Special Groups, Rings and Algebras of Continuous Functions

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Introduction

The purpose of these notes is twofold:

1) Firstly, we present a simple set of axioms for rings and q-subgroups of them (cf. Definition 1.1) sufficiently strong to ensure that, whenever R is a commutative unital ring in which 2 is invertible, and S is a q-subgroup of R, the group $S/(R^\times)^2$ of square classes of S is a special group that faithfully reflects the theory of diagonal quadratic forms with coefficients in S (cf. axioms [FQ 1], [FQ 2] and [FQ 3] in Theorems 1.17 and 1.20). These axioms substantially improve those presented in [DM5]. This is the subject matter of section 1.

These axioms can be used, in particular, to give a new and simpler proof of Theorem 3.16 of [DM5] (p.17), guaranteeing that a large class of rings R with many units give rise to special groups that faithfully reflect isometry and representation of non-singular quadratic forms with unit coefficients in R.

In general, rings of real-valued continuous functions and the real holomorphy rings of formally real fields are not rings with many units; cf. Proposition 2.14 in [DM5]. However, we show that rings of these types do verify [FQ 1], [FQ 2] and a restricted form of [FQ 3], giving rise to special groups faithfully reflecting value representation of diagonal quadratic forms with invertible entries. Thus, the class of rings possessing this property extends far beyond rings with many units. This is done in section 2 for rings of continuous functions, for which we also prove a fairly general transversality result (Theorem 2.5).

We also identify precisely the special groups associated to these classes of rings: they turn out to be Boolean algebras (of clopens of suitable topological spaces). In particular, these special groups have the K-theoretic property [SMC] – that we introduced in [DM3] – and hence by results in [DM6] and [DM7], they verify Marshall's signature conjecture and their K-theory verifies Milnor's Witt ring conjecture.

2) Our second goal, achieved in section 4, is to show that any reduced special group (RSG) is isomorphic to the special group of square classes of a q-subgroup of the algebra of continuous real-valued functions on some Boolean space, this algebra being, itself, a ring with many units (cf. Theorem 2.10 and Corollary 2.11, [DM5], pp. 9-10). This result bears on the long-standing representation problem posed by Marshall in 1974: is any RSG isomorphic to the special group of square classes of a Pythagorean **field**? Though this problem is still open – and widely believed to have a negative answer – our results show that as soon as the class of representation objects *is broadened from RSGs of fields to those of rings* – or, rather q-subgroups of them, and one may even add, rings with many units –, *the problem has a positive answer.* This issue merits, in our opinion, further examination. Observe, for example, that $\mathbb{C}(X)$ is the Dedekind reals in the topos of sheaves over the topological space X.

Notation. In all that follows, the word **ring** stands for **commutative unitary ring in which 2 is a unit.** If R is a ring, $T \subseteq R$ and $x \in R$

$* \; R^\times$ = groups of units in R; $\quad * \; T^\times = T \cap R^\times$; $\quad * \; T^2 = \{t^2 \in R : t \in T\}$;

$\Sigma T^2 = \{\sum_{i=1}^n t_i^2 : \{t_1, \ldots, t_n\} \subseteq T\}$; $\quad * \; xT = \{xt : t \in T\}$.

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1 Commutative Rings and Special Groups

Definition 1.1 *If R is a ring, a* **q-subgroup** *of R is a subset S of $R^{\times}$ such that $1, -1 \in S$, is cl under multiplication, and $(R^{\times})^2 \subseteq S$.*

1.2 Examples and Remarks. a) If S is a q-subgroup of R and $x \in S$, then $\frac{1}{x} = x \cdot \frac{1}{x^2} \in S$, bec S contains all invertible squares. Hence all q-subgroups are subgroups of $R^{\times}$.

b) The smallest q-subgroup of R is $R^{\times 2} \cup -R^{\times 2}$; the largest is, of course, $R^{\times}$ itself.

c) It is straightforward that the family of q-subgroups of R is closed under arbitrary intersections. M over, the union of any up-directed family of q-subgroups is a q-subgroup. Hence, endowed with inclusion partial order, the set of q-subgroups of R is a complete lattice.

d) If $T \subseteq R^{\times}$, the set

$$T_q = \{a_1^{k_1} a_2^{k_2} \ldots a_n^{k_n} x^2 : n \geq 0,\ x \in R^{\times},\ \{a_1, \ldots, a_n\} \subseteq T \cup \{1, -1\} \text{ and } \{k_1, \ldots, k_n\} \subseteq \mathbb{N}\}$$

is the q-subgroup of **generated by T**, the intersection of all q-subgroups of R containing T.

1.3 Diagonal S-quadratic forms in free R-modules. Let $n \geq 1$ be an integer and let R^n be the n-dimensional R-module. Let S be a q-subgroup of R.

a) To $\langle a_1, \ldots, a_n \rangle \in S^n$, we associate:

(1) A diagonal quadratic form, $q(a_1, \ldots, a_n)$, where for $x = \langle x_1, \ldots, x_n \rangle \in R^n$,

$$q(a_1, \ldots, a_n)(x) = \textstyle\sum_{i=1}^{n} a_i x_i^2.$$

(2) A diagonal matrix in $GL_n(R)$, $\mathcal{M}(a_1, \ldots, a_n)$, whose non-zero entries are precisely $a_1, \ldots, a_n$ (in order, i.e., the $\langle k, k \rangle$-entry of $\mathcal{M}$ is a_k).

(3) The discriminant of $q(a_1, \ldots, a_n)$ is the unit $a_1 \cdots a_n$ of S, exactly the determinant of $\mathcal{M}(a_1, \ldots,$

Whenever we write $\varphi = \langle a_1, \ldots, a_n \rangle$ for a n-form over S, then $q(\varphi)$ and $\mathcal{M}(\varphi)$ will stand $q(a_1, \ldots, a_n)$ and $\mathcal{M}(a_1, \ldots, a_n)$, respectively, while $d(\varphi)$ is the discriminant of φ.

b) If $\varphi = \langle a_1, \ldots, a_n \rangle$, $\psi = \langle b_1, \ldots, b_n \rangle$ are n-forms over S, define

$$q(\varphi) \approx q(\psi) \quad \text{iff} \quad \exists\, M \in GL_n(R) \text{ such that } M\mathcal{M}(\varphi)M^t = \mathcal{M}(\psi).$$

Note that $d(\varphi)\,det(M)^2 = d(\psi)$, where $det\, M$ is the determinant of M. The relation $\approx$, called ma **isometry**, is readily proven to be an equivalence relation.

c) If $\langle a_1, \ldots, a_n \rangle \in S^n$, $\langle c_1, \ldots, c_n \rangle \in R^{\times n}$ and σ is a permutation of $\{1, \ldots, n\}$, then, just as in Lem 1.28 in [DM5] we have

(1) $q(a_1, \ldots, a_n) \approx q(c_1^2 a_1, \ldots, c_n^2 a_n)$; (2) $q(a_1, \ldots, a_n) \approx q(a_{\sigma(1)}, \ldots, a_{\sigma(n)})$.

d) As usual, if φ, ψ, θ are forms over S, then

$$q(\varphi) \approx q(\psi) \quad \Rightarrow \quad q(\varphi) \oplus q(\theta) \approx q(\psi) \oplus q(\theta) \text{ and } q(\varphi) \otimes q(\theta) \approx q(\psi) \otimes q(\theta).$$

For $a, b \in S$,

$$D^S(a, b) = \{x \in S : \exists\, t, s \in R \text{ such that } x = s^2 a + t^2 b\}$$

is the set of elements of S **represented by $\langle a, b \rangle$**. If $S = R^{\times}$, write $D^R(a, b)$ for $D^{R^{\times}}(a, b)$. It is c that for all $a, b \in S$, $\{a, b\} \subseteq D^S(a, b)$. Let

$$G(S) = S/R^{\times 2} \quad \text{and} \quad q_S : S \longrightarrow G(S) \text{ be the canonical quotient morphism.}$$

Clearly, $G(S)$ is a group of exponent 2, with distinguished elements $1 = \overline{1}$ and $-1 = \overline{-1}$. To ease notat for $x \in S$, write $\overline{x}$ for $q_S(x)$ and $-\overline{x}$ for $\overline{-1} \cdot \overline{x} = \overline{-x}$. Hence,

$$G(S) = \{\overline{x} : x \in S\}.$$

Note that for $x, y \in S$, $\quad \overline{x} = \overline{y}$ iff $xy \in R^{\times 2}$ iff $\exists\, a \in R^{\times}$ such that $x = ya^2$.

The basic properties of binary representation sets are described in the result that follows, whose p is similar to those of Lemma 1.30 and Proposition 1.31 of [DM2] (cf. also Fact 8.10 in [DM7]).

Lemma 1.4 *With notation as above, let $x,\ y,\ u,\ v \in S$ and $t \in R^{\times}$.*

a) $uD^S(x,y) \;=\; D^S(ux,uy)$ *and* $D^S(x,y) \;=\; D^S(t^2x,t^2y)$.

b) $u \in D^S(x,y)$ *and* $\overline{u} = \overline{v} \;\Rightarrow\; v \in D^S(x,y)$.

c) $\overline{x} = \overline{u}$ *and* $\overline{y} = \overline{v} \;\Rightarrow\; D^S(x,y) \;=\; D^S(u,v)$.

d) $D^S(1,x)$ *is a subgroup of* S.

e) $x \in D^S(1,y) \;\Rightarrow\; D^S(x,xy) \;=\; xD^S(1,y) \;=\; D^S(1,y)$.

f) $u \in D^S(x,y) \;\Leftrightarrow\; D^S(u,\,uxy) \;=\; D^S(x,y)$.

g) The following are equivalent :

(1) $\overline{xy} = \overline{uv}$ *and* $D^S(x,y) \;=\; D^S(u,v)$;

(2) $\overline{xy} = \overline{uv}$ *and* $D^S(x,y) \cap D^S(u,v) \;\neq\; \emptyset$. $\diamondsuit$

Remark 1.5 Since the representation sets, $D^S(x,\,y)$, are invariant (or saturated) with respect to square classes (1.4.(b), (c)), they can be seen in $G(S)$, that is,

$$D^S(\overline{x},\,\overline{y}) \;=\; D^S(x,\,y)/R^{\times 2} \;=\; \{\overline{z} \in G(S) : \exists\, t_1,\, t_2 \in R \text{ such that } z = t_1^2x + t_2^2y\},$$

with $(q_S)^{-1}(D^S(\overline{x},\,\overline{y})) \;=\; D^S(x,\,y)$. Hence, for $x,\ y,\ u,\ v \in S$

$$\begin{cases} u \in D^S(x,\,y) \;\Leftrightarrow\; \overline{u} \in D^S(\overline{x},\,\overline{y}); \\ D^S(u,\,v) = D^S(x,\,y) \;\Leftrightarrow\; D^S(\overline{u},\,\overline{v}) \;=\; D^S(\overline{x},\,\overline{y}). \end{cases} \qquad \text{(rep)}$$

$\diamondsuit$

It is important to observe that $D^S(1,\,\overline{x})$ is a *subgroup* of $G(S)$.

Define a binary relation on $G(S) \times G(S)$, $\equiv^S$, called **binary isometry modulo squares**, as follows:

For $u,\ v,\ x,\ y \in S$, $\langle\,\overline{u},\overline{v}\,\rangle \equiv^S \langle\,\overline{x},\overline{y}\,\rangle \;\Leftrightarrow\; \overline{uv} \;=\; \overline{xy}$ and $D^S(u,\,v) \;=\; D^S(x,\,y)$. $\qquad (\equiv^S)$

If $S = R^{\times}$, write $\equiv$ for $\equiv^S$. Lemma 1.4 yields, with the notation in Definition 1.2 of [DM2]:

Lemma 1.6 *a)* $G(S) = \langle\,G(S), \equiv^S, -1\,\rangle$ *satisfies the following properties for all* $u,\ v,\ x,\ y,\ z \in S$

[SG 0] : $\equiv^S$ *is an equivalence relation on* $G(S) \times G(S)$.

[SG 1] : $\langle\,\overline{u},\overline{v}\,\rangle \equiv^S \langle\,\overline{v},\overline{u}\,\rangle$; $\qquad$ [SG 2] : $\langle\,\overline{u},-\overline{u}\,\rangle \;\equiv^S\; \langle\,1,-1\,\rangle$;

[SG 3] : $\langle\,\overline{u},\overline{v}\,\rangle \equiv^S \langle\,\overline{x},\overline{y}\,\rangle \;\Rightarrow\; \overline{uv} = \overline{xy}$;

[SG 5] : $\langle\,\overline{u},\overline{v}\,\rangle \equiv^S \langle\,\overline{x},\overline{y}\,\rangle \;\Rightarrow\; \langle\,\overline{zu},\overline{zv}\,\rangle \equiv^S \langle\,\overline{zx},\overline{zy}\,\rangle$.

In the terminology of [DM7], *Definition* 6.7, $G(S)$ *is a* **proto special group** (**π-SG**).

b) If representation in $G(S)$ *is* 2-transversal, *that is, if it satisfies*

$$\forall\; u,\ v,\ x \in S,\quad x \in D^S(u,v) \;\Rightarrow\; \exists\; s,\, t \in R^{\times} \text{ such that } x = s^2u + t^2v,$$

then $G(S)$ *is a pre-special group, i.e., it satisfies, in addition,*

[SG 4] : $\langle\,\overline{u},\overline{v}\,\rangle \equiv^S \langle\,\overline{x},\overline{y}\,\rangle \;\Rightarrow\; \langle\,\overline{u},\,-\overline{x}\,\rangle \equiv^S \langle\,-\overline{v},\overline{y}\,\rangle$.

c) The following are equivalent :

(1) $G(S)$ *is* **reduced**, *i.e.,* $1 \neq -1$ *and for all* $a \in S$, $\langle\,\overline{a},\overline{a}\,\rangle \equiv^S \langle\,1,1\,\rangle \;\Rightarrow\; \overline{a} = 1$;

(2) $-1 \not\in R^{\times 2}$ *and every sum of two squares in* R *is a square (and so* $\Sigma R^2 = R^2$*).*

Proof. a) We comment only on [SG 2]. Since $2 \in R^{\times}$, we may write

$$u \;=\; \left(\frac{1+u}{2}\right)^2 - \left(\frac{1-u}{2}\right)^2,$$

showing that if $u \in S$, then $u \in D^S(1,\,-1)$. Since $\overline{u(-u)} = \overline{1(-1)}$, the definition of $\equiv^S$ entails $\langle\,\overline{u},-\overline{u}\,\rangle \equiv \langle\,1,-1\,\rangle$.

b) If $\langle\,\overline{u},\overline{v}\,\rangle \equiv^S \langle\,\overline{x},\overline{y}\,\rangle$, then $\overline{uv} = \overline{xy}$ and the equivalence (*) (before the statement of 1.4, above) yields $\overline{u(-x)} = \overline{(-v)y}$. By Lemma 1.4.(g) the desired conclusion is equivalent to $u \in D^S(-v,\,y)$. Since representation in S is 2-transversal and $y \in D^S(u,\,v)$, there are $s,\,t \in R^{\times}$ such that $y \;=\; s^2u + t^2v$, and so, since $s^2 \in R^{\times 2}$, the preceding equation yields $u \;=\; \dfrac{y}{s^2} + t^2(-v)$, as needed.

c) If $\langle a, a \rangle \equiv^S \langle 1, 1 \rangle$ and in R the sum of two squares is again a square, then $a \in D^S(1, 1)$ enta[...] $a = s^2 + t^2 = w^2$, and so $\overline{a} = \overline{1} = 1$. The converse is immediate, ending the proof.

We now show that binary isometry in $G(S)$ is equivalent to matrix isometry.

Lemma 1.7 *For all* $u, v, x, y \in S$, $\langle \overline{u}, \overline{v} \rangle \equiv^S \langle \overline{x}, \overline{y} \rangle \Leftrightarrow q(u, v) \approx q(x, y)$.

Proof. ($\Rightarrow$) By the definition of $\equiv^S$,

$$\overline{uv} = \overline{xy} \quad \text{and} \quad D^S(u, v) = D^S(x, y).$$

The equations in (1) together with the definition of $\equiv^S$ yield $e \in R^\times$ and $s, t \in R$ such that

$$uxy = \frac{v}{e^2} \quad \text{and} \quad u = s^2 x + t^2 y.$$

Hence,

$$v = uxye^2 = xy^2t^2 e^2 + yx^2 s^2 e^2.$$

Set $M = \begin{pmatrix} s & t \\ -yte & xse \end{pmatrix}$; then $det(M) = xs^2e + yt^2e = ue \in R^\times$, that is, $M \in GL_2(R)$. It [...] straightforward, using the equations in (2) and (3) to show that

$$M \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} M^t = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}.$$

($\Leftarrow$) Assume there is $M \in GL_n(R)$ such that (4) holds; then $uv = det(M)^2 xy$, and so $\overline{uv} = \overline{xy}$. It [...] straightforward that $u = s^2 x + t^2 y$, where (s, t) is the first line of M. Hence, we have $\overline{uv} = \overline{xy}$ an[...] $u \in D^S(x, y) \cap D^S(u, v)$, and Lemma 1.4.(g) entails $\langle \overline{u}, \overline{v} \rangle \equiv^S \langle \overline{x}, \overline{y} \rangle$, ending the proof.

Definition 1.8 *If* G, H *are* π*-SGs, a map* $f : G \longrightarrow H$ *is a* **π-SG morphism**, *if* f *is a morphism* [...] *groups, taking* -1 *to* -1 *and such that for all* $a, b \in G$

$$a \in D^G(1, b) \quad \Rightarrow \quad f(a) \in D^H(1, f(b)).$$

A π*-SG morphism is an* **embedding** *if it is injective and for all* $a, b \in G$,

$$a \in D^G(1, b) \quad \Leftrightarrow \quad f(a) \in D^H(1, f(b)).$$

Remark 1.9 Let $f : G \longrightarrow H$ be a π-SG morphism. If G is *reduced* (cf. 1.6.(c).(1)), then f is a[...] embedding iff for all $a, b \in G$, $f(a) \in D^H(1, f(b)) \Rightarrow a \in D^G(1, b)$, because this condition implies tha[...] *f is injective*. Indeed, if $f(a) = 1$, then $f(a) \in D^H(1, f(1))$ and so $a \in D^G(1, 1)$, which by reducibilit[...] is equivalent to $a = 1$, showing that $\ker f = \{1\}$.

