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CONSISTENT ESTIMATION IN FUNCTIONAL LINEAR RELATIONSHIPS WITH REPLICATIONS

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Summary

In this paper we investigate maximum likelihood estimation in linear functional relationships with replications. The general formulation considered in Dorf and Gurland (1961) is studied. The approach is based on Mak (1982) where general results for maximum likelihood estimation in the presence of incidental parameters is considered. Since the approach allows the derivation of the asymptotic covariance matrix of the maximum likelihood estimators of the model parameters it is possible to compute the asymptotic relative efficiencies of the maximum likelihood estimators with respect to the estimators suggested in Dorf and Gurland (1961). Computation of maximum likelihood estimators is discussed. Comparisons are also reported for the situation of a small sample selected from a particular generated population.

1. Introduction

Estimation in functional relationships has been the subject of several papers in the statistical literature. As shown in Neyman and Scott (1947), maximum likelihood estimators are typically inconsistent and as considered in Patefield (1977, 1978) and Gleser (1985), the asymptotic covariance matrix is not given by the inverse of the Fisher information matrix. As shown in Solari (1969), the solution of the likelihood equations in the case of the functional relationship is not a maximum but a saddle point of the likelihood equation. Further, the likelihood function is unbounded. The functional model is important in the evaluation of the concentration of a certain mineral (copper or arsenic, for example), in samples of soil (or water) in a certain region where the distribution of the concentration certainly is not symmetrical (Ripley and Thompson, 1987) .

Maximum likelihood estimation in the unreplicated case with the ratio of variance known is considered in Mak (1992), where a general treatment is presented for maximum likelihood estimation in the presence of nuisance parameters. Consistent estimation for the case of one variance known is considered in Cheng and van Ness (1990). Both papers derive their main results under the normality assumption. Extensions for the case of elliptical models are considered in Arellano-Valle et al. (1996) and Vilca-Labra et al. (1998). In this paper we study maximum likelihood estimation considering replicated observations, which is a way of guaranteeing that no assumptions about the model variances are required to make the approach feasible. The functional linear relationship relating the variables x and y is given by $y = \alpha + \beta x$, which are not observed sure variables. We observe instead $X = x + u$ and $Y = y + e$ with α and β being unknown parameters. We consider r_i and s_i replications on x_i and y_i, respectively, so that we observe Y_{ij} and X_{ik}, where

$$Y_{ij} = y_i + e_{ij} \tag{1.1}$$

and

$$X_{ik} = x_i + u_{ik}, \tag{1.2}$$

$j = 1, \ldots, s_i$, $k = 1, \ldots, r_i$ and $i = 1, \ldots, n$. Moreover, it is also assumed that

$$\begin{pmatrix} e_{ij} \\ u_{ik} \end{pmatrix} \sim N_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_{ee} & 0 \\ 0 & \sigma_{uu} \end{pmatrix}\right),$$

so that the nonconstant part of the log-likelihood function corresponding to the observed sample (X_{ik}, Y_{ij}), $j = 1, \ldots, s_i$, $k = 1, \ldots, r_i$ and $i = 1, \ldots, n$, is given by

$$\ell = -\frac{1}{2}\sum_{i=1}^{n}[s_i \log(\sigma_{ee}) + r_i \log(\sigma_{uu})] - \frac{1}{2}\{\sigma_{uu}^{-1}\sum_{i=1}^{n}\sum_{k=1}^{r_i}(X_{ik} - x_i)^2 + \sigma_{ee}^{-1}\sum_{i=1}^{n}\sum_{j=1}^{s_i}(Y_{ij} - \alpha - \beta x_i)^2\}, \tag{1.3}$$

where α, β, σ_{ee} and σ_{uu} are structural parameters and x_i are incidental parameters. Notice that our model is similar to the model considered by Dorf and Gurland (1961), where different number of replications are considered for each x_i and y_i, $i = 1, \ldots, n$. However, Dorf and Gurland (1961) do not consider maximum likelihood estimation. A less general replication structure is considered in the structural model of Chan and Mak (1979) where the likelihood approach is implemented. In that paper Chan and Mak make reference to ANOVA type estimators but no efficiency results are reported on the comparisons between those estimators and the maximum likelihood estimators.

The assumption of equal number of replications on x and y is required for studying the asymptotic behavior of the maximum likelihood estimators. It is shown that the maximum likelihood estimators of and α and β are consistent but the maximum likelihood estimators of σ_{ee} and σ_{uu} are not. However, consistent estimators can be obtained by slightly modifying the maximum likelihood estimators of those parameters.

In the most general situation the maximum likelihood estimator of β has to be obtained numerically. An EM-type algorithm can be implemented in that situation. Simulation studies show that this algorithm typically converges and reasonably fast, except in small sample sizes in which case good starting values are required sometimes. In some less general situation, the maximum likelihood estimator of β can be obtained as the solution of a fourth degree equation. The problem of multiple roots can be circumvented by using the likelihood function or by picking the solution that follows from the solution of the EM-type algorithm when it converges. By using results in Mak (1982) on incidental parameter estimation, the asymptotic distribution of the MLE is obtained, a result so far not available in the literature. By using those results it is possible to obtain large sample (efficiency) comparisons by considering the large sample variances for the ANOVA type estimators reported in Dorf and Gurland (1961). So far only simulation studies are reported in the literature comparing the approaches as in the structural case of Schaffer and Purdy (1996).

Section 2 reviews the main results in Mak (1982). Conditions for consistency and asymptotic normality of the maximum likelihood estimators are derived. In Section 3 properties of the maximum likelihood estimators of the structural and incidental parameters are considered. It is shown that the maximum likelihood estimators of α and β are consistent while the maximum likelihood estimators of σ_{ee} and σ_{uu} are not and it is shown how to correct those estimators so that consistent ones can be obtained. The asymptotic distribution of the maximum likelihood estimators is also derived. Section 4 is dedicated to the derivation of the maximum likelihood estimators and their modified consistent estimators. In the more general situation, the maximum likelihood estimator of β can be obtained by implementing an EM type algorithm as considered in Kimura (1992). In the particular case where $r_i/s_i = k$, a constant, $i = 1, \ldots, n$, the maximum likelihood estimator of β follows by solving a fourth degree equation. Further simplifications occur if $r_i = r$ and $s_i = s$, $i = 1, \ldots, n$, and, moreover, in the case where $s = 1$, that is, only x is replicated, which is the more common situation in practical studies. In Section 5 the asymptotic covariance matrix is derived. In Section 6 two estimators proposed in Dorf and Gurland (1961) are considered. Relative efficiencies between the maximum likelihood estimators and estimators considered in Dorf and Gurland (1961) are reported in Section 7. As shown the maximum likelihood estimators can be significantly more efficient. Technical details are deferred to the Appendix.

2. Notation and preliminary results

Let $\mathbf{Z}_1, \ldots, \mathbf{Z}_n$, independent p-dimensional random vectors with log-likelihood function given by

$$\sum_{i=1}^{n} \log f_i(\mathbf{z}_i;\boldsymbol{\theta},x_i), \tag{2.1}$$

where $f_i(\mathbf{z}_i;\boldsymbol{\theta},x_i)$ is the density of $\mathbf{Z}_i$, $i=1,\ldots,n$, $\boldsymbol{\theta}=(\theta_1,\ldots,\theta_p)' \in \Theta \subset \mathcal{R}^p$ and $x_i \in \mathcal{X}_i \subset \mathcal{R}$, $i=1,\ldots,n$, are the incidental parameters. Suppose that $\boldsymbol{\theta}^0 \in \Theta$ and $x_i^0 \in \mathcal{X}_i$, $i=1,\ldots,n$, where $\boldsymbol{\theta}^0$ and $x_1^0,\ldots,x_n^0$, denote the true parameter values. The expected values are taken with respect to $\boldsymbol{\theta}^0$ and x_i^0, $i=1,\ldots,n$, which will be denoted by $E_0[.]=E[.|\boldsymbol{\theta}^0,x_1^0,\ldots,x_n^0]$. For each i and given $\boldsymbol{\theta}$, let $\tilde{x}_i=\tilde{x}_i(\mathbf{Z}_i,\boldsymbol{\theta})$, be an estimator (possibly depending on $\boldsymbol{\theta}$) of x_i, with a possibility of being the conditional maximum likelihood estimator, obtained by maximizing (2.1) with respect to x_i for fixed $\boldsymbol{\theta}$. Thus, replacing x_i by $\tilde{x}_i$ in (2.1) we obtain

$$\sum_{i=1}^{n} \log f_i(\mathbf{z}_i;\boldsymbol{\theta},\tilde{x}_i)=\sum_{i=1}^{n} h_i(\mathbf{z}_i;\boldsymbol{\theta}). \tag{2.2}$$

We also define the following functions:

$$q_{i\theta_j}(\mathbf{z}_i;\boldsymbol{\theta})=\frac{\partial h_i(\mathbf{z}_i;\boldsymbol{\theta})}{\partial \theta_j}, \quad j=1,\ldots,p, \tag{2.3}$$

$$q_{i\theta_k\theta_j}(\mathbf{z}_i;\boldsymbol{\theta})=\frac{\partial^2 h_i(\mathbf{z}_i;\boldsymbol{\theta})}{\partial \theta_j \partial \theta_k}, \quad j,k=1,\ldots,p, \tag{2.4}$$

and

$$q_{i\theta_k,\theta_j}(\mathbf{z}_i;\boldsymbol{\theta})=q_{i\theta_k}(\mathbf{z}_i;\boldsymbol{\theta})q_{i\theta_j}(\mathbf{z}_i;\boldsymbol{\theta}), \quad j,k=1,\ldots,p. \tag{2.5}$$

Morever, let $E_0[\mathbf{A}_n(\boldsymbol{\theta})]$ be the $p\times p$ random matrix with entry (j,k) given by

$$n^{-1}\sum_{i=1}^{n} E_0[q_{i\theta_k\theta_j}(\mathbf{Z}_i;\boldsymbol{\theta})], \quad j,k=1,\ldots,p. \tag{2.6}$$

In Mak (1982), Section 2, general conditions are established under which (2.2) has a maximum $\tilde{\boldsymbol{\theta}}_n=\tilde{\boldsymbol{\theta}}(\mathbf{z}_1,\ldots,\mathbf{z}_n)$, which converges in probability to some $\boldsymbol{\theta}^1$ in the interior of Θ, where $\boldsymbol{\theta}^1$ is a local maximum of the function

$$\overline{\psi}(\boldsymbol{\theta})=n^{-1}\sum_{i=1}^{n} E_0[h_i(\mathbf{Z}_i;\boldsymbol{\theta})], \tag{2.7}$$

and

$$\sqrt{n}(\mathbf{V}_n(\boldsymbol{\theta}^1))^{-1/2}(E_0[\mathbf{A}_n(\boldsymbol{\theta}^1)])(\tilde{\boldsymbol{\theta}}_n - \boldsymbol{\theta}^1) \xrightarrow{D} N_p(\mathbf{0}, \mathbf{I}_p), \tag{2.8}$$

where "$\xrightarrow{D}$" means convergence in distribution, with the (j,k)-th element of the $p \times p$ matrices $\mathbf{A}_n(\boldsymbol{\theta}^1) = (a_{jk})$ and $\mathbf{V}_n(\boldsymbol{\theta}^1) = (v_{jk})$ given, respectively, by

$$a_{jk} = \frac{1}{n}\sum_{i=1}^{n} q_{i\theta_j\theta_k}(\mathbf{Z}_i;\boldsymbol{\theta}^1) \quad \text{and} \quad v_{jk} = \frac{1}{n}\sum_{i=1}^{n} Cov[q_{i\theta_j}(\mathbf{Z}_i,\boldsymbol{\theta}^1), q_{i\theta_k}(\mathbf{Z}_i,\boldsymbol{\theta}^1)], \tag{2.9}$$

where, as pointed out before, the expected values are taken with respect to the true values $\boldsymbol{\theta}^0$ and x_i^0, $i = 1,\ldots,n$. The above results are proved in Mak (1982) and Gimenez and Bolfarine (1997). It is also noted in Mak (1982) that in some situations it is possible to obtain estimators $\tilde{x}_i$ so that $\boldsymbol{\theta}^1$ depends only on $\boldsymbol{\theta}^0$ (is independent of x_i^0), that is, there exists a function $g(.)$ such that $\boldsymbol{\theta}^1 = g(\boldsymbol{\theta}^0)$. If g is one to one then, a consistent estimator of $\boldsymbol{\theta}^0$ is given by $\hat{\boldsymbol{\theta}}_n = g^{-1}(\tilde{\boldsymbol{\theta}}_n)$.

