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ABSTRACT

An extended notion of cone K in a linear vector space is introduced and certain properties of characteristic values and eigenvectors of a linear operator mapping $K\setminus\{0\}$ into its interior are derived by proving a theorem which extends the classical result of Perron on positive matrices.

1 - INTRODUCTION

Let E, dim E = n, be a finite dimensional real vector space; let L(E) be the set of linear operators on E, r(T) the spectral radius of $T \in L(E)$ and $\sigma(T)$ the spectrum of T. The celebrated theorem of Perron [G] can be stated as follows

Theorem - If $C \subset E$ is a closed cone with nonempty interior and $T \in L(E)$ is a linear operator such that

$$T(C\setminus\{0\}) \subset \text{int } C ,$$

then T has an eigenvalue λ with is positive and equal to r(T), $\lambda > |\mu|$ for all $\mu \in \sigma(T)$, $\mu \neq \lambda$ and the algebraic multiplicity of λ is one. A corresponding unit eigenvector v can be chosen in int C; moreover v is the unique unit eigenvector of T in C .

In the statement of this theorem, a cone is, by definition, a set $C \subset E$ which satisfies the following conditions:

(i) $x \in C$, $-x \in C \Rightarrow x = 0$,

(ii) $x,y \in C$; $\alpha,\beta \geq 0 \Rightarrow \alpha x + \beta y \in C$.

If C is a closed set which satisfies these conditions, then the set $K = C \cup (-C)$, $-C = \{x \mid -x \in C\}$, satisfies:

a) $x \in K$, $\alpha \in \mathbb{R} \Rightarrow \alpha x \in K$,

b) $\max\{\dim W \mid W \text{ a subspace, } W \subset K\} = 1$.

Condition a) is obvious. To show b) we note that, if $W \neq \{0\}$ is a subspace contained in K and S is the unit sphere in W , the sets $S^+ = S^+ \cap C$, $S^- = S \cap (-C)$ are disjoint and closed because (i) implies $C \cap (-C) = \{0\}$ and C is closed. It follows that $S = S^+ \cup S^-$ is disconnected and therefore $\dim W = 1$. On the other hand it is clear that there exist closed sets K which satisfy a) and b) and cannot be written as $K = C \cup (-C)$ with C closed and satisfying (i) and (ii). However it has been shown in [AF] that one has the following theorem which implies the theorem of Perron

Theorem - If $K \subset E$ *is a closed set which satisfies* a) *and* b), $\operatorname{int} K \neq \phi$, *and* $T \in L(E)$ *is such that*

$$T(K\setminus\{0\}) \subset \operatorname{int} K ,$$

then there exist subspaces W_1, W_2, $W_1 \cap W_2 = \{0\}$, $\dim W_1 = 1$, *dim* $W_2 = n-1$ *which are invariant under* T *and such that* $W_1 \subset \{0\} \cup \text{int } K$, $W_2 \cap K = \{0\}$. *Moreover if* λ *is the eigenvalue of* T *corresponding to* W_1 *and* $\mu \in \sigma(T)$, $\mu \neq \lambda$, *then*

$$r(T) = |\lambda| > |\mu|.$$

The fact that one can prove a theorem which is essentially equivalent to Perron's theorem by defining a cone as a set which satisfies a) and b), more than in itself, it is interesting because, inusmuch the standard definition of a cone based on (i) and (ii), the definition adopted in the previous theorem can be naturally generalized to cover cones of "dimension-d" by simply changing in b) the number 1 to a generic integer $d \geq 1$ and, then, the question of a corresponding generalized version of Perron's theorem arises. The aim of this work is to give a positive answer to this question by presenting a proof of the following theorem:

Theorem -1. Let $K \subset E$ *be a closed set with non empty interior and let* $T \in L(E)$. *Assume that*

h_1) $x \in K$, $\alpha \in \mathbb{R} \Rightarrow \alpha x \in K$,

h_2) $\max\{\dim W \mid W \text{ a subspace}, W \subset K\} = d$, $1 \leq d < n$,

h_3) $T(K \setminus \{0\}) \subset \text{int } K$,

then there exist (unique) subspaces W_1, W_2 *such that*

1) $W_1 \cap W_2 = \{0\}$, $\dim W_1 = d$, $\dim W_2 = n-d$,

2) $T\, W_j \subset W_j$, $j = 1,2$,

3) $W_1 \subset \{0\} \cup \operatorname{int} K$, $W_2 \cap K = \{0\}$.