Definition 1.10 *Let* G *be a* π*-SG. Just as in the case of special groups, we can define isometry f*[...] *forms of arbitrary dimension* $n \geq 2$ *over* G *(i.e.,* $\langle a_1, \ldots, a_n \rangle$, $\langle b_1, \ldots, b_n \rangle \in G^n$*), by induction, in th*[...] *usual way: for* $n \geq 3$

$$\langle a_1, \ldots, a_n \rangle \equiv_G \langle b_1, \ldots, b_n \rangle \quad \text{iff} \quad \begin{cases} \exists\, x, y, z_3, \ldots, z_n \in G \text{ such that} \\ \langle a_1, x \rangle \equiv_G \langle b_1, y \rangle, \ \langle a_2, \ldots, a_n \rangle \equiv_G \langle x, z_3, \ldots, z_n \rangle \text{ and} \\ \langle b_2, \ldots, b_n \rangle \equiv_G \langle y, z_3, \ldots, z_n. \rangle \end{cases}$$

Corollary 1.11 *Let* S *be q-subgroup of a ring* R. *If* φ, ψ *are forms over* S *of the same dimension, the*[...] $\overline{\varphi} \equiv^S \overline{\psi} \Rightarrow q(\varphi) \approx q(\psi)$.

Proof. By induction on the common dimension, n, of φ, ψ; the case $n = 2$ is 1.7. Assume the result tru[...] for forms of dimension $n \geq 2$ and let $\varphi = \langle a \rangle \oplus \theta_1$ and $\psi = \langle b \rangle \oplus \theta_2$, with $\dim \theta_i = n$. If $\overline{\varphi} \equiv^S \overline{\psi}$, the[...] there are $x, y \in S$ and a $(n-1)$-dimensional S-form, θ, such that

(1) $\langle \overline{a}, \overline{x} \rangle \equiv^S \langle \overline{b}, \overline{y} \rangle$; (2) $\overline{\theta^P}_1 \equiv^S \langle \overline{x} \rangle \oplus \overline{\theta}$; (3) $\overline{\theta}_2 \equiv^S \langle \overline{y} \rangle \oplus \overline{\theta}$.

The induction hypothesis, Lemma 1.7 and relations (1) – (3) yield

(4) $q(a, x) \approx q(b, y)$; (5) $q(\theta_1) \approx q(x) \oplus q(\theta)$; (6) $q(\theta_2) \approx q(y) \oplus q(\theta)$.

Now, from (4) – (6) we obtain, using the associativity of $\oplus$ with respect to $\approx$ and the transitivity of the latter:

$$q(\varphi) = q(a) \oplus q(\theta_1) \approx q(a, x) \oplus q(\theta) \approx q(b, y) \oplus q(\theta) \approx q(b) \oplus (q(y) \oplus q(\theta))$$
$$\approx q(b) \oplus q(\theta_2) = q(\psi),$$

as required. $\diamondsuit$

1.12 Notation. Let A be a set and $n \geq 2$, $1 \leq k \leq n$ be integers. If $a = \langle a_1, \ldots, a_n \rangle \in A^n$, write $\langle a_1, \ldots, \check{a}_k, \ldots, a_n \rangle$ for the element of A^{n-1} obtained by forgetting the k^{th}-coordinate of a. $\diamondsuit$

In the general setting of π-SGs associated to q-subgroups in rings, there are several concepts of *representation*, which are of crucial importance to distinguish. In the field case, all these notions coincide.

Definition 1.13 *Let $n \geq 2$ be an integer, let S be a q-subgroup of a ring R, $\varphi = \langle b_1, \ldots, b_n \rangle$ be a n-form in S, and $\overline{\varphi} = \langle \overline{b}_1, \ldots, \overline{b}_n \rangle$ be the corresponding n-form in $G(S)$.*

a) $D^S(\varphi) = \{a \in S$: *There are* $a_2, \ldots, a_n$ *in* S *such that,* $\langle \overline{a}, \overline{a}_2, \ldots, \overline{a}_n \rangle \equiv^S \overline{\varphi}\}$,
is the set of elements of S **isometry-represented (iso-represented) by** φ.

b) $D^S_v(\varphi) = \{a \in S$: *There are* $x_1, \ldots, x_n \in R^2$ *such that* $a = \sum_{i=1}^n x_i b_i\}$,
is the set of elements of S **value-represented (v-represented) by** φ.

c) $D^S_{tv}(\varphi) = \{a \in S$: *There are* $z_1, \ldots, z_n \in R^{\times 2}$ *such that* $a = \sum_{i=1}^n z_i b_i\}$,
is the subset of S **transversaly v-represented (tv-represented) by** φ. *Clearly,* $D^S_{tv}(\varphi) \subseteq D^S_v(\varphi)$.

d) With notation as 1.12, let

$$\mathfrak{D}^S(\varphi) = \begin{cases} D^S_v(b_1, b_2) & \text{if } n = 2; \\ \bigcap_{k=1}^n \bigcup \{D^S_v(b_k, u) : u \in D^S_v(b_1, \ldots, \check{b}_k, \ldots, b_n)\} & \text{if } n \geq 3. \end{cases}$$

When $S = R^\times$ and φ is a n-form over $R^\times$, write $D^R_v(\varphi)$, $D^R_{tv}(\varphi)$ and $\mathfrak{D}^R(\varphi)$ for the value-representation sets defined above.

Remark 1.14 Let S be a q-subgroup of a ring R.

a) It is straightforward that all the representation sets defined in 1.13 are *invariant modulo units squares in* **R**. Hence, they may also be seen as subsets of $G(S)$.

b) The definition of binary isometry in $G(S)$ (or S) entails $D^S(s, t) = D^S_v(s, t)$, for all $s, t \in S$. It is an entirely different matter if $n \geq 3$, or if we consider transversality. $\diamondsuit$

Lemma 1.15 *Let S be a q-subgroup of a ring R and let $n \geq 2$ be an integer. Let $\varphi = \langle b_1, \ldots, b_n \rangle$ be a form over S and let σ be a permutation of $\{1, \ldots, n\}$. Let $\varphi^\sigma = \langle b_{\sigma(1)}, \ldots, b_{\sigma(n)} \rangle$.*

a) (*i*) $D^S_v(\varphi) = D^S_v(\varphi^\sigma)$; (*ii*) $D^S_{tv}(\varphi) = D^S_{tv}(\varphi^\sigma)$

(*iii*) $\mathfrak{D}^S(\varphi) = \mathfrak{D}^S(\varphi^\sigma)$; (*iv*) $G(S)$ *is a SG* $\Rightarrow$ $D^S(\varphi) = D^S(\varphi^\sigma)$.

b) $D^S(\varphi) \subseteq D^S_v(\varphi)$.

c) $D^S_{tv}(\varphi) \subseteq D^S_v(\varphi)$ *and* $\mathfrak{D}^S(\varphi) \subseteq D^S_v(\varphi)$.

d) Let $1 \leq k \leq m$ be integers, let $\varphi_1, \ldots, \varphi_m$ be forms over S and let $x_j \in D^S_v(\varphi_j)$, $1 \leq j \leq k$. Then,

$$D^S_v(x_1, \ldots, x_k) \subseteq D^S_v\left(\bigoplus_{j=1}^m \varphi_j\right).$$

In particular, if ψ is a m-form over S, then $D^S_v(\varphi) \subseteq D^S_v(\varphi \oplus \psi)$.

Proof. a) Items $(i) - (iii)$ are clear, while (iv) follows from the equivalence in Theorem 1.23 (p. 16) in [DM2].

b) The asserted inclusion holds if $\dim \varphi = 2$ (see 1.14.(b)). We proceed by induction on $2 \leq n = \dim \varphi$. If $\varphi = \langle b \rangle \oplus \psi$ and $a \in D^S(\varphi)$, with $\dim \psi = n$, according to 1.10, there are a n-form θ over S and x, $y, z_3, \ldots, z_n \in S$ such that $\overline{\varphi} = \langle \overline{a} \rangle \oplus \overline{\theta}$ and

$$\langle \overline{a}, \overline{x} \rangle \equiv^S \langle \overline{b}, \overline{y} \rangle, \quad \overline{\theta} \equiv^S \langle \overline{x}, \overline{z}_3, \ldots, \overline{z}_n \rangle \quad \text{and} \quad \psi \equiv^S \langle \overline{y}, \overline{z}_3, \ldots, \overline{z}_n \rangle.$$

But then $a = s^2\, b + t^2 y$, while the induction hypothesis guarantees that $y \in D^S_v(\psi)$. It is then immedia that $a \in D^S_v(\varphi)$, completing the induction step.

c) It is clear that $D^S_{tv}(\varphi) \subseteq D^S_v(\varphi)$. It remains to check that $\mathfrak{D}^S(\varphi) \subseteq D^S_v(\varphi)$. For dim $\varphi = 2$ this hol by definition. Assume the result true for $n \geq 2$ and let $\varphi = \langle b \rangle \oplus \psi$, where dim $\psi = n$. By items (i) a (iii) in (a), it suffices to check that if $a \in D^S_v(b, u)$, with $u \in D^S_v(\psi)$, then $a \in D^S_v(\varphi)$. But we have

$$a = s^2 b + t^2 u \quad \text{and} \quad u = \textstyle\sum_{i=1}^n x_i^2 c_i,$$

where $\psi = \langle c_1, \ldots, c_n \rangle$ and $s,\ t,\ x_1, \ldots, x_n \in R$; it is immediate from the preceding equalities th $a \in D^S_v(\varphi)$, as needed. Item (d) is straightforward.

Our next result shows that important relations between the value sets in items (b) – (d) of 1.1 particularly transversality, are consequences of very simple axioms.

Theorem 1.16 *Let S be a q-subgroup of a ring R and let $k,\ n \geq 2$ be integers. Assume that*

[FQ 1] : *For all $a,\ b \in S$, $\quad D^S_v(a, b) \ = \ D^S_{tv}(a, b)$;*

$[\text{FQ } 2]_n$: *For all $2 \leq m \leq n$ and all m-forms φ over S, $\quad \mathfrak{D}^S(\varphi) \ = \ D^S_v(\varphi)$.*

Then,

a) For all $2 \leq m \leq n$ and all m-forms φ over S, $\quad D^S_v(\varphi) \ = \ D^S_{tv}(\varphi)$.

b) If $\varphi_1, \ldots, \varphi_k$ are forms over S and $\varphi \ = \ \bigoplus_{i=1}^k \varphi_i$ is such that dim $\varphi \leq n$, then

$$\begin{aligned} D^S_v(\varphi) &= \bigcup \{D^S_v(u_1, \ldots, u_k) \ : \ u_i \in D^S_v(\varphi_i),\ 1 \leq i \leq k\} \\ &= \bigcup \{D^S_{tv}(u_1, \ldots, u_k) \ : \ u_i \in D^S_{tv}(\varphi_i),\ 1 \leq i \leq k\}. \end{aligned}$$

Proof. a) By [FQ 1], the result is true for $m = 2$. We proceed by induction on m, recalling that b 1.15.(c) it suffices to verify that $D^S_v(\varphi) \subseteq D^S_{tv}(\varphi)$. Let $\varphi = \langle a_1 \rangle \oplus \psi$, with $m =$ dim $\psi < n$ and le $x \in D^S_v(\varphi)$. By [FQ 1], $[\text{FQ } 2]_n$ and the induction hypothesis, there is $u \in D^S_v(\psi) = D^S_{tv}(\psi)$ such tha $x \in D^S_v(a_1, u) = D^S_{tv}(a_1, u)$. It is now straightforward to see that $x \in D^S_{tv}(\varphi)$. Indeed, if $\psi = \langle c_1, \ldots, c_m \rangle$ there are $s,\ t,\ z_1, \ldots, z_m \in R^{\times}$, such that $x \ = \ s^2 a_1 + t^2 u$, with $u \ = \ \sum_{i=1}^m z_i^2 c_i$, as needed.

b) It suffices to verify the first equality for $k = 2$; a straightforward induction will complete its proo while the second follows from (a) and the fact that $k \leq n$ and $\max_{1 \leq i \leq k}$ dim $\varphi_i \leq n$. Moreover, we ma assume that $n \geq 3$, otherwise there is nothing to prove. Suppose, then, that $\varphi \ = \ \varphi_1 \oplus \varphi_2$; by [FQ 2] the result is true if dim $\varphi_1 = 1$. We proceed by induction on $m =$ dim $\varphi_1 < n$, letting $\varphi_1 \ = \ \langle a_1 \rangle \oplus$ with dim $\psi = m$. Fix $a \in D^S_v(\varphi)$; by $[\text{FQ } 2]_n$, there is $x \in D^S_v(\psi \oplus \varphi_2)$ such that

$$a \in D^S_v(a_1, x). \qquad \text{(I}$$

Since dim $\psi = m$, the induction hypothesis yields $u \in D^S_v(\psi)$ and $v \in D^S_v(\varphi_2)$ such that

$$x \in D^S_v(u, v). \qquad \text{(II}$$

It follows from (I) and (II) that $a \in D^S_v(a_1, u, v)$. Because $n \geq 3$, we have

$$D^S_v(a_1, u, v) \ = \ \mathfrak{D}^S(a_1, u, v) \ \subseteq \ \bigcup \{D^S_v(z, v) \ : \ z \in D^S_v(a_1, u)\},$$

whence there is $z \in D^S_v(a_1, u)$ such that $a \in D^S_v(z, v)$. Since $u \in D^S_v(\psi)$, 1.15.(d) entails $z \in D^S_v(\langle a_1 \rangle \oplus$ $= \ D^S_v(\varphi_1)$ and so $a \in D^S_v(z, v)$, with $z \in D^S_v(\varphi_1)$ and $v \in D^S_v(\varphi_2)$, completing the induction step.

We shall now describe axioms that guarantee that the proto-special group

$$G(S) \ = \ \langle G(S), \equiv^S, -1 \rangle,$$

S a q-subgroup of a ring, is a **special group**, where $G(S) = S/R^{\times 2}$, improving the presentation in [DM5

Theorem 1.17 *Let S be a q-subgroup of a ring R.*

a) With notation as in 1.13 and 1.16, assume that representation of forms over S verifies [FQ 1] an $[\text{FQ } 2]_3$. If ψ is a form of dimension ≤ 3 over S, then $\quad D^S_v(\psi) \ = \ D^S(\psi)$, that is, an element of S i v-represented iff it is iso-represented by ψ.

b) Assume that value representation for forms over S verify [FQ 1], $[\text{FQ } 2]_3$ and that matrix isometr satisfies the following cancellation property

[FQ 3]$_3$ *For all* $a, u, v, x, y \in S$, $q(a, u, v) \approx q(a, x, y) \Rightarrow q(u, v) \approx q(x, y)$.

Then, (1) *For all 3-forms* φ, ψ *over* S, $q(\varphi) \approx q(\psi) \Leftrightarrow \overline{\varphi} \equiv^S \overline{\psi}$;

(2) $G(S) = \langle G(S), \equiv^S, -1 \rangle$ *is a special group.*

Proof. a) By Lemma 1.15.(a) and (b), it suffices to show that $D_v^S(\varphi) = \mathfrak{D}^S(\varphi) \subseteq D^S(\varphi)$. If $\dim \varphi = 2$, this is clear (see Remark 1.14.(b)). It remains to treat the case where $\dim \varphi = 3$. If $\varphi = \langle b_1, b_2, b_3 \rangle$ and $a \in \mathfrak{D}^S(\varphi)$, by [FQ 2]$_3$ and [FQ 1] there is $u \in S$ such that

$$(i) \quad a \in D_{tv}^S(b_1, u) \quad \text{and} \quad (ii) \quad u \in D_{tv}^S(b_2, b_3). \tag{I}$$

Therefore, there are s_1, s_2, s_3 in $R^\times$, such that

$$a = b_1 s_1^2 + \underbrace{b_2 s_2^2 + b_3 s_3^2}_{u}. \tag{II}$$

Let $c = b_2 b_3 u$ (clearly, $c \in S$); then, (II) entails

$$c = b_2 b_3 u = b_2 b_3 (b_2 s_2^2 + b_3 s_3^2) = b_3 (b_2 s_2)^2 + b_2 (b_3 s_3)^2,$$

and so $c \in D_{tv}^S(b_2, b_3)$ (note that $b_i s_i \in R^\times$, i = 1, 2, 3); this last relation and the very definition of c (keep in mind that $b_i \in S$, $i = 1, 2, 3$) yield

$$\overline{cu} = \overline{b_2 b_3} \quad \text{and} \quad c \in D^S(b_2, b_3),$$

and so the definition of binary isometry in $G(S)$ yields $\langle \overline{b}_2, \overline{b}_3 \rangle \equiv^S \langle \overline{u}, \overline{c} \rangle$. Now, it follows from (I).(i) that in $G(S)$ we have $\langle \overline{a}, \overline{ab_1 u} \rangle \equiv^S \langle \overline{b}_1, \overline{u} \rangle$. Putting all the above information together, we see that

$$\langle \overline{a}, \overline{ab_1 u} \rangle \equiv^S \langle \overline{b}_1, \overline{u} \rangle, \quad \langle \overline{b}_2, \overline{b}_3 \rangle \equiv^S \langle \overline{u}, \overline{c} \rangle \quad \text{and} \quad \langle \overline{c}, \overline{ab_1 u} \rangle \equiv^S \langle \overline{ab_1 u}, \overline{c} \rangle, \tag{III}$$

isometries that entail $\langle \overline{a}, \overline{c}, \overline{ab_1 u} \rangle \equiv^S \langle \overline{b}_1, \overline{b}_2, \overline{b}_3 \rangle$, establishing $a \in D^S(\varphi)$, as desired. ◇

Remark 1.18 The above proof yields precise information on how to complete $\langle \overline{a}, \cdot, \cdot \rangle$ so as to have it isometric in $G(S)$ to $\langle \overline{b}_1, \overline{b}_2, \overline{b}_3 \rangle$: having selected $u \in D_{tv}^S(b_2, b_3)$, if $c = b_2 b_3 u$ and $z = ab_1 u$, then $\langle \overline{a}, \overline{c}, \overline{z} \rangle \equiv^S \langle \overline{b}_1, \overline{b}_2, \overline{b}_3 \rangle$. □

b) (1) By 1.11, it is enough to establish ($\Rightarrow$). Let $\varphi = \langle a, x, y \rangle$ and $\psi = \langle b_1, b_2, b_3 \rangle$. It is straightforward that $q(\varphi) \approx q(\psi)$ implies $a \in D_v^S(\psi) = \mathfrak{D}^S(\psi)$. With notation as in the proof of (a), we obtain

$$\langle \overline{a}, \overline{c}, \overline{ab_1 u} \rangle \equiv^S \langle \overline{b}_1, \overline{b}_2, \overline{b}_3 \rangle,$$

and so by 1.11, it follows that $q(a, c, ab_1 u) \approx q(b_1, b_2, b_3)$. Since matrix isometry is transitive, this relation and our hypothesis yield $q(a, c, ab_1 u) \approx q(a, x, y)$. Now, the cancellation law [FQ 3]$_3$ entails $q(c, ab_1 u) \approx q(x, y)$, whence, by Lemma 1.7, $\langle \overline{x}, \overline{y} \rangle \equiv^S \langle \overline{ab_1 u}, \overline{c} \rangle$. But then, this last isometry, together with the first two in (III) above furnish

$$\langle \overline{a}, \overline{ab_1 u} \rangle \equiv^S \langle \overline{b}_1, \overline{u} \rangle, \quad \langle \overline{b}_2, \overline{b}_3 \rangle \equiv^S \langle \overline{u}, \overline{c} \rangle \quad \text{and} \quad \langle \overline{x}, \overline{y} \rangle \equiv^S \langle \overline{ab_1 u}, \overline{c} \rangle,$$

that is equivalent to $\langle \overline{a}, \overline{x}, \overline{y} \rangle \equiv^S \langle \overline{b}_1, \overline{b}_2, \overline{b}_3 \rangle$, as desired.

