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THE CHANGE OF SIGN OF DERIVATIVES OF THE DENSITY FUNCTION OF A SPIN FLIP SYSTEM WITH THE ORDER OF DERIVATION.

by

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The change of sign of derivatives of the density function of a spin flip system with the order of derivation.

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Abstract.
In a spin flip system on a regular lattice which starts from the configuration "all sites of the lattice at the state 0", we consider the average density of the sites of this lattice at the state 1 as a function of time. We prove that the sign of the n-th derivative of this function with respect to time at the point $time = 0$ is $(-1)^{n+1}$, provided the flip rate of the system satisfies certain conditions. The proof is based on a combinatorial analysis of the form which the generator of the system acquires when elevated to the n-th power. This result is used in a companion paper which discusses inference on the flip rate of a spin flip system from the dynamics of its average density.

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1. INTRODUCTION

Let $\varphi_t, t \geq 0$, denote a spin flip system on a regular lattice $\mathcal{G}$. For concreteness, call 0 and 1 the states at which a site of this lattice may stay in the process $\varphi_.$. Assume $\varphi_.$ starts from the state "all sites of $\mathcal{G}$ at 0". Denote then by ρ_t the average density of the sites of $\mathcal{G}$ which are at the state 1 at time $t \geq 0$ in $\varphi_.$. $\rho_t, t \geq 0$, will be called the density function. In Theorem 1 of this paper we prove that the n-th derivative with respect to t of the density function taken at $t = 0$ has the sign $(-1)^{n+1}$, provided the flip rate satisfies certain conditions. The proof is based on a combinatorial analysis of the form which the generator of the studied spin flip system acquires when elevated to the n-th power.

There is a class of spin flip systems that have been suggested to model certain deposition processes from the filed of physical chemistry. It has been shown that a lattice and a density function determine uniquely the flip rate of a system from this class. Theorem 1 of this paper is used to construct a procedure that infers on the flip rate from a portion of a density function for those systems of this class which satisfy the conditions of this theorem. We note that these conditions are satisfied by the Contact Process, and also, to each nearest neighbor translation invariant symmetric potential, there may be constructed a corresponding stochastic Ising model which satisfies these conditions.

The paper is organized in the following form: In Section 2 we give the definition of the studied spin flip system and we state the result. We also present there a simple calculation which reduces the proof of this result to that of a certain assertion called there the main lemma. Section 3 contain examples which are designed to indicate the ideas we employ to prove this lemma. Auxiliary results to be used in this proof form Section 4. Then, Section 5 and 6 establish the main lemma for the cases when $\mathcal{G}$ is respectively, $\mathbb{Z}$ and a torus in $\mathbb{Z}$. In Section 7, we discuss applications and extensions of our result.

2. THE MAIN DEFINITIONS AND THE RESULT

Let $\mathcal{G}$ denote a regular graph (for example, a Bethe lattice, or $\mathbb{Z}^d$, or a torus of size N in $\mathbb{Z}^d$). For simplicity, the same sign $\mathcal{G}$ will be also used to denote the set of the sites of $\mathcal{G}$. We set $\mathcal{X} := \{0,1\}^{\mathcal{G}}$ to denote the set of all functions from $\mathcal{G}$ to $\{0,1\}$. A function η from $\mathcal{X}$ is called a configuration; $\eta(x)$ means the value of the configuration η at the site $x \in \mathcal{G}$. When $\eta(x) = 1$, we say that η has a particle at the site x, or, equivalently, x is occupied by a particle in η, while when $\eta(x) = 0$, we say that the site x is empty in η. A particular configuration in which all sites are at the state 0 will be denoted $\emptyset$ throughout. Though the same $\emptyset$ denotes the empty set, there will be no confusion since the exact meaning of this symbol will be always clear from the context. Given a configuration η and a site x, we define a new configuration η^x in the following manner:

$$\eta^x(y) := \begin{cases} \eta(y) & \text{when } y \neq x \\ 1-\eta(y) & \text{when } y = x \end{cases} \qquad y \in \mathcal{G}$$

η^x is called the configuration η flipped at x. By $\varphi_t, t \geq 0$, we will denote the Markov process with the state space $\mathcal{X}$, whose initial state is $\emptyset$ and whose generator L acts on a cylinder function $\mathcal{F} : \mathcal{X} \to \mathbb{R}$ in the following form (see Liggett (1985) for the construction of a spin flip system from its generator):

$$L\mathcal{F}(\eta) = \sum_{x \in \mathcal{G}} c_x(\eta)\left[\mathcal{F}(\eta^x) - \mathcal{F}(\eta)\right], \quad \forall \eta \in \mathcal{X}, \tag{2.0}$$

where $c_\cdot(\cdot) : \mathcal{G} \times \mathcal{X} \to \mathbb{R}$ is a non-negative uniformly bounded function which is called *the flip rate* of the process $\varphi_\cdot$ and which throughout in this paper, will be assumed to have the following special form:

$$c_x(\eta) = \begin{cases} \lambda_k & \text{if } \eta(x) = 0 \\ \mu_k & \text{if } \eta(x) = 1 \end{cases} \quad \forall \eta \in \mathcal{X},\ \forall x \in \mathcal{G} \tag{2.1}$$

whereas k denotes the number of the neighbors of x which are occupied by a particle in η (two sites of $\mathcal{G}$ are said to be neighbors if they are connected by an edge in the graph $\mathcal{G}$). To exclude the trivial case when $\varphi_t = \emptyset\ \forall t \geq 0$ it will be also assumed that $\lambda_0 > 0$. The process $\varphi_\cdot$ and its flip rate are called *condensative* if

when $\eta(x) = \zeta(x)$ and $\eta(y) \geq \zeta(y)$ for all $y \in \mathcal{G} \setminus \{x\}$ then $c_x(\eta) \leq c_x(\zeta)$;
in the case (2.1) this is equivalent to $\lambda_i \geq \lambda_{i+1}, \mu_i \geq \mu_{i+1}, \quad i = 0, 1, \ldots, g-1$ (2.2)

Above in (2.2) and throughout the paper g denotes the degree of $\mathcal{G}$, i.e. the total number of the neighbors of a site of $\mathcal{G}$. The process $\varphi.$ and its flip rate are said to have *monotone differences* if

$$\lambda_0 - \lambda_1 \geq \lambda_1 - \lambda_2 \geq \ldots \geq \lambda_{g-1} - \lambda_g \text{ and } \mu_0 - \mu_1 \geq \mu_1 - \mu_2 \geq \ldots \geq \mu_{g-1} - \mu_g \quad (2.3)$$

Let 0 designate an arbitrarily distinguished site of $\mathcal{G}$ and let $I : \mathcal{X} \to \mathbb{R}$ be the indicator function of presence of a particle at the site 0, i.e., $I(\eta) := \eta(0), \forall \eta \in \mathcal{X}$. Let $\mathbf{E}$ denote the mathematical expectation with respect to the law of the process φ. For $t \geq 0$, define $\rho_t := \mathbf{E}I(\varphi_t)$. The function $\rho_t, t \geq 0$, is called *the density function*. Its particular property presented in the following theorem constitutes the main result of this paper.

Theorem 1. *Let $\varphi.$ be a condensative process with monotone differences on $\mathcal{G}$, where $\mathcal{G}$ is either $\mathbb{Z}$ or a torus in $\mathbb{Z}$ with the size $N \geq 3$. Then*

$$\frac{d^n}{dt^n}\rho_t\Big|_{t=0} \begin{cases} > 0 & \text{if } n \text{ is odd} \\ < 0 & \text{if } n \text{ is even} \end{cases} \quad (2.4)$$

Proof. For each site $B \in \mathcal{G}$, we introduce an operator Δ^B which acts on the space of the continuous functions from $\mathcal{X}$ to $\mathbb{R}$ in the following form:

$$\text{for a continuous } \mathcal{F} : \mathcal{X} \to \mathbb{R}, \qquad (\Delta^B \mathcal{F})(\eta) := c_B(\eta)\big[\mathcal{F}(\eta^B) - \mathcal{F}(\eta)\big], \quad \forall \eta \in \mathcal{X} \quad (2.5)$$

Using then the Hille-Yosida theorem (see Liggett (1985)), the form of the generator of the process $\varphi.$ and (2.5), we have that

$$\frac{d^n}{dt^n}\rho_t\Big|_{t=0} = \frac{d^n}{dt^n}\mathbf{E}I(\varphi_t)\Big|_{t=0} = (L^n I)(\emptyset) = \sum_{B_1,\ldots,B_n \in \mathcal{G}} (\Delta^{B_1}(\Delta^{B_2}\ldots(\Delta^{B_n}I)))(\emptyset)$$

$$\text{in a shorthand notation } = \sum_{B_1,\ldots,B_n \in \mathcal{G}} \Delta^{B_1}\Delta^{B_2}\ldots\Delta^{B_n}I(\emptyset) \quad (2.6)$$

where the sums in (2.6) are taken over all sequences of n not necessarily distinct sites of $\mathcal{G}$.

In the sequel we will prove the following

Lemma 1. *Let $\varphi.$ and $\mathcal{G}$ be as in Theorem 1. Then,*

(a) for any $n \geq 1$ and any sequence $B_1, \ldots, B_n$ of n not necessarily distinct sites of $\mathcal{G}$,
either the expression $\Delta^{B_1}\Delta^{B_2}\ldots\Delta^{B_n}I(\emptyset)$ equals 0 or its sign is $(-1)^{n+1}$;

(b) for any $n \geq 1$, when $B_1 = 0, \ldots, B_n = 0$ then the sign of $\Delta^{B_1}\Delta^{B_2}\ldots\Delta^{B_n}I(\emptyset)$ is $(-1)^{n+1}$.

From Lemma 1 and (2.6) Theorem 1 follows. ♣

We postpone the discussion of applications and extensions of Theorem 1 to Section 7 and here we proceed with the aim to prove Lemma 1.

Let us fix the term *sequence* to mean a sequence of not necessarily distinct sites of $\mathcal{G}$ and the sign $(B_1, \dots, B_b)$ to designate such a sequence composed from $B_1, \dots, B_b \in \mathcal{G}$ in this order, where $B_1, \dots, B_b$ are not necessarily distinct. We will use the expression $B = (B_1, \dots, B_b)$ to indicate that B will be the abbreviation of the sequence $(B_1, \dots, B_b)$; when possible we will choose this abbreviating symbol to be the same letter as the one used for the sequence' terms, also the same lowercase letter will be usually chosen to stay for the number of the terms.

A sequence $B = (B_1, \dots, B_b)$ will be called *a tower* if both $B_b = 0$ and for each $i = 1, \dots, b-1$, the site B_i either neighbors to or coincides with one of $B_{i+1}, \dots, B_b$. We then have the following

Lemma 2. *If a sequence $B = (B_1, \dots, B_b)$ is not a tower then $\Delta^{B_1} \dots \Delta^{B_b} I(\emptyset) = 0$ for any choice of $\lambda_\cdot$ and $\mu_\cdot$ in (2.1).*

The proof of Lemma 2 is omitted since this result is not new, to the best of our knowledge. For example, a somewhat general form of this result has been used by Holley (1994) in the calculations on page 135. Those interested to see its proof may consult Belitsky (1991). Unfortunately, we are not aware of another reference in which the above lemma or an equivalent result appears proven.

With the aid of Lemma 2, the item (a) of Lemma 1 will follow from the assertion presented below. It is called the main lemma since its proof is the main effort of this paper; the item (b) requires considerably less work and will be derived in Section 3.