Moreover, if $\sigma_1(T), \sigma_2(T)$ *are the spectra of* T *restricted to* W_1, W_2, *then, between* $\sigma_1(T)$ *and* $\sigma_2(T)$ *there is a gap*

$$\lambda \in \sigma_1(T),\ \mu \in \sigma_2(T) \Rightarrow |\lambda| > |\mu|.$$

It is important to remark that, as it will appear from the proof of this theorem, there is also a complex version of Theorem-1 which is obtained by assuming E is a complex vector space and by replacing the real field $\mathbb{R}$ in h_1) by the complex field $\mathbb{C}$. The proof of Theorem-1 is contained in the following paragraph and relies on a Dynamical System approach in the spirit of the proof of Perron theorem in [AF]. In the last paragraph Theorem-1 is used to derive properties of the eigenvalues and eigenvectors of some special kind of matrices.

2 - PROOF OF THEOREM-1

We shall consider first the case of a nonsingular operator with a simple spectrum and then generalize. Let G be the Grasmannian manifold of subspaces of dimension d with the standard topology and let $G^+ \subset G$ be the subset of subspaces contained in K. G^+ is a closed subset of G because K is closed.

A nonsingular operator $T \in L(E)$ induces naturally a map $\tilde{T}: G \to G$ which maps $V \in G$ into its image under T. If T satisfies $h_3)$ then $\tilde{T}$ maps G^+ into itself and actually one has

4) $$\tilde{T}\, G^+ \subset \text{int}\, G^+.$$

Our proof of Theorem-1 is based on the study of the discrete dynamical systems on G and G^+ obtained by iterating the map $\tilde{T}$. As usual [PM] we denote by $\omega(V)$ the ω-limit set of a point $V \in G$ that is the set of all limit points of the sequence $\{\tilde{T}^j V\}_{j \geq 1}$. Clearly 4) implies that $\omega(V) \subset G^+$ whenever $V \in G^+$. We shall also denote by Ω the set

5) $$\Omega = \bigcup_{V \in G} \omega(V),$$

that is the union of the ω-limit sets of all points in G. The main step in the proof is to show the existence of a d-dimensional subspace $W \subset K$ which is invariant under T or equivalently that $\tilde{T}$ has a fixed point in G^+. The basic observations in this direction are

I - As a consequence of 4) the intersection of $\omega(V)$ with the boundary ∂G^+ of G^+ is empty. This is proved in Lemma-1 below.

II - The linearity of T implies that if $C \in L(E)$ is a nonsingular operator which commutes with T and $\tilde{C} : G \to G$ is the induced map then Ω is invariant under $\tilde{C}$.

These observations enable us to show that if $V \in G^+$ then $W \in \omega(V) \subset G^+$ must be a fixed point of $\tilde{T}$. In fact if this is not the case, then as we shall see, one can construct a continuous family C_s, $s \in [0,1)$ with $C_o = I$ the identity map and such that $\sum = \bigcup_{s \in [0,1)} C_s(W) \subset E$ is a subspace of dimension > d. Then h_2) implies $\sum$ contains a vector x which lies outside K. It follows there exists $\bar{s} \in (0,1)$ such that $\tilde{C}_{\bar{s}} W \notin G^+$ and, since $\tilde{C}_o W = W$ is in G^+, and actually in int G^+ by I, there is $t \in (0,\bar{s})$ such that $\tilde{C}_t W \in \partial G^+$. This contraticts I because II implies that $\tilde{C}_t W$ belongs to Ω.

Lemma-1. Assume $T \in L(E)$ *is a nonsingular operator satisfying the hypothesis* h_3) *in Theorem-1, then if* $V \in G$ *and* $\omega(V)$ *is the* ω*-limit set of* V *with respect to the dynamical system* $\{\tilde{T}^j\}_{j \geq 1}$, *one has*

6) $$\omega(V) \cap \partial G^+ = \phi .$$

Proof - If $\omega(V)$ is in the complement of G^+, then (6) holds. Therefore we can assume there is $W \in \omega(V) \cap G^+$. Actually we can assume $W \in$ int G^+ because otherwise we take $\tilde{T} W$ which is in int G^+ by