(2) As observed in Lemma 1.6.(b), 2-transversality (i.e., [FQ 1]) guarantees that $G(S)$ is a pre-special group. To be a special group it is necessary and sufficient that the isometry relation $\equiv^S$ be transitive for 3-forms. Since matrix isometry is transitive, this is an immediate consequence of (b).(1). ◇

Remark 1.19 Since matrix isometry is preserved by scaling, property [FQ 3] in 1.17 is equivalent to

For all $a, b, c, d \in S$, $q(1, a, b) \approx q(1, c, d) \Rightarrow q(a, b) \approx q(c, d)$. ◇

Theorem 1.20 *Let* S *be a q-subgroup of a ring* R. *Notation as in* 1.16 *and* 1.17.(b), *assume that value representation of forms over* S *satisfies* [FQ 1], [FQ3]$_3$ *as well as*

[FQ 2] : [FQ 2]$_n$ *holds for all* $n \geq 2$.

Then,

a) *For all n-forms* φ *over* S, $D^S(\varphi) = D_v^S(\varphi)$, *that is, an element of* S *is value represented iff it is isometry represented.*

If, in addition, S *satisfies*

[FQ 3] : (Witt cancellation for matrix isometry) *For all forms φ, ψ of dimension $n \geq 3$ over S and all $a \in S$, $q(\langle a \rangle \oplus \varphi) \approx q(\langle a \rangle \oplus \psi) \Rightarrow q(\varphi) \approx q(\psi)$.*

then,

b) For all n-forms φ, ψ over S, $q(\varphi) \approx q(\psi) \Leftrightarrow \overline{\varphi} \equiv^S \overline{\psi}$.

c) $G(S) = \langle G(S), \equiv^S, -1 \rangle$ is a special group that faithfully represents matrix isometry and value representation of diagonal S-quadratic forms.

Proof. By Theorem 1.17(b.2), $G(S)$ is a special group. By Theorem 1.23 in [DM2], for all $n \geq 2$, the extension of $\equiv^S$ to forms of dimension n *is a transitive relation*. We shall use this below, without further comment.

a) By Lemma 1.15.(b) it is enough to verify that $D_v^S(\varphi) \subseteq D^S(\varphi)$, which will be achieved by induction on dim $\varphi \geq 2$. It follows from 1.17.(a) that the result holds true for dim $\varphi \leq 3$. Assume it valid for forms of dimension n and let $\varphi = \langle b \rangle \oplus \psi$, where dim $\psi = n$. If $a \in S$ is value-represented by $\langle b \rangle \oplus \psi$, then [FQ 2]$_n$ implies that there is $u \in S$ such that

$$a \in D_v^S(b, u) \quad \text{and} \quad u \in D_v^S(\psi).$$

The induction hypothesis yields $z_2, \ldots, z_n \in S$ such that

$$\langle \overline{u}, \overline{z}_2, \ldots, \overline{z}_n \rangle \equiv^S \overline{\psi}, \qquad (\ $$

while the first representation relation in (I) implies $\langle \overline{a}, \overline{abu} \rangle \equiv^S \langle \overline{b}, \overline{u} \rangle$. Adding $\langle \overline{b} \rangle$ to both sides of (I… Proposition 1.6.(a) in [DM2] (that holds even for pre-special groups) yields

$$\langle \overline{b}, \overline{u}, \overline{z}_2, \ldots, \overline{z}_n \rangle \equiv^S \langle \overline{b} \rangle \oplus \overline{\psi}. \qquad (I$$

Since $\langle \overline{a}, \overline{abu} \rangle \equiv^S \langle \overline{b}, \overline{u} \rangle$, from Proposition 1.6.(a) in [DM2] we obtain

$$\langle \overline{a}, \overline{abu}, \overline{z}_2, \ldots, \overline{z}_n \rangle \equiv^S \langle \overline{b}, \overline{u}, \overline{z}_2, \ldots, \overline{z}_n \rangle. \qquad (I$$

Now, (III), (IV) and the transitivity of $\equiv^S$ entail $\langle \overline{a}, \overline{abu}, \overline{z}_2, \ldots, \overline{z}_n \rangle \equiv^S \langle \overline{b} \rangle \oplus \overline{\psi}$, wherefrom we conclude $a \in D^S(\langle b \rangle \oplus \psi)$, as needed.

b) As observed in 1.11, it is enough to prove the implication ($\Rightarrow$), that we know, by 1.17.(b).(1), to hold for forms of dimension ≤ 3. We proceed by induction on dimension; assume the result holds for forms of dimension n and suppose $\varphi = \langle a \rangle \oplus \theta_1$, $\psi = \langle b \rangle \oplus \theta_2$, where dim $\theta_i = n$, $i = 1, 2$. If $q(\varphi) \approx q(\psi)$, then $a \in D_v^S(\psi)$ and so, as above, [FQ 2]$_n$ and item (a) yield $u, z_2, \ldots, z_n \in S$ such that

$$\langle \overline{a}, \overline{abu} \rangle \equiv^S \langle \overline{b}, \overline{u} \rangle \quad \text{and} \quad \langle \overline{u}, \overline{z}_2, \ldots, \overline{z}_n \rangle \equiv^S \overline{\theta}_2. \qquad (\ $$

Adding $\langle \overline{b} \rangle$ to both sides of the second isometry in (V) yields

$$\langle \overline{b}, \overline{u}, \overline{z}_2, \ldots, \overline{z}_n \rangle \equiv^S \overline{\psi}. \qquad (V$$

On the other hand, the first isometry in (V) entails, by adding $\langle \overline{z}_2, \ldots, \overline{z}_n \rangle$ to both sides,

$$\langle \overline{a}, \overline{abu}, \overline{z}_2, \ldots, \overline{z}_n \rangle \equiv^S \langle \overline{b}, \overline{u}, \overline{z}_2, \ldots, \overline{z}_n \rangle,$$

that together with (VI) and the transitivity of $\equiv^S$ yield

$$\langle \overline{a}, \overline{abu}, \overline{z}_2, \ldots, \overline{z}_n \rangle \equiv^S \overline{\psi}. \qquad (V$$

Corollary 1.11 and the assumption $q(\varphi) \approx q(\psi)$ give:

$$q(a, abu, z_2, \ldots, z_n) \approx q(\psi) \approx q(\varphi) = q(a) \oplus q(\theta_1).$$

Now, using [FQ 3] we can cancel out $q(a)$ to get $q(abu, z_2, \ldots, z_n) \approx q(\theta_1)$. Since dim $\theta_1 = n$, the induction hypothesis applies, to yield $\langle \overline{abu}, \overline{z}_2, \ldots, \overline{z}_n \rangle \equiv^S \overline{\theta}_1$, wherefrom, adding $\langle \overline{a} \rangle$ to both sides and using (VII) we get

$$\overline{\psi} \equiv^S \langle \overline{a}, \overline{abu}, \overline{z}_2, \ldots, \overline{z}_n \rangle \equiv^S \langle \overline{a} \rangle \oplus \overline{\theta}_1 \equiv^S \overline{\varphi},$$

as required. Item (c) follows immediately from (b) and Theorem 1.17.(b).

1.21 Discussion. The arguments below suggest that the axioms [FQ 1], [FQ 2] and [FQ 3] presented above are natural. Let S be a q-subgroup of a ring R.

(1) The hypothesis of 2-transversality for representation in $G(S)$, i.e., [FQ 1] in 1.16, seems to be crucial to show that $G(S)$ satisfies [SG 4]. Note that [SG 4] *entails* [SG 2] and so one might be tempted to o…

the requirement that $2 \in A^\times$, used to establish [SG 2]. However, the hypothesis that $2 \in A^\times$ has other useful consequences (e.g., a preorder P is proper iff $-1 \notin P$) and guarantees that all residue fields of A are of characteristic $\neq 2$, a classical setting for quadratic form theory.

(2) With notation as in 1.3, what would be reasonable requirements on $G(S) = \langle G(S), \equiv^S, -1 \rangle$ so that it **faithfully** represents the theory of diagonal S-quadratic forms? The first would be that isometry in $G(S)$ corresponds to matrix isometry: if φ, ψ are (diagonal) S-quadratic forms of the same dimension, then

(*) $$q(\varphi) \approx q(\psi) \quad \Leftrightarrow \quad \overline{\varphi} \equiv^S \overline{\psi}.$$

Next, since value-representation is an important ingredient in quadratic form theory, it would be natural to expect that it correspond to representation in $G(S)$, that is, for all $a \in S$ and all S-forms φ

(**) $$a \in D_v^S(\varphi) \quad \Leftrightarrow \quad \overline{a} \in D^S(\overline{\varphi}).$$

With notation as Theorems 1.16, 1.17 and 1.20, we have

Proposition 1.22 *Let S be a q-subgroup of a ring R such that* [FQ 1] *holds for binary S-representation. Then, the following are equivalent:*

(1) *$G(S)$ is a special group satisfying conditions $(*)$ and $(**)$ in 1.21.(2);*

(2) *Value representation of diagonal S-quadratic forms verify conditions* [FQ 2] *and* [FQ 3].

Proof. (2) $\Rightarrow$ (1) is the content of Theorems 1.17 and 1.20.(c). For the converse first note that since any special group satisfies Witt-cancellation (Proposition 1.6.(b), [DM2]), (*) immediately yields [FQ 3]. Let $\varphi = \langle b_1, \ldots, b_n \rangle$ be a n-ary S-form. By Lemma 1.15.(c), it is enough to check that $D_v(\varphi) \subseteq \mathfrak{D}^S(\varphi)$. Assume that for $a \in S$, we have $a \in D_v(\varphi)$; then (**) yields $\overline{a} \in D^S(\overline{\varphi})$. Since $G(S)$ is a special group, there is a S-form θ such that $\langle \overline{a} \rangle \oplus \overline{\theta} \equiv^S \overline{\varphi}$.

Fix $1 \leq k \leq m$; Theorem 1.23 in [DM2] guarantees that $\overline{\varphi} \equiv^S \langle \overline{b_k} \rangle \oplus \langle \overline{b_1}, \ldots, \check{\overline{b_k}}, \ldots, \overline{b_n} \rangle$, as well as that $\langle \overline{a} \rangle \oplus \overline{\theta} \equiv^S \langle \overline{b_k} \rangle \oplus \langle \overline{b_1}, \ldots, \check{\overline{b_k}}, \ldots, \overline{b_n} \rangle$. Hence, $\overline{a} \in D^S(\langle \overline{b_k} \rangle \oplus \langle \overline{b_1}, \ldots, \check{\overline{b_k}}, \ldots, \overline{b_n} \rangle)$. By Proposition 1.6.(c) in [DM2], there is $u \in S$ such that

$$\overline{a} \in D^S(\overline{b_k}, \overline{u}), \quad \text{with} \quad \overline{u} \in D^S(\overline{b_1}, \ldots, \check{\overline{b_k}}, \ldots, \overline{b_n}).$$

Now, (**) entails $a \in D_v^S(b_k, u)$ and $u \in D_v^S(b_1, \ldots, \check{b_k}, \ldots, b_n)$, as needed to conclude the proof. $\diamondsuit$

2 The Special Group of a Ring of Continuous Functions

Definition 2.1 *Let Y be a topological space.*

a) *Write $\mathbb{C}(Y)$ for the $\mathbb{R}$-algebra of all continuous, real-valued maps on Y. It is clear that 2 (the map with constant value 2) is a unit in $\mathbb{C}(Y)$.*

b) *For $f, g \in \mathbb{C}(Y)$, write $f \wedge g$ and $f \vee g$ for the meet and join of $\{f, g\}$ in $\mathbb{C}(Y)$, which are, as usual, computed pointwise.*

c) *For $f \in \mathbb{C}(Y)$, $|f| = (f \vee 0) + (-f \vee 0)$ is the absolute value of f. Clearly, $|f| \in \mathbb{C}(Y)$.*

d) *Write $B(Y)$ for the Boolean algebra (BA) of clopens in Y.*

d) *For $f, g \in \mathbb{C}(Y)$, define $[\![f < g]\!] = \{x \in Y : f(x) < g(x)\}$. Similarly one defines $[\![f > g]\!]$, $[\![f \leq g]\!]$, $[\![f \geq g]\!]$ and $[\![f = g]\!]$.*

Remark 2.2 a) Notation as above, observe that for $f, g \in \mathbb{C}(Y)$,

$[\![f < g]\!]$ and $[\![f > g]\!]$ are open, while $[\![f = g]\!]$, $[\![f \geq g]\!]$ and $[\![f \leq g]\!]$ are closed in Y.

Moreover, for all $f \in \mathbb{C}(Y)$, we have the disjoint union $Y = [\![f < 0]\!] \cup [\![f = 0]\!] \cup [\![f > 0]\!]$.

b) Note that $f \in \mathbb{C}(Y)^\times$ iff $[\![f = 0]\!] = \emptyset$. Hence, for all $f \in \mathbb{C}(Y)^\times$,

$$Y = [\![f < 0]\!] \cup [\![f > 0]\!],$$

and so $[\![f > 0]\!]$ and $[\![f < 0]\!]$ are *disjoint clopens* in Y. Moreover, it is clear that

$$f \in \mathbb{C}(Y)^\times \quad \text{iff} \quad |f| \in \mathbb{C}(Y)^\times.$$

c) Since $\mathbb{R}$ is a Pythagorean field, we have

$$\begin{cases} \Sigma\, \mathbb{C}(Y)^2 = \mathbb{C}(Y)^2 = \{f \in \mathbb{C}(Y) : [\![f \geq 0]\!] = Y\}; \\ \mathbb{C}(Y)^{\times 2} = (\mathbb{C}(Y)^2)^\times = \{f \in \mathbb{C}(Y) : [\![f > 0]\!] = Y\}. \end{cases}$$

Lemma 2.3 *a) For $f,\ g \in \mathbb{C}(Y)$ and $d \in \mathbb{R}$,*

$$(1)\quad d(f \wedge g) = \begin{cases} df \wedge dg & \text{if } d \geq 0; \\ df \vee dg & \text{if } d \leq 0. \end{cases} \qquad (2)\quad d(f \vee g) = \begin{cases} df \vee dg & \text{if } d \geq 0; \\ df \wedge dg & \text{if } d \leq 0. \end{cases}$$

b) $\mathbb{C}(Y)^\times$ is closed under absolute value, as well as finite meets and joins.

c) For all $f \in \mathbb{C}(Y)^\times$, the **sign of f**, *defined as $\mathfrak{s}(f) = \dfrac{f}{|f|}$, is a unit in $\mathbb{C}(Y)$. Moreover,*

$$[\![f < 0]\!] = [\![\mathfrak{s}(f) = -1]\!], \quad [\![f > 0]\!] = [\![\mathfrak{s}(f) = 1]\!] \quad \text{and} \quad \mathfrak{s}(f) \equiv f \ mod\ \mathbb{C}(Y)^{\times 2}.$$

Proof. Straightforward. ◊

2.4 The Proto-Special Group Structure of $\mathbb{C}(Y)$. a) Taking into account item (c) in 2.2, the definition of representation (see paragraphs right after Definition 1.1) yields, for $f,\ g,\ h \in \mathbb{C}(Y)^\times$,

$$f \in D^Y(g,h) \quad \text{iff} \quad \exists\, s, t \in \mathbb{C}(Y) \text{ such that } [\![s \geq 0]\!] = [\![t \geq 0]\!] = Y \quad \text{and} \quad f = sg + th, \tag{*}$$

with $D^Y(\cdot,\ \cdot)$ standing for representation sets with respect to $\mathbb{C}(Y)^\times$ (instead of $D^{\mathbb{C}(Y)}$); we shall also write the associated isometry of forms as $\equiv^Y$; thus, for $f_1,\ f_2,\ g_1,\ g_2 \in \mathbb{C}(Y)^\times$,

$$\langle f_1, f_2 \rangle \equiv^Y \langle g_1, g_2 \rangle \quad \Leftrightarrow \quad f_1 f_2 g_1 g_2 \in \mathbb{C}(Y)^{\times 2} \text{ and } D^Y(f_1,\ f_2) = D^Y(g_1,\ g_2). \tag{**}$$

Lemmas 1.4 and 1.6.(a) apply to our present situation and may be used without further comment. In particular, if $G(Y) = \mathbb{C}(Y)^\times/\mathbb{C}(Y)^{\times 2}$, then items (a) and (c) in 1.6, together with the first equality in 2.2.(c), entail that $G(Y) = \langle G(Y), \equiv^Y, -1 \rangle$ is a *reduced* π-SG; it will be shown shortly that, in fact, it is a *reduced special group*.

b) Regarding v-representation (see 1.13), if φ is a n-form over $\mathbb{C}(Y)^\times$, write $D^Y_v(\varphi)$, $D^Y_{tv}(\varphi)$ and $\mathfrak{D}^Y(\varphi)$ for the sets of value-representation defined in 1.13. ◊

The next result shows that value representation in $\mathbb{C}(Y)$ has properties [FQ 1] and [FQ 2], stated in 1.17 and 1.20. We actually get somewhat sharper results, namely items (a) and (c) in the following

Theorem 2.5 (Transversality of v-representation) *Let Y be a topological space and let $n \geq 2$ be an integer. Let $f,\ g_1, \ldots, g_n$ be elements of $\mathbb{C}(Y)^\times$. With notation as in Definition 1.13 and in 2.4,*

a) If

$$[\![f > 0]\!] \cap \textstyle\bigcap_{j=1}^n [\![g_j < 0]\!] = [\![f < 0]\!] \cap \bigcap_{j=1}^n [\![g_j > 0]\!] = \emptyset, \tag{*}$$

then there are $h_1, \ldots, h_n$ in $\mathbb{C}(Y)^{\times 2}$ such that $f = \sum_{i=1}^n h_i g_i$, that is, $f \in D^Y_{tv}(g_1, \ldots, g_n)$

b) For all n-forms φ over $\mathbb{C}(Y)^\times$, $D^Y_v(\varphi) = D^Y_{tv}(\varphi)$.

c) Fix $1 \leq k \leq n$, and assume that

$$[\![f < 0]\!] \cap \textstyle\bigcap_{j \neq k} [\![g_j > 0]\!] \subseteq [\![g_k < 0]\!] \ \text{and}\ [\![f > 0]\!] \cap \bigcap_{j \neq k} [\![g_j < 0]\!] \subseteq [\![g_k > 0]\!]. \tag{**}$$

Then, there is $\alpha \in \mathbb{C}(Y)^{\times 2}$ such that

(i) $u = f - \alpha g_k \in \mathbb{C}(Y)^\times$;

(ii) $[\![u > 0]\!] \cap \bigcap_{j \neq k} [\![g_j < 0]\!] = \emptyset = [\![u < 0]\!] \cap \bigcap_{j \neq k} [\![g_j > 0]\!]$.

d) For all n-forms φ over $\mathbb{C}(Y)^\times$, $\mathfrak{D}^Y(\varphi) = D^Y_v(\varphi)$, that is, $\mathbb{C}(Y)^\times$ satisfies property [FQ 2] in 1.20.