3. Asymptotic behavior of the MLE

In this section the maximum likelihood estimator of x_i is obtained and it is shown that the maximum likelihood estimators $\hat{\boldsymbol{\theta}}$ of the true parameter value $\boldsymbol{\theta}^0 = (\alpha^0, \beta^0, \sigma_{ee}^0, \sigma_{uu}^0)'$ is not consistent that is, $\hat{\boldsymbol{\theta}} \xrightarrow{P} \boldsymbol{\theta}^1 \neq \boldsymbol{\theta}^0$. Specifically, we show that $\hat{\boldsymbol{\theta}} \xrightarrow{P} \boldsymbol{\theta}^1 = g(\boldsymbol{\theta}^0)$ with the function $g(.)$ presenting a simple form which can be inverted so that we can obtain $g^{-1}(.)$ and then obtain consistent estimators $\hat{\boldsymbol{\theta}}_c$ by computing $\hat{\boldsymbol{\theta}}_c = g^{-1}(\hat{\boldsymbol{\theta}})$.

Lemma 3.1. *Given $\boldsymbol{\theta} = (\alpha, \beta, \sigma_{ee}, \sigma_{uu})'$ the maximum likelihood estimator of x_i is given by*

$$g((\mathbf{X}_i, \mathbf{Y}_i), \boldsymbol{\theta}) = \frac{\sigma_{uu}^{-1} r_i \overline{X}_i + s_i \beta \sigma_{ee}^{-1}(\overline{Y}_i - \alpha)}{(r_i \sigma_{uu}^{-1} + s_i \beta^2 \sigma_{ee}^{-1})}, \tag{3.1}$$

where $\mathbf{X}_i = (X_{i1}, \ldots, X_{ir_i})'$, $\mathbf{Y}_i = (Y_{i1}, \ldots, Y_{ir_i})'$, $\overline{X}_i = \sum_{j=1}^{r_i} X_{ij}/r_i$ and $\overline{Y}_i = \sum_{j=1}^{s_i} Y_{ij}/r_i$, $i = 1, \ldots, n$.

The proof of the above lemma follows by directly maximizing the log-likelihood (1.3) with respect to x_i and keeping $\boldsymbol{\theta}$ fixed. Replacing $g_i = g((\mathbf{X}_i, \mathbf{Y}_i), \boldsymbol{\theta})$ given in (3.1) in the expression for the likelihood given in (1.3) we obtain, after disregarding unimportant constants, the function

$$h_i((\mathbf{X}_i, \mathbf{Y}_i); \boldsymbol{\theta}) = -\frac{r_i}{2}\log\sigma_{uu} - \frac{s_i}{2}\log\sigma_{ee} - \frac{1}{2}\{\sigma_{uu}^{-1}\sum_{j=1}^{r_i}(X_{ij} - g_i)^2 \tag{3.2}$$

$$+\sigma_{ee}^{-1}\sum_{j=1}^{s_i}(Y_{ij}-\alpha-\beta g_i)^2\},$$

from where we obtain

$$h((\mathbf{X},\mathbf{Y}),\boldsymbol{\theta})=\sum_{i=1}^{n}h_i((\mathbf{X}_i,\mathbf{Y}_i),\boldsymbol{\theta}), \tag{3.3}$$

which upon maximized yields the maximum likelihood estimator of $\boldsymbol{\theta}$.

Theorem 3.1. *Given $\boldsymbol{\theta}^0=(\alpha^0,\beta^0,\sigma_{ee}^0,\sigma_{uu}^0)'$ the true parameter value, the maximum likelihood estimator $\hat{\boldsymbol{\theta}}$ converges in probability to*

$$\boldsymbol{\theta}^1=(\alpha^0,\beta^0,\frac{(\overline{r}+\overline{s}-1)}{\overline{r}+\overline{s}}\sigma_{ee}^0,\frac{(\overline{r}+\overline{s}-1)}{\overline{r}+\overline{s}}\sigma_{uu}^0)',$$

where $\overline{r}=\sum_{k=1}^{n}r_k/n$ and $\overline{s}=\sum_{j=1}^{n}s_j/n$, for large n, which are expected to be finite.

Proof. After lengthy algebraic manipulations and disregarding unimportant constants, we obtain replacing (3.1) in (3.2) the following expression

$$\begin{aligned} h_i((\mathbf{X}_i,\mathbf{Y}_i);\boldsymbol{\theta}) &= -\frac{r_i}{2}\log\sigma_{uu}-\frac{s_i}{2}\log\sigma_{ee} \\ &-\frac{1}{2}\{\sigma_{uu}^{-1}\sum_{j=1}^{r_i}(X_{ij}-\overline{X}_i)^2+\sigma_{ee}^{-1}\sum_{j=1}^{s_i}(Y_{ij}-\overline{Y}_i)^2 \\ &+\frac{r_i\sigma_{ee}^{-1}\sigma_{uu}^{-1}}{s_i(r_i\sigma_{uu}^{-1}+s_i\beta^2\sigma_{ee}^{-1})}[\sum_{j=1}^{s_i}(Y_{ij}-\beta\overline{X}_i-\alpha)]^2\}, \end{aligned} \tag{3.4}$$

where $\overline{X}_i$ and $\overline{Y}_i$ are given in (3.1). Now, with respect to the true $\boldsymbol{\theta}^0=(\alpha^0,\beta^0,\sigma_{ee}^0,\sigma_{uu}^0)'$ it follows that

$$E[\sum_{j=1}^{r_i}(X_{ij}-\overline{X}_i)^2]=(r_i-1)\sigma_{uu}^0,\qquad E[\sum_{j=1}^{s_i}(Y_{ij}-\overline{Y}_i)^2]=(s_i-1)\sigma_{ee}^0$$

and

$$E[\sum_{j=1}^{s_i}(Y_{ij}-\beta\overline{X}_i-\alpha)]^2=s_i\sigma_{ee}^0+\frac{\beta^2s_i}{r_i}\sigma_{uu}^0+s_i^2[(\alpha-\alpha^0)+x_i^0(\beta-\beta^0))]^2,$$

which leads, under $\boldsymbol{\theta}^0$, to

$$\overline{\psi}(\boldsymbol{\theta})=\frac{1}{n}\sum_{i=1}^{n}E[h_i((\mathbf{X}_i,\mathbf{Y}_i),\boldsymbol{\theta})]=-\frac{\overline{r}}{2}\log\sigma_{uu}-\frac{\overline{s}}{2}\log\sigma_{ee} \tag{3.5}$$

$$-\frac{1}{2n}\{\sigma_{uu}^0\sigma_{uu}^{-1}\sum_{i=1}^{n}(r_i-1)+\sigma_{ee}^0\sigma_{ee}^{-1}\sum_{i=1}^{n}(s_i-1)+\sigma_{ee}^{-1}\sigma_{uu}^{-1}\sum_{i=1}^{n}\frac{(r_i\sigma_{ee}^0+s_i\beta^2\sigma^0)}{r_i\sigma_{uu}^{-1}+s_i\beta^2\sigma_{ee}^{-1}}$$

$$-\sigma_{ee}^{-1}\sigma_{uu}^{-1}\sum_{i=1}^{n}\frac{r_is_i}{r_i\sigma_{uu}^{-1}+s_i\beta^2\sigma_{ee}^{-1}}[(\alpha-\alpha^0)+(\beta-\beta^0)x_i^0]^2.$$

Clearly, expression (3.5) is maximized by taking

$$\alpha=\alpha^0 \qquad \text{and} \qquad \beta=\beta^0. \tag{3.6}$$

Moreover, differentiating (3.5) with respect to β, and defining $\lambda^0=\sigma_{ee}^0/\sigma_{uu}^0$ and $\lambda=\sigma_{ee}^1/\sigma_{uu}^1$, we arrive at the following equation

$$\sum_{i=1}^{n}\frac{1}{r_i\lambda+s_i\beta^2}=\sum_{i=1}^{n}\frac{r_i\lambda^0+s_i\beta^2}{(r_i\lambda+s_i\beta^2)^2},$$

leading to the solution $\lambda=\lambda^0$ that is

$$\frac{\sigma_{ee}^1}{\sigma_{uu}^1}=\frac{\sigma_{ee}^0}{\sigma_{uu}^0}=\lambda. \tag{3.7}$$

Notice that (3.7) is an assumption in Mak (1982) and not a consequence of the model assumptions. The meaning of (3.7) is that the ratio of the error variances at the maximum and at the true values are the same. Further, differentiating (3.5) with respect to σ_{uu} and σ_{ee} and equating the derivatives to zero, we arrive at the following equations:

$$-\bar{r}\sigma_{uu}+\bar{r}\sigma_{uu}^0=\frac{\sigma_{uu}^0\lambda}{n}\sum_{i=1}^{n}\frac{r_i}{r_i\lambda+s_i\beta^2}, \tag{3.8}$$

$$\bar{s}\sigma_{uu}-(\bar{s}-1)\sigma_{uu}^0=\frac{\sigma_{uu}^0}{n}\sum_{i=1}^{n}\frac{\lambda r_i}{\lambda r_i+s_i\beta^2}. \tag{3.9}$$

Thus, from (3.7) and (3.8), it follows that

$$-\bar{r}\sigma_{uu}+\bar{r}\sigma_{uu}^0-\bar{s}\sigma_{uu}+(\bar{s}-1)\sigma_{uu}^0=0,$$

which leads to

$$\sigma_{uu}=\frac{(\bar{r}+\bar{s}-1)}{\bar{r}+\bar{s}}\sigma_{uu}^0.$$

Furthermore, from (3.7) and (3.9), it follows that

$$\sigma_{ee}=\frac{(\bar{r}+\bar{s}-1)}{\bar{r}+\bar{s}}\sigma_{ee}^0,$$

which in conjunction with (3.6) concludes the proof.