The main lemma. *Let $\varphi_\cdot$ and $\mathcal{G}$ be as in Theorem 1. If $B = (B_1, \dots, B_b)$ is a tower on $\mathcal{G}$ then*

$$\text{either } \Delta^{B_1}\Delta^{B_2} \dots \Delta^{B_b} I(\emptyset) = 0 \text{ or its sign is } (-1)^{n+1} \tag{2.7}$$

The rest of this section is devoted to introducing the basic concepts and notations which we will use in the sequel.

A set $S \subseteq \mathcal{G}$ is called *the projection* of a sequence $B = (B_1, \dots, B_b)$ on $\mathcal{G}$ if it consists of those and only those sites of $\mathcal{G}$ which are listed at least once in $B_1, \dots, B_b$. For a site x from the projection S of a sequence $B = (B_1, \dots, B_b)$, the value $f(x)$ defined by

$$f(x) := \max\{\, i \in \{1, 2, \dots, b\} \text{ such that } B_i = x \,\} \tag{2.8}$$

is called *the roof above x* or simply, *the x-th roof*. The x-th roof is said to be *higher* than the y-th roof if $f(x) > f(y)$. The x-th roof is said to be *immediately above* the y-th roof if there is no z from the projection of B for which $f(y) < f(z) < f(x)$.

The terms "roof" and "tower" were suggested by the visual image of sequences which we used when working out the proof of the main lemma. We now present it for the case $\mathcal{G} = \mathbb{Z}$. The extension to other lattices should then be clear though not necessary since Theorem 1 concerns exclusively the cases when $\mathcal{G}$ is either $\mathbb{Z}$ or a torus in $\mathbb{Z}$. Let $B = (B_1, \ldots, B_b)$ be an arbitrary sequence. Let $\mathbb{R}^2$ be the usual Euclidean plane. For each $i = 1, \ldots, b$, we draw a side-1 square centered at the point $(B_i, i) \in \mathbb{R}^2$. Project all these squares on the axis of the first coordinate. The set of its integer points covered by this projection is what we have called the projection of B. Each site from this projection has the square that is the highest one among all the squares that project on this site. The height of the center of this square is what has been called the roof above this site. The terms "higher" and "immediately above" are then naturally suggested when one imagines the roofs as the heights of certain squares in our picture. To understand where the term "tower" came from, take all the squares of a tower B and move them downwards till they meet the axis of the first coordinate. The set of the points of $\mathbb{R}^2$ swept by this motion has a silhouétte of a tower, which suggested this name.

Let $\mathcal{G} = \mathbb{Z}$ and let q and p be respectively the leftmost and the rightmost sites of the projection of B. Observe (the visual image presented above may be helpful in this observation) that if B is a tower then $f(0)$ is the highest roof, and for $0 \leq i < j \leq p$, the i-th roof is higher than the j-th one, while for $q \leq m < k \leq 0$, the k-th roof is higher the m-th one. Observe also that if $q = 0$ then for each $i = 0, 1, \ldots, p-1$, the i-th roof is immediately above the $(i+1)$-st one; a similar property holds if $p = 0$. To distinguish between these cases, which is necessary because they will be treated in different ways, we introduce a particular terminology: A tower whose projection is solely $\{0\}$ will be called *a column*. A tower which is not a column and whose projection either does not contain negative integers or does not contain positive integers will be called *a one-side tower*. A tower that is neither a column nor a one-side tower is called *a two-side tower*.

We proceed with more designations. For $x \in \mathcal{G}, \eta \in \mathcal{X}$ and $\epsilon \in \{0, 1\}$, we define $\eta_\epsilon^x \in \mathcal{X}$ by

$$\eta_\epsilon^x := \begin{cases} \eta & \text{if } \epsilon = 0 \\ \eta^x & \text{if } \epsilon = 1 \end{cases}$$

For a configuration $\eta \in \mathcal{X}$, a sequence $B = (B_1, \ldots, B_n)$ and a vector $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ consisting of n components such that each one of them may be either 0 or 1 (which will be indicated by writing $\epsilon \in \{0,1\}^n$), we introduce a new configuration

$$\eta_{\epsilon_1 \ldots \epsilon_n}^{B_1 \ldots B_n} (= \text{ in a shorthand notation } \eta_\epsilon^B) := (((\eta_{\epsilon_1}^{B_1})_{\epsilon_2}^{B_2}) \ldots)_{\epsilon_n}^{B_n} \tag{2.9}$$

The rank of a site $x \in \mathcal{G}$ in a sequence $B = (B_1, \ldots, B_n)$ is denoted by $\text{rank}_{(B_1 \ldots B_n)} x$ (observe, the commas in the subscript are omitted for typographical convenience) and defined as the number of the members of this sequence which coincide with x:

$$\text{rank}_{(B_1 \ldots B_n)} x := \sum_{i=1}^{n} I_{\{B_i = x\}}$$

Throughout we will use the following abbreviating notations:

for $x \in \mathcal{G}$ and for a sequence $B = (B_1, \ldots, B_b)$, $\quad \text{rank}_B x := \text{rank}_{(B_1 \ldots B_b)} x;$
for a set $S = \{S_1, \ldots, S_s\} \subseteq \mathcal{G}$ and a sequence $B = (B_1, \ldots, B_b)$,
$$\text{rank}_B S := \text{rank}_B S_1 + \ldots + \text{rank}_B S_s;$$
for $x \in \mathcal{G}$ and $n \in \mathbb{Z}^+$, $\quad \Delta_n^x := \Delta^x \ldots \Delta^x \quad n$ times;
for a vector $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$, $\quad (-1)^{|\epsilon|} := (-1)^{\epsilon_1 + \ldots + \epsilon_n};$
for $\eta_1, \ldots, \eta_n \in \mathcal{X}$, $a_1, \ldots, a_n \in \mathbb{R}$, and $\mathcal{F} : \mathcal{X} \to \mathbb{R}$,
$$\mathcal{F}\{a_1\eta_1 + \ldots + a_n\eta_n\} := \sum_{i=1}^{n} a_i \mathcal{F}(\eta_i);$$
for a sequence $B = (B_1, \ldots, B_b)$, $\quad \Delta^B := \Delta^{B_1}\Delta^{B_2} \ldots \Delta^{B_b}$.

3. ILLUSTRATIVE EXAMPLES

Both examples of this section use the following

Lemma 3. *Let $\eta \in \mathcal{X}$ and $x \in \mathcal{G}$ be arbitrary and let j denote the number of the neighbors of x which are occupied in η. Then, for any $n \in \mathbb{Z}^+$ and any $\mathcal{F} : \mathcal{X} \to \mathbb{R}$,*

$$\Delta_n^x \mathcal{F}(\eta) = \begin{cases} (-1)^n \lambda_j (\lambda_j + \mu_j)^{n-1} \mathcal{F}\{\eta_0^x - \eta_1^x\}, & \text{if } \eta(x) = 0 \\ (-1)^n \mu_j (\lambda_j + \mu_j)^{n-1} \mathcal{F}\{\eta_1^x - \eta_0^x\}, & \text{if } \eta(x) = 1 \end{cases} \tag{3.1}$$

Proof. For $n = 1$, the result follows directly from the definition (2.5). Assume the lemma holds for some $n \geq 1$. Let us consider the case $\eta(x) = 0$, the opposite case follows in a similar manner. Using consequently our assumption and (2.5) we derive that

$$\begin{aligned} \Delta_{n+1}^x \mathcal{F}(\eta) &= (-1)^n \lambda_j (\lambda_j + \mu_j)^{n-1} \Delta^x \mathcal{F}\{\eta_0^x - \eta_1^x\} \\ &= (-1)^n \lambda_j (\lambda_j + \mu_j)^{n-1} [-\lambda_j \mathcal{F}\{\eta_0^x - \eta_1^x\} + \mu_j \mathcal{F}\{\eta_1^x - \eta_0^x\}] \\ &= (-1)^{n+1} \lambda_j (\lambda_j + \mu_j)^n \mathcal{F}\{\eta_0^x - \eta_1^x\} \end{aligned} \tag{3.2}$$
♣

The above lemma leads directly to

The proof of the assertion (b) of Lemma 1: $\Delta_n^0 I(\emptyset) = (-1)^n I\{\emptyset_0^0 - \emptyset_1^0\} = (-1)^{n+1}$. ♣

EXAMPLE 1. For arbitrarily three positive integers n_1, n_2, n_3 and for $b = n_1 + n_2 + n_3$, consider the one-side tower

$$B = (B_1, \ldots, B_b) = (2, \ldots, 2, 1, \ldots, 1, 0, \ldots, 0) \tag{3.3}$$

where 2, 1 and 0 are repeated respectively, n_1, n_2 and n_3 times. We will show here that the above tower satisfies (2.7).

Expanding $\Delta^{B_1}\dots\Delta^{B_{n_1}} = \Delta^2_{n_1}$ in accordance to (3.1), we find that

$$\begin{aligned}\Delta^B I(\emptyset) =&\Delta^{B_1}\dots\Delta^{B_k} I(\emptyset) = \Delta^2_{n_1}\Delta^1_{n_2}\Delta^0_{n_3} I(\emptyset)\\ =&(-1)^{n_1}\lambda_0(\lambda_0+\mu_0)^{n_1-1}\Delta^1_{n_2}\Delta^0_{n_3} I\{\emptyset^2_0-\emptyset^2_1\}\end{aligned} \tag{3.4}$$

Using again (3.1) to expand $\Delta^{B_{n_1+1}}\dots\Delta^{B_{n_1+n_2}} = \Delta^1_{n_2}$ in $\Delta^1_{n_2}\Delta^0_{n_3} I\{\emptyset^2_0\}$ and in $\Delta^1_{n_2}\Delta^0_{n_3} I\{\emptyset^2_1\}$ and combining the obtained results with (3.4), we have that

$$\begin{aligned}\Delta^B I(\emptyset) =&(-1)^{n_1}\lambda_0(\lambda_0+\mu_0)^{n_1-1}\\ &\big[(-1)^{n_2}\lambda_0(\lambda_0+\mu_0)^{n_2-1}\Delta^0_{n_3} I\{\emptyset^{12}_{00}-\emptyset^{12}_{10}\}\\ &-(-1)^{n_2}\lambda_1(\lambda_1+\mu_1)^{n_2-1}\Delta^0_{n_3} I\{\emptyset^{12}_{01}-\emptyset^{12}_{11}\}\big]\end{aligned} \tag{3.5}$$

An important observation (that follows from (2.5) and the independence of the value of $c_x(\zeta)$ on the value of ζ at any y which is not a neighbor of x) is that $\Delta^0_{n_3} I(\eta)$ does not depend on the values of η outside of the set $\{-1,0,1\}$. We thus substitute $\emptyset^{12}_{\alpha\beta}$ in (3.5) by $\emptyset^1_\alpha$ for each $\alpha,\beta\in\{0,1\}$ and obtain the following relation which is equivalent to (3.5):

$$\begin{aligned}\Delta^{B_1}\dots\Delta^{B_k} I(\emptyset) =&(-1)^{n_1}\lambda_0(\lambda_0+\mu_0)^{n_1-1}\\ &\times(-1)^{n_2}\big[\lambda_0(\lambda_0+\mu_0)^{n_2-1}-\lambda_1(\lambda_1+\mu_1)^{n_2-1}\big]\Delta^0_{n_3} I\{\emptyset^1_0-\emptyset^1_1\}\end{aligned} \tag{3.6}$$

A one more application of (3.1) gives that the last line in (3.6) equals

$$\begin{aligned}&(-1)^{n_3}\lambda_0(\lambda_0+\mu_0)^{n_3-1} I\{\emptyset^{10}_{00}-\emptyset^{10}_{01}\} - (-1)^{n_3}\lambda_1(\lambda_1+\mu_1)^{n_3-1} I\{\emptyset^{10}_{10}-\emptyset^{10}_{11}\}\\ &= -(-1)^{n_3}\big[\lambda_0(\lambda_0+\mu_0)^{n_3-1}-\lambda_1(\lambda_1+\mu_1)^{n_3-1}\big]\end{aligned} \tag{3.7}$$

The final conclusion that (2.7) holds for the considered tower B is obtained by plugging (3.7) in (3.6) and using the property (2.2) to establish that each term in the square brackets in (3.6) and (3.7) is non-negative.