Proof. a) Since $f,\ g_1, \ldots, g_n$ are units in $\mathbb{C}(Y)$, there is a *clopen* partition of Y, $\{V_1, \ldots, V_m\}$ such that f, $g_1, \ldots, g_n$ have constant sign in each V_k, $1 \leq k \leq m$ (take the common refinement of the clopen partitions $\{[\![f < 0]\!],\ [\![f > 0]\!]\}$, $\{[\![g_i < 0]\!],\ [\![g_i > 0]\!]\}$, $1 \leq i \leq n$). For $1 \leq j \leq n$, we shall define h_j as the unique gluing of *positive* units constructed over each V_k. Henceforth, fix V_k, $1 \leq k \leq m$, and set

$$\mathfrak{p}_k \;=\; \{j : 1 \leq j \leq n \text{ and } g_j > 0 \text{ on } V_k\} \quad \text{and} \quad p_k = \text{cardinal of } \mathfrak{p}_k.$$

Case 1. $V_k \subseteq [\![f > 0]\!]$: Note that $\mathfrak{p}_k \neq \emptyset$, otherwise $[\![f > 0]\!] \cap \bigcap_{j=1}^n [\![g_j < 0]\!] \neq \emptyset$. Hence, $p_k \geq 1$.

Case 1.1 : $n - p_k < p_k$. For $1 \leq j \leq n$, set $h_j \upharpoonright V_k = \dfrac{|f|}{(2p_k - n)|g_j|}$. Clearly, each $h_j \upharpoonright V_k$ is strictly positive at all points of V_k. Now, on V_k, we have, recalling that $f > 0$,

$$\begin{aligned}\sum_{j=1}^n h_j g_j &= \sum_{j\in\mathfrak{p}_k} \frac{|f|}{(2p_k - n)|g_j|}\, g_j + \sum_{j\notin\mathfrak{p}_k} \frac{|f|}{(2p_k - n)|g_j|}\, g_j = p_k \frac{|f|}{2p_k - n} - (n - p_k)\frac{|f|}{2p_k - n} \\ &= \frac{2p_k - n}{2p_k - n}\,|f| = f.\end{aligned}$$

Case 1.2. $n - p_k > p_k$. For $1 \leq j \leq n$, set

$$h_j \upharpoonright V_k = \begin{cases} \dfrac{(n - p_k + 1)|f|}{p_k|g_j|} & \text{if } j \in \mathfrak{p}_k; \\ \dfrac{|f|}{|g_j|} & \text{if } j \notin \mathfrak{p}_k. \end{cases}$$

Then, on V_k we get

$$\begin{aligned}\sum_{j=1}^n h_j g_j &= \sum_{j\in\mathfrak{p}_k} \frac{(n - p_k + 1)|f|}{p_k|g_j|}\, g_j + \sum_{j\notin\mathfrak{p}_k} \frac{|f|}{|g_j|}\, g_j = p_k \frac{(n - p_k + 1)|f|}{p_k} - (n - p_k)|f| \\ &= |f| = f.\end{aligned}$$

Case 1.3. $n - p_k = p_k$. In this case set, for $1 \leq j \leq n$,

$$h_j \upharpoonright V_k = \begin{cases} \dfrac{2|f|}{p_k|g_j|} & \text{if } j \in \mathfrak{p}_k; \\ \dfrac{|f|}{p_k|g_j|} & \text{if } j \notin \mathfrak{p}_k. \end{cases}$$

It is straightforward that on V_k, $\sum_{j=1}^n h_j g_j = p_k \dfrac{2|f|}{p_k} - p_k \dfrac{|f|}{p_k} = |f| = f$, ending the discussion of Case 1.

Case 2. $V_k \subseteq [\![f < 0]\!]$. Since $[\![f < 0]\!] \cap \bigcap_{j=1}^n [\![g_j > 0]\!] = \emptyset$, we must have $\mathfrak{p}_k \neq \{1, \ldots, n\}$, whence $n - p_k \geq 1$.

Case 2.1. $n - p_k > p_k$. For $1 \leq j \leq n$, set $h_j \upharpoonright V_k = \dfrac{|f|}{(n - 2p_k)|g_j|}$. Then, since $f < 0$ on V_k, it is straightforward that

$$\sum_{j=1}^n h_j g_j = |f|\left(\frac{p_k}{n - 2p_k} - \frac{n - p_k}{n - 2p_k}\right) = -|f| = f.$$

Case 2.2. $n - p_k \leq p_k$. For $1 \leq j \leq n$, set

$$h_j \upharpoonright V_k = \begin{cases} \dfrac{|f|}{|g_j|} & \text{if } j \in \mathfrak{p}_k; \\ \dfrac{(p_k + 1)|f|}{(n - p_k)|g_j|} & \text{if } j \notin \mathfrak{p}_k. \end{cases}$$

Then, on V_k,

$$\begin{aligned}\sum_{j=1}^n h_j g_j &= \sum_{j\in\mathfrak{p}_k} \frac{|f|}{|g_j|}\, g_j + \sum_{j\notin\mathfrak{p}_k} \frac{(p_k + 1)|f|}{(n - p_k)|g_j|}\, g_j = p_k\,|f| - \left((n - p_k)\frac{(p_k + 1)|f|}{n - p_k}\right) \\ &= |f|(p_k - (p_k + 1)) = -|f| = f.\end{aligned}$$

Clearly, the gluing of the pieces constructed above on each V_k will yield positive units, $h_1, \ldots, h_n$, in $C(Y)$, satisfying the conclusion of item (a).

(b) Let $\varphi = \langle g_1, \ldots, g_n \rangle$; if $f \in D_v^Y(\varphi)$, there are squares $\alpha_j \in C(Y)$ such that $f = \sum_{j=1}^n \alpha_j g_j$. Since the α_j are non-negative, $[\![f < 0]\!] \cap \bigcap_{j=1}^n [\![g_j > 0]\!] = \emptyset = [\![f > 0]\!] \cap \bigcap_{j=1}^n [\![g_j < 0]\!]$, and the desired assertion follows immediately from (a).

(c) Let $V^+ = \bigcap_{j\neq k} [\![g_j > 0]\!]$ and $V^- = \bigcap_{j\neq k} [\![g_j < 0]\!]$, both clopens in Y. We have

$$\begin{cases} V^+ = (V^+ \cap [\![f < 0]\!]) \cup (V^+ \cap [\![f > 0]\!]) = (V^+ \cap [\![g_k < 0]\!]) \cup (V^+ \cap [\![g_k > 0]\!]); \\ V^- = (V^- \cap [\![f > 0]\!]) \cup (V^- \cap [\![f < 0]\!]) = (V^- \cap [\![g_k > 0]\!]) \cup (V^- \cap [\![g_k < 0]\!]). \end{cases}$$

V^+ and V^- can be written as a disjoint union of clopens in Y, as follows:

$$\begin{cases} V^+ = (V^+ \cap [\![f < 0]\!]) \cup (V^+ \cap [\![f > 0]\!] \cap [\![g_k < 0]\!]) \cup (V^+ \cap [\![f > 0]\!] \cap [\![g_k > 0]\!]); \\ V^- = (V^- \cap [\![f > 0]\!]) \cup (V^- \cap [\![f < 0]\!] \cap [\![g_k > 0]\!]) \cup (V^+ \cap [\![f < 0]\!] \cap [\![g_k < 0]\!]). \end{cases} \quad \text{(I}$$

We shall construct $\alpha > 0$ on Y, satisfying requirements (i) and (ii) in the statement, by giving its value on V^+, V^- and $U = (V^+ \cup V^-)^c$, which constitute a clopen partition of Y.

(1) On U, we set $\alpha = \dfrac{|f|}{2|g_k|}$; since $\left|\dfrac{g_k}{|g_k|}\right| = 1$, it is clear that $u = f - \alpha g_k \neq 0$ on U;

(2) On V^+, we shall choose $\alpha > 0$ such that $u = f - \alpha g_k > 0$, and so $[\![u < 0]\!] \cap V^+ = \emptyset$, entailing (i and the corresponding part of (ii) in the statement. Again, we shall proceed by giving the values of α o each clopen of the partition of V^+ appearing in (I) above:

<u>On $V^+ \cap [\![f < 0]\!]$</u> : Recalling that this set is contained in $[\![g_k < 0]\!]$, set $\alpha = \dfrac{2|f|}{|g_k|}$; then,

$$f - \alpha g_k = f - \frac{2|f|}{|g_k|} g_k = f - (-2|f|) = f + 2|f| = |f| > 0.$$

<u>On $V^+ \cap [\![f > 0]\!] \cap [\![g_k < 0]\!]$</u> : Just take $\alpha = 1$ to get $f - g_k > 0$.

<u>On $V^+ \cap [\![f > 0]\!] \cap [\![g_k > 0]\!]$</u> : Set $\alpha = \dfrac{|f|}{2|g_k|}$; then

$$f - \alpha g_k = f - \frac{|f|}{2|g_k|} g_k = f - \frac{|f|}{2} = \frac{f}{2} > 0.$$

(3) On V^-, $\alpha > 0$ must be chosen such that we have $u < 0$, whence $[\![u > 0]\!] \cap V^- = \emptyset$, as needed; as above, we give its value on each clopen of the partition of V^- appearing in (I).

<u>On $V^- \cap [\![f > 0]\!]$</u> : Recalling that this clopen is contained in $[\![g_k > 0]\!]$, set $\alpha = \dfrac{2|f|}{|g_k|}$. Then,

$$f - \alpha g_k = f - 2f \frac{g_k}{|g_k|} = f - 2f = -f < 0.$$

<u>On $V^- \cap [\![f < 0]\!] \cap [\![g_k > 0]\!]$</u> : Set $\alpha = 1$, to get $f - g_k < 0$.

<u>On $V^- \cap [\![f < 0]\!] \cap [\![g_k < 0]\!]$</u> : Here, take $\alpha = \dfrac{|f|}{2|g_k|}$; then

$$f - \alpha g_k = f - \frac{|f|}{2} \frac{g_k}{|g_k|} = f + \frac{|f|}{2} = \frac{f}{2} < 0,$$

completing the construction of α and the proof of (c).

d) Let $\varphi = \langle g_1, \ldots, g_n \rangle$ be a form over $\mathbb{C}(Y)^\times$. By 1.15.(c), it suffices to check that $D_v^Y(\varphi) \subseteq \mathfrak{D}^Y(\varphi)$. For $f \in D_v^Y(\varphi)$ and $1 \leq k \leq n$, let

$$V^+ = \textstyle\bigcap_{j \neq k} [\![g_j > 0]\!] \quad \text{and} \quad V^- = \textstyle\bigcap_{j \neq k} [\![g_j < 0]\!].$$

Since $f = \sum_{j=1}^n x_i g_i$, with $x_i \geq 0$ on all of Y, in $[\![f < 0]\!] \cap V^+$, g_k cannot be positive, and so this set must be contained in $[\![g_k < 0]\!]$; similarly, we have $[\![f > 0]\!] \cap V^- \subseteq [\![g_k > 0]\!]$. Consequently, item (c) yields $\alpha \in \mathbb{C}(Y)^{\times 2}$ such that

$$u = f - \alpha g_k \in \mathbb{C}(Y)^\times, \text{ with } [\![u > 0]\!] \cap V^- = \emptyset = [\![u < 0]\!] \cap V^+. \quad \text{(II)}$$

The first equation in (II) entails $f \in D_{tv}^Y(g_k, u)$, while the last equalities in (II) and item (a) guarantee that $u \in D_{tv}^Y(g_1, \ldots, \check{g}_k, \ldots, g_n)$. Since k is arbitrary in $\{1, \ldots, n\}$, we obtain $f \in \mathfrak{D}^Y(\varphi)$. ◇

Remark 2.6 Note that 2.5.(a) and (b) hold for **any** q-subgroup of $\mathbb{C}(Y)$, with the same proof. ◇

Corollary 2.7 *Let Y be a topological space and let S be a q-subgroup of $\mathbb{C}(Y)$.*

a) With notation as above and for $f, g \in S$, the following are equivalent:

(1) $f \in D^S(1, g)$; (2) $\mathfrak{s}(f) \in D^S(1, \mathfrak{s}(g))$;

(3) $[\![f < 0]\!] \subseteq [\![g < 0]\!]$; (4) *There are* $h_1, h_2 \in \mathbb{C}(Y)^{\times 2}$ *such that* $f = h_1 + h_2 g$.

b) $G(S)$ *is a reduced pre-special group.*

Proof. As registered in 2.6, items 2.5.(a) and (b) hold true for all q-subgroups of $\mathbb{C}(Y)$.

a) Since a function is congruent to its sign modulo invertible squares (2.3.(c)), 1.4.(c) entails (1) $\Leftrightarrow$ (2), while (4) $\Rightarrow$ (1) is obvious. The remaining equivalences are immediate consequences of 2.5. For instance, (1) $\Rightarrow$ (3) is clear (the argument is the same as that in the proof of 2.5.(b)) and, since $[\![1 > 0]\!] = Y$, $[\![f > 0]\!] \cap ([\![1 < 0]\!] \cap [\![g < 0]\!]) = \emptyset$, while

$$[\![f < 0]\!] \cap ([\![1 > 0]\!] \cap [\![g > 0]\!]) = [\![f < 0]\!] \cap [\![g > 0]\!] = \emptyset \quad \Leftrightarrow \quad [\![f < 0]\!] \subseteq [\![g < 0]\!],$$

whence, 2.5.(a) yields (3) $\Rightarrow$ (4).

b) By condition (4) above, $\equiv^S$ is 2-transversal (see 1.6.(b)) and so, by this same result, $G(S)$ is a pre-special group, i.e., verifies axiom [SG 4]. ◊

Proposition 2.8 *a)* (3-cancellation for $\mathbb{C}$(Y)) *Let* Y *be a topological space and let* S *be a* q*-subgroup of* $\mathbb{C}(Y)$. *With notation as in* 1.3 *and* 1.17, $G(S)$ *verifies* $[\mathrm{FQ3}]_3$.

b) $G(Y) = \mathbb{C}(Y)^{\times}/\mathbb{C}(Y)^{\times 2}$ *is a RSG, whose representation faithfully reflect value representation of diagonal quadratic forms with unit coefficients over* $\mathbb{C}(Y)$.

Proof. Item (b) is an immediate consequence of (a) and Theorems 1.17.(b), 1.20.(c) and 2.5.(a). To establish (a), we first register the following

Fact 2.9 For $x, y, u, v \in \mathbb{C}(Y)^{\times}$, $\overline{x}\,\overline{y} = \overline{u}\,\overline{v} \Rightarrow [\![x < 0]\!] \triangle [\![y < 0]\!] = [\![u < 0]\!] \triangle [\![v < 0]\!]$, *where* $\triangle$ *is symmetric difference.*

Proof. Clearly, if $f, g \in \mathbb{C}(Y)^{\times}$ are in the same square class, then $[\![f < 0]\!] = [\![g < 0]\!]$. Since for $h_1, h_2 \in \mathbb{C}(Y)^{\times}$ we have $[\![h_1 h_2 < 0]\!] = [\![h_1 < 0]\!] \triangle [\![h_2 < 0]\!]$, the Fact's conclusion follows immediately.

By Remark 1.19, it is enough to show that

$$q(1, u, v) \approx q(1, x, y) \quad \Rightarrow \quad q(u, v) \approx q(x, y), \tag{I}$$

or equivalently, by 1.7, that $\langle \overline{u}, \overline{v} \rangle \equiv^S \langle \overline{x}, \overline{y} \rangle$. Since the hypothesis in (I) implies $\overline{u}\,\overline{v} = \overline{x}\,\overline{y}$, Fact 2.9 entails

$$[\![u < 0]\!] \triangle [\![v < 0]\!] = [\![x < 0]\!] \triangle [\![y < 0]\!]. \tag{II}$$

The hypothesis in (I) yields $M, N \in GL_3(\mathbb{C}(Y))$ such that

$$M\mathcal{M}(1, u, v)M^t = \mathcal{M}(1, x, y) \quad \text{and} \quad N\mathcal{M}(1, x, y)N^t = \mathcal{M}(1, u, v). \tag{III}$$

Hence, there $t_1, t_2, t_3 \in \mathbb{C}(Y)$ such that $u = t_1^2 + t_2^2 x + t_3^2 y$; whence, $[\![u < 0]\!] \subseteq [\![x < 0]\!] \cup [\![y < 0]\!]$. Similarly, $[\![v < 0]\!] \subseteq [\![x < 0]\!] \cup [\![y < 0]\!]$, and so $[\![u < 0]\!] \cup [\![v < 0]\!] \subseteq [\![x < 0]\!] \cup [\![y < 0]\!]$. Symmetry in (III) will then imply

$$[\![u < 0]\!] \cup [\![v < 0]\!] = [\![x < 0]\!] \cup [\![y < 0]\!]. \tag{IV}$$

Equalities (II) and (IV) yield

$$[\![u < 0]\!] \cap [\![v < 0]\!] = [\![x < 0]\!] \cap [\![y < 0]\!]. \tag{V}$$

From (V), we conclude that

$$[\![u > 0]\!] \cap ([\![x < 0]\!] \cap [\![y < 0]\!]) = \emptyset. \tag{VI}$$

Equality (IV) also yields, taking complements and recalling that $u, v, x, y \in S \subseteq \mathbb{C}(Y)^{\times}$:

$$[\![u > 0]\!] \cap [\![v > 0]\!] = [\![x > 0]\!] \cap [\![y > 0]\!],$$

wherefrom we obtain $[\![u < 0]\!] \cap ([\![x > 0]\!] \cap [\![y > 0]\!]) = \emptyset$. This last relation, (VI) and 2.5.(a) guarantee that $u \in D^S(x, y)$. Now, $\overline{u}\,\overline{v} = \overline{x}\,\overline{y}$, together with Lemma 1.4.(g) and the definition of $\equiv^S$ (see ($\equiv^S$), right before the statement of 1.6) entail $\langle \overline{u}, \overline{v} \rangle \equiv^S \langle \overline{x}, \overline{y} \rangle$, as needed. ◊

2.10 Observation. By a gluing argument which we omit, it can be proved that the ring $\mathbb{C}(Y)$ veri [FQ 3], Witt cancellation for matrix isometry of forms *of arbitrary dimension*. Hence, by Theorem 1.20 (the special group $G(Y)$ faithfully reflects matrix isometry in $\mathbb{C}(Y)$ as well.