Notice that the dominated convergence theorem permits to write

$$\frac{\partial\psi(\boldsymbol{\theta})}{\partial\theta_j} = E_\theta[q_{i\theta_j}(\mathbf{Z}_i;\boldsymbol{\theta})],$$

so that conditions under which the above solution is a local maximum of the equation

$$\frac{\partial}{\partial\theta_j}\overline{\psi}(\boldsymbol{\theta}) = 0,$$

with $\overline{\psi}(.)$ in (2.7), are the conditions under which the matrix $E_0[\mathbf{A}_n(\boldsymbol{\theta}^1)]$ is negative definite. See Vilca-Labra et al. (1998). Such conditions are considered in Appendix 4.

Thus, given the maximum likelihood estimator $\hat{\boldsymbol{\theta}} = (\hat{\alpha}, \hat{\beta}, \hat{\sigma}_{uu}, \hat{\sigma}_{ee})'$ of $\boldsymbol{\theta}$, which follows by maximizing the likelihood function that follows from (3.2) (or (1.3)), a consistent estimator of $\boldsymbol{\theta}^0$ is given by

$$\hat{\boldsymbol{\theta}}_c = (\hat{\alpha}, \hat{\beta}, \frac{(\overline{r}+\overline{s})}{\overline{r}+\overline{s}-1}\hat{\sigma}_{uu}, \frac{(\overline{r}+\overline{s})}{\overline{r}+\overline{s}-1}\hat{\sigma}_{ee})'. \tag{3.10}$$

Some special cases of the above general result are in order.

Corollary 3.1. *Under the assumptions of Theorem 3.1, we have that:*

(a) If $r_i/s_i = k$, $i = 1, \ldots, n$, then the maximum likelihood estimator $\hat{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$ converges in probability to

$$\boldsymbol{\theta}^1 = (\alpha^0, \beta^0, (\frac{r+k-1/\overline{r}}{1+k})\sigma^0_{uu}, (\frac{r+k-1/\overline{r}}{1+k})\sigma^0_{ee})';$$

(b) If $r_i = r$, and $s_i = s$, $i = 1, \ldots, n$, it follows that

$$\boldsymbol{\theta}^1 = (\alpha^0, \beta^0, \frac{(r+s-1)}{r+s}\sigma^0_{uu}, \frac{(r+s-1)}{r+s}\sigma^0_{ee})';$$

(c) If $r_i = r$ and $s_i = 1$, $i = 1, \ldots, n$, it follows that

$$\boldsymbol{\theta}^1 = (\alpha^0, \beta^0, \frac{r}{r+1}\sigma^0_{uu}, \frac{r}{r+1}\sigma^0_{ee})'.$$

Case (c) in Corollary 3.1 is the one typically considered in practice. More general replication schemes, as the one considered in Fuller (1995), for example, may also be entertained.

We discuss next the asymptotic distribution of the maximum likelihood estimators and then derive the asymptotic distribution of the consistent estimator $\hat{\boldsymbol{\theta}}_c$.

4. Computing consistent estimators

The maximum likelihood estimator of $\boldsymbol{\theta}$ follows by maximizing the function $h((\mathbf{X}', \mathbf{Y}'), \boldsymbol{\theta})$ in (3.3) or, equivalently, the likelihood function in (1.3). Using the same notation as in Mak (1982), the following expressions are obtained after differentiating the function h_i given in (3.4):

$$(4.1) \qquad q_{i\alpha} = \frac{\partial h_i}{\partial \alpha} = \frac{r_i \sigma_{ee}^{-1} \sigma_{uu}^{-1}}{(s_i \beta^2 \sigma_{ee}^{-1} + r_i \sigma_{uu}^{-1})} \sum_{j=1}^{r_i} (Y_{ij} - \beta \overline{X}_i - \alpha),$$

$$(4.2) \qquad q_{i\beta} = \frac{\partial h_i}{\partial \beta} = \frac{r_i \sigma_{ee}^{-2} \sigma_{uu}^{-1}}{(s_i \beta^2 \sigma_{ee}^{-1} + r_i \sigma_{uu}^{-1})^2} \sum_{j=1}^{s_i} (Y_{ij} - \beta \overline{X}_i - \alpha) \sum_{j=1}^{r_i} (\beta Y_{ij} + \frac{r_i}{s_i} \lambda \overline{X}_i - \beta \alpha),$$

with $\lambda = \sigma_{ee}/\sigma_{uu}$, as before,

$$(4.3) \quad q_{i\sigma_{uu}} = \frac{\partial h_i}{\partial \sigma_{uu}} = -\frac{r_i}{2} \sigma_{uu}^{-1} + \frac{\sigma_{uu}^{-2}}{2} \sum_{j=1}^{r_i} (X_{ij} - \overline{X}_i)^2 + \frac{r_i \beta^2 \sigma_{ee}^{-2} \sigma_{uu}^{-2}}{2(s_i \beta^2 \sigma_{ee}^{-1} + r_i \sigma_{uu}^{-1})^2} [\sum_{j=1}^{r_i} (Y_{ij} - \beta \overline{X}_i - \alpha)]^2$$

and, finally,

$$(4.4) \qquad q_{i\sigma_{ee}} = \frac{\partial h_i}{\partial \sigma_{ee}} = -\frac{r_i}{2} \sigma_{ee}^{-1} + \frac{\sigma_{ee}^{-2}}{2} \sum_{j=1}^{r_i} (Y_{ij} - \overline{Y}_i)^2$$

$$+ \frac{\sigma_{ee}^{-2} \sigma_{uu}^{-2}}{2 r_i (\beta^2 \sigma_{ee}^{-1} + \sigma_{uu}^{-1})^2} [\sum_{j=1}^{r} (Y_{ij} - \beta X_{ij} - \alpha)]^2.$$

Equating derivatives (4.1)-(4.4) to zero, we obtain the following equations:

$$(4.5) \qquad \sum_{i=1}^{n} \sum_{j=1}^{s_i} Y_{ij} - \beta \sum_{i=1}^{n} s_i \sum_{j=1}^{r_i} \frac{X_{ij}}{r_i} = 0,$$

$$(4.6) \qquad \sum_{i=1}^{n} \frac{r_i \sigma_{uu}^{-1} \sigma_{ee}^{-2}}{(r_i \sigma_{uu}^{-1} + s_i \beta^2 \sigma_{ee}^{-1})^2} \sum_{i=1}^{s_i} (Y_{ij} - \beta \overline{X}_i - \alpha) \sum_{i=1}^{s_i} (\beta Y_{ij} + \frac{r_i}{s_i} \lambda \overline{X}_i - \beta \alpha) = 0,$$

$$-\frac{\sigma_{uu}^{-1}}{2} \sum_{i=1}^{n} r_i + \frac{\sigma_{uu}^{-2}}{2} \sum_{i=1}^{n} \sum_{j=1}^{r_i} (X_{ij} - \overline{X}_i)^2$$

$$(4.7) \qquad +\frac{1}{2}\sum_{j=1}^{n}\frac{r_i\beta^2\sigma_{ee}^{-2}\sigma_{uu}^{-2}}{2(r_i\sigma_{uu}^{-1}+s_i\beta^2\sigma_{ee}^{-1})^2}[\sum_{j=1}^{s_i}(Y_{ij}-\beta\overline{X}_i-\alpha)]^2=0$$

and finally

$$-\frac{\sigma_{ee}^{-1}}{2}\sum_{i=1}^{n}s_i+\frac{\sigma_{ee}^{-2}}{2}\sum_{i=1}^{n}\sum_{j=1}^{s_i}(Y_{ij}-\overline{Y}_i)^2$$

$$(4.8) \qquad +\sum_{i=1}^{n}\{\frac{r_i^2\sigma_{uu}^{-2}\sigma_{ee}^{-2}}{s_i(r_i\sigma_{uu}^{-1}+s_i\beta^2\sigma_{ee}^{-1})^2}[\sum_{j=1}^{s_i}(Y_{ij}-\beta\overline{X}_i-\alpha)]^2\}=0.$$

Solving the above equations to obtain the maximum likelihood estimators of $\boldsymbol{\theta}$ is not simple and has to be done numerically. One possibility is to consider an EM type algorithm as considered in Kimura (1992), working directly with the likelihood function (1.3). Great simplifications occur when

$$(4.9) \qquad \frac{r_i}{s_i}=k, \qquad i=1,\ldots,n,$$

in which case equations (4.5)-(4.8) can be written as

$$(4.10) \qquad \hat{\alpha}=\overline{Y}-\hat{\beta}\overline{X},$$

$$(4.11) \qquad \hat{\lambda}=\frac{k\hat{\beta}(\hat{\beta}S_{XY}^*-S_{YY}^*)}{S_{XY}^*-\hat{\beta}S_{XX}^*},$$

$$(4.12) \qquad \hat{\sigma}_{uu}=W_{XX}^*+\frac{\hat{\beta}^2k^2}{(\hat{\lambda}+k\hat{\beta}^2)}(S_{YY}^*+\hat{\beta}^2S_{XX}^*-2\hat{\beta}S_{XY}^*),$$

and

$$(4.13) \qquad \hat{\sigma}_{ee}=W_{YY}^*+\frac{\hat{\lambda}^2}{(\hat{\lambda}+\hat{\beta}^2)^2}(S_{YY}^*+\hat{\beta}^2S_{XX}^*-2\hat{\beta}S_{XY}^*),$$

where

$$S_{XX}^*=\frac{\sum_{i=1}^n r_i(\overline{X}_i-\overline{X})^2}{\sum_{i=1}^n r_i}, \qquad S_{YY}^*=\frac{\sum_{i=1}^n s_i(\overline{Y}_i-\overline{Y})^2}{\sum_{i=1}^n s_i},$$

$$S_{XY}^*=\sum_{i=1}^{n}\frac{s_i(\overline{X}_i-\overline{X})(\overline{Y}_i-\overline{Y})}{\sum_{i=1}^n s_i}, \qquad W_{YY}^*=\frac{\sum_{i=1}^n\sum_{j=1}^{s_i}(Y_{ij}-\overline{Y}_i)^2}{\sum_{i=1}^n s_i},$$

with W^*_{XX} as W^*_{YY} with Y replaced by X,

$$\overline{Y} = \frac{1}{S}\sum_{i=1}^{n}\sum_{j=1}^{s_i} Y_{ij}, \qquad \overline{X} = \frac{1}{R}\sum_{i=1}\sum_{j=1}^{r_i} X_{ij},$$

with $S = \sum_{i=1}^n s_i$ and $R = \sum_{i=1}^n r_i$.