(4) and is in $\omega(V)$ by the invariance of the ω-limit set. This and the definition of ω-limit set imply there is j such that $\tilde{T}^j V \in \text{int } G^+$. Then by (4) it follows that $\tilde{T}^i V \in \text{int } G^+$ for all $i \geq j$ and therefore that $\omega(V) \subset G^+$. This and the invariance of the ω-limit set, $\tilde{T}\omega(V) = \omega(V)$, imply by (4) that in fact one has $\omega(V) \subset \text{int } G^+$. □

Lemma-2. Assume $T \in L(E)$ *is a nonsingular operator, and let* $C \in L(E)$ *be a nonsingular operator which commutes with* T. *Then*

7) $$W \in \Omega \Rightarrow \tilde{C} W \in \Omega.$$

Proof. Let $\{e_j\}_{j=1}^n \subset E$ be a basis and τ, γ be the matrix representations of T, C with respect to this basis. With respect to the basis $\{e_j\}_{j=1}^n$, a d-dimensional subspace $V \subset E$, that is a point $V \in G$, is determined by a $d \times n$ matrix v of maximal rank and two such matrices v, v' correspond to the same subspace if and only if there is a nonsingular $d \times d$ matrix σ such that $\sigma v = v'$. Therefore to say that $W \in G$ is in $\omega(V)$ is equivalent to the existence of a sequence σ^k, σ^k a nonsingular $d \times d$ matrix, and of a subsequence i_k such that

8) $$\lim_{k \to \infty} \sigma^k \, v \, \tau^{i_k} = w.$$

From this and the fact τ and γ commute it follows

$$\lim_{k \to \infty} \sigma^k \, v \, \gamma \, \tau^{i_k} = \lim_{k \to \infty} \sigma^k \, v \, \tau^{i_k} \, \gamma = w\gamma$$

which concludes the proof. □

The following lemma gives information on the ω-limit sets of points in G under the hypothesis that T has a simple spectrum and satisfies

9) $\lambda_i, \lambda_j \in \sigma(T),\ |\lambda_i| = |\lambda_j| \Rightarrow \lambda_i, \lambda_j$ *complex conjugate.*

We now assume $\{e_j\}_{j=1}^{n} \subset E$ to be a basis such that the matrix representation τ of T with respect to that basis is in real diagonal form with diagonal blocks (of size 1×1 or 2×2) ordered by decreasing modulus of the corresponding characteristic values.

Lemma-3. If $T \in L(E)$ *is a nonsingular operator with a simple spectrum which satisfies 9), then there is an open dense set* $\Gamma \subset G$ *such that for* $V \in \Gamma$ *and* $W \in \omega(V)$ *one of the following alternatives holds:*

(i) *there is an integer* k *such that* d *is exactly equal to the sum of the sizes of the first* k *blocks on the diagonal of* τ *and*

$$W = \text{span}\{e_1, \dots, e_d\},$$

$$\omega(V) = \{W\}.$$

(ii) *there exists* $r \in [0,1)$ *such that if* $\alpha = \cos 2\pi r$, $\beta = \sin 2\pi r$, *then*

$$W = \text{span}\{e_1, \dots, e_{d-1}, \alpha e_d + \beta\, e_{d+1}\}.$$

Proof - Let $\Gamma \subset G$ be the set of $V \in G$ such that their matrix representation v with respect to the basis $\{e_j\}_{j=1}^n$ has all minors different from zero. It is a routine to check that Γ is open and dense in G. Assume there is k such that d is exactly equal to the sum of the sizes of the first k blocks on the diagonal of τ and consider τ as a 2×2 block matrix

$$\tau = \left[\begin{array}{c|c} \tau_1 & 0 \\ \hline 0 & \tau_2 \end{array}\right]$$

where τ_1 is a d×d matrix. Similarly let

$$v = \left[\begin{array}{c|c} v_1 & v_2 \end{array}\right] ,$$

where v_1 is a d×d matrix. When $V \in \Gamma$, as we assume, v_1 is nonsingular. To prove that, under the above assumptions, alternative (i) occurs it is the same as to show there is a sequence of nonsingular d×d matrices σ^i such that

$$\lim_{i\to\infty} \sigma^i \, v \, \tau^i = \left[\begin{array}{c|c} I_d & 0 \end{array}\right],$$

where I_d is the identity matrix in $\mathbb{R}^d$.