Our next order of business is to describe the reduced special group $G(Y)$. To this end we recall so basic facts concerning Boolean algebras (BAs) and their relationship with reduced special groups. T comments below apply, in particular to the BA of clopens of a topological space.

2.11 Boolean Algebras and Special Groups. a) In the sequel we assume basic knowledge of Boole algebras (BA), in particular Stone's Representation Theorem. The Boolean algebraic operations denoted by $\wedge$ (meet), $\vee$ (join), while complement of an element a is written $-a$. Symmetric differe is defined by $a \bigtriangleup b = (a \wedge -b) \vee (b \wedge -a)$. $\top$ (top) and $\bot$ (bottom) denote the largest and small elements in the partial order $\leq$ of a BA. Recall that the set of clopens in a topological space form a under the set-theoretic operations.

b) The group of exponent 2, $\langle\, B, \bigtriangleup, \bot \,\rangle$, underlies a natural structure of reduced special group (RSG) (details, see section 1 of Chapter 4 of [DM2]). In this case, $1 = \bot$ and $-1 = \top$, and for all a, $b \in B$,

$$a \in D_B(1, b) \quad \text{iff} \quad a \leq b, \qquad (D$$

where $\leq$ is the partial order in B. When B is considered as a RSG, we write $B = \langle\, B, \equiv_B, -1 \,\rangle$.

2.12 The Boolean Hull of a Reduced Special Group. The basic reference for this topic is sectio of Chapter 4, [DM2]. Let $G = \langle\, G, \equiv_G, -1 \,\rangle$ be a reduced special group (RSG), with binary representati denoted by $D_G(\cdot, \cdot)$.

a) Let B_G the BA of clopens of the Boolean space X_G of $\{\pm 1\}$ special group characters of G, and f each $g \in G$, let $\varepsilon_G(g) \;=\; \{\sigma \in X_G : \sigma(g) = -1\}$, a sub-basic clopen in X_G.

b) The Boolean algebra B_G and the map $\varepsilon_G : G \longrightarrow B_G$ have the following properties:

(1) ε_G is an injective group morphism from G to $\langle\, B_G, \bigtriangleup, \bot \,\rangle$, taking -1 in G to $\top$ in B. Hence, f x_1, $x_2 \in G$, we have $\varepsilon_G(x_1 x_2) = \varepsilon_G(x_1) \bigtriangleup \varepsilon_G(x_2)$;

(2) ε_G is an **embedding of RSGs**, that is for all x, $y \in G$, $\;x \in D_G(1, y) \;\Leftrightarrow\; \varepsilon_G(x) \subseteq \varepsilon_G(y)$.

(3) B_G is generated by Im ε_G as a lattice : for every $b \in B_G$, there is a family $\{F_i : 1 \leq i \leq n\}$ of fini subsets of G such that $b \;=\; \bigvee_{1 \leq i \leq n} \bigwedge_{a \in F_i} \varepsilon_G(a)$. The diagram $\varepsilon_G : G \longrightarrow B_G$ is called the **Boolea hull of G**.

Let Y be a topological space and $B(Y)$ be its BA of clopens. Define a map

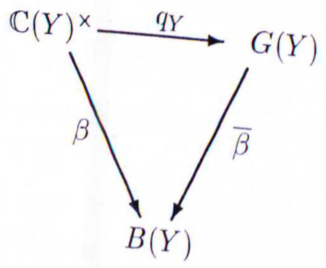
$$\beta : \mathbb{C}(Y)^{\times} \longrightarrow B(Y), \quad \text{given by} \quad \beta(f) \;=\; [\![f < 0]\!]. \qquad (\beta$$

By 2.2.(b), $[\![f < 0]\!]$ is indeed in $B(Y)$. We now have

Proposition 2.13 *a) β is a surjective group morphism, taking $-1 \in \mathbb{C}(Y)^{\times}$ to $Y \in B(Y)$, whose kern is $\mathbb{C}(Y)^{\times 2}$.*

b) For all f, $g \in \mathbb{C}(Y)^{\times}$, $\quad f \in D^Y(1, g) \;\Leftrightarrow\; \beta(f) \in D_{B(Y)}(1,\ \beta(g))$.

c) There is a unique isomorphism of RSGs, $G(Y) \xrightarrow{\overline{\beta}} B(Y)$, making the following diagram commutativ where q_Y is the canonical quotient morphism:



roof. a) Clearly, β takes 1 to $\emptyset$ ($= \bot$) $\in B(Y)$, and $\beta(-1) = Y$ ($= \top$) $\in B(Y)$. To see that β is onto, st observe that if U is a clopen in Y, the map defined by

$$h(x) \;=\; \begin{cases} -1 & \text{if } x \in U; \\ 1 & \text{if } x \notin U, \end{cases}$$

a unit in $\mathbb{C}(Y)$, with $\beta(h) = [\![h < 0]\!] = U$. Since $\beta(fg) \;=\; [\![fg < 0]\!] \;=\; [\![f < 0]\!] \,\triangle\, [\![g < 0]\!]$, we see at β is a group morphism. To finish the proof of (a), note that for all $f \in \mathbb{C}(Y)^{\times}$,

$$\beta(f) \;=\; [\![f < 0]\!] \;=\; \emptyset \quad\Leftrightarrow\quad [\![f > 0]\!] \;=\; Y \quad\Leftrightarrow\quad f \in \mathbb{C}(Y)^{\times 2},$$

e., $\ker \beta = \mathbb{C}(Y)^{\times 2}$, as needed.

By 2.11.(b) and the equivalence between (1) and (3) in 2.7.(a), we have, for $f, g \in \mathbb{C}(Y)^{\times}$:

$$f \in D^{Y}(1, g) \quad\Leftrightarrow\quad \beta(f) = [\![f < 0]\!] \subseteq [\![g < 0]\!] = \beta(g) \quad\Leftrightarrow\quad \beta(f) \in D_{B(Y)}(1,\ \beta(g)).$$

Since β is onto, with kernel $\mathbb{C}(Y)^{\times 2}$, there is a unique group morphism,

$$\overline{\beta} : G(Y) \;=\; \mathbb{C}(Y)^{\times}/\mathbb{C}(Y)^{\times 2} \;\longrightarrow\; B(Y),$$

aking the displayed diagram commutative. Since representation sets are saturated with respect to uare classes (as observed right after the statement of Lemma 1.4), items (a) an (b) above imply that $\overline{\beta}$ an isomorphism of reduced special groups, completing the proof. $\diamondsuit$

orollary 2.14 *$G(Y)$ is a* [SMC]*-group and so verifies Marshall's signature conjecture and the mod 2 -theory of $G(Y)$ satisfies Milnor's Witt ring conjecture.*

roof. This follows from Theorem 2.9 in [DM6] and Lemma 1.2 in [DM7]. $\diamondsuit$

emark 2.15 In Theorem 4.12 of [DM7] it is shown that if R is a ring with many units in which 2 is a it and every residue field has at least 7 elements, then Milnor's mod 2 K-theory of R is *isomorphic* to e K-theory of the special group naturally associated to R. In general, $\mathbb{C}(Y)$ *is not* a ring with many its (Proposition 2.14, [DM5]) and so we do not know if the K-theory of $G(Y)$ is naturally isomorphic the ring-theoretic mod 2 K-theory of $\mathbb{C}(Y)$, an interesting question in its own right. $\diamondsuit$

For later use, we describe the basic properties of the map β in 2.13, starting with the following

oposition 2.16 *Let Y be a topological space and let S be a q-subgroup of $\mathbb{C}(Y)$. Let $L(S)$ be the blattice of $\mathbb{C}(Y)^{\times}$ generated by S. Then,*

$h \in L(S)$ and $k \in \mathbb{C}(Y)^{\times 2} \;\Rightarrow\; kh \in L(S)$.

For all $f \in \mathbb{C}(Y)^{\times}$, $f \in L(S) \;\Leftrightarrow\; \mathfrak{s}(f) \in L(S)$, where $\mathfrak{s}(f) = \dfrac{f}{|f|}$ is the sign of f (2.3.(c)).

roof. a) By induction on $n \geq 0$ we construct a double chain of subsets of $\mathbb{C}(Y)^{\times}$, $\{\Sigma_n : n \geq 0\}$ and $\Delta_n : n \geq 0\}$, as follows:

(1) $\Sigma_0 \;=\; \Delta_0 \;=\; S$; (2) $\Delta_{n+1} \;=\; \{\bigwedge F : F$ is a finite subset of $\Sigma_n\}$;

(3) $\Sigma_{n+1} \;=\; \{\bigvee G : G$ is a finite subset of $\Delta_n\}$.

aking F (in (2)) and G (in (3)) to be singletons, it follows that for all $n \geq 0$,

$$\Sigma_n \;\subseteq\; \Delta_{n+1} \;\subseteq\; \Sigma_{n+2} \quad \text{and} \quad \Delta_n \;\subseteq\; \Sigma_{n+1} \;\subseteq\; \Delta_{n+2}.$$

et $K \;=\; \bigcup_{n \geq 0} \Delta_n \;=\; \bigcup_{n \geq 0} \Sigma_n$. Clearly, $S \subseteq K$ and K is closed under finite joins and meets. oreover, induction on $n \geq 0$ shows that any sublattice of $\mathbb{C}(Y)^{\times}$ including S must contain $\Delta_n \cup \Sigma_n$, r all $n \geq 0$ and thus, also K, establishing that $K \;=\; L(S)$. Hence, to finish the proof, it suffices to rify that for all $n \geq 0$,

$$\begin{cases} k \in \mathbb{C}(Y)^{\times 2} \text{ and } h \in \Delta_n \;\Rightarrow\; kh \in \Delta_n; \\ k \in \mathbb{C}(Y)^{\times 2} \text{ and } h \in \Sigma_n \;\Rightarrow\; kh \in \Sigma_n, \end{cases} \tag{I}$$

be proved by induction on $n \geq 0$. Since $\Delta_0 = \Sigma_0 = S$ and $S \supseteq \mathbb{C}(Y)^{\times 2}$, (I) holds for $n = 0$. Assume true for $n \geq 0$ and let $h \in \Delta_{n+1}$. Then, there are $f_1, \ldots, f_m$ in Σ_n such that $h \;=\; f_1 \wedge \ldots \wedge f_m$. nce, $k(y) > 0$ for all $y \in Y$, Lemma 2.3.(a).(1) guarantees that

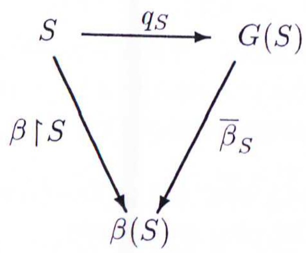
$$k(y)h(y) \;=\; k(y)f_1(y) \wedge \ldots \wedge k(y)f_m(y),$$

for all $y \in Y$, entailing, $kh = kf_1 \wedge \cdots \wedge kf_m$. Now the induction hypothesis implies $kf_i \in \Sigma_n$ a so $kh \in \Delta_{n+1}$. A similar argument, using 2.3.(a).(2) and the induction hypothesis shows that $h \in \Sigma_n$ implies $hk \in \Sigma_{n+1}$, as needed to prove (a). Since for all $f \in \mathbb{C}(Y)^{\times}$, we have $|f|$, $\dfrac{1}{|f|} \in \mathbb{C}(Y)^{\times 2}$, item is an immediate consequence of (a).

Proposition 2.17 *With notation as in 2.13, let Y be a topological space and let S, T be q-subgroups $\mathbb{C}(Y)$. Let X be the Stone space of the BA of clopens of Y, $B(Y) =_{def} B$.*

a) For all $s, t \in \mathbb{C}(Y)^{\times}$, $\beta(s) = \beta(t) \;\Leftrightarrow\; st \in \mathbb{C}(Y)^{\times 2}$.

b) $\beta(S)$, with the isometry induced by B, is a reduced pre-special group. Moreover, there is an isomorphi of pre-special groups, $\overline{\beta}_S : \langle G(S), \equiv^S, -1 \rangle \longrightarrow \beta(S)$, making the following diagram commutative, whe $q_S : S \longrightarrow G(S)$ is the canonical quotient map:



c) $\beta(S) \subseteq \beta(T) \;\Leftrightarrow\; S \subseteq T$.

d) S is a sublattice of $\mathbb{C}(Y)^{\times}$ $\Leftrightarrow$ $\beta(S)$ is a Boolean sub-algebra of B.

e) If A is a sub-algebra of $C(Y)^{\times}$, then $\beta(A^{\times})$ is a Boolean sub-algebra of B.

f) Let $L(S)$ be the sublattice of $\mathbb{C}(Y)^{\times}$ generated by S. Then, $L(S) = \mathbb{C}(Y)^{\times} \;\Leftrightarrow\; L(S)$ is dense $\mathbb{C}(Y)$ in the topology of uniform convergence.

Proof. a) Since $\ker \beta = \mathbb{C}(Y)^{\times 2}$ (2.13.(a)), ($\Leftarrow$) is clear. For the converse, assume that $\beta(s) = [\![s <$ $= [\![t < 0]\!] = \beta(t)$; because $s, t \in \mathbb{C}(Y)^{\times}$, this entails $[\![s > 0]\!] = [\![t > 0]\!]$, and $st \in C(Y)^{\times 2}$, as claimed.

b) By 2.13.(a), $\beta : \mathbb{C}(Y)^{\times} \longrightarrow B$ is a group morphism (the group operation in B being symmetr difference, Δ), taking -1 to X ($= \top$ in B). Hence, if S is a q-subgroup of $\mathbb{C}(Y)$, $\beta(S)$ is a subgroup $\langle B, \Delta, \emptyset \rangle$, containing $\top$, and so, with the binary isometry and representation induced by B, a reduce pre-special subgroup of B. Since $\ker \beta = \mathbb{C}(Y)^{\times 2} \subseteq S$, it follows that $\beta\lceil S : S \longrightarrow \beta(S)$ facto uniquely through $G(S)$, to yield a group isomorphism, $\overline{\beta}_S : G(S) \longrightarrow \beta(S)$, taking -1 to -1 and makin the displayed diagram commutative. Since binary representation in S is that induced by $\mathbb{C}(Y)^{\times}$, th argument for 2.13.(c) (using 2.13.(b)) shows that $\overline{\beta}_S$ is an isomorphism of reduced pre-special groups.

c) The implication ($\Leftarrow$) being clear, we shall only verify ($\Rightarrow$). Given $s \in S$, there is $t \in T$ such tha $\beta(s) = [\![s < 0]\!] = [\![t < 0]\!] = \beta(t)$. By (a) we get $st = h \in C(Y)^{\times 2} \subseteq T$, and so $s = \dfrac{h}{t} \in T$.

d) ($\Rightarrow$) : Since $\beta(S)$ is a subgroup of B (under Δ) and closed under complements ($-\beta(s) = \beta(-s)$), suffices to check that if S is a sublattice of $\mathbb{C}(Y)^{\times}$, then $\beta(S)$ is closed under finite meets [1]. For $s, t \in$ note that $\beta(s \vee t) = [\![(s \vee t) < 0]\!] = [\![s < 0]\!] \cap [\![t < 0]\!] = \beta(s) \cap \beta(t)$, as needed.

($\Leftarrow$) : By 2.3.(a), it is enough to show that S is closed under finite joins. If $s, t \in S$, just as above we have $\beta(s \vee t) = \beta(s) \wedge \beta(t) \in \beta(S)$, and so there is $z \in S$ such that $\beta(z) = \beta(s \vee t)$. By (a $z(s \vee t) = h \in \mathbb{C}(Y)^{\times 2} \subseteq S$, entailing $s \vee t = \dfrac{h}{z} \in S$. Item (e) is immediate from (d) and 2.3.(b).

f) It suffices to prove ($\Leftarrow$); for $g \in \mathbb{C}(Y)^{\times}$, let $\mathfrak{s}(g) = \dfrac{g}{|g|}$ be the sign of g (as in 2.3.(c)). Then, $\mathfrak{s}(g)$ take on only the values ± 1, and for all $y \in Y$,

$$\mathfrak{s}(g)(y) = -1 \;\text{ iff }\; y \in [\![g < 0]\!] \quad \text{and} \quad \mathfrak{s}(g)(y) = 1 \;\text{ iff }\; y \in [\![g > 0]\!].$$

[1] Recall that in a BA, $x \vee y = (x \,\Delta\, y) \,\Delta\, (x \wedge y)$.

ince $L(S)$ is dense in $\mathbb{C}(Y)$, there is $h \in L(S)$ such that $\sup_{y \in Y} |\mathfrak{s}(g) - h| < \frac{1}{2}$. Hence, h and $\mathfrak{s}(g)$ have he same sign on all of Y, that is, $\mathfrak{s}(h) = \mathfrak{s}(g)$. Since, $h \in L(S)$, 2.16.(b) entails $\mathfrak{s}(g) \in L(S)$, that in turn ields (again by 2.16.(b)) $g \in L(S)$, as needed. ◇

Representable $\mathbb{Q}$-Algebras

ecall that a ring R is a $\mathbb{Q}$-algebra if there is a unitary ring morphism from $\mathbb{Q}$ to R; since any such orphism is injective, we may identify $\mathbb{Q}$ with a subfield of R and all strictly positive rationals are unit ums of squares in R.

efinition 3.1 *Let Y be a topological space. A $\mathbb{Q}$-algebra R is* **representable in $\mathbb{C}(Y)$ modulo sums f squares** *or simply* **representable in $\mathbb{C}(Y)$**, *if there is a $\mathbb{Q}$-algebra morphism, $R \xrightarrow{\varphi} \mathbb{C}(Y)$, such that*

[RQA 1] : *The image of φ is dense in $\mathbb{C}(Y)$ in the topology of uniform convergence;*

[RQA 2] : *For all $a \in R$, $[\![\varphi(a) > 0]\!] = Y \quad \Leftrightarrow \quad a \in (\Sigma R^2)^{\times} = R^{\times} \cap \Sigma R^2$.*

he $\mathbb{Q}$-algebra morphism φ is called a **representation of R in $\mathbb{C}(Y)$**.

emark 3.2 Definition 3.1 originates in (and generalizes) the proof of Thm. 2.2.9 and the statement of hm. 2.2.10 (pp. 28 and 29) in [Be]; as will be shown below, the real holomorphy ring of a formally real ld is a representable $\mathbb{Q}$-algebra (Lemma 3.3) and some of the important properties of real holomorphy ngs proved in [Be] in fact hold true in this general context (Lemma 3.4). ◇

emma 3.3 *If K is a formally real field and $M(K)$ is its space of real places, then the real holomorphy ng of K, $H(K)$, is a $\mathbb{Q}$-algebra, representable in $\mathbb{C}(M(K))$.*

roof. Let $H = H(K)$ be the real holomorphy ring of the formally real field K. It shown in [Be]:

) Thm. *2.1.11* (p. 20). *H is the smallest completely real Prüfer ring $A \subseteq K$, whose fraction field is K.*

) Prop. *2.1.9.(v)* (p. 19). *If A is a completely real Prüfer ring, then for all t, $n \in \mathbb{N}$, $A \cap \sum_{i=1}^{t} K^{2n}$ $= \sum_{i=1}^{t} A^{2n}$. In particular, $A \cap \Sigma K^2 = \Sigma A^2$.*

From (1) and (2) we immediately conclude

) $\Sigma K^2 \cap H = \Sigma H^2$ and $\Sigma K^2 \cap H^{\times} = \Sigma H^2 \cap H^{\times} = (\Sigma H^2)^{\times}$.

et $M(K)$ be the compact Hausdorff space of real places of K (Theorem 2.2.5, p. 26, [Be]). By property) in Theorem 2.2.9 of [Be] (pp. 28-29), H satisfies [RQA 1], while property (*iv*) of that same result and) above entail that H verifies [RQA 2], that is, H is $\mathbb{Q}$-algebra, representable in $\mathbb{C}(M(K))$. ◇

emma 3.4 *Let $\varphi : R \longrightarrow \mathbb{C}(Y)$ be a representation of the $\mathbb{Q}$-algebra R. Then,*

) *For all $a \in R$, $a \in R^{\times} \Leftrightarrow \varphi(a) \in \mathbb{C}(Y)^{\times}$.*

) *For all $a \in R$, $[\![\varphi(a) \geq 0]\!] = Y \Leftrightarrow$ For all rationals $r > 0$, $(a + r) \in (\Sigma R^2)^{\times}$.*

) $\ker \varphi = \{x \in R :$ *For all rationals $r > 0$, $(r \pm x) \in (\Sigma R^2)^{\times}\}$.*

roof. a) The implication ($\Rightarrow$) is clear; for the converse, if $\varphi(a) \in \mathbb{C}(Y)^{\times}$, then $Y = [\![(\varphi(a))^2 > 0]\!]$ and RQA 2] entails $a^2 \in (\Sigma R^2)^{\times}$; in particular, a^2 is a unit in R and hence so is a.