By manipulating with the above equations, the maximum likelihood estimator of β under the assumption (4.9) is given by the solution of the following equation:

$$A_4\beta^4 + A_3\beta^3 + A_2\beta^2 + A_1\beta + A_0 = 0, \tag{4.14}$$

where

$$A_4 = kS^*_{XX}S^*_{XY}W^*_{XX} + k(S^*_{XX})^2S^*_{XY},$$

$$A_3 = -2k(S^*_{XY})^2W^*_{XX} - kS^*_{XX}S^*_{YY}W^*_{XX} + (S^*_{XX})^2W^*_{YY} - (2k-1)S^*_{XX}(S^*_{XY})^2 - k(S^*_{XX})^2S^*_{YY},$$

$$A_2 = 3kS^*_{XY}S^*_{YY}W^*_{XX} - 3S^*_{XX}S^*_{XY}W^*_{YY} + (k-1)(S^*_{XY})^3 + (2k-2)S^*_{XX}S^*_{YX}S^*_{YY},$$

$$A_1 = -k(S^*_{YY})^2W^*_{XX} + 2(S^*_{XY})^2W^*_{YY} + S^*_{XX}S^*_{YY}W^*_{YY} + (2-k)(S^*_{XY})^2S^*_{YY} + S^*_{XX}(S^*_{YY})^2,$$

and

$$A_0 = -S^*_{XY}S^*_{YY}W^*_{YY} - S^*_{XY}(S^*_{YY})^2.$$

Some special cases of particular interest are:

(*i*) $r_i = s_i = r$. In this special case, it follows that the coefficients of the fourth degree equation (4.14) are given by

$$A_4 = S_{XX}S_{XY}W_{XX} + S^2_{XX}S_{XY},$$

$$A_3 = -2S^2_{XY}W_{XX} - S_{XX}S_{YY}W_{XX} + S^2_{XX}W_{YY} - S_{XX}S^2_{XY} - S^2_{XX}S_{YY},$$

$$A_2 = 3S_{XY}S_{YY}W_{YY} - 3S_{YY}S_{XY}W_{YY},$$

$$A_1 = -S^2_{YY}W_{YY} + 2S^2_{XY}W_{YY} + S_{XX}S_{YY}W_{YY} + S^2_{XY} + S_{XX}S^2_{YY},$$

and

$$A_0 = -S_{XY}S_{YY}W_{YY} - S_{XY}S^2_{YY},$$

where

$$S_{XX} = \frac{\sum_{i=1}^n(\overline{X}_i - \overline{X})^2}{n}, \qquad S_{YY} = \frac{\sum_{i=1}^n(\overline{Y}_i - \overline{Y})^2}{n},$$

$$S_{XY} = \sum_{i=1}^{n}\frac{(\overline{X}_i - \overline{X})(\overline{Y}_i - \overline{Y})}{n}, \qquad W_{YY} = \frac{\sum_{i=1}^n\sum_{j=1}^r(Y_{ij} - \overline{Y}_i)^2}{nr},$$

with W_{XX} as W_{YY} with Y replaced by X.

(*ii*) $r_i = r$, $s_i = 1$. In this case, it follows that the fourth degree equation (4.14) becomes

$$(\beta S_{XY} - S_{YY})\{\beta^3 S_{XX} t_{XX} + \beta^2 (r S_{XY} S_{XX} - 2 S_{XY} t_{XX})$$
$$+\beta[S_{YY} t_{XX} - (r-1) S_{XY}^2 - (r+1) S_{YY} S_{XX}] + r S_{XY} S_{YY}\} = 0,$$

where

$$t_{XX} = \frac{\sum_{i=1}^n \sum_{j=1}^r (X_{ij} - \overline{X})^2}{nr}.$$

Thus, in this special case the fourth degree equation reduces to a third degree equation. In both situations multiple roots may result so that to pick the correct root the likelihood function can be used.

5. The asymptotic covariance matrix

As seen in Section 2, the asymptotic covariance matrix of the maximum likelihood estimators is given by $n^{-1}\{E_0[\mathbf{A}_n(\boldsymbol{\theta}^1)]\}^{-1}\mathbf{V}_n(\boldsymbol{\theta}^1)\{E_0[\mathbf{A}_n(\boldsymbol{\theta}^1)]\}^{-1}$. The matrix $\mathbf{V}_n$ depends on the first derivatives of the function h_i, given by $q_{i\theta_j}$, which are given in Section 4, where $\theta_1 = \alpha$, $\theta_2 = \beta$, $\theta_3 = \sigma_{ee}$, and $\theta_4 = \sigma_{uu}$. The second derivatives of the function h_i and its expected values used to compute $E_0[\mathbf{A}_n(\boldsymbol{\theta}^1)]$ are considered in Appendices 2 and 3. Thus, we can write:

(5.1) $$E_0[\mathbf{A}_n(\boldsymbol{\theta}^1)] =$$

$$= \left(\frac{w}{n\sigma_{uu}^0}\right) \begin{bmatrix} -\sum_{i=1}^n v_i & -\sum_{i=1}^n v_i x_i^0 & 0 & 0 \\ -\sum_{i=1}^n v_i x_i^0 & -\sum_{i=1}^n v_i x_i^{0^2} & \lambda^0 w \beta^0 \Sigma_{rs} & -\beta^0 w \Sigma_{rs} \\ 0 & -\lambda\beta^0 w \Sigma_{rs} & \frac{w^2}{\sigma_{uu}^0}(nc - \beta^{04}\Sigma_{ss}) & \frac{-\beta^{02} w^2}{\sigma_{uu}}\Sigma_{rs} \\ 0 & -\beta^0 w \Sigma_{rs} & \frac{-w^2\beta^{02}}{\sigma_{uu}}\Sigma_{rs} & \frac{w^2}{\lambda^2\sigma_{uu}^0}(nb - \lambda^2\Sigma_{rr}) \end{bmatrix},$$

where

$$v_i = \frac{r_i s_i}{\Delta_i}; \quad \Delta_i = r_i\lambda + s_i\beta^{02}; \quad \Sigma_{rs} = \sum_{i=1}^n \frac{r_i s_i}{\Delta_i^2}; \quad \Sigma_{rr} = \sum_{i=1}^n \frac{r_i^2}{\Delta_i^2}; \quad \Sigma_{ss} = \sum_{i=1}^n \frac{s_i^2}{\Delta_i^2};$$
$$\overline{\Sigma}_{rs} = \frac{\Sigma_{rs}}{n}; \quad c = \frac{\overline{r}}{2w} - \overline{r} + 1; \quad b = \frac{\overline{s}}{2w} - \overline{s} + 1; \quad w = \frac{\overline{r} + \overline{s}}{\overline{r} + \overline{s} - 1}.$$

Now, considering the second derivatives and expected values which are given in Appendix 3, we have:

(5.2) $$\mathbf{V}_n(\boldsymbol{\theta}^1) = \left(\frac{1}{n\sigma_{uu}^0}\right)$$

$$
\times \begin{bmatrix} w^2\sum_{i=1}^{n} v_i & w^2\sum_{i=1}^{n} v_i x_i^0 & 0 & 0 \\ w^2\sum_{i=1}^{n} v_i x_i^0 & w^2\sum_{i=1}^{n} v_i(x_i^{02} + \frac{\lambda^0\sigma_{uu}^0}{\Delta_i}) & 0 & \\ 0 & 0 & \frac{w^4}{2\sigma_{uu}^0}(n(\bar{r}-1) + \beta^{04}\Sigma_{ss}) & \frac{\beta^{02}w^4}{2\sigma_{uu}}\Sigma_{rs} \\ 0 & 0 & \frac{\beta^{02}w^4}{2\sigma_{uu}}\Sigma_{rs} & \frac{w^4}{2\lambda^2\sigma_{uu}^0}(n(\bar{s}-1) + \lambda^2\Sigma_{rr}) \end{bmatrix}.
$$

Thus, after extensive algebraic manipulations, the asymptotic covariance matrix can be written as

$$
n^{-1}\{E^0[\mathbf{A}_n(\boldsymbol{\theta}^1)]\}^{-1}\mathbf{V}_n(\boldsymbol{\theta}^1)\{E^0[\mathbf{A}_n(\boldsymbol{\theta}^1)]\}^{-1} = \left(\frac{w^{-1}\sigma_{uu}^0}{D_1^2}\right) \tag{5.3}
$$

$$
\times \begin{bmatrix} w(F_1 + F_2B_2^2) & w(C_1F_3 - F_2B_1B_2) & B_2(C_3F_5 + C_4F_6) - C_3F_3 & -B_2(C_3F_7 + C_4F_8) - C_4F_3 \\ w(C_1F_3 - F_2B_1B_2) & w(F_2B_1^2 - F_4C_1) & C_3F_4 + B_2(C_3F_5 + C_4F_6) & B_1(C_3F_7 + C_4F_8) + C_4F_4 \\ - & - & \frac{1}{w}\left[\sigma_{uu}^0(D_3F_5 - D_4F_6) - \frac{C_3^2F_4}{C_1}\right] & \frac{1}{w}\left[\sigma_{uu}^0(D_4F_8 - D_3F_7) - \frac{C_3C_4F_4}{C_1}\right] \\ - & - & - & \frac{1}{w}\left[\sigma_{uu}^0(D_4F_7 - D_5F_8) - \frac{C_4^2F_4}{C_1}\right] \end{bmatrix},
$$

where

$$
\begin{aligned}
v_i &= \frac{r_i s_i}{\Delta_i}, \quad \Delta_i = (r_i\lambda + s_i\beta^{02}), \quad \lambda = \frac{\sigma_{ee}}{\sigma_{uu}} = \frac{\sigma_{ee}^0}{\sigma_{uu}^0}; \\
B_1 &= -\sum_{i=1}^{n} v_i, \quad B_2 = -\sum_{i=1}^{n} v_i x_i^0, \quad B_3 = n\bar{r}(\frac{1}{2w} - 1) + n - \sum_{i=1}^{n}\frac{s_i^2\beta^{04}}{\Delta_i^2}, \\
B_4 &= n\bar{s}(\frac{1}{2w} - 1) + n - \sum_{i=1}^{n}\frac{r_i^2\lambda^2}{\Delta_i^2};
\end{aligned}
$$

$$
\begin{aligned}
C_1 &= \frac{1}{\lambda^2\sigma_{uu}^0}\Big\{B_3B_4 - \Big(\sum_{i=1}^{n}\frac{v_i\lambda\beta^{02}}{\Delta_i}\Big)^2\Big\}, \quad C_2 = \Big(\sum_{i=1}^{n} v_i\Big)\Big(\sum_{i=1}^{n} v_i x_i^{02}\Big) - \Big(\sum_{i=1}^{n} v_i x_i^0\Big)^2, \\
C_3 &= \frac{\beta^0}{\lambda}\Big(\sum_{i=1}^{n}\frac{v_i}{\Delta_i}\Big)\Big(B_4 - \lambda\beta^{02}\sum_{i=1}^{n}\frac{v_i}{\Delta_i}\Big), \quad C_4 = -\beta^0(\sum_{i=1}^{n}\frac{v_i}{\Delta_i})(B_3 - \lambda\beta^{02}\sum_{i=1}^{n}\frac{v_i}{\Delta_i}); \\
D_1 &= C_1C_2 - \frac{n(\bar{r}+\bar{s}-1)}{2}\beta^{02}(\sum_{i=1}^{n} v_i)(\sum_{i=1}^{n}\frac{v_i}{\Delta_i})^2, \\
D_2 &= -C_1\sum_{i=1}^{n} v_i x_i^{02} + \frac{n\beta^{02}(\bar{r}+\bar{s}-1)}{2}(\sum_{i=1}^{n}\frac{v_i}{\Delta_i})^2,
\end{aligned}
$$