What we want to demonstrate in Example 1 is the method of reduction of configurations which allowed us to pass from (3.5) to (3.6). We now recall the definition (2.8) of $f(\cdot)$ and observe that $f(1) = n_1 + n_2$ in the case (3.3), so that we may say that the reduction was applied after the expansion of $\Delta^{B_{f(1)}}$ and was made possible by the fact that each of $B_{f(1)+1},\dots,B_k$ neither neighbors to nor coincides with the site 2. Looking from this angle at the argument we applied above to the tower (3.3), it is easy to see that this argument may be extended so that it will demonstrate that the property (2.7) holds for any tower of the form

$$B = (B_1,\dots,B_k) = (p,\dots,p,p-1,\dots,p-1,\dots,1,\dots,1,0,\dots,0), \tag{3.8}$$

where $p \geq 1$ and the numbers of repetitions of $p, p-1, \ldots, 1, 0$ are arbitrary. Indeed, the relation (2.7) for a tower of the form (3.8) can be derived expanding $\Delta^{B_1}, \ldots, \Delta^{B_k}$ consequently due to (2.5) and applying the reduction of configurations after the expansions of $\Delta^{B_{f(p-1)}}, \Delta^{B_{f(p-2)}}, \ldots, \Delta^{B_{f(1)}}$ consequently in the manner similar to the one just discussed. We remark that in order to prove the main lemma for a general one-side tower B, we will show that a similar sequence of reductions may be done even when B does not have the particular structure (3.8). What is worthwhile to be stressed is that in the general case, similarly as in the case (3.8), these reductions will be also done after the expansions of $\Delta^{B_{f(p-1)}}, \Delta^{B_{f(p-2)}}, \ldots, \Delta^{B_{f(1)}}$. We finally remark that the reduction method we will apply is provided by Lemma 5 of Section 4.

EXAMPLE 2. Here we will show that the tower $B = (B_1, B_2, B_3, B_4) = (-1, 0, 1, 0)$ satisfies (2.7).

After expanding $\Delta^{B_1}, \Delta^{B_2}, \Delta^{B_3}$ recurrently in accordance to (2.5), one finds that $\Delta^B I(\emptyset)$ is a linear combination of $\Delta^{B_4} I(\emptyset^{0(-1)1}_{\epsilon\ \alpha\ \beta})$, $\epsilon, \alpha, \beta \in \{0,1\}$. Namely, one has that

$$\Delta^{B_1} \ldots \Delta^{B_4} I(\emptyset) = (-1)^3 \sum_{\epsilon \in \{0,1\}} (-1)^{\epsilon} \left(\sum_{\alpha,\beta \in \{0,1\}} (-1)^{\alpha+\beta} C_{\alpha\beta}(\epsilon) \Delta^{B_4} I(\emptyset^{0(-1)1}_{\epsilon\ \alpha\ \beta}) \right) \tag{3.9}$$

where

$$\begin{gathered} C_{01}(0) = C_{00}(0) = \lambda_0^3,\ C_{10}(0) = C_{11}(0) = \lambda_0^2 \lambda_1, \\ C_{01}(1) = C_{00}(1) = \lambda_0^2 \lambda_1,\ C_{10}(1) = C_{11}(10) = \lambda_0 \lambda_1^2 \end{gathered} \tag{3.10}$$

A rather strange form of the right-hand-side in (3.9) is chosen consciously. Such a form will be used in the proof of the main lemma for a general tower, and here we would like to demonstrate the manner this form is composed through considering a tower with a simple structure. From (2.2) and (3.10) we have an important relation to be used below

$$C_{01}(\epsilon) = C_{00}(\epsilon) \geq C_{10}(\epsilon) = C_{11}(\epsilon) \geq 0, \qquad \forall \epsilon \tag{3.11}$$

We now consider the second sum in (3.9) that corresponds to the value 0 of ϵ. Expanding Δ^{B_4} due to (2.5) in each term of this sum (the resulting expression appears in the square bracket in (3.12)), we find this sum equals (below we use that $I\{\emptyset^{0-11}_{0\alpha\beta} - \emptyset^{0-11}_{1\alpha\beta}\} = I\{\emptyset^0_0 - \emptyset^0_1\} = \{-1\}$ for all α, β)

$$\begin{aligned} &\sum_{\alpha,\beta \in \{0,1\}} (-1)^{\alpha+\beta} C_{\alpha\beta}(0) [-\lambda_{\alpha+\beta} I\{\emptyset^{0-11}_{0\ \alpha\ \beta} - \emptyset^{0-11}_{1\ \alpha\ \beta}\}] \\ &= \text{due to the equalities in (3.11)} = -[C_{00}(0)(\lambda_0 - \lambda_1) - C_{11}(0)(\lambda_1 - \lambda_2)]\{-1\} \geq 0 \end{aligned} \tag{3.12}$$

where the last inequality is because of the inequalities in (3.11) and because of the monotonicity of differences (2.3). A similar argument shows that the term of (3.9) that corresponds to $\epsilon = 1$ is ≤ 0. Consequently, (2.7) is true for the considered B. Observe that in Example 2 we used that the flip rates have monotone differences while in Example 1 we did not need this fact. A careful analysis of the proof of Theorem 1 reveals that in general, the property (2.3) is utilized solely to treat a two-side tower.

4. AUXILIARY RESULTS

We start with a set of definitions.

For $m \geq 1$, a set of constants $\{C_\delta, \delta = (\delta_1, \ldots, \delta_m) \in \{0,1\}^m\}$ is said to be *monotone in the i-th entry of its subscript*, if

$$C_{\delta_1,\ldots,\delta_{i-1},0,\delta_{i+1},\ldots,\delta_m} \geq C_{\delta_1,\ldots,\delta_{i-1},1,\delta_{i+1},\ldots,\delta_m} \quad \text{for all possible values of } \delta_1, \ldots, \delta_{i-1}, \delta_{i+1}, \ldots, \delta_m \tag{4.1}$$

and is said to be *insensitive in its i-th entry*, if the strict equality holds in (4.1).

For $m \geq 2$, a set of constants $\{C_\delta, \delta \in \{0,1\}^m\}$ is said to have *monotone differences in the pair i, j ($i < j$) of the entries of its subscript*, if for all possible values of $\delta_1, \ldots, \delta_{i-1}, \delta_{i+1}, \ldots, \delta_{j-1}, \delta_{j+1}, \ldots, \delta_m$, one of the following, equivalent among themselves, relations holds:

$$C_{\delta_1,\ldots,\delta_{i-1},0,\delta_{i+1},\ldots,\delta_{j-1},0,\delta_{j+1},\ldots,\delta_m} - C_{\delta_1,\ldots,\delta_{i-1},0,\delta_{i+1},\ldots,\delta_{j-1},1,\delta_{j+1},\ldots,\delta_m} \geq C_{\delta_1,\ldots,\delta_{i-1},1,\delta_{i+1},\ldots,\delta_{j-1},0,\delta_{j+1},\ldots,\delta_m} - C_{\delta_1,\ldots,\delta_{i-1},1,\delta_{i+1},\ldots,\delta_{j-1},1,\delta_{j+1},\ldots,\delta_m}; \tag{4.2}$$

$$C_{\delta_1,\ldots,\delta_{i-1},0,\delta_{i+1},\ldots,\delta_{j-1},0,\delta_{j+1},\ldots,\delta_m} - C_{\delta_1,\ldots,\delta_{i-1},1,\delta_{i+1},\ldots,\delta_{j-1},0,\delta_{j+1},\ldots,\delta_m} \geq C_{\delta_1,\ldots,\delta_{i-1},0,\delta_{i+1},\ldots,\delta_{j-1},1,\delta_{j+1},\ldots,\delta_m} - C_{\delta_1,\ldots,\delta_{i-1},1,\delta_{i+1},\ldots,\delta_{j-1},1,\delta_{j+1},\ldots,\delta_m} \tag{4.3}$$

We say a set has *monotone differences* (there are certain reasons to use here the same term as for the property (2.3)) if (4.2), (4.3) are satisfied for any pair of entries of its subscript. The property of monotonicity of differences is not attributed to the case when $m = 1$.

For $m \geq 2$, a set of constants $\{C_\delta, \delta \in \{0,1\}^m\}$ is said to be *twice monotone* if

$$(i)\ C_\delta \geq 0 \text{ for all } \delta; \quad (ii) \text{ the set is monotone in each entry of its subscript; } (iii) \text{ the set has monotone differences.} \tag{4.4}$$

A set $\{C_\delta, \delta \in \{0,1\}\}$ is said to be *twice monotone* if it satisfies *(i), (ii)* of (4.4). For convenience in the further argument, we would like to give a meaning to the symbol $\{C_\delta, \delta \in \{0,1\}^0\}$. We postulate the latter should be understood as a set consisting of just one constant C. We will call this set *twice monotone* if this constant is non-negative.

Consider an arbitrary sequence $B = (B_1, \ldots, B_b)$ and an arbitrary $n \leq b$. Let $S = \{S_1, \ldots, S_s\}$ denote the projection of $(B_1, \ldots, B_n)$ on $\mathcal{G}$. Let $Y = \{Y_1, \ldots, Y_y\}$ and $X = \{X_1, \ldots, X_x\}$ be two disjoint subsets of $\mathcal{G}$ and such that $X \subseteq S$, $Y \subseteq S$.