) For $a \in R$, recalling that φ is a $\mathbb{Q}$-algebra morphism, [RQA 2] yields

$$Y = [\![\varphi(a) \geq 0]\!] \quad \Leftrightarrow \quad \forall\, r > 0 \text{ in } \mathbb{Q}, \quad Y = [\![\varphi(a) + r > 0]\!] = [\![\varphi(a + r) > 0]\!]$$
$$\Leftrightarrow \quad \forall\, r > 0 \text{ in } \mathbb{Q}, \quad a + r \in (\Sigma R^2)^{\times}.$$

) If $x \in R$, then $\varphi(x) = 0 \quad \Leftrightarrow \quad Y = [\![\varphi(x) \geq 0]\!] = [\![\varphi(-x) \geq 0]\!]$, whence (c) follows immediately rom (b) applied to $\varphi(x)$ and $-\varphi(x) = \varphi(-x)$. ◇

.5 **Notation and Remarks.** Let $\langle A, P \rangle$ be a p-ring as in Definition 6.1 in [DM7], i.e., A is a unital ommutative ring such that $2 \in A^{\times}$ and P is a preorder of A. If P is a *proper* preorder of A (equivalently, $1 \notin P$), we say that $\langle A, P \rangle$ is a *proper p-ring*. Hence, the only improper p-ring is the pair $\langle A, A \rangle$.

(1) We shall employ the notation and results in Definition 6.3 and of section 8.B in [DM7].

(2) If $\langle A, P\rangle$ is a proper p-ring,

$$G_P(A) = A^\times/P^\times = \{a^P : a \in A^\times\}$$

is a group of exponent two, that by Fact 8.11 in [DM7], underlies a reduced proto special grou Definition 6.3, [DM7]), whose binary isometry is given by

$$\langle a^P, b^P\rangle \equiv_P \langle c^P, d^P\rangle \quad \text{iff} \quad a^P b^P = c^P d^P \text{ and } D_P(a, b) = D_P(c, d),$$

where (see formula (D_T) in 8.9 of [DM7])

$$D_P(a, b) = \{c \in A^\times : \exists\, t_1, t_2 \in P \text{ such that } c = t_1 a + t_2 b\}.$$

(3) If $n \geq 1$ is an integer, as usual, a (non-singular, diagonal) quadratic form over A, φ, is a $\varphi = \langle a_1, \ldots, a_n\rangle$, where $a_j \in A^\times$, $1 \leq j \leq n$. Let $\varphi^P = \langle a_1^P, \ldots, a_n^P\rangle$ be the corresponding r $G_P(A)$. Since $G_P(A)$ is a π-SG, 1.10 also applies here, furnishing an extension of the binary iso $G_P(A)$ to forms of arbitrary dimensions.

(4) If $P = \Sigma A^2$ and $a \in A^\times$, write $G_{red}(A)$ for $G_P(A)$, a^Σ for a^P and $\equiv_{red}$ for isometry in G_{red}

The analogue of Definition 1.13 for p-rings is the following

Definition 3.6 *Let $\langle A, P\rangle$ be a proper p-ring, let $\varphi = \langle b_1, \ldots, b_n\rangle$ be a n-form over A and $\langle b_1^P, \ldots, b_n^P\rangle$ be the corresponding n-form in $G_P(A)$.*

a) $D_P(\varphi) = \{a \in A^\times : \exists\, a_2, \ldots, a_n \in A^\times$ *such that* $\varphi^P \equiv_P \langle a^P, a_2^P, \ldots, a_n^P\rangle\}$
is the set of elements of $A^\times$ ***P-isometry represented (P-iso-represented) by φ.***

b) $D_{Pv}(\varphi) = \{a \in A^\times :$ *There are* $x_1, \ldots, x_n \in P$ *such that* $a = \sum_{i=1}^n x_i b_i\}$,
is the set of elements of $A^\times$ **value-represented mod P (v-represented mod P) by φ.**

c) $D_{Ptv}(\varphi) = \{a \in A^\times :$ *There are* $z_1, \ldots, z_n \in P^\times$ *such that* $a = \sum_{i=1}^n z_i b_i\}$
is the set of elements of $A^\times$ **transversaly v-represented mod P (tv-represented mod P**
Clearly, $D_{Pv}(\varphi) \subseteq D_{Ptv}(\varphi)$.

d)
$$\mathfrak{D}_P(\varphi) = \begin{cases} D_{Pv}(b_1, b_2) & \text{if } n = 2; \\ \bigcap_{k=1}^n \bigcup \{D_{Pv}(b_k, u) : u \in D_{Pv}(b_1, \ldots, \check{b}_k, \ldots, b_n)\} & \text{if } n \geq 3. \end{cases}$$

If there is need of displaying the ring A in which these representation sets are defined, write D D_{Ptv}^A *and* $\mathfrak{D}_P^A$ *for the representation sets defined above. Moreover, if $P = \Sigma A^2$ is a proper preord (i.e., A is semi-real), the corresponding representation sets will be written* D_Σ^A, $D_{\Sigma v}^A$, $D_{\Sigma tv}^A$ *and* $\mathfrak{D}$

Remark 3.7 All of the representation sets defined in 3.6 are *invariant modulo units in P*. Hen may be also seen as subsets of $G_P(A)$. Moreover, the definition of binary isometry in $G_P(A)$ $D_P(s, t) = D_{Pv}(s, t)$, for all $s, t \in A^\times$.

There are analogues of Lemma 1.15 and Theorem 1.16 for the value sets defined in 3.6, whose are a straightforward adaptation of those of 1.15 and 1.16. We state only the result correspon 1.15, that will be used subsequently, omitting the proof.

Lemma 3.8 *Let $\langle A, P\rangle$ be a proper p-ring and let $n \geq 2$ be an integer. Let $\varphi = \langle b_1, \ldots, b_n\rangle$ be over $A^\times$ and let σ be a permutation of $\{1, \ldots, n\}$. Let $\varphi^\sigma = \langle b_{\sigma(1)}, \ldots, b_{\sigma(n)}\rangle$.*

a) (i) $D_{Pv}(\varphi) = D_{Pv}(\varphi^\sigma)$; (ii) $D_{Ptv}(\varphi) = D_{Ptv}(\varphi^\sigma)$

(iii) $\mathfrak{D}_P(\varphi) = \mathfrak{D}_P(\varphi^\sigma)$; (iv) $G_P(A)$ *is a SG* $\Rightarrow D_P(\varphi) = D_P(\varphi^\sigma)$.

b) $D_P(\varphi) \subseteq D_{Pv}(\varphi)$. *c)* $\mathfrak{D}_P(\varphi) \cup D_{Ptv}(\varphi) \subseteq D_{Pv}(\varphi)$.

d) Let $1 \leq k \leq m$ be integers, $\varphi_1, \ldots, \varphi_m$ be forms over $A^\times$ and let $x_j \in D_{Pv}(\varphi_j)$, $1 \leq j \leq k$. T

$$D_{Pv}(x_1, \ldots, x_k) \subseteq D_{Pv}\left(\bigoplus_{j=1}^m \varphi_j\right).$$

In particular, if ψ is a m-form over S, then $D_{Pv}(\varphi) \subseteq D_{Pv}(\varphi \oplus \psi)$.

Theorem 3.9 (Transversality for representable $\mathbb{Q}$-algebras) *Let R be $\mathbb{Q}$-algebra and let X be a compact Hausdorff space. If R is representable in $\mathbb{C}(X)$ and $\varphi = \langle b_1, \ldots, b_n \rangle$ is a n-form over $R^{\times}$, then*

a) $D^R_{\Sigma v}(\varphi) \;=\; D^R_{\Sigma tv}(\varphi)$.   *b)* $D^R_{\Sigma v}(\varphi) \;=\; \mathfrak{D}^R_{\Sigma}(\varphi)$.

Proof. Let $\|f\|_\infty = \sup_{x \in X} |f(x)|$ be the uniform convergence norm in $\mathbb{C}(X)$; recall that for $g, h \in \mathbb{C}(X)$, $\|gh\|_\infty \leq \|g\|_\infty \, \|h\|_\infty$. To simplify exposition, write $a \in R \longmapsto \widehat{a} \in \mathbb{C}(X)$ for a representation of R in $\mathbb{C}(X)$. Let $\widehat{R} = \{\widehat{a} \in \mathbb{C}(X) : a \in R\}$.

a) By Lemma 3.8.(c), it suffices to check that $D^R_{\Sigma v}(\varphi) \subseteq D^R_{\Sigma tv}(\varphi)$. Fix $a \in D^R_{\Sigma v}(\varphi)$; then there are $y_1, \ldots, y_n \in \Sigma R^2$ such that

$$a \;=\; y_1 b_1 + \ldots + y_n b_n. \tag{I}$$

Since representation is a ring morphism, $\widehat{a}, \widehat{b}_1, \ldots, \widehat{b}_n \in \mathbb{C}(X)^{\times}$, $\widehat{y}_j$ is a non-zero sum of squares in $\mathbb{C}(X)$ and (I) entails

$$\widehat{a} \;=\; \widehat{y}_1 \widehat{b}_1 + \ldots + \widehat{y}_n \widehat{b}_n, \tag{II}$$

with $X \;=\; [\![\widehat{y}_j \geq 0]\!]$, $1 \leq j \leq n$; (II) and the fact that all $\widehat{y}_j$ are ≥ 0 on X immediately imply that

$$[\![\widehat{a} > 0]\!] \;\cap\; \textstyle\bigcap_{j=1}^n [\![\widehat{b}_j < 0]\!] \;=\; \emptyset \;=\; [\![\widehat{a} < 0]\!] \;\cap\; \textstyle\bigcap_{j=1}^n [\![\widehat{b}_j > 0]\!].$$

These equalities and Theorem 2.5.(a) yield $f_1, \ldots, f_n \in \mathbb{C}(X)^{\times 2}$ such that

$$\widehat{a} \;=\; f_1 \widehat{b}_1 + \ldots + f_n \widehat{b}_n. \tag{III}$$

Note that:

- Since X is compact and $X = [\![f_j > 0]\!]$, $\forall\, 1 \leq j \leq n$, we have $m_j \;=\; \min\,\{f_j(p) \in \mathbb{R} : p \in X\} \;>\; 0$;
- For all $1 \leq j \leq n$, $\|\widehat{b}_j\|_\infty > 0$;
- Since $\widehat{a} \in \mathbb{C}(X)^{\times}$, $[\![|\widehat{a}| > 0]\!] = X$ and compactness yields $m \;=\; \min\,\{|\widehat{a}(p)| \in \mathbb{R} : p \in X\} \;>\; 0$.

With notation as above, let $\varepsilon \;=\; \min\,\{m, m_1, \ldots, m_n\}$. Since $\widehat{R}$ is a dense subalgebra of $\mathbb{C}(X)$ in the topology of uniform convergence,

$$\text{For each } 1 \leq j \leq n, \text{ there is } c_j \in R \text{ such that } \|\widehat{c}_j \,-\, f_j\|_\infty \;<\; \frac{\varepsilon}{2n(1 + \|\widehat{b}_j\|_\infty)}. \tag{IV}$$

Fact 1. For $1 \leq j \leq n$, $\;c_j \in (\Sigma R^2)^{\times}$.

Proof. Fix $1 \leq j \leq n$; since $0 < \varepsilon \leq m_j = \min_{p \in X} f_j(p)$, (IV) implies $|\widehat{c}_j(p) - f_j(p)| \;<\; \dfrac{m_j}{2}$, for all $p \in X$; hence,

$$\widehat{c}_j(p) \;>\; f_j(p) - \frac{m_j}{2} \;\geq\; f_j(p) - \frac{f_j(p)}{2} \;=\; \frac{f_j(p)}{2} \;>\; 0,$$

and $\widehat{c}_j \in \mathbb{C}(X)^{\times 2}$. Now, property [RQA 2] in Definition 3.1 yields $c_j \in (\Sigma R^2)^{\times}$, as needed.

$\Diamond$ Let $z \in R$ be defined by $z \;=\; c_1 b_1 + \ldots + c_n b_n$. Then,

Fact 2. $az \in (\Sigma R^2)^{\times}$.

Proof. We have $\widehat{z} \;=\; \widehat{c}_1 \widehat{b}_1 + \ldots + \widehat{c}_n \widehat{b}_n$; (III) and (IV) entail

$$\begin{aligned} \|\widehat{a} - \widehat{z}\|_\infty &= \|(f_1 - \widehat{c}_1)\widehat{b}_1 + \ldots + (f_n - \widehat{c}_n)\widehat{b}_n\|_\infty \;\leq\; \textstyle\sum_{j=1}^n \|f_j - \widehat{c}_j\|_\infty \, \|\widehat{b}_j\|_\infty \\ &< \textstyle\sum_{j=1}^n \dfrac{\varepsilon}{2n(1 + \|\widehat{b}_j\|_\infty)} \, \|\widehat{b}_j\|_\infty \;<\; n\, \dfrac{\varepsilon}{2n} \;=\; \dfrac{\varepsilon}{2}. \end{aligned} \tag{V}$$

Since $\dfrac{\varepsilon}{2} < \dfrac{m}{2} = \dfrac{\min_{p \in X} |\widehat{a}(p)|}{2}$, (V) yields

$$[\![\widehat{az} > 0]\!] \;=\; [\![\widehat{a}\,\widehat{z} > 0]\!] \;=\; X. \tag{VI}$$

Indeed, for $p \in X$, (V) implies $|\widehat{a}(p) - \widehat{z}(p)| < \dfrac{\varepsilon}{2} < \dfrac{m}{2}$, that is,

$$\widehat{a}(p) - \frac{m}{2} \;<\; \widehat{z}(p) \;<\; \widehat{a}(p) + \frac{m}{2}.$$

Hence, recalling that $X \;=\; [\![\widehat{a} > 0]\!] \cup [\![\widehat{a} < 0]\!]$:

- If $\widehat{a}(p) > 0$, then $\;\widehat{z}(p) \;>\; \widehat{a}(p) - \dfrac{m}{2} \;\geq\; \widehat{a}(p) - \dfrac{\widehat{a}(p)}{2} \;=\; \dfrac{\widehat{a}(p)}{2} > 0$;

$*$ If $\widehat{a}(p) < 0$, then $\widehat{z}(p) < \widehat{a}(p) + \frac{m}{2} \le \widehat{a}(p) + \frac{|\widehat{a}(p)|}{2} = \frac{\widehat{a}(p)}{2} < 0$,

establishing (VI). Now, (VI) and property [RQA 2] in 3.1 guarantee that $az \in (\Sigma R^2)^\times$, as desired.

From Fact 2 we conclude that $z \in R^\times$ and $\frac{a}{z} = \frac{az}{z^2} \in (\Sigma R^2)^\times$. Now, Fact 1 entails $z \in D^R_{\Sigma tv}(\varphi)$ a so $a = z\,\frac{a}{z} \in D^R_{\Sigma tv}(\varphi)$, ending the proof of (a).

