $n_\pm(L) = n_\pm(\widetilde{L}) = 3$ and $D(L^*) = D(\widetilde{L}^*)$. Then by the von Neumann decomposition (see Theorem 6.1) we have

$$D(L^*) = \{v \in H^2(\mathbb{R} - \{0, \pm\mu\}) \cap H^1(\mathbb{R} - \{0\}) : v'(0+) = v'(0-)\}.$$

Thus, all the self-adjoint extensions of $L$ are given by a 9-parameter family. Here we restrict ourselves to the case of separated boundary conditions at each point $0, \pm\mu$. Therefore, there exists a 3-parameter family of self-adjoint operators $(L_{\nu,Z_\pm}, D(L_{\nu,Z_\pm}))$ depending on $\nu, Z_\pm \in \mathbb{R}$, and given by

$$\begin{cases} L_{\nu,Z_\pm} = -\frac{d^2}{dx^2} + \omega - p|\varphi_{\omega,\beta}|^{p-1} \\ D(L_{\nu,Z_\pm}) = \left\{ \begin{array}{c} H^2(\mathbb{R} - \{0, \pm\mu\}) \cap H^1(\mathbb{R} - \{0\}) : v(0+) - v(0-) = -\nu v'(0), \\ v'(0+) = v'(0-),\ v'(\pm\mu+) - v'(\pm\mu-) = -Z_\pm v(\pm\mu) \end{array} \right\}. \end{cases}$$

It is easily seen that we arrive at $(L^\beta_{1,\omega}, D_\beta)$ for $Z_\pm = 0, \nu = \beta$. This finishes the proof. □

6.2. **Perron-Frobenius property for the repulsive $\delta'$-interactions.** In this subsection we show that the Schrödinger operators with a repulsive $\delta'$-interaction defined in (2.3) satisfy the following Perron-Frobenius property.

**Lemma 6.5.** *Let $\beta < 0$, $\omega > \frac{4}{\beta^2}$. Let also $L^\beta_{j,\omega}$, $j \in \{1, 2\}$, be defined by (2.3). Assume that $\lambda_{j,\omega,\beta} = \inf \sigma(L^\beta_{j,\omega})$ is the smallest eigenvalue. Then $\lambda_{j,\omega,\beta}$ is simple, and its corresponding eigenfunction $\psi_{j,\omega,\beta}$ is positive (after replacing $\psi_{j,\omega,\beta}$ by $-\psi_{j,\omega,\beta}$ if necessary) and even.*

*Remark* 6.6. For $\beta > 0$, the profile $\varphi^{odd}_{\omega,\beta}$ is of tail-type and Theorem 5.1 shows that the Perron-Frobenius property for the associated operators $L^{\beta}_{j,\omega}$, $j \in \{1,2\}$, remains to be false on the domain $D_\beta$ defined by (2.3).

*Proof.* This result follows by a slight twist of standard abstract Perron-Frobenius arguments. We prove the assertion for $\lambda_{1,\omega,\beta}$, the proof for $\lambda_{2,\omega,\beta}$ is similar. We divide the proof into several steps.

1) Let $\mu > 0$. Denote $-\Delta_\beta = -\frac{d^2}{dx^2} - \beta\langle\cdot,\delta'\rangle\delta'(x)$. From the Krein formula follows the representation for the resolvent $(-\Delta_\beta+\mu)^{-1}$ as $\mu$ is sufficiently large (see [10, 11])

$$(-\Delta_\beta+\mu)^{-1}f = (-\Delta_0+\mu)^{-1}\,f + \frac{-2\beta\mu}{2-\beta\sqrt{\mu}}\langle f,\bar{J}_\mu\rangle J_\mu,$$

where $-\Delta_0 = -\frac{d^2}{dx^2}$ denotes the classical 1-dimensional Laplacian with domain $H^2(\mathbb{R})$ and $(-\Delta_0+\mu)^{-1}$ denotes its resolvent which exists for any $\mu > 0$. $J_\mu$ is defined by

$$J_\mu(x) = \frac{1}{2\sqrt{\mu}}sign(x)e^{-\sqrt{\mu}|x|}.$$

Thus, since $(-\Delta_0+\mu)^{-1}f = \frac{1}{2\sqrt{\mu}}e^{-\sqrt{\mu}|\cdot|} * f$, we obtain

$$(-\Delta_\beta+\mu)^{-1}f(x) = \int_{\mathbb{R}} K(x,y)f(y)dy,$$

with

$$K(x,y) = \frac{1}{2\sqrt{\mu}}e^{-\sqrt{\mu}|x-y|} - \frac{\beta}{2}\frac{1}{2-\beta\sqrt{\mu}}\,\mathrm{sign}(xy)e^{-\sqrt{\mu}(|x|+|y|)}.$$

Moreover, for every $x$ fixed, $K(x,\cdot) \in L^2(\mathbb{R})$. Thus, the existence of the integral is guaranteed by Holder's inequality.