We say that B admits *an n-representation with the first class Y and the second class X* if there exists a set of constants $\{C_\delta(\epsilon), \epsilon \in \{0,1\}^y, \delta \in \{0,1\}^x\}$ satisfying

$$\text{for each } \epsilon, \quad \{C_\delta(\epsilon), \delta \in \{0,1\}^x\} \text{ is twice monotone} \tag{4.5}$$

such that (below we start to use the abbreviating notations introduced in the end of Section 2; also, in the vein of (2.9), the sign $\emptyset^{YX}_{\epsilon\delta}$ below should be understood as an abbreviation for $\emptyset^{Y_1 \ldots Y_y X_1 \ldots X_x}_{\epsilon_1 \ldots \epsilon_y \delta_1 \ldots \delta_x}$)

$$\Delta^{B_1} \ldots \Delta^{B_b} I(\emptyset) =$$
$$(-1)^{\text{rank}(B_1 \ldots B_n) S} \sum_{\epsilon \in \{0,1\}^y} (-1)^{|\epsilon|} \Delta^{B_{n+1}} \ldots \Delta^{B_b} I\{ \sum_{\delta \in \{0,1\}^x} (-1)^{|\delta|} C_\delta(\epsilon) \emptyset^{YX}_{\epsilon\delta}\} \tag{4.6}$$

The expression in the right-hand-side of (4.6) will be called *an n-representation of B with the first class Y and the second class X* or shortly *an n-representation of B.* When the second class $X \neq \emptyset$ in an n-representation (4.6), we say this representation is *insensitive at $X_i \in X$* if

$$\text{for each } \epsilon, \quad \{C_\delta(\epsilon), \delta \in \{0,1\}^x\} \text{ is insensitive in the } i\text{-th entry of its subscript} \tag{4.7}$$

It may be possible that either the first class or the second class (but never both!) is the empty set. To avoid possible confusion, we now state explicitly the form which the above definitions acquire in these cases: If $Y = \emptyset$, then there is no first sum in (4.6), $(-1)^{|\epsilon|}$ should be substituted by 1 and $\emptyset^{YX}_{\epsilon\delta}$ should be substituted by $\emptyset^{X}_{\delta}$ in (4.6), also $C_\delta(\epsilon)$ should be substituted by C_δ throughout in (4.5) - (4.7), and finally, the words "for each ϵ" should be omitted in (4.5), (4.7). If $X = \emptyset$, then there is no second sum in (4.6), $(-1)^{|\delta|}$ should be substituted by 1 and $\emptyset^{YX}_{\epsilon\delta}$ should be substituted by $\emptyset^{Y}_{\epsilon}$ in (4.6), also in this case for each ϵ, the corresponding set in (4.5) and in (4.7) has the form $\{C_\delta(\epsilon), \delta \in \{0,1\}^0\}$ and due to our convention adopted in the above paragraph, this set thus consists of just one constant which de denote $C(\epsilon)$.

Lemma 4. The stability of representation.
Assume that for some X, Y and $n < b$, a sequence $B = (B_1, \ldots, B_b)$ admits an n-representation with the first class Y and the second class X. Assume these classes, the site B_{n+1}, and two sets $Y', X' \subseteq \mathcal{G}$ relate between themselves in one of the following two ways;

Way 1 consists of the following cases:
(a) $B_{n+1} \in Y$, $Y' = Y$, and $X' = X$;
(b) $B_{n+1} \notin Y \cup X$, $Y' = Y \cup \{B_{n+1}\}$, and $X' = X$.

Way 2 consists of the following cases:
(c) $B_{n+1} \in Y$, $Y' = Y \setminus \{B_{n+1}\}$, and $X' = X \cup \{B_{n+1}\}$;
(d) $B_{n+1} \notin Y \cup X$, $Y' = Y$ and $X' = X \cup \{B_{n+1}\}$;

(e) $B_{n+1} \in X$, $Y' = Y$, and $X' = X$.

Then B admits an $(n+1)$-representation with the first class Y' and the second class X'. Moreover, in any case from Way 2, this $(n+1)$-representation is insensitive at B_{n+1}.

Proof. Fix arbitrarily a sequence $B = (B_1, \ldots, B_b)$ and an $n < b$. Let $S = \{S_1, \ldots, S_s\}$ denote the projection of $(B_1, \ldots, B_n)$ on $\mathcal{G}$. Assume that for two subsets of $\mathcal{G}$, $Y = \{Y_1, \ldots, Y_y\}$ and $X = \{X_1, \ldots, X_x\}$, the sequence B admits an n-representation with the first class Y and the second class X.

We will first prove the lemma assertion for the case (a). In this case $B_{n+1} \in Y$ and thus $B_{n+1} \in S$ because $Y \subseteq S$. Without a loss of generality, we assume $B_{n+1} = Y_y = S_s$ (observe, this is legitimate since we are free in choosing the order of indexing of the members of the sets Y, X, S). For brevity, we introduce

$$\mathcal{F} := \Delta^{B_{n+2}} \ldots \Delta^{B_b} I \tag{4.8}$$

We then rewrite the n-representation of B, whose existence we have assumed, in the following form (in (4.9), $C_\delta(\epsilon_1, \ldots, \epsilon_{y-1}, 0)$ means $C_\delta(\epsilon)$ with $\epsilon = (\epsilon_1, \ldots, \epsilon_{y-1}, 0)$; the same notational convention regards other constants in the text below)

$$\begin{aligned}(-1)^{\mathrm{rank}(B_1 \ldots B_n)S} & \sum_{\epsilon = (\epsilon_1, \ldots, \epsilon_{y-1}) \in \{0,1\}^{y-1}} (-1)^{|\epsilon|} \Delta^{B_{n+1}} \\ & \times \mathcal{F} \Big\{ \sum_{\delta \in \{0,1\}^x} (-1)^{|\delta|} \big[C_\delta(\epsilon_1, \ldots, \epsilon_{y-1}, 0) \theta^{Y_1 \ldots Y_{y-1} Y_y X}_{\epsilon_1 \ldots \epsilon_{y-1}\, 0\, \delta} \\ & \qquad - C_\delta(\epsilon_1, \ldots, \epsilon_{y-1}, 1) \theta^{Y_1 \ldots Y_{y-1} Y_y X}_{\epsilon_1 \ldots \epsilon_{y-1}\, 1\, \delta} \big] \Big\}\end{aligned} \tag{4.9}$$

Expanding $\Delta^{B_{n+1}}$ due to (2.5), we find that (4.9) equals

$$\begin{aligned}(-1)^{\mathrm{rank}(B_1 \ldots B_n)S} & \sum_{\epsilon = (\epsilon_1, \ldots, \epsilon_{y-1}) \in \{0,1\}^{y-1}} (-1)^{|\epsilon|} \\ & \times \mathcal{F} \Big\{ \sum_{\delta \in \{0,1\}^x} (-1)^{|\delta|} \Big[-C_\delta(\epsilon_1, \ldots, \epsilon_{y-1}, 0) c_{B_{n+1}} (\theta^{Y_1 \ldots Y_{y-1} Y_y X}_{\epsilon_1 \ldots \epsilon_{y-1}\, 0\, \delta}) \\ & \qquad \times \left(\theta^{Y_1 \ldots Y_{y-1} Y_y X}_{\epsilon_1 \ldots \epsilon_{y-1}\, 0\, \delta} - \theta^{Y_1 \ldots Y_{y-1} Y_y X}_{\epsilon_1 \ldots \epsilon_{y-1}\, 1\, \delta} \right) \\ & \qquad + C_\delta(\epsilon_1, \ldots, \epsilon_{y-1}, 1) c_{B_{n+1}} (\theta^{Y_1 \ldots Y_{y-1} Y_y X}_{\epsilon_1 \ldots \epsilon_{y-1}\, 1\, \delta}) \\ & \qquad \left(\theta^{Y_1 \ldots Y_{y-1} Y_y X}_{\epsilon_1 \ldots \epsilon_{y-1}\, 1\, \delta} - \theta^{Y_1 \ldots Y_{y-1} Y_y X}_{\epsilon_1 \ldots \epsilon_{y-1}\, 0\, \delta} \right) \Big] \Big\}\end{aligned} \tag{4.10}$$

For each $(\epsilon_1,\ldots,\epsilon_{y-1}) \in \{0,1\}^{y-1}$ and each $\delta \in \{0,1\}^x$ define

$$C'_\delta(\epsilon_1,\ldots,\epsilon_{y-1}) := C_\delta(\epsilon_1,\ldots,\epsilon_{y-1},0)c_{B_{n+1}}(\theta^{Y_1\ldots Y_{y-1}Y_yX}_{\epsilon_1\ldots\epsilon_{y-1}\,0\,\delta})$$
$$C''_\delta(\epsilon_1,\ldots,\epsilon_{y-1}) := C_\delta(\epsilon_1,\ldots,\epsilon_{y-1},1)c_{B_{n+1}}(\theta^{Y_1\ldots Y_{y-1}Y_yX}_{\epsilon_1\ldots\epsilon_{y-1}\,1\,\delta}) \tag{4.11}$$

and then, for each $\epsilon = (\epsilon_1,\ldots,\epsilon_{y-1},\epsilon_y) \in \{0,1\}^y$ and each $\delta \in \{0,1\}^x$ define

$$K_\delta(\epsilon) := C'_\delta(\epsilon_1,\ldots,\epsilon_{y-1}) + C''_\delta(\epsilon_1,\ldots,\epsilon_{y-1}) \tag{4.12}$$

With the definitions (4.11) and (4.12), the expression (4.10) acquires the following form

$$\begin{aligned}(-1)^{\mathrm{rank}_{(B_1\ldots B_n)}S} &\sum_{\epsilon=(\epsilon_1,\ldots,\epsilon_{y-1})\in\{0,1\}^{y-1}} (-1)^{|\epsilon|}\\ &\times\mathcal{F}\Big\{\sum_{\delta\in\{0,1\}^x}(-1)^{|\delta|}\Big[-K_\delta(\epsilon_1,\ldots,\epsilon_{y-1},0)\theta^{Y_1\ldots Y_{y-1}Y_yX}_{\epsilon_1\ldots\epsilon_{y-1}\,0\,\delta}\\ &\qquad\qquad +K_\delta(\epsilon_1,\ldots,\epsilon_{y-1},1)\theta^{Y_1\ldots Y_{y-1}Y_yX}_{\epsilon_1\ldots\epsilon_{y-1}\,1\,\delta}\Big]\Big\}\\ = -(-1)^{\mathrm{rank}_{(B_1\ldots B_n)}S} &\sum_{\epsilon=(\epsilon_1,\ldots,\epsilon_y)\in\{0,1\}^y}(-1)^{|\epsilon|}\mathcal{F}\Big\{\sum_{\delta\in\{0,1\}^x}(-1)^{|\delta|}[K_\delta(\epsilon)\theta^{YX}_{\epsilon\,\delta}]\Big\}\end{aligned} \tag{4.13}$$

Since $B_{n+1} = S_s$, we have that $\mathrm{rank}_{(B_1\ldots B_n)}S + 1 = \mathrm{rank}_{(B_1\ldots B_{n+1})}S$. Thus, if we define $Y' := Y$ and $X' := X$ then the last line of (4.13) is an $(n+1)$-representation of B with the first class Y' and the second class X', provided

$$\text{for each } \epsilon \in \{0,1\}^y, \text{ the set } \{K_\delta(\epsilon), \delta \in \{0,1\}^x\} \text{ is twice monotone} \tag{4.14}$$

We now present the reasoning which establishes (4.14) for the case $x \geq 2$. Fix an arbitrary $\epsilon = (\epsilon_1,\ldots,\epsilon_{y-1}) \in \{0,1\}^{y-1}$. By the lemma assumption, the set $\{C_\delta(\epsilon_1,\ldots,\epsilon_{y-1},0), \delta \in \{0,1\}^x\}$ (henceforth to be referred to as "the set $\{C\}$") satisfies *(i)*, *(ii)* and *(iii)* of (4.4). From *(i)* and the non-negativity of the flip rate $c.(\cdot)$, we have that

$$C'_\delta(\epsilon) \geq 0, \qquad \forall\delta \in \{0,1\}^x \tag{4.15}$$

Next, for arbitrarily fixed $\delta_2,\ldots,\delta_x \in \{0,1\}$, it holds that

$$C_{0\delta_2\ldots\delta_x}(\epsilon) \geq C_{1\delta_2\ldots\delta_x}(\epsilon)$$

by the mentioned above property *(ii)*, and it also holds that

$$c_{B_{n+1}}(\theta^{Y_1\ldots Y_{y-1}Y_yX_1X_2\ldots X_x}_{\epsilon_1\ldots\epsilon_{y-1}\,0\,0\,\delta_2\ldots\delta_x}) \geq c_{B_{n+1}}(\theta^{Y_1\ldots Y_{y-1}Y_yX_1X_2\ldots X_x}_{\epsilon_1\ldots\epsilon_{y-1}\,0\,1\,\delta_2\ldots\delta_x})$$

due to the property (2.2) of the flip rate and the fact that the configuration in the left hand side of the above inequality has one particle less than that in its right hand side. Consequently, from the definition (4.11), we have that

$$C'_{0\delta_2\ldots\delta_x}(\epsilon) \geq C'_{1\delta_2\ldots\delta_x}(\epsilon).$$