Take $\sigma^i = \tau_1^{-i} \, v_1^{-1}$, then

$$\sigma^i \, v \, \tau^i = \left[I_d \,|\, \tau_1^{-i} \, v_1^{-1} \, v_2 \, \tau_2^i \right]$$

and

$$\lim_{i\to\infty} \|\tau_1^{-i} \, v_1^{-1} \, v_2 \, \tau_2^i\| \le \lim_{i\to\infty} \|v_1^{-1} \, v_2\| \, (\|\tau_1^{-1}\| \, \|\tau_2\|)^i =$$

$$\lim_{i\to\infty} \|v_1^{-1} \, v_2\| \left(\frac{\nu_2}{\nu_1} \right)^i = 0 \; ,$$

where ν_1 is the minimum modulus for the characteristic values of τ_1 and ν_2 is the maximum modulus for the characteristic values of τ_2 and $\nu_1 > \nu_2$ by assumption.

We remark that in this proof we have only used the fact that v_1 is nonsingular. We shall use this observation in the remaining part of the proof. Assume now that there is k such that the sum of the sizes of the first k blocks is d-1 and that the (k + 1)th block is a 2×2 block so that the sum of the sizes of the first k+1 blocks is d+1. Assume $V \in \Gamma$ and let v^o be the (d-1)×n matrix obtained from v by canceling the last row and let V^o the corresponding (d-1) dimensional subspace. Clearly the (d-1) × (d-1) matrix formed by the first (d-1) columns of v^o is nonsingular. On the other hand it is obvious that we can construct a (d+1)×n matrix v^* by adding a row to the matrix v in such a way that the (d+1) × (d+1) matrix formed by the first (d+1) columns of v^* be nonsingular. Let V^* be the (d+1) dimensional subspace corresponding to v^*. Let $\bar{T}_o$,

$\tilde{T}_*$ the maps induced by T in the Grasmannian manifolds of order d-1, d+1, then from the first part of the proof we have

$$\lim_{j\to\infty} \tilde{T}_o^j V^o = W^o = \text{span}\{e_1,\ldots,e_{d-1}\} ,$$

$$\lim_{j\to\infty} \tilde{T}_*^j V^* = W^* = \text{span}\{e_1,\ldots,e_{d+1}\} .$$

This and the fact that $V^o \subset V \subset V^*$ imply that any $W \in \omega(V)$ satisfies $W^o \subset W \subset W^*$. □

Proposition-1. Assume T *and* K *satisfy the hypothesis of Theorem-1. Assume moreover that* T *is nonsingular with a simple spectrum which verifies condition* 9). *Then the conclusions of theorem-1 hold.*

Proof. Since the set $\Gamma \subset G$ in Lemma 3 is open and dense and G^+ has nonempty interior, there exists $V \in \Gamma \cap G^+$. Therefore, by Lemma-1, $W \in \omega(V)$ is in the interior of G^+. We claim that W is invariant under T. To prove this we shall show that alternative (ii) in Lemma-3 is in contradiction with h_2). If alternative (ii) holds, the operator $C_s \in L(E)$ defined with respect to $\{e_j\}_{j=1}^n$ by the block matrix

$$\gamma_s = \left[\begin{array}{c|cc|c} I_{d-1} & 0 & & 0 \\ \hline 0 & \cos 2\pi s & \sin 2\pi s & 0 \\ & -\sin 2\pi s & \cos 2\pi s & \\ \hline 0 & 0 & & I_{n-d-1} \end{array}\right],$$

commutes with T for all s and is nonsingular. It follows that the subspace

$$\tilde{C}_s W = \text{span}\{e_1, \ldots, e_{d-1}, (\cos 2\pi(r+s))e_d + (\sin 2\pi(r+s))e_{d+1}\},$$

is in Ω by Lemma-2. Then we also have $\{\tilde{C}_s W \mid s \in [0,1)\} \subset \Omega$. On the other hand if Σ is the union of $\tilde{C}_s W$ for $s \in [0,1)$ it results