$$D_3 = \frac{B_4C_2}{\lambda^2\sigma^0_{uu}} + (\sum_{i=1}^n v_i)(\beta^0\sum_{i=1}^n \frac{v_i}{\Delta_i})^2,$$

$$D_4 = -(\beta^{02}\sum_{i=1}^n \frac{v_i}{\Delta_i})[\frac{C_2}{\sigma^0_{uu}} + \lambda(\sum_{i=1}^n v_i)(\sum_{i=1}^n \frac{v_i}{\Delta_i})],$$

$$D_5 = \frac{B_3C_2}{\sigma^0_{uu}} + (\sum_{i=1}^n v_i)(\lambda\beta^0\sum_{i=1}^n \frac{v_i}{\Delta_i})^2;$$

$$F_1 = D_2^2\sum_{i=1}^n v_i + C_1\{2D_2 + C_1\sum_{i=1}^n[v_i(x_i^{02} + \frac{\lambda\sigma^0_{uu}}{\Delta_i})]\}\left(\sum_{i=1}^n v_i x_i^0\right)^2,$$

$$F_2 = \frac{1}{2\sigma^0_{uu}}\left(\frac{\beta^0}{\lambda}\sum_{i=1}^n \frac{v_i}{\Delta_i}\right)^2\Big\{\left(\frac{n\bar{r}}{2w} - B_3\right)\left(B_4 - \lambda\beta^{02}\sum_{i=1}^n \frac{v_i}{\Delta_i}\right)^2$$
$$-\left(2\lambda\beta^{02}\sum_{i=1}^n \frac{v_i}{\Delta_i}\right)\left(B_4 - \lambda\beta^{02}\sum_{i=1}^n \frac{v_i}{\Delta_i}\right)\left(B_3 - \lambda\beta^{02}\sum_{i=1}^n \frac{v_i}{\Delta_i}\right)$$
$$+\left(\frac{n\bar{s}}{2w} - B_4\right)\left(B_3 - \lambda\beta^{02}\sum_{i=1}^n \frac{v_i}{\Delta_i}\right)^2\Big\},$$

$$F_3 = -C_1\left(C_2 + (\sum_{i=1}^n v_i)\lambda\sigma^0_{uu}\sum_{i=1}^n \frac{v_i}{\Delta_i}\right)\sum_{i=1}^n v_i x_i^0,$$

$$F_4 = -C_1(\sum_{i=1}^n v_i)\left(C_2 + (\sum_{i=1}^n v_i)\lambda\sigma^0_{uu}\sum_{i=1}^n \frac{v_i}{\Delta_i}\right),$$

$$F_5 = \frac{1}{2}(\frac{n\bar{r}}{2w} - B_3)[\frac{B_4C_2}{\lambda^2\sigma^0_{uu}} + \beta^{02}(\sum_{i=1}^n v_i)(\sum_{i=1}^n \frac{v_i}{\Delta_i})^2]$$
$$+\frac{1}{2}(\beta^{02}\sum_{i=1}^n \frac{v_i}{\Delta_i})^2[\frac{C_2}{\sigma^0_{uu}} + \lambda(\sum_{i=1}^n v_i)(\sum_{i=1}^n \frac{v_i}{\Delta_i})],$$

$$F_6 = \frac{1}{2}(\beta^{02}\sum_{i=1}^n \frac{v_i}{\Delta_i})\{(\sum_{i=1}^n v_i)(\sum_{i=1}^n \frac{v_i}{\Delta_i})[\beta^{02}\sum_{i=1}^n \frac{v_i}{\Delta_i} +$$
$$+\frac{1}{\lambda}(\frac{n\bar{s}}{2w} - B_4)] + \frac{n\bar{s}C_2}{2\lambda^2 w\sigma^0_{uu}}\},$$

$$F_7 = -\frac{1}{2}(\beta^{02}\sum_{i=1}^n \frac{v_i}{\Delta_i})\{\lambda(\sum_{i=1}^n v_i)(\sum_{i=1}^n \frac{v_i}{\Delta_i})[(\frac{n\bar{r}}{2w} - B_3) + \lambda\beta^{02}\sum_{i=1}^n \frac{v_i}{\Delta_i}] + \frac{n\bar{r}C_2}{2w\sigma^0_{uu}}\}$$

and

$$F_8 = -\frac{1}{2}(\beta^0\sum_{i=1}^{n}\frac{v_i}{\Delta_i})^2\Big[(\frac{\beta^{02}C_2}{\sigma^0_{uu}}) + \sum_{i=1}^{n} v_i(\lambda\beta^{02}\sum_{i=1}^{n}\frac{v_i}{\Delta_i} + \frac{n\bar{s}}{2w} - B_4)\Big]$$
$$-\frac{1}{2\lambda^2}(\frac{n\bar{s}}{2w} - B_4)\frac{B_3C_2}{\sigma^0_{uu}}.$$

A consistent estimator of the covariance matrix in (5.3) follows by replacing $\boldsymbol{\theta}$ by the consistent estimator $\hat{\boldsymbol{\theta}}_c$. Another consistent estimator, which may be simple to implement computationally is obtained by following the "sandwich" approach in Gimenez and Bolfarine (1997). Considering

$$\mathbf{q}_i(\boldsymbol{\theta}) = \frac{\partial h_i(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \quad \text{and} \quad \mathbf{I}_i(\boldsymbol{\theta}) = \frac{\partial^2 h_i(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'},$$

a consistent estimator of the asymptotic covariance matrix in (5.3) is given by

$$(\bar{\mathbf{I}}_n(\hat{\boldsymbol{\theta}}_c))^{-1}\bar{\mathbf{S}}_n(\hat{\boldsymbol{\theta}}_c)(\bar{\mathbf{I}}_n(\hat{\boldsymbol{\theta}}_c))^{-1}, \tag{5.4}$$

where

$$\bar{\mathbf{S}}_n = \frac{1}{n}\sum_{i=1}^{n}\mathbf{q}_i(\boldsymbol{\theta})\mathbf{q}_i(\boldsymbol{\theta})', \quad \text{and} \quad \bar{\mathbf{I}}_n(\boldsymbol{\theta}) = \frac{1}{n}\sum_{i=1}^{n}\mathbf{I}_i(\boldsymbol{\theta}),$$

with $\boldsymbol{\theta}$ replaced by the maximum likelihood estimator $\hat{\boldsymbol{\theta}}$ considered in Section 4.

6. Dorf and Gurland's estimators

Dorf and Gurland (1961), propose using the following estimators for estimating the parameter β in the general replicated situation:

$$\hat{\beta}_{DG1} = \frac{B_{XY}}{B_{XX} - W_{XX}} \quad \text{and} \quad \hat{\beta}_{DG2} = \frac{B_{YY} - W_{YY}}{B_{XY}}, \tag{6.1}$$

where

$$B_{XX} = \sum_{i=1}^{n}\frac{r_i(\overline{X}_i - \overline{X})^2}{n-1}, \quad W_{XX} = (R-n)^{-1}\sum_{i=1}^{n}\sum_{j=1}^{r_i}(X_{ij} - \overline{X}_i)^2,$$

$$B_{YY} = \sum_{i=1}^{n}\frac{s_i(\overline{Y}_i - \overline{Y})^2}{n-1} \quad \text{and} \quad W_{YY} = (S-n)^{-1}\sum_{i=1}^{n}\sum_{j=1}^{s_i}(Y_{ij} - \overline{Y}_i)^2,$$

with

$$R = \sum_{i=1}^{n} r_i \quad \text{and} \quad S = \sum_{i=1}^{n} s_i.$$

Approximations for their asymptotic variances can be written as

$$\begin{aligned} AVAR(\hat{\beta}_{DG1}) &= \frac{\beta^2 \sigma_{uu}}{\sum_{i=1}^{n} r_i (x_i - \overline{x})^2} + \frac{\sigma_{ee} \sum_{i=1}^{n} (r_i^2 / s_i)(x_i - \overline{x})^2}{(\sum_{i=1}^{n} r_i (x_i - \overline{x})^2)^2} \\ &+ \frac{n-1}{n} \frac{\sigma_{uu}\sigma_{ee} \sum_{i=1}^{n} (r_i/s_i)}{(\sum_{i=1}^{n} r_i (x_i - \overline{x})^2)^2} \\ &+ \frac{\beta^2 \{Q_1(r_H, R)\sigma_{uu}^2 + Q_2(r_H, R)(3\sigma_{uu}^2)\}(n-1)^2}{(\sum_{i=1}^{n} r_i (x_i - \overline{x})^2)^2}, \end{aligned}$$

$$\begin{aligned} AVAR(\hat{\beta}_{DG2}) &= \frac{\beta^2 \sigma_{uu} \sum_{i=1}^{n} (s_i^2 / s r_i)(x_i - \overline{x})^2}{(\sum_{i=1}^{n} s_i (x_i - \overline{x})^2)^2} + \frac{\sigma_{ee}}{\sum_{i=1}^{n} s_i (x_i - \overline{x})^2} \\ &+ \frac{n-1}{n} \frac{\sigma_{uu}\sigma_{ee} \sum_{i=1}^{n} (s_i/r_i)}{(\sum_{i=1}^{n} s_i (x_i - \overline{x})^2)^2} \\ &+ \frac{\{Q_1(s_H, S)\sigma_{ee}^2 + Q_2(s_H, s)(3\sigma_{ee}^2)\}(n-1)^2}{\beta^2 (\sum_{i=1}^{n} s_i (x_i - \overline{x})^2)^2}, \end{aligned}$$

where

$$r_H^{-1} = n^{-1} \sum_{i=1}^{n} r_i^{-1}, \quad s_H^{-1} = n^{-1} \sum_{i=1}^{n} s_i^{-1},$$

$$\begin{aligned} Q_1(r_H, R) &= \frac{1}{(n-1)^2} \left(2n - 2 - \frac{3n}{r_H} + \frac{6n}{R} - \frac{3}{R} \right) + \frac{1}{(R-n)^2} \left(4n - R - \frac{3n}{r_H} \right) \\ &+ \frac{6}{R-n} - \frac{6n}{(R-n)(n-1)} \left(\frac{1}{r_H} - \frac{1}{R} \right), \end{aligned}$$

$$\begin{aligned} Q_2(r_H, R) &= \frac{1}{(n-1)^2} \left(\frac{n}{r_H} - \frac{2n}{R} + \frac{3}{R} \right) + \frac{1}{(R-n)^2} \left(\frac{n}{r_H} + R - 2n \right) \\ &+ \frac{2}{R-n} - \frac{2n}{(R-n)(n-1)} \left(\frac{1}{r_H} - \frac{1}{R} \right), \end{aligned}$$

with corresponding definitions for $Q_1(s_H, S)$ and $Q_2(s_H, S)$.