Repeating then the reasoning that brought to the above inequality, for every entry of the subscript of $C'_\delta(\epsilon)$ we conclude that

$$\{C'_\delta(\epsilon), \delta \in \{0,1\}^x\} \text{ is monotone in each entry of its subscript} \tag{4.16}$$

We will now show that

$$\{C'_\delta(\epsilon), \delta \in \{0,1\}^x\} \text{ has monotone differences} \tag{4.17}$$

For fixed $\delta_3, \ldots, \delta_x$, call (below, $C'_{\gamma\sigma\delta_3,\ldots,\delta_x}(\epsilon)$ stands for $C'_\delta(\epsilon)$ with $\delta = (\gamma, \sigma, \delta_3, \ldots, \delta_x) \in \{0,1\}^x$)

$$V_{\gamma\sigma} := C'_{\gamma\sigma\delta_3\ldots\delta_x}(\epsilon), \qquad \gamma, \sigma \in \{0,1\} \tag{4.18}$$

Assume B_{n+1} neighbors both X_1 and X_2. We then have that

$$\begin{aligned} &\text{if for some } j, \quad c_{B_{n+1}}(\theta^{Y_1\ldots Y_{y-1}Y_yX_1X_2X_3\ldots X_x}_{\epsilon_1\ldots\epsilon_{y-1}\,0\;0\;0\;\delta_3\ldots\delta_x}) = \lambda_j \\ &\text{then } c_{B_{n+1}}(\theta^{Y_1\ldots Y_{y-1}Y_yX_1X_2X_3\ldots X_x}_{\epsilon_1\ldots\epsilon_{y-1}\,0\;0\;1\;\delta_3\ldots\delta_x}) \\ &\quad = c_{B_{n+1}}(\theta^{Y_1\ldots Y_{y-1}Y_yX_1X_2X_3\ldots X_x}_{\epsilon_1\ldots\epsilon_{y-1}\,0\;1\;0\;\delta_3\ldots\delta_x}) = \lambda_{j+1} \\ &\text{and } c_{B_{n+1}}(\theta^{Y_1\ldots Y_{y-1}Y_yX_1X_2X_3\ldots X_x}_{\epsilon_1\ldots\epsilon_{y-1}\,0\;1\;1\;\delta_3\ldots\delta_x}) = \lambda_{j+2} \end{aligned} \tag{4.19}$$

Call $\alpha := \lambda_j - \lambda_{j+1}$, $\beta := \lambda_{j+2} - \lambda_{j+1}$. Then,

$$\lambda_j V_{00} - \lambda_{j+1} V_{01} = \lambda_{j+1}(V_{00} - V_{01}) + \alpha V_{00} \geq \lambda_{j+1}(V_{10} - V_{11}) + \beta V_{11} = \lambda_{j+1} V_{10} - \lambda_{j+2} V_{11} \tag{4.20}$$

where the inequality above is because $\alpha \geq \beta \geq 0$ (follows from the properties (2.2) and (2.3) of the flip rate), $V_{00} \geq V_{11}$ (a consequence of $V_{00} \geq V_{01}$ and $V_{01} \geq V_{11}$ which hold due to the monotonicity of the set $\{C\}$) and $V_{00} - V_{01} \geq V_{10} - V_{11}$ (because the set $\{C\}$ has monotone differences). Thus, due to (4.18), $\{C'_\delta(\epsilon), \delta \in \{0,1\}^x\}$ has monotone differences in the first two entries of its subscript in the case when B_{n+1} neighbors both X_1 and X_2. Clearly, there may be three other cases: (1) B_{n+1} neighbors neither X_1 nor X_2; (2) B_{n+1} neighbors X_1 but not X_2; (3) B_{n+1} neighbors X_2 but not X_1. In the case 1, all flip rates in (4.19) equal λ_j; in the case 2, the first and the second rates in (4.19) equal λ_j while the third and the fourth ones equal λ_{j+1}; in the case 3, the first and the third rates are λ_j and the second and the fourth ones are λ_{j+1}. We leave it to the reader to verify that in each case $\{C'_\delta(\epsilon), \delta \in \{0,1\}^x\}$

has monotone differences in the first two entries. The reasoning is similar to the one we presented above. After all possible cases have been considered, the conclusion is that $\{C'_\delta(\epsilon), \delta \in \{0,1\}^x\}$ has monotone differences in the first and the second entries of the subscript. The same is true for any pair of entries of δ by a similar argument. Thus, (4.17) is established and so is the twice monotonicity of $\{C'_\delta(\epsilon), \delta \in \{0,1\}^x\}$. The twice monotonicity of $\{C''_\delta(\epsilon), \delta \in \{0,1\}^x\}$ follows by the same argument (one has only to change λ. to μ. in this argument). By (4.12) this provides (4.14) in the case $x \geq 2$. In the case when $x = 1$ or $x = 0$, the latter case happens when $X = \emptyset$, the proof is similar to that presented above for $x \geq 2$. It is even more simple because in order to demonstrate that the set $\{K\}$ from (4.14) is twice monotone one does not need to show that this set satisfies *(iii)* of (4.4), when $x = 1$, and both *(ii)*, *(iii)* of (4.4), when $x = 0$.

We will now establish the lemma in the case (c). In this case, $B_{n+1} \in Y$, and, similarly as in the argument for the case (a), we assume that $B_{n+1} = Y_y = S_s$. We define $Y' = \{Y'_1, \ldots, Y'_{y-1}\}$, whereas $Y'_i := Y_i$, $i = 1, \ldots, y-1$; we then define $X' = \{X'_1, \ldots, X'_{x+1}\}$, where $X'_i := X_i, i = 1, \ldots, x$, and $X'_{x+1} := Y_y$. Then, for each $\epsilon = (\epsilon_1, \ldots, \epsilon_{y-1}) \in \{0,1\}^{y-1}$ and each $\delta' = (\delta'_1, \ldots, \delta'_x, \delta'_{x+1}) \in \{0,1\}^{x+1}$ we use the constants introduced in (4.11) to define

$$K_{\delta'}(\epsilon) := C'_\delta(\epsilon) + C''_\delta(\epsilon), \quad \text{where } \delta := (\delta'_1, \ldots, \delta'_x) \tag{4.21}$$

With the above notations, we rewrite (4.10), which as we have shown is equivalent to the n-representation of B, in the following form:

$$-(-1)^{\text{rank}_{(B_1 \ldots B_n)} S} \sum_{\epsilon = (\epsilon_1, \ldots, \epsilon_{y-1}) \in \{0,1\}^{y-1}} (-1)^{|\epsilon|} \mathcal{F} \left\{ \sum_{\delta' \in \{0,1\}^{x+1}} (-1)^{|\delta'|} [K_{\delta'}(\epsilon) \theta^{Y' X'}_{\epsilon\, \delta'}] \right\} \tag{4.22}$$

As in the case (a), we have that $\text{rank}_{(B_1 \ldots B_n)} S + 1 = \text{rank}_{(B_1 \ldots B_{n+1})} S$. Thus, (4.22) is an $(n+1)$-representation of B with the first class Y' and the second class X', provided

$$\text{for each } \epsilon \in \{0,1\}^{y-1}, \text{ the set } \{K_{\delta'}(\epsilon), \delta' \in \{0,1\}^{x+1}\} \text{ is twice monotone} \tag{4.23}$$

The argument that proves (4.23) is as follows. Fix an arbitrary $\epsilon = (\epsilon_1, \ldots, \epsilon_{y-1}) \in \{0,1\}^{y-1}$. Directly from the definition,

$$K_{\delta'_1 \ldots \delta'_x 0}(\epsilon) = K_{\delta'_1 \ldots \delta'_x 1}(\epsilon) \quad \text{for any } \delta'_1 \ldots, \delta'_x \in \{0,1\} \tag{4.24}$$

which shows that the set corresponding to the fixed ϵ, is monotone in the last entry of its subscript. The fact that this set is monotone in any other entry follows from (4.16), an analogous property for $\{C''_\delta(\epsilon), \delta \in \{0,1\}^x\}$, and the definition (4.21). Choose now any pair of entries of the subscript of the set $\{K_{\delta'}(\epsilon), \delta' \in \{0,1\}^{x+1}\}$ but such that this pair does not contain the last entry. The fact that this set has monotone differences in this pair follows from (4.17), an analogous property for $\{C''_\delta(\epsilon), \delta \in \{0,1\}^x\}$ and the definition (4.21). Fix now an arbitrary pair of entries of the subscript such that one of them is the last entry. For this pair, the monotonicity in differences holds automatically because of the equality (4.24).

The proof of the lemma in any of the rest of the cases utilizes the ideas shown above and requires minor modifications which we leave to the reader. ♣

Remark 1. The following may help to understand the role of the next result in establishing the main lemma.

Observe that if for a tower $B = (B_1, \ldots, B_b)$ we succeed to show that $\Delta^B I(\emptyset)$ equals

$$(-1)^{\text{rank}_B S} I\{C_0 \emptyset_0^0 - C_1 \emptyset_1^0\} \tag{4.25}$$

where S is the projection of B on $\mathcal{G}$ and $C_0 = C_1 \geq 0$ then we immediately have the conclusion (2.7) for this B (this is because $I(\emptyset_0^0) = 0, I(\emptyset_1^0) = 1$ and $\text{rank}_B S = b$). Observe also that we may consider (4.25) as a b-representation of B with the first class $\emptyset$ and the second class $X = \{0\}$ which is insensitive at 0. The above two observation suggest the following way of proving the main lemma: to find a 1-representation of B and to use Lemma 4 recurrently trying to derive from it a b-representation of B of the form (4.25). However, a simple analysis of the relations (a)-(e) of Lemma 4 shows that this way will bring us to a b-representation of B such that the union of its first and second classes coincides with the projection of B. Thus, the b-representation (4.25) cannot be obtained in the way suggested above, unless B is a column. For this to be possible, we need a method that allows us to pass from a representation to representation reducing the union of the first and the second classes. This method is provided in the following Lemma 5. We observe that this result allows to reduce the second class; if a point from the first class is desired to be reduced, this point should be first passed to the second class using (c) of Lemma 4.

Lemma 5. The method of reduction of the second class.
Assume that for some $m \in [1, b]$ a sequence $B = (B_1, \ldots, B_b)$ admits an m-representation with the first class Y and the second class X and that the following properties are satisfied:

(a) X contains at most three sites;

(b) there is a site in X, which we denote by t, such that the representation is insensitive at t;

(c) if $m < b$ then there is a site in X different from t which we call u, such that any of $B_{m+1}, \ldots, B_b$ neither neighbors nor equals u; if $m = b$ then there is a site in X different from t and 0, which we call u.

Then B admits an m-representation with the same first class Y and the second class $X' = X \setminus \{u\}$ and this representation is insensitive at t.