$$\Sigma = \text{span}\{e_1, \ldots, e_{d+1}\}.$$

Since Σ is a d+1 dimensional subspace, $h_2)$ implies $\Sigma \cap (E \setminus K) \neq \phi$. It follows that there is $\bar{s} \in (0,1)$ such that $\tilde{C}_{\bar{s}} W \not\subset G^+$. This, $\tilde{C}_0 W = W \in \text{int } G^+$, and the fact that $\tilde{C}_s W$ is continuous in s imply the existence of $t \in (0,\bar{s})$ such that $\tilde{C}_t W \in \partial G^+$ in contradiction with Lemma-1. This contradiction proves our claim. The subspace W can therefore be identified with the subspace W_1 in the statement of Theorem 1. To complete the proof we only need to show that the subspace $W_2 = \text{span}\{e_{d+1}, \ldots, e_n\}$ satisfies $W_2 \cap K = \{0\}$. For the sake of contradiction assume there is $x \neq 0$, $x \in W_2 \cap K$. We can also assume $x \in \text{int } K$ because otherwise we can take Tx which is in int K by h_3). It follows that there is $y = \sum_{j=1}^{n} y_j e_j \in \text{int } K$ such that $y_{d+1} \neq 0$. Then the arguments in the proof of Lemma-3 imply that, there are two possibilities corresponding to the fact that the size of the first block on the diagonal of τ_2 be 1 or 2. In the first case the sequence of one-dimensional subspaces $T^i(Y)$, $Y = \text{span}\{y\}$, converges to $\text{span}\{e_{d+1}\}$ and $e_{d+1} \in \text{int } K$. In the second case there is a subsequence $\{T^{i_h}\}$ of $\{T^i\}$ and numbers r, s, $r^2 + s^2 = 1$, such that $T^{i_h}(Y)$ converges to $\text{span}\{r\, e_{d+1} + s e_{d+2}\}$ and $(r\, e_{d+1} + s\, e_{d+2}) \in \text{int } K$. We shall discuss only the first case because the proof in the second case is essentially the same. Let $\widehat{W}_1$ be the linear variety of dimension d defined by $\widehat{W}_1 = \{e_{d+1} + x \mid x \in W_1\}$ and $I = \widehat{W}_1 \cap (E \setminus K)$. I is a bounded set. In fact assume there is a sequence $\{y_i\} \subset W_1$ such that $(e_{d+1} + y_i) \in I$ and $\lim_{i\to\infty} \|y_i\| = \infty$, then we can also assume that, as $i \to \infty$, $y_i/\|y_i\|$ converges to some $v \in W_1$, and therefore, if $V \subset W_1$ is a d-1 dimensional subspace which does not contain v and $\Sigma_i = \text{span}\{V, e_{d+1} + y_i\}$ we have $\Sigma_i \in (G \setminus G^+)$ and

$$\lim_{i\to\infty} \Sigma_i = W_1 ,$$

in contradiction with the fact that $W_1 \in \text{int } G^+$. Therefore there is $\sigma < \infty$ such that $I \subset B_\sigma =: \{e_{d+1} + y \mid \|y\| < \sigma\}$. Also I is nonempty because otherwise K would contain the d+1 dimensional subspace $\text{span}\{W_1, e_{d+1}\}$ which is in contrast with h_3). By considering, instead of T, the operator $\pm\nu_2 T$, if necessary, we can suppose the eigenvalue corresponding to e_{d+1} be equal to 1 so that the variety $\hat{W}_1$ be invariant under T and e_{d+1} be a fixed point. Then the assumptions on the spectrum of T imply that the minimum ν_1 of the modulii of characteristic values of $T|W_1$ is > 1. Thus, for any given $s>0$, $T(B_s)$ contains $B_{\nu_1 s}$ and therefore $\bigcup_i T^i(B_s) = \hat{W}_1$. Since e_{d+1} is in int K we can choose s so small that $B_s \cap I = \phi$ then by h_3) $T^i(B_s) \cap I = \phi$. This contradiction concludes the proof. □

Now we turn to generic operators satisfying the hypothesis of Theorem-1.

Lemma-4. Let $K \subset E$ *be a closed set with nonempty interior and let* $T \in L(E)$. *Assume* K *and* T *satisfy the hypothesis* h_1),h_3) *in Theorem-1. Then there is* $\varepsilon > 0$ *such that:* $S \in L(E), \|S-T\| < \varepsilon \Rightarrow S$ *satisfies* h_3).