Remark 3.10 As in Theorem 2.5, the proof of (a) above shows that if $x, z_1, \ldots, z_n \in R^\times$, then $[\![\widehat{x} > 0]\!] \cap \bigcap_{j=1}^n [\![\widehat{z}_j < 0]\!] = \emptyset = [\![\widehat{x} < 0]\!] \cap \bigcap_{j=1}^n [\![\widehat{z}_j > 0]\!] \Rightarrow x \in D^R_{\Sigma tv}(z_1, \ldots, z_n)$.

b) By Lemma 3.8.(c) it is enough to check that $D^R_{\Sigma v}(\varphi) \subseteq \mathfrak{D}^R_\Sigma(\varphi)$. Fix $a \in D^R_{\Sigma v}(\varphi)$; just as above, have equality (II)

$$\widehat{a} = \widehat{y}_1\widehat{b}_1 + \ldots + \widehat{y}_n\widehat{b}_n,$$

with $X = [\![\widehat{y}_j \ge 0]\!]$, $1 \le j \le n$. Fix k between 1 and n; the preceding equality (i.e., (II)) then implies

$$[\![\widehat{a} < 0]\!] \cap \bigcap_{j \ne k} [\![\widehat{b}_j > 0]\!] \subseteq [\![\widehat{b}_k < 0]\!] \text{ and } [\![\widehat{a} > 0]\!] \cap \bigcap_{j \ne k} [\![\widehat{b}_j < 0]\!] \subseteq [\![\widehat{b}_k > 0]\!]. \quad \text{(V}$$

By Theorem 2.5.(c) there is $\alpha \in \mathbb{C}(X)^{\times 2}$ such that

(*i*) $u = \widehat{a} - \alpha\widehat{b}_k \in \mathbb{C}(X)^\times$;

(*ii*) $[\![u > 0]\!] \cap \bigcap_{j \ne k} [\![\widehat{b}_j < 0]\!] = \emptyset = [\![u < 0]\!] \cap \bigcap_{j \ne k} [\![\widehat{b}_j > 0]\!]$. (VI

Let $\varepsilon = \min\{\min_{p\in X} \alpha(p), \min_{p \in X} |u(p)|\}$; the density if $\widehat{R}$ in $\mathbb{C}(X)$ yields $\zeta \in R$ such that

$$\|\widehat{\zeta} - \alpha\|_\infty < \frac{\varepsilon}{2(1 + \|\widehat{b}_k\|_\infty)}.$$

Define $w = a - \zeta b_k \in R$; then:

Fact 3. a) $\zeta \in (\Sigma R^2)^\times$.

b) $X = [\![u\widehat{w} > 0]\!]$. *In particular,* $w \in R^\times$ *and* $\begin{cases} [\![\widehat{w} > 0]\!] = [\![u > 0]\!]; \\ [\![\widehat{w} < 0]\!] = [\![u < 0]\!]. \end{cases}$

Proof. a) We have $\|\widehat{\zeta} - \alpha\|_\infty < \frac{\varepsilon}{2} \le \frac{\min_{p \in X} \alpha(p)}{2}$, and so for all $p \in X$

$$\widehat{\zeta}(p) > \alpha(p) - \frac{\alpha(p)}{2} = \frac{\alpha(p)}{2} > 0,$$

showing that $[\![\widehat{\zeta} > 0]\!] = X$. By property [RQA 2] in 3.1, $\zeta \in (\Sigma R^2)^\times$, as claimed.

b) We have

$$\|u - \widehat{w}\|_\infty = \|(\widehat{\zeta} - \alpha)\widehat{b}_k\|_\infty \le \|\widehat{\zeta} - \alpha\|_\infty \|\widehat{b}_k\|_\infty < \frac{\varepsilon\|\widehat{b}_k\|_\infty}{2(1 + \|\widehat{b}_k\|_\infty)} < \frac{\varepsilon}{2} \le \frac{\min_{p\in X} |u(p)|}{2}.$$

Hence, for all $p \in X$, $|u(p) - \widehat{w}(p)| < \frac{\min_{p \in X} |u(p)|}{2} \le \frac{u(p)}{2}$. Now, the same argument used in the proof of Fact 2 will show that $X = [\![u\widehat{w} > 0]\!]$. Since u is a unit in $\mathbb{C}(X)$ and $u\widehat{w} > 0$ on all of X, conclude that

$$[\![\widehat{w} > 0]\!] = [\![u > 0]\!],\ [\![\widehat{w} < 0]\!] = [\![u < 0]\!] \text{ and } \widehat{w} \text{ is a unit in } \mathbb{C}(X).$$

The last assertion and item (a) in Lemma 3.4 entail $w \in R^\times$, ending the proof of Fact 3.

Since $w \in R^\times$, Fact 3.(a) implies that $a \in D^R_{\Sigma tv}(b_k, w)$; moreover, Fact 3.(b), together with (*ii*) (VIII) and Remark 3.10 guarantee that $w \in D^R_{\Sigma tv}(b_1, \ldots, \check{b}_k, \ldots, b_n)$. Since, $1 \le k \le n$ is arbitrary, obtain $a \in \mathfrak{D}^R_\Sigma(\varphi)$, ending the proof.

Corollary 3.11 *Let R be a $\mathbb{Q}$-algebra, representable in $\mathbb{C}(X)$, where X is a compact Hausdorff spac Let $B(X)$ be the BA of clopens in X. Then, with notation as in the proof of Theorem 3.9,*

a) For $u, w \in R^\times$, the following are equivalent:

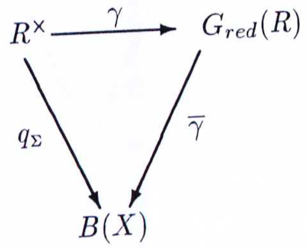
(1) $u \in D^R_\Sigma(1, w)$; (2) $u \in D^R_{\Sigma tv}(1, w)$; (3) $[\![\widehat{u} < 0]\!] \subseteq [\![\widehat{w} < 0]\!]$.

$G_{red}(R)$ *is a reduced pre-special group.*

Let $\gamma : R^{\times} \longrightarrow B(X)$ *be defined by* $\gamma(u) = [\![\widehat{u} < 0]\!]$. *Then:*

(1) γ *is a surjective group homomorphism, taking* -1 *to* X, *whose kernel is* $(\Sigma R^2)^{\times}$;

(2) γ *factors uniquely through* $G_{red}(R)$, *yielding an isomorphism of pre-special groups,* $\overline{\gamma} : G_{red}(R) \longrightarrow B(X)$, *making the following diagram commutative:*



where $q_{\Sigma} : R^{\times} \longrightarrow G_{red}(R) = R^{\times}/(\Sigma R^2)^{\times}$ *is the canonical quotient morphism.*

$G_{red}(R)$ *is a* [SMC]*-special group, verifying Marshall's signature conjecture and whose* K*-theory satis-*
es Milnor's Witt ring conjecture.

roof. The proofs of (a) and (b) are a straightforward adaptation of those of items (a) and (b) of 2.7. to $\widehat{R}$ with the

(1) Since γ is the composition of the restriction of the map β of Proposition 2.13 to $\widehat{R}$ with the
striction of the representation morphism of R into $\mathbb{C}(X)$, it is clear that it is a group morphism, taking
in $R^{\times}$ to $X \in B(X)$. For $u \in R^{\times}$, we have

$$\gamma(u) = \emptyset \quad \Leftrightarrow \quad X = [\![\widehat{u} > 0]\!],$$

d property [RQA 2] in Definition 3.1 insures that $u \in (\Sigma R^2)^{\times}$. It remains to check that γ is onto $B(X)$.
t V be a clopen set in X and let $\chi_V : X \longrightarrow \mathbb{R}$ be given by

$$\chi_V(p) = \begin{cases} -1 & \text{if } p \in V; \\ 1 & \text{if } p \notin V. \end{cases}$$

hen, $\chi_V \in \mathbb{C}(X)^{\times}$ and the density of $\widehat{R}$ in $\mathbb{C}(X)$ yields $v \in R$ such that $\|\widehat{v} - \chi_V\|_{\infty} < \frac{1}{2}$. It is then clear
at $[\![\widehat{v} < 0]\!] = V$, $[\![\widehat{v} > 0]\!] = X \setminus V$; hence, $\widehat{v} \in \mathbb{C}(X)^{\times}$ and so Lemma 3.4.(a) implies that $v \in R^{\times}$, with
$(v) = V$, as needed.

) Clearly, there is a unique bijective group morphism, $\overline{\gamma} : G_{red}(R) \longrightarrow B(X)$, making the displayed
agram commute, with $\overline{\gamma}(-1) = X$. It follows immediately from (a) and 2.11.(b) that for $u, w \in R^{\times}$,
$\in D^R_{\Sigma}(1, w) \quad \Leftrightarrow \quad \gamma(u) \in D_{B(X)}(1, \gamma(w))$, and $\overline{\gamma}$ is an isomorphism of pre-special groups. Item (d)
an immediate consequence of (c), Theorem 2.9 in [DM6] and Lemma 1.2 in [DM7]. ◊

emark 3.12 By Lemma 3.3, Theorem 3.9 and Corollary 3.11 apply, *ipsis litteris*, to the real holomor-
y ring of any formally real field. ◊

Representation of Reduced Special Groups by Algebras of Continuous Functions

ur aim is to represent any reduced special group in the form

$$G(S) = \langle G(S), \equiv^S, -1 \rangle,$$

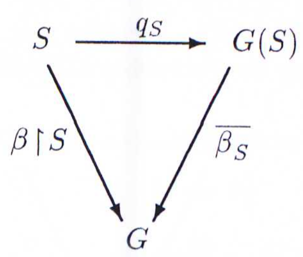
here S is a q-subgroup of $\mathbb{C}(X)$, with X a Boolean space. Notation is as in section 1.

heorem 4.1 *Let* $G = \langle G, \equiv_G, -1 \rangle$ *be a reduced special group. Then, there is a Boolean space* X *and*
q-subgroup, S, *of* $\mathbb{C}(X)$ *with the following properties, where* $f \in \mathbb{C}(X)^{\times} \longmapsto \beta(f) = [\![f < 0]\!] \in B(X)$
the map defined just before the statement of 2.13:

) *The restriction of* β *to* S, $\beta\lceil S$, *is a surjective group morphism,* $\beta\lceil S : S \longrightarrow G$, *taking* -1 *in* S *to*
1 in G *and whose kernel is* $\mathbb{C}(X)^{\times 2}$;

(2) *For all* $f, g \in S$, $\; f \in D^S(1,g) \;\Leftrightarrow\; \beta(f) \in D_G(1, \beta(g))$;

(3) *If* $G(S) = S/\mathbb{C}(X)^{\times 2}$, *there is* unique *isomorphism of reduced pre-special groups,* $\overline{\beta_S}$, *making t following diagram commutative:*



In particular, $G(S)$ *is a reduced special group, isomorphic to* G;

(4) *If* T *is a q-subgroup of* $\mathbb{C}(X)$ *such that* $\beta(T) = G$ [2], *then* $T = S$.

(5) *If* $L(S)$ *is the sublattice of* $\mathbb{C}(X)^{\times}$ *generated by* S, *then* $L(S) \;=\; \mathbb{C}(X)^{\times}$.

Proof. Since G will remain fixed throughout the proof, write $\varepsilon : G \longrightarrow B$ for the Boolean hull of (instead of $\varepsilon_G : G \longrightarrow B_G$, as in 2.12.(b)). Let X be the Stone space of B (a Boolean space) and, before, let $B(X)$ be the BA of clopens in X. By Stone duality and 2.12.(b), there is a SG-embeddi $a \in G \longmapsto \varepsilon(a) \in B(X) = B$. Thus, without loss of generality, we may forthwith identify G with t reduced special subgroup $\{\varepsilon(a) \in B(X) : a \in G\}$ of clopens in X.

For each $a \in G$, set $S_a \;=\; \{f \in \mathbb{C}(X)^{\times} : [\![f < 0]\!] = \varepsilon(a)\}$; since $\varepsilon(-a) = -\varepsilon(a)$, it follows that f all $f \in S_a$,

$$[\![f > 0]\!] \;=\; X \setminus \varepsilon(a) \;=\; \varepsilon(-a).$$

Moreover,

a) Since $\varepsilon(1) = \emptyset$, we have, by 2.2.(c),

$$S_1 \;=\; \{f \in \mathbb{C}(X)^{\times} : [\![f < 0]\!] = \emptyset\} \;=\; \{f \in \mathbb{C}(X)^{\times} : [\![f > 0]\!] = X\} \;=\; \mathbb{C}(X)^{\times 2}.$$

In particular, all strictly positive maps on X are in S_1.

b) Since $\varepsilon(-1) = X$, we get

$$S_{-1} \;=\; \{f \in \mathbb{C}(X)^{\times} : [\![f < 0]\!] = X\}.$$

In particular, all strictly negative maps on X are is S_{-1}.

Now define

$$S \;=\; \bigcup_{a \in G} S_a.$$

Proof of (1). By (a) and (b) above, we have $\mathbb{C}(X)^{\times 2} \subseteq S$ and $-1 \in S$. To establish that S is q-subgroup of $\mathbb{C}(X)$ it remains to check that it is closed under products. Let $f \in S_a$ and $g \in S_b$, whe $a, b \in G$. Then,

$$[\![fg < 0]\!] \;=\; [\![f < 0]\!] \,\Delta\, [\![g < 0]\!] \;=\; \varepsilon(a) \,\Delta\, \varepsilon(b) \;=\; \varepsilon(ab),$$

and so $fg \in S_{ab} \subseteq S$. Since β is a group morphism, taking -1 to X, with kernel $\mathbb{C}(X)^{\times 2} \subseteq S$, i restriction to S will have these same properties [3]. Furthermore, for each $a \in G$ and $f \in S_a$, we hav $\beta(f) \;=\; [\![f < 0]\!] \;=\; \varepsilon(a)$, showing that $\beta \restriction S$ takes values in G and is surjective.

Proof of (2). By the definition of D^S (see paragraph right after Definition 1.1), for $f, g \in S$, we have

$$f \in D^S(1, g) \;\Leftrightarrow\; D^X(1, g),$$

i.e., binary representation in S is that induced by $\mathbb{C}(X)^{\times}$ (cf. 2.4). Now, (I), equivalence (D_B) in 2.11.(2.13.(b) and 2.12.(b)(2) yield, for $f, g \in S$

$$f \in D^S(1, g) \;\Leftrightarrow\; \beta(f) \;\subseteq\; \beta(g) \;\Leftrightarrow\; \beta(f) \in D_G(1, \beta(g)),$$

as desired. Item (3) follows immediately from (1) and (2), with the same argument used in establishi item (c) in Proposition 2.13, while item (4) is an immediate consequence of 2.17.(c).

[2] Or equivalently, by 2.17.(b), $\overline{\beta}_T : G(T) \longrightarrow G$ is an isomorphism of pre-special groups.

[3] $\ker \beta \restriction S = \ker \beta \cap S = \mathbb{C}(X)^{\times 2} \cap S = \mathbb{C}(X)^{\times 2}$.

Proof of (5). Let $L = L(S)$ be the sub-lattice of $\mathbb{C}(X)$ generated by S. By 2.17.(f), it suffices to show that L is dense in $\mathbb{C}(Y)$. The facts below are tailored to that purpose. Note that 2.3.(b) entails $L \subseteq \mathbb{C}(X)^{\times}$.

For $U \in B(X)$, let χ_U, given by $\chi_U(x) = \begin{cases} -1 & \text{if } x \in U; \\ 1 & \text{if } x \notin U, \end{cases}$ be the characteristic function of U. When $U = \varepsilon(a)$, $a \in G$, we write χ_a in place of $\chi_{\varepsilon(a)}$. We now have,

Fact 4.2 *a) For* $U, V \in B(X)$,

(i) $\chi_U \wedge \chi_V = \chi_{U \cup V}$. (ii) $\chi_U \vee \chi_V = \chi_{U \cap V}$. (iii) $\chi_{X \setminus U} = -\chi_U$.

(iv) $-\chi_{U \cup V} = -\chi_U \vee -\chi_V$. (v) $-\chi_{U \cap V} = -\chi_U \wedge -\chi_V$.

b) For all $a \in G$ *and all* $d \in \mathbb{R}^{\times}$, $d\chi_a \in S$.

c) For all $U \in B(X)$ *and all* $d \in \mathbb{R}^{\times}$, $d\chi_U \in L$.

d) For $d_1, d_2 \in \mathbb{R}^{\times}$ *and* $U \in B(X)$, *let* $f \in \mathbb{C}(X)^{\times}$ *be defined by* $f(x) = \begin{cases} d_1 & \text{if } x \in U; \\ d_2 & \text{if } x \notin U. \end{cases}$ *Then,* f *is in* L.

Proof. Item (a) is straightforward.

(b) $\varepsilon(-a) = (X \setminus \varepsilon(a))$ entails $[\![d\chi_a < 0]\!] = \begin{cases} \varepsilon(a) & \text{if } d > 0; \\ \varepsilon(-a) & \text{if } d < 0, \end{cases}$ and so $d\chi_a \in S_a \cup S_{-a} \subseteq S$.

(c) Given $U \in B(X)$, Stone duality, the identifications of G with $\{\varepsilon(a) \in B(X) : a \in G\}$ and of B with $B(X)$, together with 2.12.(b)(3), guarantee that there is a finite family, $F = \{F_i : 1 \leq i \leq n\}$, of finite subsets of G such that [4]

$$U = \bigcup_{i=1}^{n} \bigcap_{a \in F_i} \varepsilon(a).$$

Hence, (a) implies $\chi_U = \bigwedge_{i=1}^{n} \bigvee_{a \in F_i} \chi_a$. Now, if $d \in \mathbb{R}^{\times}$, Lemma 2.3.(a) yields

$$d\chi_U = \begin{cases} \bigwedge_{i=1}^{n} \bigvee_{a \in F_i} d\chi_a & \text{if } d > 0; \\ \bigvee_{i=1}^{n} \bigwedge_{a \in F_i} d\chi_a & \text{if } d < 0, \end{cases}$$

and (b) entails $d\chi_U \in L$.

(d) If d_1, d_2 are both strictly positive or strictly negative, then $f \in S_1 \cup S_{-1}$, since these sets contain all units of $\mathbb{C}(X)$ of constant sign. It remains to check the statement when $d_1 d_2 < 0$. Without loss of generality, assume $d_2 < 0 < d_1$. As above, we write U^c for $X \setminus U$. We discuss three cases:

(i) If $|d_1| = |d_2|$, then $d_1\chi_{U^c}$ is equal to d_1 in U and $-d_1 = d_2$ in U^c;

(ii) If $|d_2| < d_1$, set $f = d_1\chi_{U^c} \vee d_2$, where d_2 is the constant map with value d_2 on X (note that since $d_2 < 0$, this map is in $S_{-1} \subseteq S$). Clearly, $f \in L$ and we have:

* If $x \in U$, $f(x) = \sup\,\{d_1, d_2\} = d_1$;
* If $x \in U^c$, then $f(x) = \sup\,\{-d_1, d_2\} = d_2$, because $d_2 = -|d_2| > -d_1$;

(iii) If $d_1 < |d_2|$, let $g \in \mathbb{C}(X)^{\times 2} \subseteq S$ be given by

$$g(x) = \begin{cases} d_1 & \text{if } x \in U; \\ 1 & \text{if } x \in U^c, \end{cases}$$

and set $f = |d_2|\chi_{U^c} \wedge g$, which is clearly in L. Then,

* If $x \in U$, then $f(x) = \inf\,\{|d_2|, d_1\} = d_1$;
* If $x \in U^c$, then $f(x) = \inf\,\{-|d_2|, 1\} = \inf\,\{d_2, 1\} = d_2$,

completing the proof of Fact 4.2. □

We are now in a position to apply the following seminal result of Marshall Stone:

Theorem. *Let K be a compact Hausdorff space, let D be a dense subset of $\mathbb{R}$ and let L be a sub-lattice*

[4] Recall finite joins and meets in B are taken to finite unions and intersections in $B(X)$.

of $\mathbb{C}(K)$ *such that if* $d_1, d_2 \in D$ *and* x, y *are distinct points in* K, *then there is* $f \in L$ *such that* $f(x)$ *and* $f(y) = d_2$. *Then,* L *is dense in* $\mathbb{C}(K)$ *in the topology of uniform convergence.*

We take $D = \mathbb{R}^{\times}$; given $d_1, d_2 \in D$ and $p \neq q$ in X, because the clopens are a basis for the topo of X, there is $U \in B(X)$ such that $p \in U$ and $q \in U^c$. By Fact 4.2.(d), there is $f \in L$ such that $f(p)$ and $f(q) = d_2$. Hence, the above Theorem guarantees that L is dense in $\mathbb{C}(X)$, and 2.17.(f) implies $L = \mathbb{C}(Y)^{\times}$, ending the proof of Theorem 4.1.