Proof. Let X consists of three sites, $X = \{t, v, u\}$, let $m < b$ and assume $B = (B_1, \ldots, B_b)$ admits an n-representation with the first class $Y = \{Y_1, \ldots, Y_y\}$ and the second class X. Fix arbitrarily $\epsilon \in \{0,1\}^y$ and introduce

$$\eta := \theta^Y_\epsilon$$

The term that corresponds to the above fixed ϵ in the m-representation of B whose existence we have assumed, has the following form (below, $C_{\gamma 00}(\epsilon)$ means $C_\delta(\epsilon)$ with $\delta = (\gamma, 0, 0)$; similar convention on notation applies to other constants in (4.26) as well as to the constants of the set $\{K\}$ to be defined later):

$$\begin{aligned}(-1)^{|\epsilon|}\Delta^{B_{m+1}}\ldots\Delta^{B_b}I\Big\{\sum_{\gamma\in\{0,1\}}(-1)^\gamma\big[&C_{\gamma 00}(\epsilon)\eta^{vtu}_{\gamma 00} - C_{\gamma 01}(\epsilon)\eta^{vtu}_{\gamma 01} \\ &- C_{\gamma 10}(\epsilon)\eta^{vtu}_{\gamma 10} + C_{\gamma 11}(\epsilon)\eta^{vtu}_{\gamma 11}\big]\Big\}\end{aligned} \tag{4.26}$$

To continue, we need the set of constants $\{K_\sigma(\epsilon), \sigma \in \{0,1\}^2\}$ which will be referred to in the sequel as "the set $\{K\}$" and which we define in the following manner:

$$\text{for each } \gamma \in \{0,1\}, \quad K_{\gamma 0}(\epsilon) := C_{\gamma 00}(\epsilon) - C_{\gamma 01}(\epsilon), K_{\gamma 1}(\epsilon) := C_{\gamma 10}(\epsilon) - C_{\gamma 11}(\epsilon) \tag{4.27}$$

Due to the assumption *(c)* of the lemma, for any $\zeta \in \mathcal{X}$, the value of $\Delta^{B_{m+1}}\ldots\Delta^{B_b}I(\zeta)$ does not depend on the value of ζ at u. Using this independence and utilizing the set $\{K\}$ just defined and the set $X' = X \setminus \{u\} = \{t, v\}$ we rewrite (4.26) as below:

$$\begin{aligned}&(-1)^{|\epsilon|}\Delta^{B_{m+1}}\ldots\Delta^{B_b}I\Big\{\sum_{\gamma\in\{0,1\}}(-1)^\gamma\big[K_{\gamma 0}(\epsilon)\eta^{vt}_{\gamma 0} - K_{\gamma 1}(\epsilon)\eta^{vt}_{\gamma 1}\big]\Big\} \\ &= (-1)^{|\epsilon|}\Delta^{B_{m+1}}\ldots\Delta^{B_b}I\Big\{\sum_{\sigma\in\{0,1\}^2}(-1)^{|\sigma|}K_\sigma(\epsilon)\eta^{X'}_\sigma\Big\}\end{aligned} \tag{4.28}$$

We will now show that $\{K\}$ is twice monotone. Recall that the set $\{C_\delta(\epsilon), \delta \in \{0,1\}^3\}$ of the constants involved in (4.26) is twice monotone since (4.26) is a part of an m-representation. We will abbreviate the notation of this set to just $\{C\}$. Recall also $\{C\}$ is insensitive in the second entry of its subscript due to the assumption *(b)* of the lemma. This insensitivity yields that

$$\text{for each } \gamma \in \{0,1\}, \quad K_{\gamma 0}(\epsilon) = K_{\gamma 1}(\epsilon) \tag{4.29}$$

which proves the insensitivity of $\{K\}$ in the second entry of its subscript. Then, from the definition (4.27) and the monotonicity of differences of $\{C\}$ in the first and the third entries of its subscript, we have that

$$K_{00}(\epsilon) \geq K_{10}(\epsilon), \quad K_{01}(\epsilon) \geq K_{11}(\epsilon) \tag{4.30}$$

which proves $\{K\}$ is monotone in the first entry of its subscript. Next, the non-negativity of each member of $\{K\}$ follows from the monotonicity of $\{C\}$ in the third entry of its subscript. Finally, the monotonicity of differences of $\{K\}$ is automatically guaranteed by the equalities (4.29).

We thus have shown that for any $\epsilon \in \{0,1\}^{Y}$, (4.26) equals (4.28) and $\{K_\sigma(\epsilon), \sigma \in \{0,1\}^2\}$ is twice monotone and insensitive in the second entry of its subscript. This establishes the lemma for the case when $m < b$ and X consists of three sites. The case when it consists of just two sites follows in the same manner; the treatment is even more simple since in this case, the subscript of $\{K\}$ has only one entry and consequently, one has to verify solely the properties *(i)*, *(ii)* from (4.4) in order to establish the twice monotonicity of $\{K\}$. The proof of the lemma for the case $m = b$ will be omitted since it utilizes the tools which we have demonstrated above for the case $m < b$. ♣

Remark 2. Let $\{C\}$ and $\{K\}$ have the meanings as prescribed in the proof of Lemma 5. We observe that the monotonicity of differences of the set $\{C\}$ was used in this proof to establish the monotonicity of the set $\{K\}$ in each entry of its subscript. As for the monotonicity of differences of $\{K\}$, we remark it automatically followed from the fact that the subscript of $\{K\}$ has no more than two entries and from the insensitivity of $\{K\}$ in one of these entries. Thus, for Lemma 5 to hold it is crucial to require that X has at most three sites. Unfortunately, when we tried to adapt our proofs to a Bethe lattice of degree ≥ 3, we obtained representations whose second class contained more than 3 sites. This prevents us from extending Theorem 1 to a class of lattices wider than just $\mathbb{Z}$ and tori in $\mathbb{Z}$. Our conjecture is that for this theorem to hold for a Bethe lattice of degree ≥ 3, the definition of a representation should be modified so that the set of constants $\{C_b(\epsilon)\}$ satisfies a condition stronger than (4.5). This will then alterate appropriately the condition imposed on the flip rate in the formulation of the theorem.

5. THE PROOF OF THE MAIN LEMMA FOR ONE DIMENSIONAL INFINITE LATTICE

We recall that the assertion (b) of the main lemma has been established in Section 3. Here we will establish its assertion (a). Throughout this section, $\mathcal{G} = \mathbb{Z}$.

Let q and p denote respectively, the leftmost and the rightmost point of the projection of a tower B on $\mathcal{G}$. We will conduct the proof for the case

$$q \leq -1 \text{ and } f(q) < f(p) \tag{5.1}$$

Since B is a tower, there may be only two other cases: either B is a column or $p \geq 1$, and $f(p) < f(q)$. The former case is treated in (b) of the main lemma, the latter can be transformed to (5.1) by the reflection $\mathbb{Z} \to -\mathbb{Z}$ without alterating the value of $\Delta^B I(\emptyset)$.

Let $S = \{S_1\}$ where $S_1 := B_1$ so that S is the projection of (B_1) and $\text{rank}_{(B_1)}S = 1$. From (2.5) applied for Δ^{B_1} we then have that

$$\Delta^{B_1}\Delta^{B_2}\ldots\Delta^{B_k}I(\emptyset) = (-1)^{\text{rank}_{(B_1)}S}\sum_{\delta\in\{0,1\}}(-1)^{|\delta|}C_\delta\Delta^{B_2}\ldots\Delta^{B_k}I(\emptyset_\delta^S) \tag{5.2}$$

where

$$C_0 = C_1 = c_{B_1}(\emptyset) = \lambda_0 \tag{5.3}$$

If $f(q) = 1$ (which means, in particular, that $B_1 = q$) then (5.2) and (5.3) say that

$$\begin{aligned}&B \text{ admits a } f(q)\text{−representation with the second class } \{B_1\} = \{q\}\\ &\qquad\qquad\text{and the first class } \emptyset\end{aligned} \tag{5.4}$$

If to the contrary, $f(q) \neq 1$ then we modify (5.2) and (5.3) in the following way: we write $C(\epsilon)$ instead of C_δ, and $C(0)$ and $C(1)$ instead of respectively C_0 and C_1, and finally we write ϵ instead of δ throughout. The modified relations (5.2), (5.3) now say that

$$B \text{ admits a 1−representation with the second class } \emptyset \text{ and the first class } \{B_1\} \tag{5.5}$$

In the case $f(q) \neq 1$, we conduct the following procedure: we start from (5.5) and apply Lemma 4 successively with $n = 1, \ldots, f(q) - 2$, each time constructing Y' and X' in the second way. After the $(f(q)-2)$-nd application of Lemma 4, the conclusion is that (Remark 4 below will explain the way the second line in (5.6) is concluded)

$$\begin{aligned}&B \text{ admits an } (f(q)-1)\text{−representation with the second class } \emptyset\\ &\text{and an appropriate first class that does not contain points to the left of } q\end{aligned} \tag{5.6}$$

Remark 3. Here and below, the phrase "we apply Lemma 4 with a certain n" should be understood in the following way: Firstly, this phrase assumes there has been already established the fact that B admits an n-representation. Then, "to apply the lemma" means to take this representation as the lemma's assumption and to use the lemma's assertion to conclude that B admits an $(n+1)$-representation. We remark that the way the first and the second classes of these two representations relate between themselves is always implicitly stated by us, and in such statement, Y' and X' denote respectively, the first and the second class of the gotten $(n+1)$-representation.

Remark 4. Observe that (a)-(e) of Lemma 4 guarantee that $X' \cup Y' = X \cup Y \cup B_{n+1}$ where Y and X are the first and the second classes of an n-representation of $B = (B_1, \ldots, B_k)$ and Y' and X' are the first and the second classes of the $(n+1)$-representation of B obtained from this n-representation by application of Lemma 4. This observation together with the fact that the fist and the second classes of a representation are disjoint will be usually used below to derive that certain points of $\mathbb{Z}$ do not belong to the first class of representations of B. The second line in (5.6) is an example of such a conclusion. It follows from the fact that the first class of the 1-representation in (5.5) was $\{B_1\}$, the manner (5.6) is obtained from (5.5), the definition of q and the observation in this remark.

The conclusion (5.6) and Lemma 4 applied with $n = f(q) - 1$ and X', Y' constructed in the second way, provide that

$$\begin{array}{l} B \text{ admits an } f(q)\text{-representation with the second class } \{q\} \\ \text{and an appropriate first class that contains no points to the left of } q+1 \end{array} \tag{5.7}$$

Comparing the latter with (5.4) shows that (5.7) may be taken as a general statement that holds independently of whether $f(q) = 1$ or not.

Remark 5. We will pause to give the precise meaning to the words "appropriate first class" which a reader has met in (5.6) and (5.7) and also will meet below. Consider, for example, the first class of the $f(q)$-representation in (5.7) for the case $f(q) \neq 1$. Tracking the way this representation has been obtained, it is easy to conclude this class equals the projection of $(B_1, \dots, B_{f(q)})$ without the site q. In general, the first class in any n-representations which we construct, can be described as the projection of $(B_1, \dots, B_n)$ without certain sites. The reason which prevented us from describing precisely the first classes is our aim to present the argument in its most general form. For example, we want (5.7) to be true independently of whether $f(q) = 1$ or not. But in the case $f(q) = 1$ the first class is $\emptyset$, while in the opposite case it is as described above. Thus, taking into account that we do not need the exact form of the first class to continue, we restrict ourselves to referring to it as "an appropriate class". What however, we do need is that it does not contain points to the left of $q + 1$, and this fact is stated explicitly in (5.7).