Proof - Let P be the intersection of K with the unit sphere. Since P is compact, T(P) is compact and contained in int K by h_3). Therefore if $d(T(P),\partial K)$ is the distance between T(P) and ∂K we have $d(T(P),\partial K) = \delta > 0$. Let $S \in L(E)$, $\|S-T\| < \varepsilon =: \delta/2$ and let $x \in P$, then

$$d(Sx,\partial K) = \inf_{y\in\partial K} \|Sx-y\| = \inf_{y\in\partial K} \|Tx-y-(T-S)x\| \geq$$

$$\inf_{y\in\partial K} \|Tx-y\| - \|(T-S)x\| \geq \delta/2. \ \square$$

Lemma-5 - Let $T_j \in L(E)$, $j=1,\ldots,$ *be a sequence of operators and* $W_j \subset E$, $j=1,\ldots,$ *a corresponding sequence of d-dimensional subspaces. Assume that* $T_j \to T$; $W_j \to W$ *as* $j \to \infty$ *and* $T_j W_j \subset W_j$. *Then* $T W \subset W$.

Proof - We can assume E is an Euclidean space. Then we can associate to W and W_j the corresponding orthogonal projections P , P_j and $P_j \to P$ as $j \to \infty$. If $T W \not\subset W$ there is $x \in W$, $\|x\| = 1$, such that $\|Tx - P\,Tx\| = \delta > 0$. The hypothesis in the Lemma imply that, given $\varepsilon > 0$, there exist an integer k and $y \in W_k$ such that

$$\|T_k - T\| < \varepsilon \ , \ \|y - x\| < \varepsilon \ , \ \|P_k - P\| < \varepsilon .$$

It follows that we can also assume

$$\|T_k\, y - P\,Tx\| = \|P_k\, T_k\, y - P\,Tx\| < \varepsilon$$

then if ε is sufficiently small we can write

$$0 = \|T_k\, y - P_k\, T_k\, y\| = \|(T_k\, y - Ty) + (Ty - Tx) + (Tx - P\,Tx) +$$

$$(P\,Tx - P_k\, T_k\, y)\| \geq \|Tx - P\,Tx\| - (\|(T_k - T)y\| +$$

$$\|T(y - x)\| + \|P\,Tx - P_k\, T_k\, y\|) \geq \delta - \varepsilon(\|y\| + \|T\| + 1) > 0$$

which is a contradiction. □

We are now in the position of completing the proof of Theorem-1. It is a well known fact that any given operator $T \in L(E)$ can be approximated by a sequence $T_j \in L(E)$, $T_j \to T$ as $j \to \infty$, such that T_j is nonsingular and has a simple spectrum. It is also clear that we can choose the sequence T_j so that also condition 9) is fulfilled. By Lemma-4, for j sufficiently large, T_j satisfies the hypothesis of Theorem-1 provided T does. We can therefore apply Proposition-1 to each T_j and get corresponding subspace W_{1j}, W_{2j} that satisfy, with respect to T_j, conditions 1), 2), 3) in Theorem-1. By taking subsequences, if necessary, we can assume that the sequences W_{1j}, W_{2j}, $j=1,\ldots,$ converge to subspaces W_1, W_2 and $W_1 \subset K$, $W_2 \subset \overline{(E \setminus K)}$. By Lemma-5 W_1 and W_2 are invariant under T. This and h_3) imply that $W_1 \subset \{0\} \cup \text{int } K$ and $W_2 \cap K = \{0\}$. Therefore W_1 and W_2 satisfy, with respect to T, the conditions 1), 2), 3). Therefore to conclude the proof we only need to show that also the statement about the spectra of the restrictions $T|W_1$, $T|W_2$ holds. By Proposition-1 the above statement holds for the spectra of the restrictions $T_j|W_{1j}$, $T_j|W_{2j}$. Thus, since the spectrum $\sigma(S)$ of $S \in L(E)$ depends continuously on S [K] we immediately get

$$\lambda \in \sigma_1(T),\ \mu \in \sigma_2(T) \Rightarrow |\lambda| \geq |\mu|.$$