7. Numerical Evaluations

The parameter values used for computing the (approximate) asymptotic variances of MLE and the estimators $\hat{\beta}_{DG1}$ and $\hat{\beta}_{DG2}$ are $\alpha = 10$, $\beta = 0.5$, $\sigma_{ee} = 50$ and several values of σ_{uu}. It was also considered that the true $x_i \sim N(\mu_x, \sigma_{xx})$, with $\mu_x = 30$ and $\sigma_{xx} = 64$. Other values were also considered for all the parameters but the values obtained for the efficiencies were similar to the ones obtained with the above values. In the table presented below, ARE is the ratio of the asymptotic variances. Table 6.1 presents the situation where only x_i is replicated, that is, $r_i = r = 4$, and $s_i = 1$. In this case, only $\hat{\beta}_{DG1}$ can be computed.

Table 6.1: ARE considering $n = 100$.

σ_{uu}	4	9	16	60	100	144	240
$AVAR(\hat{\beta}_{MV})$	0,0085	0,0098	0,0096	0,0080	0,0048	0,0033	0,0025
$AVAR(\hat{\beta}_{DG1})$	0,0095	0,0112	0,0118	0,0345	0,0642	0,1584	0,4146
$ARE(\hat{\beta}_{MV}, \hat{\beta}_{DG1})$	1,1112	1,1359	1,2261	4,3128	13,442	47,585	165,56.

Thus, it follows that the performance of the maximum likelihood estimator improves as the error variance σ_{uu} increases. It also follows, as expected, that if the variance σ_{uu} gets too large then the sufficient condition for $E[\mathbf{A}_n(\boldsymbol{\theta}^1)]$ to be negative definite may not hold.

We also conducted simulations study with the estimator $\hat{\beta}_{DG2}$ and the results were similar to the ones obtained with $\hat{\beta}_{DG1}$ and so are not reported.

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Appendix 1. Computing second derivatives

We compute now first derivatives of the functions $q_{i\gamma}$, $\gamma = \alpha, \beta, \sigma_{ee}, \sigma_{uu}$, given in (4.1)-(4.4). After standard algebraic manipulations, it follows that

$$
\begin{aligned}
q_{i\alpha\alpha} &= \frac{\partial q_{i\alpha}}{\partial \alpha} = -\frac{r_i s_i \sigma_{ee}^{-1}\sigma_{uu}^{-1}}{r_i\sigma_{uu}^{-1} + s_i\beta^2\sigma_{ee}^{-1}};\\
q_{i\alpha\beta} &= \frac{\partial q_{i\alpha}}{\partial \beta} = r_i\sigma_{ee}^{-1}\sigma_{uu}^{-1}\Big\{ -\frac{2\beta s_i\sigma_{ee}^{-1}}{(r_i\sigma_{uu}^{-1} + s_i\beta^2\sigma_{ee}^{-1})^2} \times \sum_{j=1}^{s_i}(Y_{ij} - \beta\overline{X}_i - \alpha)\\
&\quad -\frac{s_i\overline{X}_i}{(r_i\sigma_{uu}^{-1}s_i\beta^2\sigma_{ee}^{-1})^2}\Big\}\\
&= \frac{-r_i\sigma_{uu}^{-1}\sigma_{ee}^{-2}}{(r_i\sigma_{uu}^{-1} + s_i\beta^2\sigma_{ee}^{-1})^2} \times \Big[s_i\beta\sum_{j=1}^{s_i}(Y_{ij} - \beta\overline{X}_i - \alpha)\\
&\quad + s_i\sum_{j=1}^{s_i}(\beta Y_{ij} + \frac{r_i}{s_i}.\alpha\overline{X}_i - \beta\alpha)\Big];\\
q_{i\alpha\sigma_{uu}} &= \frac{\partial q_{i\alpha}}{\partial \sigma uu} = \frac{-r_i s_i\beta^2\sigma_{ee}^{-2}\sigma_{uu}^{-2}}{(r_i\sigma_{uu}^{-1} + s_i\beta^2\sigma_{ee}^{-1})^2} \times \sum_{j=1}^{s_i}(Y_{ij} - \beta\overline{X}_i - \alpha);\\
q_{i\alpha\sigma_{ee}} &= \frac{\partial q_{t\alpha}}{\partial \sigma ee} = \frac{-r_i^2\sigma_{uu}^{-2}\sigma_{ee}^{-2}}{(r_i\sigma_{uu}^{-1} + s_i\beta^2\sigma_{ee}^{-1})^2} \times \sum_{j=1}^{s_i}(Y_{ij} - \beta\overline{X}_i - \alpha).
\end{aligned}
$$

$$
\begin{aligned}
q_{i\beta\beta} &= \frac{-r_i\sigma_{uu}^{-1}\sigma_{ee}^{-1}(4s_i\beta\sigma_{ee}^{-1})}{(r_i\sigma_{uu}^{-1} + s_i\beta^2\sigma_{ee}^{-1})^3}[\sum_{j=1}^{s_i}(Y_{ij} - \beta\overline{X}_i - \alpha)] \times [\sum_{j=1}^{s_i}(\beta Y_{ij} + \frac{r_i}{s_i}\lambda\overline{X}_i - \beta\alpha)]\\
&\quad + \frac{r_i\sigma_{uu}^{-1}\sigma_{ee}^{-1}}{(r_i\sigma_{uu}^{-1} + s_i\beta^2\sigma_{ee}^{-1})^2} \times [-s_i\overline{X}_i\sum_{j=1}^{s_i}(\beta Y_{ij} + \frac{r_i}{s_i}\lambda\overline{X}_i - \beta\alpha)]\\
&\quad + [\sum_{j=1}^{s_i}(Y_{ij} - \alpha)] \times [\sum_{j=1}^{s_i}(Y_{ij} - \beta\overline{X}_i - \alpha)]\\
&= \frac{-4s_i\beta\sigma_{ee}^{-1}}{(r_i\sigma_{uu}^{-1} + s_i\beta^2\sigma_{ee}^{-1})} \times q_{i\beta} - \frac{s_i\sigma_{uu}^{-1}\sigma_{ee}^{-1}}{(r_i\sigma_{uu}^{-1} + s_i\beta^2\sigma_{ee}^{-1})^2} \times (\sum_{j=1}^{r_i} X_{ij})\\
&\quad \times[\sum_{j=1}^{s_i}(\beta Y_{ij} + \frac{r_i}{s_i}\lambda\overline{X}_i - \beta\alpha)] + \frac{r_i\sigma_{uu}^{-1}\sigma_{ee}^{-2}}{(r_i\sigma_{uu}^{-1} + s_i\beta^2\sigma_{ee}^{-1})^2}[\sum_{j=1}^{s_i}(Y_{ij} - \alpha)]\\
&\quad \times[\sum_{j=1}^{s_i}(Y_{ij} - \beta\overline{X}_i - \alpha)];
\end{aligned}
$$

(1)

$$q_{i\beta\sigma_{uu}} = \frac{\partial q_{i\beta}}{\partial \sigma_{uu}} = \frac{r_i\beta\sigma_{ee}^{-2}\sigma_{uu}^{-2}}{(r_i\sigma_{uu}^{-1}+s_i\beta^2\sigma_{ee}^{-1})^3}\Big\{r_i\sigma_{uu}^{-1}\times[\sum_{j=1}^{s_i}(\beta Y_{ij}-\beta\overline{X}_i-\alpha)]^2$$

$$-s_i\beta\sigma_{ee}^{-1}[\sum_{j=1}^{s_i}(Y_{ij}-\beta\overline{X}_i-\alpha)][\sum_{j=1}^{s_i}(\beta Y_{ij}+\frac{r_i}{s_i}\lambda\overline{X}_i-\beta\alpha)]\Big\}$$

$$= \frac{r_i\beta\sigma_{ee}^{-2}\sigma_{uu}^{-1}}{(r_i\sigma_{uu}^{-1}+s_i\beta^2\sigma_{ee}^{-1})^3}[\sum_{j=1}^{s_i}(Y_{ij}-\beta\overline{X}_i-\alpha)]^2 - \frac{s_i\beta^2\sigma_{uu}^{-1}\sigma_{ee}^{-1}}{(r_i\sigma_{uu}^{-1}+s_i\beta^2\sigma_{ee}^{-1})}\times q_{i\beta};$$

(2)

$$q_{i\beta\sigma_{ee}} = \frac{\partial q_{i\beta}}{\partial \sigma_{ee}} = \frac{(r_i\sigma_{uu}^{-1})(r_i\sigma_{uu}^{-1}\sigma_{ee}^{-2})}{(r_i\sigma_{uu}^{-1}+s_i\beta^2\sigma_{ee}^{-1})^3}[\sum_{j=1}^{s_i}(Y_{ij}-\beta\overline{X}_i-\alpha)]\times$$

$$\times\{-2\sigma_{ee}^{-1}\sum_{j=1}^{s_i}(\beta Y_{ij}-\frac{r_i}{s_i}\lambda\overline{X}_i-\beta\alpha)+(r_i\sigma_{uu}^{-1}+s_i\beta^2\sigma_{ee}^{-1}\overline{X}_i\}$$

$$= \frac{-r_i^2\sigma_{uu}^{-2}\sigma_{ee}^{-3}}{(r_i\sigma_{uu}^{-1}+s_i\beta^2\sigma_{ee}^{-1})^3}\Big\{\beta\Big[\sum_{j=1}^{s_i}(Y_{ij}-\beta\overline{X}_i-\alpha)\Big]^2$$

$$+\Big[\sum_{j=1}^{s_i}(Y_{ij}-\beta\overline{X}_i-\alpha)\Big]\Big[\sum_{j=1}^{s_i}(\beta Y_{ij}+\frac{r_i}{s_i}\lambda\overline{X}_i-\beta\alpha)\Big]\Big\};$$

$$q_{i\sigma_{uu}\alpha} = q_{i\alpha\sigma_{uu}};$$

$$q_{i\sigma_{uu}\beta} = q_{i\beta\sigma_{uu}};$$

$$q_{i\sigma_{uu}\sigma_{uu}} = \frac{\partial q_{i\sigma_{uu}}}{\partial \sigma_{uu}} = \frac{r_i}{2}\sigma_{uu}^{-2} - \sigma_{uu}^{-3}\sum_{j=1}^{r_i}(X_{ij}-\overline{X}_i)^2$$

$$-\frac{r_is_i\beta^4\sigma_{ee}^{-3}\sigma_{uu}^{-3}}{(r_i\sigma_{uu}^{-1}+s_i\beta^2\sigma_{ee}^{-1})^3}[\sum_{j=1}^{s_i}(Y_{ij}-\beta\overline{X}_i-\alpha)]^2;$$

