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A note on derivations with power central values in prime rings

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DERIVATIONS WITH POWER CENTRAL VALUES IN PRIME RINGS

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Recently, I. N. Herstein [2] described prime rings R having a non zero derivation such that a fixed power of each image is central. This result was extended to the case of variable powers by C. Lanski who proved the following [3, theorems 4 and 7].

Theorem - Let R be a prime ring with no non zero nil right ideal and let $Z = Z(R)$ denote the center of R. Assume that there exists a non zero derivation d of R such that $d(x)^{n(x)} \in Z$ for each element x of a non zero ideal I of R. If Z is uncountable, then R_Z is a simple algebra at most 4-dimensional over its center.

In this note, we shall show that when $\mathrm{char}(R) = p > 0$ the theorem holds regardless of the size of Z.

We beguin with an easy result that shall be needed in the sequel.

Lemma 1 - Let $\mathbb{Z}\langle X_1,\dots,X_n\rangle$ be the ring of polynomials over $\mathbb{Z}$ in the non-commuting indeterminates $X_1,\dots,X_n$ and let S be the subring generated by all possible elements of the form $(\sum_{i \in I} X_i)^n$, where I is ani subset of the set of integers $\{1,2,\ldots,n\}$. Then

$$\sum_{\sigma\in S_n} X_{\sigma(1)},\ldots,X_{\sigma(n)} \in S .$$

Proof - Write:

$$(X_1+\ldots+X_n)^n = \sum_{i=1}^{n} \theta_i(X_1,\ldots,X_n)$$

where $\theta_k(X_1,\ldots,X_n)$ denotes the sum of all terms envolving exactly k different variables, $1\le k\le n$.

Thus, $\theta_1(X_1,\ldots,X_n) = X_1^n+\ldots+X_n^n \in S$.

It is easily seen that:

$$\theta_k(X_1,\ldots,X_n) = \sum_{j_1<\ldots<j_k} (X_{j_1}+\ldots+X_{j_k})^n - \sum_{i=1}^{k-1} \binom{n-i}{k-i} \theta_i(X_1,\ldots,X_n)$$

Hence, it follows, by induction, that

$$\theta_n(X_1,\ldots,X_n) = \sum_{\sigma\in S_n} X_{\sigma(1)}\ldots X_{\sigma(n)} \in S . \qquad \Delta$$

Now, we return to our problem. In what follows, R will denote a prime ring with no non zero nil right ideals and characteristic $p\neq 0$; $I\neq 0$ an ideal of R and $d\neq 0$ a derivation in R such that, for each element $x\in I$ there exists an integer $n(x)$ verifying $d(x)^{n(x)}\in Z$, the center of R.

Let G be an uncountable, ordered, abelian group (for example, we can take G to be isomorphic to the additive group of the real number system). We denote by RG the group ring of G over R. For an element $\alpha = \sum\alpha(g).g$ we set, as usual, $supp(\alpha) = \{g \mid \alpha(g) \neq 0\}$ and we denote by $\tau(\alpha)$ the coefficient of the greatest element in $supp(\alpha)$.

If J is an ideal in RG it is easily seen that $\{\tau(\alpha) \mid \alpha \in J\}$ is an ideal of R. Hence, it follows readily that RG in also a prime ring with no non zero nil right ideals.

We define a derivation δ in RG by:

$$\delta(\sum\alpha(g)g) = \sum d(\alpha(g))g.$$

With the notation above, we have:

Lemma 2 - The set $IG = \{\sum\alpha(g)g \mid \alpha(g) \in I, \ \forall g \in G\}$ is a non zero ideal of RG and, for each element $\alpha \in IG$, there exists a positive integer $n(\alpha)$ such that $\delta(\alpha)^{n(\alpha)} \in Z(RG)$.

Proof - Let $\alpha = \sum\alpha(g)g$ be an arbitrary element in IG, write:

$$supp(\alpha) = \{g_1, g_2, \dots, g_t\}$$

and denote, for briefness, $\alpha_i = \alpha(g_i)$ $\quad 1 \leq i \leq t$.

Since $char(R) = p \neq 0$, the set:

$$\{ \sum_{i=1}^{t} m_i \alpha_i \mid m_i \in Z \; , \; 1 \le i \le t \} \subset R$$

is finite. Hence, we can determine a positive integer n such that

$$(\sum_{i=1}^{t} m_i d(\alpha_i))^n \in Z$$

for any choice of integers $m_i \in Z$, $1 \le i \le t$. Also, we can choose n such that $n > t$.

We shall now show that $\delta(\alpha)^n \in Z(RG) = Z.G$. We have that:

$$\delta(\alpha)^n = (\sum_{i=1}^{t} d(\alpha_i) g_i)^n = \sum \beta(g) g$$

where

$$\beta(g) = \sum_{g_{i_1} \cdots g_{i_n} = g} d(\alpha_{i_1}) \ldots d(\alpha_{i_n}) =$$

$$= \sum_{\substack{g_{i_1} \cdots g_{i_n} = g \\ i_1 \le \cdots \le i_n}} \sum_{\sigma \in S_n} d(\alpha_{i_{\sigma(1)}}) \ldots d(\alpha_{i_{\sigma(n)}}) \; .$$

For a given n-tuple $\alpha_{i_1}, \ldots, \alpha_{i_n}$ we consider the homomorphism $\pi_i : Z\langle X_1, \ldots, X_n \rangle \longrightarrow R$ given by $\pi(X_j) = d(\alpha_{i_j})$. $1 \le j \le n$. Then, using lemma 1 we see that:

$$\sum_{\sigma \in S_n} d(\alpha_{i_{\sigma(1)}}) \ldots d(\alpha_{i_{\sigma(n)}}) = \pi(\sum_{\sigma \in S_n} X_{\sigma(1)} \cdots X_{\sigma(n)} \in \pi(S) \subset Z.$$

Hence, $\delta(\alpha)^n$ belongs to ZG, the center of RG. Δ

We are now ready to prove:

<u>Theorem</u> - Let R be a prime ring with no non zero nil right ideals, with $\mathrm{char}(R) = p > 0$. Assume that there exists a non zero derivation d of R such that, for every element x in a given ideal $I \neq 0$ of R there exists an integer $n(x)$ verifying $d(x)^{n(x)} \in Z$, the center of R. Then R_Z is a simple algebra, at most 4-dimensional over its center.

<u>Proof</u> - After lemma 2, the theorem of Lanski quoted above shows that, in any case, the localization RG_{ZG} of the group ring RG on its center ZG is a simple algebra, at most 4-dimensional over its own center, which we shall denote by $\underline{\mathbf{Z}}$.

Since we have the inclusions:

$$\begin{tikzcd} R \arrow[r, hook] \arrow[d, dash] & RG \arrow[d, dash] \\ Z \arrow[r, hook] & ZG \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} R_Z \arrow[r, hook] \arrow[d, dash] & RG_{ZG} \arrow[d, dash] \\ Z(R_Z) \arrow[r, hook] & \underline{\mathbf{Z}} \end{tikzcd}$$

a straightforward computation shows that any set of five elements $\alpha_i \in R_Z$, $1 \leq i \leq 5$, being linearly dependent over $\underline{\mathbf{Z}}$, when considered as elements of RG_{ZG}, must also be linearly dependent over $Z(R_Z)$.

Thus,

$$\dim_{Z(R_Z)} R_Z \leq 4 \quad .$$

Finally, since R_Z is prime and artinian, it is a simple algebra over its center. Δ

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