To show that actually we have the strict inequality $|\lambda| > |\mu|$ we let $x = x_1 + x_2$ be the unique decomposition of $x \in E$ with $x_1 \in W_1$, $x_2 \in W_2$ and let $I \in L(E)$ defined by $Ix = x_2$. Clearly W_1 and W_2 are invariant under the operator $T_\varepsilon =: T + \varepsilon I$, and $\sigma_1(T_\varepsilon) = \sigma_1(T)$, $\sigma_2(T_\varepsilon) = \sigma_2(T) + \varepsilon$. On the other hand Lemma-4 implies there is $\bar{\varepsilon} > 0$ such that T_ε satisfies the hypothesis of Theorem-1 for $\varepsilon \in (-\bar{\varepsilon}, \bar{\varepsilon})$. From the previous part of the proof we then have

$$\lambda \in \sigma_1(T_\varepsilon) = \sigma_1(T) \;,\; \mu_\varepsilon \in \sigma_2(T_\varepsilon) \Rightarrow |\lambda_\varepsilon| \geq |\mu| \;,$$

for $\varepsilon \in (-\bar{\varepsilon}, \bar{\varepsilon})$. This and the fact that $\mu_\varepsilon = \mu + \varepsilon$ for some $\mu \in \sigma_2(T)$ prove the claimed strict inequality. □

3 - APPLICATIONS AND REMARKS

We begin by observing that Theorem-1 can be inverted. We have in fact

Proposition-2 - Let $T \in L(E)$ *and* $W_1, W_2 \subset E$ *subspaces of dimension* d *and* $n-d$ *such that* $W_1 \cap W_2 = \{0\}$. *Assume that*

a) $T\, W_j \subset W_j$, $j = 1,2$,

b) $\lambda \in \sigma(T|W_1)$, $\mu \in \sigma(T|W_e) \Rightarrow |\lambda| > |\mu|$,

then there is a closed set K *with nonempty interior such that* K *and and* T *satisfy* h_1), h_2), h_3).

Proof - By a suitable choice of the norm in E we may assume that

10) $$\|(T|W_1)^{-1}\| = 1/\nu_1,\ \|T|W_2\| = \nu_2$$

where $\nu_1 = \min\{|\lambda|, \lambda \in \sigma(T|W_1)\}$, $\nu_2 = \max\{|\mu|, \mu \in \sigma(T|W_2)\}$ and $\nu_1 > \nu_2$ by b). Let $x = x_1 + x_2$ be the unique decomposition of x with $x_1 \in W_1$, $x_2 \in W_2$ and let $K = \{x \mid \|x_2\| \leq \|x_1\|\}$. Then K is closed, has nonempty interior and satisfy h_1). From 10) it follows that for $x \in K\setminus\{0\}$ we have

$$\|T\, x_2\| \leq \nu_2\|x_2\| \leq \nu_2\|x_1\| \leq \frac{\nu_2}{\nu_1}\|T\, x_1\| < \|T\, x_1\| .$$

which proves h_3). Let $q = \max\{\dim W | W \text{ a subspace, } W \subset K\}$. Clearly $q \geq d$ because $W_1 \subset K$. Let V be a subspace of dimension $d+1$ and let $z^i = x_1^i + x_2^i$, $i=1,\dots,d+1$ be a basis in V. Since $x_1^i \in W_1$, a d dimensional subspace, there exist numbers a_i, $i = 1, \dots, d+1$, $\sum_{i=1}^{d+1} |a_i| > 0$, such that $\bar{x}_1 =: \sum_{i=1}^{d+1} a_i x_1^i = 0$. On the other hand it results $\bar{x}_2 =: \sum_{i=1}^{d+1} a_i x_2^i \neq 0$ because the z^i's are linearly independent. It follows $\|\bar{x}_2\| > \|\bar{x}_1\| = 0$. Thus $V \not\subset K$. □

As a first application of the results described above we give an alternative proof of a Theorem [G],[G,K] which concerns eigenvalues and eigenvectors of a tridiagonal matrix

11) $$I = \begin{bmatrix} a_1 & b_1 & \cdots & 0 \\ c_2 & a_2 & b_2 & \\ & \ddots & \ddots & b_{n-1} \\ 0 & & c_n & a_n \end{bmatrix}$$

satisfying the condition

12) $$b_i \, c_{i+1} > 0 \, , \; i=1,\dots,n-1.$$

In the statement of the theorem we only consider the special case when

13) $$b_i > 0 \ , \ c_{i+1} > 0 \ , \ i=1,\dots,n-1.$$

In fact any matrix of type 11) satisfying 12) can be written as $I = P\hat{I}P$ where P is a diagonal matrix with diagonal elements equal to 1 or -1 and $\hat{I}$ is a "positive Jacobi matrix" that is a matrix of type 11) which satisfies 13). To state the theorem we need some notation. If $x = (x_1,\dots,x_n) \in \mathbb{R}^n$ has non zero components we let $N(x)-1$ be the number of sign changes in the sequence $x_1,x_2,\dots,x_n$. Define N_m , $N_M : \mathbb{R}^n \to \{1,\dots,n\}$ by letting $N_m(x)$, $N_M(x)$ be the minimum and the maximum value of $N(x')$ when x' ranges in a small neighborhood of x and has non zero components. Let N be extended to the (open and dense) set $\mathcal{N} =: \{x \mid N_m(x) = N_M(x)\}$ by setting $N(x) = N_m(x) = N_M(x)$ for $x \in \mathcal{N}$.