$$q_{i\sigma_{uu}\sigma_{ee}} = \frac{\partial q_{i\sigma_{uu}}}{\partial \sigma_{ee}} = \frac{rt^2\beta^2\sigma_{uu}^{-3}\sigma_{ee}^{-3}}{(r_i\sigma_{uu}^{-1}+s_i\beta^2\sigma_{ee}^{-1})^3}[\sum_{j=1}^{s_i}(Y_{ij}-\beta\overline{X}_i-\alpha)]^2;$$

$$q_{i\sigma_{ee}\sigma_{ee}} = -\frac{s_i}{2}\sigma^{-2} - \sigma_{ee}^{-3}[\sum_{j=1}^{s_i}(Y_{ij}-\overline{Y}_i)^2 - \frac{r_i^3\sigma_{uu}^{-3}\sigma_{ee}^{-3}}{s_i(r_i\sigma_{uu}^{-1}+s_i\beta^2\sigma_{ee}^{-1})^3}$$

$$\times[\sum_{j=1}^{s_i}(Y_{ij} - \beta\overline{X}_i - \alpha)]^2.$$

Appendix 2. Computing $E_0[\mathbf{A}_n(\boldsymbol{\theta}^1)]$

In the following the notation $E_0[\left(\mathbf{A}_n(\boldsymbol{\theta}^1)\right)_{ij}]$ is used to denote the (i,j)-th element of the matrix $E[\mathbf{A}_n(\boldsymbol{\theta})^1]$. Using the second derivatives given above, it follows that

$$\begin{aligned}
E_0[\left(\mathbf{A}_n(\boldsymbol{\theta}^1)\right)_{11}] &= \left[E_0\left(\frac{\sum_{i=1}^n q_{i\alpha\alpha}}{n}\right)\right]|_{\boldsymbol{\theta}=\boldsymbol{\theta}^1} \\
&= -\frac{(\overline{r}+\overline{s})}{n(\overline{s}+\overline{r}-1)\sigma_{uu}^0}\sum_{i=1}^n\left(\frac{r_i s_i}{r_i\lambda^0 + s_i(\beta^0)^2}\right); \\
E_0[\left(\boldsymbol{A}_n(\boldsymbol{\theta}^1)\right)_{12}] &= \frac{1}{n}[\sum_{i=1}^n E_0(q_{i\alpha\beta})]|_{\boldsymbol{\theta}=\boldsymbol{\theta}^1} = -\frac{(\overline{r}+\overline{s})}{n(\overline{s}+\overline{r}-1)\sigma_{uu}^0}\sum_{i=1}^n\frac{s_i r_i x_i^0}{(r_i\lambda^0 + s_i(\beta^0)^2)};
\end{aligned}$$

$$\begin{aligned}
E_0[(\boldsymbol{A}_n(\boldsymbol{\theta}^1))_{13}] &= [\sum_{i=1}^n \frac{E^0(q_{i\alpha\sigma_{uu}})}{n}]|_{\boldsymbol{\theta}=\boldsymbol{\theta}^1} = 0; \\
E_0[(\mathbf{A}_n(\boldsymbol{\theta}^1))_{14}] &= [\sum_{i=1}^n \frac{E^0(q_{i\alpha\sigma_{ee}})}{n}]|_{\boldsymbol{\theta}=\boldsymbol{\theta}^1} = 0; \\
E_0[\left(\mathbf{A}_n(\boldsymbol{\theta}^1)\right)_{22}] &= [\sum_{i=1}^n \frac{E_0(q_{i\beta\beta})}{n}]|_{\boldsymbol{\theta}=\boldsymbol{\theta}^1} = -\frac{\overline{r}+\overline{s}}{n\sigma_{uu}^0(\overline{r}+\overline{s}-1)}\sum_{i=1}^n\frac{r_i s_i x_i^{02}}{(r_i\lambda^0 + s_i(\beta^0)^2)}; \\
E_0[\left(\boldsymbol{A}_n(\boldsymbol{\theta}^1)\right)_{23}] &= [\frac{\sum_{i=1}^n E_0(q_{i\beta\sigma_{uu}})}{n}]|_{\boldsymbol{\theta}=\boldsymbol{\theta}^1} \\
&= \frac{\lambda^0 + \beta^0(\sigma_{uu}^0)^{-1}}{n}\left(\frac{\overline{r}+\overline{s}}{\overline{r}+\overline{s}-1}\right)^2\sum_{i=1}^n\frac{r_i s_i}{(r_i\lambda^0 + s_i(\beta^0)^2)^2};
\end{aligned}$$

$$E_0[\left(\mathbf{A}_n(\boldsymbol{\theta}^1)\right)_{24}] = [\sum_{i=1}^n \frac{E_0(q_{i\beta\sigma_{ee}})}{n}]|_{\boldsymbol{\theta}=\boldsymbol{\theta}^1} = -\frac{\beta^0(\sigma_{uu}^0)^{-1}}{n}\left(\frac{\overline{r}+\overline{s}}{\overline{r}+\overline{s}-1}\right)^2\sum_{i=1}^n\frac{r_i s_i}{(r_i\lambda^0 + s_i\beta^{02})^2};$$

(3)

$$E_0[\left(\mathbf{A}_n(\boldsymbol{\theta}^1)\right)_{33}] = [\frac{\sum_{i=1}^n E_0(q_{i\sigma_{uu}\sigma_{uu}})}{n}]|_{\boldsymbol{\theta}=\boldsymbol{\theta}^1} =$$

$$
\begin{aligned}
&= \frac{1}{n}\Big(\frac{\bar r+\bar s}{\bar r+\bar s-1}\Big)^3(\sigma_{uu}^0)^{-2}\Big\{\Big(\frac{\bar r+\bar s-1}{\bar r+\bar s}\Big)\frac{\sum_{i=1}^n r_i}{2} - \sum_{i=1}^n (r_i-1) \\
&\quad -\beta^{04}\sum_{i=1}^n \frac{s_i^2}{(r_i\lambda^0+s_i\beta^{02})^2}\Big\};
\end{aligned}
$$

$$
\begin{aligned}
E_0[\big(\mathbf{A}_n(\boldsymbol\theta^1)\big)_{34}] &= \Big[\frac{\sum_{i=1}^n E_0(q_{i\sigma_{uu}\sigma_{ee}})}{n}\Big]\Big|_{\boldsymbol\theta=\boldsymbol\theta^1} = \\
&= -\frac{\beta^{02}(\sigma_{uu}^0)^{-2}(\bar r+\bar s)^3}{n(\bar r+\bar s-1)^3}\sum_{i=1}^n \frac{r_i s_i}{(r_i\lambda^0+s_i\beta^{02})^2};
\end{aligned}
$$

$$
\begin{aligned}
E_0[\big(\mathbf{A}_n(\boldsymbol\theta^1)\big)_{44}] &= \Big[\frac{\sum_{i=1}^n E_0(q_{i\sigma_{ee}\sigma_{ee}})}{n}\Big]\Big|_{\boldsymbol\theta=\boldsymbol\theta^1} = \\
&= \frac{1}{n}\Big(\frac{\bar r+\bar s}{\bar r+\bar s-1}\Big)^3(\sigma_{ee}^0)^{-2}\Big\{\frac{\bar r+\bar s-1}{\bar r+\bar s}\frac{\sum_{i=1}^n s_i}{2} \\
&\quad -\sum_{i=1}^n (s_i-1) - \sum_{i=1}^n \frac{r_i^2\lambda^2}{(r_i\lambda+s_i\beta^{02})^2}\Big\}.
\end{aligned}
$$

Thus, the matrix $E[\mathbf{A}_n(\boldsymbol\theta^1)]$ given in (5.1) follows from the above derivations.

Appendix 3. Derivation of $\mathbf{V}_n(\boldsymbol\theta^1)$

In the following, $(\mathbf{V}_n)_{ij}$ denotes the (i,j)-th element of the matrix $\mathbf{V}_n(\boldsymbol\theta^1)$. As mentioned above

$$
\begin{aligned}
V_n(\boldsymbol\theta^1) &= \sum_{i=1}^n \frac{cov(q_i((X_i,Y_i),\boldsymbol\theta^1)|\boldsymbol\theta^0, x_i^0)}{n} \\
&= \frac{\sum_{i=1}^n cov_0[\mathbf{q}_i(\boldsymbol\theta^1)]}{n},
\end{aligned}
$$

where

$$\mathbf{q}_i(\boldsymbol\theta^1) = \{q_{i\alpha}(\boldsymbol\theta^1), q_{i\beta}(\boldsymbol\theta^1), q_{i\sigma_{uu}}(\boldsymbol\theta^1), q_{i\sigma_{ee}}(\boldsymbol\theta^1)\}.$$

After some algebraic manipulations, we obtain

$$
\begin{aligned}
(\mathbf{V}_n)_{11} &= \Big[\frac{1}{n}\sum_{i=1}^n cov^0(q_{i\alpha}, q_{i\alpha})\Big]_{\boldsymbol\theta=\boldsymbol\theta^1} = \frac{1}{n}\sum_{i=1}^n\Big[\frac{\sigma_{uu}^{-2}s_i r_i\sigma_{uu}^0}{(r_i\lambda+s_i\beta^{02})}\Big]_{\boldsymbol\theta=\boldsymbol\theta^1} \\
&= \frac{1}{n}(w)^2(\sigma_{uu}^0)^{-1}\sum_{i=1}^n \frac{s_i r_i}{\Delta_i} = \frac{w^2}{n\sigma_{uu}^0}\sum_{i=1}^n \frac{s_i r_i}{\Delta_i}; \\
(\mathbf{V}_n)_{12} &= \Big[\frac{1}{n}\sum_{i=1}^n cov^0(q_{i\alpha}, q_{i\beta})\Big]_{\boldsymbol\theta=\boldsymbol\theta^1} = \Big[\frac{1}{n}\frac{r_i s_i\sigma_{uu}^{-2}}{(r_i\lambda+s_i\beta^{02})}x_i^0\sigma_{uu}^0\Big]_{\boldsymbol\theta=\boldsymbol\theta^1}
\end{aligned}
$$

$$
\begin{aligned}
&= (\frac{1}{n\sigma_{uu}^0})(\frac{\bar{r}+\bar{s}}{\bar{r}+\bar{s}-1})^2 \sum_{i=1}^{n} \frac{s_i r_i x_i^0}{(r_i\lambda + s_i\beta^{02})} = \frac{1}{n\sigma_{uu}^0} w^2 \sum_{i=1}^{n} \frac{s_i r_i x_i^0}{\Delta_i};\\
(\mathbf{V}_n)_{13} &= [\frac{1}{n}\sum_{i=1}^{n} cov^0(q_{i\alpha}, q_{i\sigma_{uu}})]_{\boldsymbol{\theta}=\boldsymbol{\theta}^1} = 0;\\
(\mathbf{V}_n)_{14} &= [\frac{1}{n}\sum_{i=1}^{n} cov_0(q_{i\alpha}, q_{i\sigma_{ee}})]_{\boldsymbol{\theta}=\boldsymbol{\theta}^1} = 0;
\end{aligned}
$$