It is well-known that $\mathbb{C}(X)$ is the ring of global sections of a sheaf of local rings over X. Hence, is Boolean, then $\mathbb{C}(X)$ is a *ring with many units* (Theorem 2.10 and Corollary 2.11.(b) in [DM5]). Theorem 4.1 yields

Corollary 4.3 *If* G *is a reduced special group, there are a ring with many units,* R, *and a q-subgro of* R *such that* G *is isomorphic to* $G(S)$.

There is yet another form of *uniqueness* of representation in the construction, besides that of 4. that we now discuss. As a preliminary step, we register

Proposition 4.4 *Notation as in* 2.12, *let* G, H *be RSGs and let* $\varepsilon_G : G \longrightarrow B_G$ *be the Boolean h* G. *Let* $f : H \longrightarrow B_G$ *and* $\alpha : G \longrightarrow H$ *be complete embeddings. Theorem* 4.17, [DM2], *applied to* *yields a unique BA-morphism,* $\tau : B_G \longrightarrow B_G$, *making the following diagram commutative:*

$$\begin{array}{ccc} G & \xrightarrow{\ \alpha\ } & H \\ {\scriptstyle \varepsilon_G}\downarrow & & \downarrow{\scriptstyle f} \\ B_G & \xrightarrow[\ \tau\ ]{} & B_G \end{array}$$

Then, τ *is injective. Moreover, if* $\mathrm{Im}\ f$ *generates* B_G *(as a lattice) and* α *is a SG-isomorphism, th is an automorphism of* B_G.

Proof. For $b \in B_G$, assume that $\tau(b) = \bot$. By 2.12.(b)(3), there is a family $\{F_i : 1 \leq i \leq n\}$ of subsets of G such that

$$b = \textstyle\bigvee_{i=1}^{n} \bigwedge_{a \in F_i} \varepsilon_G(a).$$

Since τ is a BA-morphism and the above diagram commutes, we get

$$\bot \;=\; \tau\left(\textstyle\bigvee_{i=1}^{n} \bigwedge_{a \in F_i} \varepsilon_G(a)\right) \;=\; \textstyle\bigvee_{i=1}^{n} \bigwedge_{a \in F_i} \tau(\varepsilon_G(a)) \;=\; \textstyle\bigvee_{i=1}^{n} \bigwedge_{a \in F_i} f(\alpha(a)),$$

and so, for all $1 \leq i \leq n$, $\bot \;=\; \bigwedge_{a \in F_i} f(\alpha(a))$, or equivalently, recalling that any SG-morphism prese multiplication by -1,

$$\text{For all } 1 \leq i \leq n, \;\; \textstyle\bigvee_{a \in F_i} f(\alpha(-a)) \;=\; \top.$$

By Theorem 7.12 (p. 149) in [DM2], this is equivalent to

$$\text{For all } 1 \leq i \leq n, \;\; \text{the Pfister form } \textstyle\bigotimes_{a \in F_i} \langle 1, f(\alpha(-a)) \rangle \text{ is hyperbolic in } B_G.$$

Since the compositions of complete embeddings is again a complete embedding, we conclude that for the Pfister form $\bigotimes_{a \in F_i} \langle 1, -a \rangle$ is hyperbolic *in* G. Hence, $\bigotimes_{a \in F_i} \langle 1, \varepsilon_G(-a) \rangle$ is hyperbolic in B_G, another application of Theorem 7.12 in [DM2] yields

$$\text{For all } 1 \leq i \leq n, \;\; \textstyle\bigvee_{a \in F_i} \varepsilon_G(-a) \;=\; \bigvee_{a \in F_i} -\varepsilon_G(a) \;=\; \top,$$

that in turn entails that for all $1 \leq i \leq n$, $\bigwedge_{a \in F_i} \varepsilon_G(a) = \bot$; whence (I) guarantees that $b = \bot$, sho that $\ker \tau = \{\bot\}$ and that τ is injective.

Suppose α is an isomorphism and that $\mathrm{Im}\ f$ generates B_G as a lattice, i.e, for each $z \in B_G$, t are finite subsets of H, $\{A_i : 1 \leq k \leq m\}$, such that $z \;=\; \bigvee_{k=1}^{m} \bigwedge_{h \in A_k} f(h)$; for $1 \leq k \leq n$ $C_k = \alpha^{-1}(A_k) \subseteq G$. Clearly,

$$\textstyle\bigvee_{k=1}^{m} \bigwedge_{c \in C_k} f(\alpha(c)) \;=\; \bigvee_{k=1}^{m} \bigwedge_{h \in A_k} f(h) \;=\; z,$$

and the commutativity of the diagram in the statement implies that if $b = \bigvee_{k=1}^{m} \bigwedge_{c \in C_k} \varepsilon_G(c)$, then $\tau(b) = z$, establishing the surjectivity of τ, as needed. ◊

Proposition 4.5 *Let G be a RSG and let X and S be the Boolean space and q-subgroup of $\mathbb{C}(X)$ constructed in 4.1. Let $\beta : \mathbb{C}(X)^{\times} \longrightarrow B(X)$ be the map defined before 2.13, and let T be a q-subgroup of $\mathbb{C}(X)$ such that:*

(1) *$\beta(T)$ is a complete subgroup of B_G* [5]*, that generates B_G as a lattice;*

(2) *G and $G(T)$ are isomorphic reduced special groups.*

Then, there is an automorphism ξ of $\mathbb{C}(X)$ such that $\xi(S) = T$.

Proof. The hypotheses, 2.17 and 4.4 yield a commutative square

$$\begin{array}{ccc} \beta(S) = G & \xrightarrow{\ \ \alpha\ \ } & G(T) \\ \Big\downarrow \varepsilon_G & (\mathrm{D}) & \Big\downarrow \overline{\beta}_T \\ B_G & \xrightarrow[\ \ \tau\ \ ]{} & B_G \end{array}$$

here α is a SG-isomorphism and τ is an automorphism of B_G. Since τ is an automorphism, Stone duality plies at once that its dual $S(\tau)$, given by $S(\tau)(\mathcal{F}) = \tau^{-1}(\mathcal{F})$, $\mathcal{F} \in S(B_G) = X$, is a *homeomorphism* X onto X. Let $\xi : \mathbb{C}(X) \longrightarrow \mathbb{C}(X)$ be the $\mathbb{R}$-algebra *isomorphism* induced by $S(\tau)$ (see 4.7, below), at is,

$$\xi : \mathbb{C}(X) \longrightarrow \mathbb{C}(X), \text{ given by } \xi(f) \ = \ f \circ S(\tau).$$

nce ξ is a ring isomorphism, preserving square units (4.7.(b).(3)), $\xi(S)$ is a q-subgroup of $\mathbb{C}(X)$. It mains to check that $\xi(S) = T$. We first note that for all $f \in S$

$$[\![\xi(f) < 0]\!] \ = \ \tau([\![f < 0]\!]). \qquad \text{(I)}$$

deed, the definition of $S(\tau)$ and the fact that τ is bijective, give, for $\mathcal{F} \in X$:

$$\begin{aligned} \mathcal{F} \in [\![\xi(f) < 0]\!] \ &\Leftrightarrow \ \xi(f)(\mathcal{F}) \ = \ f(S(\tau)(\mathcal{F})) \ = \ f(\tau^{-1}(\mathcal{F})) < 0 \ \Leftrightarrow \ \tau^{-1}(\mathcal{F}) \in [\![f < 0]\!] \\ &\Leftrightarrow \ \mathcal{F} \in \tau([\![f < 0]\!]), \end{aligned}$$

ablishing (I). The commutativity of diagram (D) and (I) entail $[\![\xi(f) < 0]\!] \in \beta(T) = \mathrm{Im}\, \overline{\beta}_T$, wherefrom follows that

$$[\![\xi(f) < 0]\!] \ = \ \beta(\xi(f)) \ \in \ \beta(T).$$

nce, f is arbitrary in S, we conclude that $\beta(\xi(S)) \subseteq \beta(T)$, and so 2.17.(c) implies $\xi(S) \subseteq T$. For the verse inclusion, let $g \in T$; then, $\beta(g) = [\![g < 0]\!] \in \mathrm{Im}\, \overline{\beta}_T$, whence the commutativity of diagram (D) d (I) above yield $f \in S$ such that

$$\beta(g) \ = \ \tau(\beta(f)) \ = \ [\![\xi(f) < 0]\!] \ = \ \beta(\xi(f)),$$

d 2.17.(a) entails $\xi(f)g \in \mathbb{C}(X)^{\times 2} \subseteq \xi(S)$. It is now immediate that $g \in \xi(S)$, as needed. ◊

emark 4.6 In Proposition 2.13 the space Y was *arbitrary*. Theorem 3.9 (p.41) in [GJ], shows that Y ay be taken to be Tychonoff (i.e., Hausdorff and completely regular), without changing $\mathbb{C}(Y)$:

For every topological space Z, there is a Tychonoff space Y and a continuous map, $\tau : Z \longrightarrow Y$, such that $f \in \mathbb{C}(Y) \longmapsto f \circ \tau \in \mathbb{C}(Z)$ is a $\mathbb{R}$-algebra isomorphism. ◊

view of this, **henceforth the word "space" will stand for "Tychonoff space"**.

Since we shall be dealing with several spaces and to keep notation straight, if K is a topological space d $f \in \mathbb{C}(K)$, write $[\![f < 0]\!]_K$ for the (open) subset of K where f is negative.

emma 4.7 *Let $h : K \longrightarrow K'$ be a continuous map of topological spaces. Let $\mathfrak{h} : \mathbb{C}(K') \longrightarrow \mathbb{C}(K)$ be ven by $\mathfrak{h}(f) \ = \ f \circ h$. Then, $\mathfrak{h}$ is a $\mathbb{R}$-algebra morphism. Moreover,*

[5] I.e., with the notation of 2.17.(b), $\overline{\beta}_T : G(T) \longrightarrow B_G$ is a complete embedding.

a) For all $f \in \mathbb{C}(K')$, $h^{-1}([\![f < 0]\!]_{K'}) = [\![\mathfrak{h}(f) < 0]\!]_K$.

b) If Im h *is dense in* K'*, then:*

(1) $\mathfrak{h}$ *is a* $\mathbb{R}$*-algebra embedding;* (2) *For all clopen* U *in* K', $\overline{h(h^{-1}(U))} = U$.

(3) *For all* $f \in \mathbb{C}(K')^{\times}$, $\mathfrak{h}(f) \in \mathbb{C}(K)^{\times 2} \Leftrightarrow f \in \mathbb{C}(K')^{\times 2}$.

Proof. Clearly, $\mathfrak{h}$ is a morphism of $\mathbb{R}$-algebras. For (a), note that for all $y \in K$,

$$\mathfrak{h}(f)(y) < 0 \quad\Leftrightarrow\quad f(h(y)) < 0 \quad\Leftrightarrow\quad h(y) \in [\![f < 0]\!]_{K'} \quad\Leftrightarrow\quad y \in h^{-1}([\![f < 0]\!]_{K'}).$$

b)(1) If f_1, f_2 are such that $\mathfrak{h}(f_1) = \mathfrak{h}(f_2)$, then f_1 and f_2 coincide on the dense set Im $h \subseteq K'$, and so since $\mathbb{R}$ is Hausdorff, we conclude that they must be equal on all of K'.

(2) Fix a clopen U in K'. Since $h^{-1}(U) = h^{-1}(U \cap h(K))$, we have $h(h^{-1}(U)) = U \cap h(K)$, and so given the density of Im h in K' and the fact that U is clopen, we obtain

$$\overline{h(h^{-1}(U))} = \overline{U \cap h(K)} = U,$$

as claimed.

(3) Since $\mathfrak{h}$ is a ring embedding, ($\Leftarrow$) is clear. Conversely, assume that $\mathfrak{h}(f)$ is strictly positive in K and $[\![f < 0]\!]_{K'} \neq \emptyset$. Then, $[\![f < 0]\!]_{K'} \cap h(K) \neq \emptyset$, and (a) yields $[\![\mathfrak{h}(f) < 0]\!]_K = h^{-1}([\![f < 0]\!]_{K'}) \neq \emptyset$ which is impossible. $\diamondsuit$

Theorem 4.8 *Let* $\omega : Y \longrightarrow Z$ *be a continuous map of Tychonoff spaces, whose image is dense in* Z*. Then, for each q-subgroup* P *of* $\mathbb{C}(Z)$*, there is a q-subgroup* Q *of* $\mathbb{C}(Y)$ *such that* $\langle G(P), \equiv^P, -1\rangle$ *is isomorphic to* $\langle G(Q), \equiv^Q, -1\rangle$.

Proof. Let $\mathfrak{o} : \mathbb{C}(Z) \longrightarrow \mathbb{C}(Y)$, given by $\mathfrak{o}(f) = f \circ \omega$. By 4.7.(b).(1), $\mathfrak{o}$ is a $\mathbb{R}$-algebra embedding and hence $\mathfrak{o}(P)$ is a subgroup of $\mathbb{C}(Y)^{\times}$, containing -1. Let Q be the saturation of $\mathfrak{o}(P)$ modulo square units, that is,

$$Q = \mathfrak{o}(P) \cdot \mathbb{C}(Y)^{\times 2} = \{\mathfrak{o}(f)h \in \mathbb{C}(Y)^{\times} : f \in P \text{ and } h \in \mathbb{C}(Y)^{\times 2}\}.$$

Then, Q is a subgroup of $\mathbb{C}(Y)^{\times}$, containing $\mathbb{C}(Y)^{\times 2}$ and -1, whence a q-subgroup of $\mathbb{C}(Y)$. Clearly, $\mathfrak{o}\restriction P$ is a group embedding of P into Q, taking -1 to -1. We now have, for $f, g \in P$:

$$f \in D^P(1, g) \quad\Leftrightarrow\quad \mathfrak{o}(f) \in D^Q(1, \mathfrak{o}(g)). \tag{*}$$

Indeed, since $\mathfrak{o}$ is a ring embedding, ($\Rightarrow$) is clear (2.7). For the converse, because $[\![h < 0]\!]_Z$ is a clopen in Z for all $h \in \mathbb{C}(Z)^{\times}$, items (a).(1) and (a).(2) in 4.7, together with the equivalence in 2.7.(a) yield

$$\begin{aligned}
\mathfrak{o}(f) \in D^Q(1, \mathfrak{o}(g)) &\Rightarrow [\![\mathfrak{o}(f) < 0]\!]_Y \subseteq [\![\mathfrak{o}(g) < 0]\!]_Y \Rightarrow \omega^{-1}([\![f < 0]\!]_Z) \subseteq \omega^{-1}([\![g < 0]\!]_Z)\\
&\Rightarrow \omega\left(\omega^{-1}([\![f < 0]\!]_Z)\right) \subseteq \omega\left(\omega^{-1}([\![g < 0]\!]_Z)\right)\\
&\Rightarrow \overline{\omega\left(\omega^{-1}([\![f < 0]\!]_Z)\right)} \subseteq \overline{\omega\left(\omega^{-1}([\![g < 0]\!]_Z)\right)}\\
&\Rightarrow [\![f < 0]\!]_Z \subseteq [\![g < 0]\!]_Z \Rightarrow f \in D^P(1, g),
\end{aligned}$$

as asserted.

Let $\mu : P \longrightarrow G(Q) = Q/\mathbb{C}(Y)^{\times 2}$, be given by $\mu = q_Q \circ \mathfrak{o}\restriction P$, where $q_Q : Q \longrightarrow G(Q)$ is the natural quotient morphism. Since every element of Q is equivalent, modulo $\mathbb{C}(Y)^2$ to an element of $\mathfrak{o}(P)$, it is clear that μ is surjective, while 4.7.(b).(3) implies that ker $\mu = \mathbb{C}(Z)^{\times 2}$. The same argument used to establish 2.13.(c) or 4.1.(3) (that depends only on (*) above) shows that μ factors uniquely through $G(P)$, to yield an isomorphism of reduced π-SGs, $\overline{\omega} : G(P) \longrightarrow G(Q)$. $\diamondsuit$

The last step in our discussion is determining, for each reduced special group G, a projective compact K such that G is isomorphic to $G(S)$ for some q -subgroup S of $\mathbb{C}(K)^{\times}$. The interest in this lies in the fact that the Boolean algebra of clopens in K is *complete*. In fact, it will be shown that there is such a K universal for *countable* special groups.

It is well-known that a

compact Hausdorff space is projective iff it is extremally disconnected

iff it is Boolean and its BA of clopens is complete.

The category of compact Hausdorff spaces has enough projectives (for background on projective compacts, the reader may consult [Mag] or Chapter 21 in [Mir]). This fact is a consequence of a construction due to Gleason. To fix notation, let K be a compact space and let $\mathfrak{G}(K)$ be the Stone space of the complete BA of regular opens in K, i.e., those open U in K such that $U = \text{int}\,\overline{U}$ (int is the interior operation). Then, there is a continuous *surjection*, $\mathfrak{g} : \mathfrak{G}(K) \longrightarrow K$, such that no restriction of $\mathfrak{g}$ to a proper closed subset of $\mathfrak{G}(K)$ is **onto** K. The diagram $\mathfrak{g} : \mathfrak{G}(K) \longrightarrow K$ is the **Gleason projective cover of K**.

Theorem 4.9 *a) If G is a RSG, then there is a projective compact space Z and a q-subgroup S of $\mathbb{C}(Z)$ such that G is isomorphic to $G(S)$.*

b) Any countable RSG is of the form $G(S)$, where S is a q-subgroup of $\mathbb{C}(2^\omega)$.

c) There is a projective compact P such that if G is countable RSG, then there is a q-subgroup of S of $\mathbb{C}(P)$ such that G is isomorphic to $G(S)$.

Proof. a) By Theorem 4.1, there is a Boolean space X and a q-subgroup T of $\mathbb{C}(X)$ such that G is isomorphic to $G(T)$. Let $\mathfrak{g} : \mathfrak{G}(X) \longrightarrow X$ be the Gleason cover of X. Then, Theorem 4.8 yields a q-subgroup S of $\mathbb{C}(\mathfrak{G}(X))$ such that $G(T) \approx G(S) \approx G$, as desired.

b) Let G be a countable RSG; then its Boolean hull, B_G, is also countable (this follows straightforwardly from 2.12.(b).(3)). Hence, the Stone space of B_G, X_G, is a second countable Boolean space and thus metrizable. Now, a result of Kuratowski asserts that X_G is the continuous image of 2^ω, that is, there is a continuous surjection, $\theta : 2^\omega \longrightarrow X_G$. By Theorem 4.1, G is isomorphic to $G(T)$ for some q-subgroup T of $\mathbb{C}(X_G)$ and the desired conclusion follows from Theorem 4.8.

c) Let $\mathfrak{g} : P = \mathfrak{G}(2^\omega) \longrightarrow 2^\omega$ be the Gleason cover of 2^ω. The same arguments used in the proofs of items (a) and (b), yield a q-subgroup S of $\mathbb{C}(P)$ such that $G(S) \approx G$. $\diamondsuit$

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