2) Let us show that $K(x,y) > 0$ for $(x,y) \in \mathbb{R}\times\mathbb{R}$. Indeed, since $K(x,y) = K(y,x)$, it is sufficient to consider the following cases.

   a) Let $x > 0$ and $y > 0$ or $x < 0$ and $y < 0$. Since $-\frac{\beta}{2}\frac{1}{2-\beta\sqrt{\mu}} > 0$, we obtain

$$K(x,y) = \frac{1}{2\sqrt{\mu}}e^{-\sqrt{\mu}|x-y|} - \frac{\beta}{2}\frac{1}{2-\beta\sqrt{\mu}}e^{-\sqrt{\mu}|x+y|} > 0.$$

   b) Let $x > 0$ and $y < 0$. Since $\frac{1}{2\sqrt{\mu}} + \frac{\beta}{2}\frac{1}{2-\beta\sqrt{\mu}} > 0$, we obtain

$$K(x,y) = \Big[\frac{1}{2\sqrt{\mu}} + \frac{\beta}{2}\frac{1}{2-\beta\sqrt{\mu}}\Big]e^{-\sqrt{\mu}(x-y)} > 0.$$

   Moreover, by $K(x,y) = K(y,x)$, we get $K(x,y) > 0$ for $x < 0$ and $y > 0$.

3) Applying standard Perron-Frobenious-type arguments (see, e.g., Lemma 5 in [8]), we conclude that there is $\mu_1 > 0$ sufficiently large such that the operator $R = (L^{\beta}_{1,\omega}+\mu_1)^{-1}$ exists, is bounded on $L^2(\mathbb{R})$, and is *positivity improving*, i.e., if $f \in L^2(\mathbb{R})$ and $f(x) \geqq 0$ almost everywhere in $\mathbb{R}$, and $f \neq 0$, then $Rf(x) > 0$ almost everywhere in $\mathbb{R}$.

   Note the spectrum $\sigma(R)$ of $R$ is the image of $\sigma(L^{\beta}_{1,\omega})$ under the mapping $\lambda \to (\lambda+\mu_1)^{-1}$. Denote the greatest eigenvalue of $R$ by $\lambda_0 = (\lambda_{1,\omega,\beta}+\mu_1)^{-1}$, and let $\psi_0$ be an eigenfunction corresponding to $\lambda_0$. Thus,

$$\langle R\psi_0,\psi_0\rangle \geqq \langle R|\psi_0|,|\psi_0|\rangle \geqq \langle R\psi_0,\psi_0\rangle,$$

where in the last inequality was used the positivity improving property of $R$. Therefore, $\langle R|\psi_0|, |\psi_0|\rangle = \langle R\psi_0, \psi_0\rangle$. Hence for $\psi_0^+ = \frac{|\psi_0|+\psi_0}{2}$ and $\psi_0^- = \frac{|\psi_0|-\psi_0}{2}$ being the positive and negative parts of $\psi_0$ follows $\langle R\psi_0^+, \psi_0^-\rangle = 0$. Therefore, $\psi_0^-$ must vanish almost everywhere. Indeed, suppose that $\psi_0^-(x) > 0$ for all $x \in E$ with $|E| > 0$, while $R\psi_0^+(x) > 0$ almost everywhere in $\mathbb{R}$. In particular, we can assume that there exists $\epsilon > 0$ such that $R\psi_0^+(x) > \epsilon$ for all $x \in E$. Thus,

$$0 = \langle R\psi_0^+, \psi_0^-\rangle \geqq \int_E \psi_0^-(x) R\psi_0^+(x)dx > 0,$$

but this is a contradiction. Therefore, we have that any eigenfunction $\psi_0$ of $R$ corresponding to $\lambda_0$ is positive almost everywhere.

It is easy to see that $\psi_0$ is the eigenfunction of $L^\beta_{1,\omega}$ corresponding to the smallest eigenvalue $\lambda_{1,\omega,\beta}$.

Further, suppose that $\psi_1$ and $\psi_2$ are two different eigenfunctions corresponding to $\lambda_{1,\omega,\beta}$, then the preceding analysis shows that

$$\langle \psi_1, \psi_2\rangle = \int_{\mathbb{R}} \psi_1(x)\psi_2(x)dx \neq 0,$$

because $\psi_1(x)\psi_2(x) > 0$ for a.e. $x \in \mathbb{R}$. Therefore, $\lambda_{1,\omega,\beta}$ is simple.

4) From Lemma 3.4 follows immediately that the eigenfunction corresponding to $\lambda_{1,\omega,\beta}$ is even.

□

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STABILITY PROPERTIES OF STANDING WAVES FOR NLS EQUATIONS WITH THE $\delta'$-INTERACTION

Highlights

- The nonlinear Schrödinger model with the *repulsive* $\delta'$-interaction on the line.
- Orbital (in)stability of standing waves with discontinuous bump-like profile.
- Extension theory of symmetric operators by Krein-von Neumann.
- Morse index of self-adjoint operators, Sturm oscillation results and analytic perturbation theory.
- Perron-Frobenius property for the repulsive $\delta'$-interaction.

**Declaration of interests**

[XXX] The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

☐The authors declare the following financial interests/personal relationships which may be considered as potential competing interests:

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Author Contribution Statement

1. Jaime Angulo Pava: Conceptualization, Methodology, Formal analysis, Investigation, Writing-Original Draft, Supervision.
2. Nataly Goloshchapova: Investigation, Draft Writing-Review&Editing, Visualization, Resources.