$$
\begin{aligned}
(\mathbf{V}_n)_{22} &= [\frac{1}{n}\sum_{i=1}^{n} cov^0(q_{i\beta}, q_{i\beta})]|_{\boldsymbol{\theta}=\boldsymbol{\theta}^1}\\
&= \frac{1}{n}\sum_{i=1}^{n}\left\{\frac{r_i\sigma_{ee}^{-2}s_i\sigma_{ee}^0 x_i^0}{(r_i\lambda+s_i\beta^{02})} + \frac{r_i\sigma_{ee}^{-2}s_i\sigma_{ee}^0\sigma_{uu}^0}{(r_i\lambda+s_i\beta^{02})^2}\right\};\\
&= \frac{1}{n}\{w^2(\sigma_{uu}^0)^{-1}\sum_{i=1}^{n}\frac{s_i r_i x_i^{02}}{\Delta_i} + w^2(\sigma_{uu}^0)^{-1}\sigma_{ee}^0\sum_{i=1}^{n}\frac{s_i r_i}{\Delta_i^2}\}\\
&= \frac{w^2}{n\sigma_{uu}^0}\{\sum_{i=1}^{n}\frac{s_i r_i x_i^{02}}{\Delta_i} + \sigma_{ee}^0\sum_{i=1}^{n}\frac{s_i r_i}{\Delta_i}\};\\
(\mathbf{V}_n)_{23} &= [\frac{1}{n}\sum_{i=1}^{n} cov^0(q_{i\beta}, q_{i\sigma_{uu}})]|_{\boldsymbol{\theta}=\boldsymbol{\theta}^1} = 0;\\
(\mathbf{V}_n)_{24} &= [\frac{1}{n}\sum_{i=1}^{n} cov^0(q_{i\beta}, q_{i\sigma_{ee}})]|_{\boldsymbol{\theta}=\boldsymbol{\theta}^1} = 0;
\end{aligned}
$$

$$
\begin{aligned}
(\mathbf{V}_n)_{33} &= [\frac{1}{n}\sum_{i=1}^{n} cov_0(q_{i\sigma_{uu}}, q_{i\sigma_{uu}})]|_{\boldsymbol{\theta}=\boldsymbol{\theta}^1} = \frac{w^4}{2n(\sigma_{uu}^0)^4}\sum_{i=1}^{n}\{\frac{s_i^2\beta^{04}\sigma_{uu}^{02}}{(r_i\lambda+s_i\beta^{02})^2} + (r_i-1)\sigma_{uu}^{02}\}\\
&= \frac{1}{2n}\frac{w^4}{\sigma_{uu}^{02}}\{\beta^{04}\sum_{i=1}^{n}\frac{s_i^2}{\Delta_i^2} + \sum_{i=1}^{n}(r_i-1)\};\\
(\mathbf{V}_n)_{34} &= [\frac{1}{n}\sum_{i=1}^{n} cov_0(q_{i\sigma_{uu}}, q_{i\sigma_{ee}})]|_{\boldsymbol{\theta}=\boldsymbol{\theta}^1} = \frac{1}{n}\sum_{i=1}^{n}\left\{\frac{r_i s_i\beta^{02}\sigma_{uu}^{-4}\sigma_{uu}^{02}}{2(r_i\lambda+s_i\beta^{02})^2}\right\}\\
&= \frac{1}{2n}\frac{w^4\beta^{04}}{\sigma_{uu}^{02}}\sum_{i=1}^{n}\frac{s_i r_i}{\Delta_i^2};\\
(\mathbf{V}_n)_{44} &= [\frac{1}{n}\sum_{i=1}^{n} cov_0(q_{i\sigma_{ee}}, q_{i\sigma_{ee}})]|_{\boldsymbol{\theta}=\boldsymbol{\theta}^1}
\end{aligned}
$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left\{\frac{1}{2}\sigma_{ee}^{-4}[(s_i-1)\sigma_{ee}^{02}+\frac{r_i^2\sigma_{uu}^{02}\lambda^4}{(r_i\lambda+s_i\beta^{02})^2}]\right\}.$$

Thus, the matrix $\mathbf{V}_n(\boldsymbol{\theta}^1)$ given in (5.2) and the asymptotic covariance matrix in (5.3) follows from the above derivations.

Appendix 4. Conditions under which the matrix $E[\mathbf{A}_n(\boldsymbol{\theta}^1)]$ is negative definite

In the following conditions are stated under which the matrix $E_0[\mathbf{A}_n(\boldsymbol{\theta}^1)]$ is negative definite. We recall that a necessary conditions for a symmetric matrix $\mathbf{A}$ to be negative definite is that all its odd principal minors are negative and all its even principal minors are positive (Fleming, 1966). Thus, after extensive algebraic manipulations, we arrive at the following conditions:

CM-1) $\bar{r} > 1 + \dfrac{\bar{s}}{\bar{r}+\bar{s}} - 2\beta^{04}\overline{\Sigma}_{rs}$;

CM-2) $\bar{s} > 1 + \dfrac{\bar{r}}{\bar{r}+\bar{s}} - 2\lambda^2\overline{\Sigma}_{rr}$

and

CM-3) $S_{xx}^{(v)} > \max\{S_{11}, S_{22}\}$,

where

$$S_{xx}^{v} = \frac{\sum_{i=1}^{n} v_i(x_i^0-\bar{x}_v)^2}{\sum_{i=1}^{n} v_i}, \qquad \bar{x}_v = \frac{\sum_{i=1}^{n} v_i x_i^0}{\sum_{i=1}^{n} v_i}, \qquad v_i = \frac{r_i s_i}{\lambda r_i + \beta^{02} s_i},$$

$$\overline{\Delta} = \frac{1}{n}\sum_{i=1}^{n}\Delta_i, \qquad S_{11} = \frac{(\lambda\beta^0\overline{\Sigma}_{rs})^2\sigma_{uu}}{\overline{\Delta}(\beta^{04}\overline{\Sigma}_{ss}-c)},$$

with

$$c = \frac{\bar{r}}{2w} - \bar{r} + 1, \quad w = \frac{\bar{r}+\bar{s}}{\bar{r}+\bar{s}-1},$$

and

$$S_{22} = \frac{\lambda^2\beta^{02}\sigma_{uu}^0\overline{\Sigma}_{rs}^2\{\beta^{02}(\lambda\overline{\Sigma}_{rs}+\beta^{02}\overline{\Sigma}_{ss})+\nu(\beta^{02}\overline{\Sigma}_{rs}+\lambda\overline{\Sigma}_{rr})-(b+c)\}}{\bar{v}[(b-\lambda^2\Sigma_{rr})c-b\beta^{04}\overline{\Sigma}_{ss}+\lambda^2\beta^{04}(\overline{\Sigma}_{rr}\overline{\Sigma}_{ss}-\overline{\Sigma}_{rs}^2)]}$$

$$= \frac{\{\lambda^2\beta^{02}\sigma_{uu}^0\overline{\Sigma}_{rs}^2\{\beta^{02}(\lambda\overline{\Sigma}_{rs}+\beta^{02}\Sigma_{ss})+\nu(\beta^{02}\Sigma_{rs}+\lambda\overline{\Sigma}_{rr})-(b+c)\}}{\bar{v}[bc-(c\lambda^2\overline{\Sigma}_{rr}+b\beta^{04}\overline{\Sigma}_{ss})+\lambda^2\beta^{04}(\Sigma_{rr}\Sigma_{ss}-\Sigma_{rs}^2)]},$$

using the notation considered in Section 5.

Special Case

We consider now the case where $r_i = r \geq 2$ e $s_i = 1$. In this special case, it follows that

$$v_i = \frac{r_i s_i}{\Delta_i} = \frac{r}{(r\lambda + \beta^{02})} = \frac{r}{\Delta_r};$$
$$\Delta_i = \Delta_r = r\lambda + \beta^{02};$$
$$\sum_{i=1}^{n} \frac{v_i}{\Delta_i} = \frac{nr}{\Delta_r^2}; \quad \sum_{i=1}^{n} \frac{r_i^2}{\Delta_i^2} = \frac{nr^2}{\Delta_r^2};$$
$$\sum_{i=1}^{n} \frac{s_i^2}{\Delta_i^2} = \frac{n}{\Delta_r^2}; \quad w = \frac{\overline{r} + \overline{s}}{\overline{r} + \overline{s} - 1} = \frac{r+1}{r} \quad \text{e}$$
$$c = \frac{\overline{r}}{2w} - \overline{r} + 1 = \frac{-r^2 + 2}{2(r+1)} < 0; \text{ since, } r \geq 2 \quad \text{and} \quad b = \frac{r}{2(r+1)} > 0.$$

Thus, it follows that condition CM-1 is always satisfied. With respect to condition CM-2, notice that
$\overline{s} = s_i = 1 > 2 - \frac{1}{r+1} - \frac{2\lambda^2 r^2}{\Delta_r^2}$.
Thus, condition (CM-2) is satisfied provided

$$1 - \frac{1}{r+1} - \frac{2r^2}{r + \frac{\beta^{02}}{\lambda}} < 0 \Rightarrow r > (\frac{\beta^{02}}{\lambda} - 1) + \sqrt{2(\frac{\beta^{02}}{\lambda} - \frac{1}{2})^2 + \frac{1}{2}}.$$

Finally, with respect to condition CM-3, notice that

$$S_{11} = \frac{r\lambda^2\beta^{02}\sigma_{uu}^0}{(r\lambda + \beta^{02})[\beta^{04} - c(\lambda + r\beta^{02})^2]} = \frac{r\lambda^2\beta^{02}\sigma_{uu}^0}{\Delta_r(\beta^{04} - c\Delta_r^2)}.$$

Since, $c < 0$, it follows that, $0 < S_{11} < \infty$. Thus, condition (CM-3) is satisfied provided $S_{xx} \geq \max\{S_{11}, S_{22}\}$, where

$$S_{22} = \frac{\lambda^2\beta^{02}\sigma_{uu}^0 r^2}{2\Delta_r[bc\Delta_r^2 - (cr^2\lambda^2 + b\beta^{04})]} = \frac{\lambda^2\beta^{02}\sigma_{uu}^0 r^2}{2\Delta_r[-c(r^2\lambda^2 - b\Delta_r^2) - b\beta^{04}]},$$

with

$$S_{xx} = \sum_{i=1}^{n} \frac{(x_i - \overline{x})^2}{n}, \qquad 0 < b = \frac{r}{2(r+1)} < \frac{1}{2} \qquad \text{e} \qquad c = \frac{-r^2 + 2}{2(r+1)} < 0.$$

The main conclusion is that S_{xx} being large enough implies that $E[A_n(\boldsymbol{\theta}^{(1)})|\boldsymbol{\theta}^{(0)}, x_i^0]$ is negative definite. Similar results hold for the other special cases. As shown in Bolfarine and Galea-Rojas (1995), S_{xx} also plays an important role on the asymptotic variance of the maximum likelihood estimator in the unreplicated case.

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