Assume B is such that the $(q+1)$-st roof is immediately above the q-th roof. We then denote by i the maximal integer such that

$$\forall j = 0, 1, \dots, i-1, \text{ the } (q+j+1)\text{-the roof is immediatly above the } (q+j)\text{-the roof} \tag{5.8}$$

We then stem from (5.7), which as we have shown holds in general, and apply recurrently Lemma 4 with $n = f(q), \dots, f(q+1) - 2$, each time constructing X' and Y' in the first way. To the obtained result we apply Lemma 4 with $n = f(q+1) - 1$ but now we construct X', Y' in the second way. The resulting $f(q+1)$-representation of B has the second class $\{q, q+1\}$ and is insensitive at $q+1$ as guaranteed by Lemma 4. We now proceed using essentially the tower structure of B. This structure provides that any one of $B_{f(q+1)+1}, \dots, B_k$ neither neighbors to nor equals q. This allows to apply Lemma 5 to the obtained $f(q+1)$-representation taking $t = q + 1$ and $u = q$ in this lemma. The result of this application is as follows

$$\begin{aligned}&B \text{ admits an } f(q+1)\text{-representation with the second class } X = \{q+1\}\\ &\text{and an appropriate first class that contains no points to the left of } q+2;\\ &\text{moreover, this representation is insensitive at } q+1\end{aligned} \tag{5.9}$$

If i from (5.8) is 1 then (5.9) is what we need and we proceed directly to the next paragraph. If not, we conduct the following procedure: In the argument from the above paragraph which we used to derive (5.9) from (5.7), we substitute q and $q+1$ by respectively, $q+1$ and $q+2$. We then apply the modified argument to (5.9). This gives (5.10) below with $j = 2$.

$$\begin{aligned}&B \text{ admits an } f(q+j)\text{-representation with the second class } X = \{q+j\}\\ &\text{and an appropriate first class that contains no points to the left of } q+j+1;\\ &\text{moreover, this representation is insensitive at } q+j\end{aligned} \tag{5.10}$$

It is straightforward to verify that if (5.10) holds for some $j = m < i$ then the application of the same argument with appropriate changes of q and $q+1$ gives that (5.10) holds for $j = m+1$ as well. By induction, we then have that

$$\begin{aligned}&B \text{ admits an } (f(q+i))\text{-representation with the second class } X = \{q+i\}\\ &\text{and an appropriate first class that contains no points to the left of } q+i+1\\ &\text{moreover, this representation is insensitive at } q+i\end{aligned} \tag{5.11}$$

The tools that have been developed by now, enable us to conclude the proof of the main lemma when B is a one-side tower. Let us do so in this paragraph. For a one-side tower B satisfying (5.1), we have that (5.8) holds for i such that $q+i = 0$. For this i, the $f(q+i)$-representation whose existence is guaranteed by (5.11), must have the first class $\emptyset$ (because of what stated in the second line of (5.11)) and the second class $\{0\}$, and also, this representation is insensitive at 0. Thus, this representation has the form (4.25) and as it has been shown in Remark 1, this fact proves that the property (2.7) holds for the considered B.

We now consider the case when B is a two side tower. If the $(q+1)$-st roof is immediately above the q-th one, we define i as the maximal integer for which (5.8) holds, in the opposite case, we set $i := 0$. With such a definition of i, we have that the p-th roof is immediately above the $(q+i)$-th one independently of the value of i. We then proceed in the following way: if $i \geq 1$ then the basis of our argument is the assertion (5.11); if $i = 0$ then it is (5.7). To an appropriately chosen assertion, we apply Lemma 4 recurrently with $n = f(q+i), \ldots, f(p) - 2$, each time constructing X', Y' in the first way. To the obtained result, we apply Lemma 4 with $n = f(p) - 1$

but now we construct X', Y' in the second way. The result is as follows:

B admits an $(f(p))-$representation with the second class $X = \{q+i, p\}$
and an appropriate first class that contains no points (5.12)
to the left of $q+i+1$ and to the right of $p-1$

Let us now define k as the integer such that all $B_{k+1}, B_{k+2}, \ldots, B_b$ equal 0 but B_k does not. Observe such k necessarily exists and also $k \geq f(p)$ due to the tower structure of B and the assumption (5.1). We now start from the conclusion (5.12) and recurrently for each $n = f(p), \ldots, k-1$, we conduct the following procedure: If $n+1$ is not a roof (to say x is a roof is a short form to say that $x = f(y)$ for some y from the projection of B on $\mathcal{G}$) then at this step of our procedure we apply Lemma 4 with n and construct X' and Y' the first way; if, to the contrary, $n+1$ is a roof then we apply Lemma 4 with n but we construct X' and Y' in the second way, and then, we apply Lemma 5 to the $(n+1)$-representation obtained. In this lemma, we take $t = B_{n+1}$ and u to be that site of the second class for which it holds that the t-th roof is immediately above the u-th roof. Observe that if $n+1$ is a roof then $n+1 = f(B_{n+1})$ and thus, the application of Lemma 4 gives an $(n+1)$-representation of B which is insensitive at B_{n+1}. Observe also that every time when $n+1$ is not a roof, our application of Lemma 4 does not increase the cardinality of the second class, while when $n+1$ is a roof, this set acquires one more site after the application of Lemma 4 but then, looses one site after the application of Lemma 5. It is then clear that after each step of our procedure, the cardinality of second class is 2. There is one more important observation to be done: Let $\ell < 0$ and $r > 0$ be two sites from the projection of a two-side tower B. Consider the roofs $f(\ell)$ and $f(r)$ and consider the roof which is immediately above the highest one of them. Due to the tower structure of B we have that the latter is either the $(\ell+1)$-st roof or the $(r-1)$-st one. Employing again the tower structure of B we also have that in the first case, any of $B_{f(\ell+1)+1}, \ldots, B_b$ neither neighbors nor equals to ℓ, while in the second case any of $B_{f(r-1)+1}, \ldots, B_b$ neither neighbors nor equals to r. The above three observations help to understand why and how we apply Lemma 5. With the aid of the signs introduced in the third observation, we even may be more specific: when using this lemma, we take $u = \ell$, $t = \ell+1$, if $\ell+1 = B_{n+1}$, and we take $u = r$, $t = r-1$, if $r-1 = B_{n+1}$. By induction in n, with (5.12) being the basis of the induction, it is easy to see that (in this inductive argument, one uses the property observed in Remark 4 to control the transformation of the first class)

for each $n = f(p)+1, f(p)+2, \ldots, k$, the tower B admits an $n-$representation
with the second class $\{\ell, r\}$ and the first class contained inteiraly in
$\{\ell+1, \ell+2, \ldots, -1, 0, 1, \ldots, r-2, r-1\}$where ℓ is the maximal among
$\{q+i, q+i+1, \ldots, -1\}$ such that the $\ell-$th roof is not greater than n,

and r is the minimal among $\{1, 2 \ldots, p\}$ such that the r−th roof is not greater that n (observe, such ℓ and r exist due to the tower structure of B and because of our definition of k);

(5.13)

For $n = k$, let us consider the n-representation of B provided by (5.13). If $n = k$ then necessarily ℓ and r in (5.13) are -1 and 1, respectively. This is because of our definition of k and because B is a two-side tower. Thus, the first and the second classes in this k-representation of B are either *(a)* $Y = \{0\}$, $X = \{-1, 1\}$, or *(b)* $Y = \emptyset$, $X = \{-1, 1\}$, or *(c)* $Y = \emptyset$, $X = \{-1, 0, 1\}$. The continuation of our argument suits any of these three cases. To this k-representation we apply recurrently Lemma 4 with $n = k, k+1, \ldots, b-1$ each time constructing X', Y' in the second way. After the last application, the result is as follows:

B admits a b−representation with the second class $X = \{-1, 0, 1\}$ and the first class $\emptyset$ and this representation is insensitive at 0 (5.14)

To (5.14) we apply Lemma 5 with $t = 0$ and $u = 1$. The result is a b representation of B with the second class $\{-1, 0\}$ and the first class $\emptyset$, moreover, this representation is insensitive at 0. The latter allows to apply once more Lemma 5 but now with $t = 0$, $u = -1$. The obtained b-representation has the form (4.25) and, as we have shown in Remark 1, this yields that the property (2.7) holds for the considered B. ♣

6. THE PROOF OF THE MAIN LEMMA FOR ONE DIMENSIONAL FINITE LATTICE WITH PERIODIC BOUNDARY

Throughout this section N will be a fixed natural greater than 3, though arbitrary, and $\mathcal{G}$ will denote the torus in $\mathbb{Z}$ of size N (called in the literature also as the one-dimensional lattice of size N with periodic boundary), that is, $\mathcal{G}$ is the graph with the vertex set $\{0, 1, \ldots, N-1\}$ and the edge set such that an edge connects two vertices i, j if and only if either $|i - j| = 1$ or $|i - j| = N - 1$.

Let $B = (B_1, \ldots, B_b)$ be a tower on $\mathcal{G}$ such that there exists a site $k \in \mathcal{G}$ such that none of $B_1, \ldots, B_b$ equals k. Define the mapping $\mathcal{M}_k : \mathcal{G} \to \mathbb{Z}$ by

$$\mathcal{M}_k(x) := \begin{cases} x & \text{if } x \leq k \\ x - N & \text{otherwise} \end{cases} \tag{6.1}$$

and construct the sequence $\bar{B} = (\bar{B}_1, \ldots, \bar{B}_b)$ of sites of $\mathbb{Z}$ by $\bar{B}_i := \mathcal{M}_k(B_i), i = 1, \ldots, b$. It is straightforward to see that this mapping guarantees that firstly, $\bar{B}_b = 0$ and secondly, $\bar{B}_i$ and $\bar{B}_j$ neighbor each other (coincide) if and only if B_i and B_j neighbor each other (resp., coincide). Thus $\Delta^B I(\emptyset) = \Delta^{\bar{B}} I(\emptyset)$. (Note: I and $\emptyset$ in the left hand side are defined on $\mathcal{G}$ while those from the right hand side are defined on $\mathbb{Z}$.) Using the result of the previous section, we then have that the assertion of the main lemma holds for the considered B.

We now consider a tower B whose projection is the whole $\mathcal{G}$. Call q the site of $\mathcal{G}$ whose roof is the lowest one among all the roofs of B. Observe $q \neq 0$ because B is a tower. Repeat then the argument that brought to (5.7) in Section 5. In the case considered in the present section, this argument gives that

$$\begin{array}{l} B \text{ admits an } (f(q))\text{–representation with the second class } \{q\} \\ \text{and an appropriate first class that does not contain the site } q \end{array} \tag{6.2}$$

We continue our argument under the assumption the $(q+1)$-st roof is immediately above the q-th roof. To treat the opposite case, one just inverts the directions of numbering of sites of the torus $\mathcal{G}$ ($0 \to 0$, $i \to N-i$, $i = 1, \ldots, N-1$) in the argument to be presented below. We start from (6.2) and apply successively Lemma 4 with $n = f(q), \ldots, f(q+1)-2$, each time constructing X', Y' in the first way. To the final result, we apply Lemma 4 with $n = f(q+1)-1$, but now we construct X', Y' in the second way. The conclusion is that

$$\begin{array}{l} B \text{ admits an } (f(q+1))\text{–representation with the second class } \{q, q+1\} \\ \text{and an appropriate first class that contains neither } q \text{ nor } q+1 \end{array} \tag{6.3}$$

The reasoning presented in this paragraph applies solely to the case when the $(q+2)$-nd roof is immediately above the $(q+1)$-st one. In this case, we define i as the maximal integer for which (5.8) holds. We then apply to (6.3) Lemma 4 recurrently with $n = f(q+1), \ldots, f(q+2)-2$, each time constructing X', Y' in the first way. To the obtained result, we apply Lemma 4 with $n = f(q+2)-1$, but now we construct X', Y' in the second way. This procedure gives an $f(q+2)$-representation of B with the second class $\{q, q+1, q+2\}$ and an appropriate first class that does not contain any of $q, q+1, q+2$, moreover, this representation is insensitive at $q+2$. Due to the tower structure of B, any of $B_{f(q+2)+1}, \ldots, B_b$ neither neighbors to nor equals $q+1$. This allows to apply to the obtained representation Lemma 5 with $t = q+2$ and $u = q+1$. The result reads:

$$\begin{array}{l} B \text{ admits an } (f(q+2))\text{–representation with the second class } \{q, q+2\} \\ \text{and an appropriate first class that does not contain } q, q+1, q+2 \end{array} \tag{6.4}$$

Recall the argument from the previous section which we used to derive (5.11) from (5.9). A similar argument being applied to (6.4) gives that

$$\begin{array}{l} B \text{ admits an } (f(q+i))\text{–representation with the second class } \{q, q+i\} \\ \text{and an appropriate first class that does not contain the sites } q, q+1, \ldots, q+i \end{array} \tag{6.5}$$

If $(q+2)$-nd roof is immediately above the $(q+1)$-st one we define i to be the maximal integer for which (5.8) holds and we proceed stemming from (6.5). In the opposite case, we define $i = 1$ and proceed stemming from (6.3). Observe, i is defined in such a way that the $(q-1)$-st roof is immediately above the $(q+i)$-th one in both cases. To the assertion appropriately chosen among (6.3) and (6.5), we apply Lemma 4 recurrently with $n = f(q+i), \ldots, f(q-1)-2$, each time constructing X', Y' in the first way. To the obtained result we apply the same Lemma 4 with $n = f(q-1)-1$ and X', Y' constructed in the second way. The result is

B admits an $(f(q-1))$–representation with the second class $\{q+i, q, q-1\}$ and an appropriate first class that does not contain the points $q-1, q, q+1, \ldots, q+i$, moreover, this representation is insensitive at $q-1$ (6.6)

In the case $q-1=0$ we terminate the proof similar to how it has been done after (5.14): we apply to (6.6) Lemma 5 twice, the first time with $t = q-1 = 0$ and $u = q = 1$ and the second time with $t = q-1 = 0$ and $u = q+i = N-1$. These applications give the b-representation of the form (4.25), which, as we have shown, completes the proof.

In the case $q-1 \neq 0$, we apply to (6.6) Lemma 5 with $t = q-1$ and $u = q$. The result says:

B admits an $(f(q-1))$–representation with the second class $\{q+i, q-1\}$ and an appropriate first class that does not contain the points $q-1, q, q+1, \ldots, q+i$, (6.7)

The argument which we presented in Section 5 starting right after (5.12) and finishing at the end of that section, will be now repeated literally with the following modifications: (5.12) in that argument should be substituted by (6.7), q and i are exactly q and i defined in the present section, p should be substituted by $q-1$, and any site z from $\mathbb{Z}$ should be substituted by the site $\gamma \in \mathcal{G}$ such that $\mathcal{M}_q(\gamma) = z$ (the transformation $\mathcal{M}_q$ has been defined in (6.1)). With such modifications, this argument leads to a b-representation of the form (4.25), which as we have shown in Remark 1, completes the proof. ♣

7. APPLICATIONS AND EXTENSIONS

The following explains from where the question about the change of the sign of derivatives of a density function came, and how an affirmative answer to this question may be utilized.

Mann et al. (1988, 1989) and Granovsky et al. (1989) suggested that a particular spin flip system is capable to emulates appropriately the nucleation phenomenon that is observed for a class of deposition processes from the field of physical chemistry. This system is exactly the process $\varphi.$ that has been defined by (2.0), (2.1) in Section 1 of the present paper, with an additional condition on the flip rate that is as following:

$$\lambda_0 = \lambda_1 = \ldots = \lambda_g \tag{7.1}$$

$$\mu_0 \geq \mu_1 \geq \dots \geq \mu_g \tag{7.2}$$

Belitsky (1991) then showed that

Theorem 2. *If $\mathcal{G}$ is an arbitrarily fixed regular lattice and $\varphi.$ is the spin flip system on $\mathcal{G}$ defined by (2.0), (2.1) and satisfying (7.1), then the density function of this process determines uniquely the values of its flip rate.*

The above result, the application of $\varphi.$ as suggested by Mann and collaborators, and the fact that the density of a deposit can be easily measured in experiments, motivated us to search for an algorithm that infers on the values of the flip rate of $\varphi.$ from the values of ρ_t^{exp} for $t \in [t_0, t_1] \subset [0, \infty)$, provided we know that $\rho_t^{exp}, t \geq 0$, is the density function of $\varphi.$ satisfying (2.0), (2.1),(7.1) on a lattice $\mathcal{G}$, and provided the structure of this lattice is known to us.

Unfortunately, the argument that proves Theorem 2 cannot be taken as a basis to construct such an algorithm. This argument is as follows: $d/dt\,\rho_0 = \lambda$, and for each $k = 2, 3, \dots, g+2$, $d^k/dt^k \rho_0$ is a polynomial in $\lambda, \mu_0, \dots, \mu_{k-2}$ whose coefficients depend on $\mathcal{G}$, and this polynomial is linear in μ_{k-2}. This provides that the first $g+2$ derivatives of ρ_t at $t = 0$ determine the flip rates uniquely. However, any numerical differentiation method will provide $d^k/dt^k \rho_0^{exp}$, $k = 1, \dots, g+2$, with certain errors and these errors will affect essentially the inferential values, if the inference procedure uses the polynomials from the argument above. Another reason that prevents one from applying this inference procedure is that the values of ρ_t^{exp}, which we regard as experimental data, are usually known for $t \in [t_0, t_1]$ where $t_0 > 0$.

Belitsky and Kohayakawa (1995) suggested to take for the inferential values the flip rate $\lambda, \mu_0, \dots, \mu_g$ which minimize the distance between the density function of $\varphi.$ with this flip rate and the function ρ_t^{exp} on $t \in [t_0, t_1]$. In this work, ρ_t is suggested to be approximated by

$$\sum_{k=0}^{n} \frac{t^k\, L^k I(\emptyset)}{k!} \tag{7.3}$$

In order to estimate the error of the suggested approximation, Belitsky and Kohayakawa give a bound on the growth of $|L^k I(\emptyset)|$. They also suggest to use a certain result of Belitsky (1991) which allows us to minimize the amount of calculation needed to find $L^k I(\emptyset)$. We must say, however, that in order to get a good approximation, the value of n in (7.3) should be taken quite large. When we used a computer to calculate $L^k I(\emptyset)$ we saw it will require an unreachable amount of time to get this expressions for k close to n. We are currently working on improving the algorithm we used for the computer programme so that it gives $L^k I(\emptyset)$ approximately but fast.

Actually, the phenomenon (2.4) was discovered when we tested the above mentioned programme. Clearly, when (2.4) holds, the difference between ρ_t and (7.3) is estimated in the most simple manner.

In the view of the above mentioned application, it is desirable to extend Theorem 1 to a wider class of lattices. In this respect we note that the reasoning we used in Section 5 can be adapted to prove this theorem for any Bethe lattice. By the reason explained in Remark 2, this adaptation must impose a stronger conditions on the flip rate than those required by Theorem 1 in this paper. This alteration is technical however. The main idea of the proof remains the same as for the lattice $\mathbb{Z}$, and the reason for this is that the projection of a tower on $\mathbb{Z}$ as well as on a Bethe lattice does not contain loops. It is the presence of such loops that makes difficult the proof of Theorem 1 for the case $\mathbb{Z}^2$ (and $\mathbb{Z}^d$, in general). If the projection contains just one loop, the argument of Section 6 works. Meanwhile, we do not possess tools to treat the case when there are several loops. We thus resorted to computer simulations. We tested the process $\varphi_\cdot$ on $\mathbb{Z}^d$ with $d = 2$ for which

$$\lambda_k = 1 \text{ and } \mu_k = \Lambda(2d - k), \quad k = 0, 1, \ldots, 2d, \quad \text{for some } \Lambda > 0 \tag{7.4}$$

The result is that $\text{sign}(d^n/dt^n \rho_0) = (-1)^{n+1}, n = 1, 2, \ldots, 8$, for each $\Lambda > 0$.

Finally, we would like to remark that the class of processes to which Theorem 1 applies, contains the Contact Process which starts from the state "all individuals infected" and also, a particular stochastic Ising model that starts from the state "all spins down"(the definition of both processes can be found in the book of Liggett (1985)). To bring the Contact Process into our setting, put 0 to denote the state "healthy" and 1 to denote the state "infected". With such an interpretation of 0 and 1, the flip rate of the Contact Process has the form (7.4) where Λ is the parameter of this process that is customarily called "the rate of the infection spread". The value ρ_t corresponds then to the density of the healthy individuals in the Contact Process at time t given it started from the state "all individuals infected". Consider then any nearest neighbor symmetric translation invariant potential. To construct a stochastic Ising model whose flip rate $c_\cdot(\cdot)$ satisfies (2.1), (2.2), (2.3) and which corresponds to this potential, one takes 0 and 1 to stand for respectively, the states "a spin down" and "a spin up" and constructs the flip rate using the methods utilized in Example 5 of Neves and Schonmann (1989). Clearly, in this case $2\rho_t - 1$ expresses the quantity that is customarily called "the magnetization at time t", if -1 and $+1$ denoted "down" and "up" respectively.

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References

V. Belitsky, Dynamics of spin systems on finite and infinite lattices. Ph. D. thesis. Technion - The Israel Institute of Technology, Haifa, Israel, 1991.

V. Belitsky and Y. Kohayakawa, Inference on the flip rates from the density dynamics for a class of spin-flip systems. Technical Report No. 9514 of the Department of

Statistics of the Institute of Mathematics and Statistics at the University of São Paulo, São Paulo, Brazil, 1995. To be submitted to *The Journal of Statistical Physics.*

B. L. Granovsky, J. A. Mann, T. Rolski and W. A. Woyczynski, A general stochastic model of adsorption–desorption: transient behavior. *Chemometrics and Intelligent Laboratory Systems.* Vol. 6, 1989, pp. 273–280.

R. Holley, Rapid convergence to equilibrium in ferromagnetic Stochastic Ising Model. *Resenhas do Instituto de Matemática e Estatística da Universidade de São Paulo,* Vol. 1, No. 2/3, 1994, p.131.

T. Liggett, *Interacting particle systems.* Springer, 1985.

J. A. Mann, J. Szulga, S. Reazor, W. A. Woyczynski and B. Ycart, Monte-Carlo simulation of some interacting particle systems in the plane: towards the theory of underpotential deposition. *Chemometrics and Intelligent Laboratory Systems,* vol.3, 1988, pp.141–149.

J. A. Mann, J. Szulga, S. Reazor, W. A. Woyczynski and B. Ycart, An interacting particle model of adsorption. *Applicationes Mathematicae.* vol.20, No3, 1989.

E. J. Neves and R. H. Schonmann, Behavior of droplets for a class of Glauber dynamics at a very law temperature. Probab. Theory Relat. Fields, Vol. 91, 1992, pp. 331-354.

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