Theorem-2 (Gantmacher α Krein). *Let I be a positive Jacobi matrix. Then*

(i) *I has simple real eigenvalues* $\lambda_1 > \lambda_2 > \dots > \lambda_n$.

(ii) *If* v_j *is an eigenvector corresponding to* λ_j *, then*

$$v_j \in \mathcal{N} \text{ and } N(v_j) = j \ , \ j=1,\dots,n \ .$$

(iii) *If* $\alpha_h, \alpha_{h+1},\dots,\alpha_k$ *are real numbers and* $v = \sum_{j=h}^{k} \alpha_j v_j \neq 0$,
then $h \leq N_m(v) \leq N_M(v) \leq k$.

Proof. Since I and its exponential e^{tI} have the same eigenvectors and $\lambda \in \sigma(I) \Leftrightarrow e^{\lambda t} \in \sigma(e^{tI})\ \forall t$, it sufficies to show that the eigenvalues and the eigenvectors of e^{tI} satisfy (i),(ii),(iii). for any integer $1 \leq d \leq n$ let K_d be the closure of $\{x|N_M(x) \leq d\}$. Then K_d is closed, has non empty interior and satisfies h_1) in Theorem-1 . Moreover we claim that $q =: \max\{\dim W|W \text{ a subspace, } W \subset K_d\}$ is equal to d. To see this let $p^j(s) = s^{j-1}$ and let $x_i^j = p^j(i), 1 \leq j \leq d$. The d vectors $x^j = (x_1^j,\ldots,x_n^j)$, $1 \leq j \leq n$, are linearly independent because otherwise $1,\ldots,n$ would be roots of a polinomium $p = \sum_{j=1}^{d} \beta_j p^j \neq 0$ which is at most of degree $d-1 < n$. Therefore $W = \text{span}\{x_1^j,\ldots,x_n^j\}$ is of dimension d. Since K_d is closed, to show that $W \subset K_d$ it sufficies to show a dense subset of W is contained in K_d. For any $x = \sum_{j=1}^{d} \xi_j x^j$ let p_x the corresponding polinomium $p_x = \sum_{j=1}^{d} \xi_j p^j$ and p_x' the derivative of p_x and consider the open and dense subset $D \subset W$ defined by $D =: \{x|x \in W,\ p_x'(i) \neq 0,\ 1 \leq i \leq n\}$. From the definition of D it follows that for any $x \in D$ there is a neighborhood U_x of x in $\mathbb{R}^n$ with the property that, for any $y \in U_x$, there exist numbers $\delta_1,\ldots,\delta_n$ satisfying $y_i = p_y(i + \delta_i)$. Therefore the number of sign changes in the sequence $y_1,\ldots,y_n$ for any $y \in U_x$ with all non zero components cannot exceed the degree of $p_y \neq 0$ which is at most $d-1$. This proves $N_M(y) \leq d$ and therefore $W \subset K_d$ which implies $q \geq d$. We now prove that $q \leq d$. This is obvious if d=n, therefore we assume $d < n$ and let $x^1,\ldots,x^d,x^{d+1} \in \mathbb{R}^n$ be d+1 vectors. If these vectors are linearly independent, by a suitable choice of d+1 numbers $\gamma_1,\ldots,\gamma_d, \gamma_{d+1}$ we can make d+1 of the components of the vector $y = \sum_{j=1}^{d+1} \gamma_j x^j$ to be equal to 1 or -1 alternatively and this implies $y \notin K_d$ because any vector z with all non zero components near y will have at least d sign changes. This shows any subspace of dimension d+1 has points outside K_d thus implying $q \le d$. By using the differential equation $\dot{x} = Ix$ one can prove (see [F,01] Theorem-1) that $T =: e^{tI}$ and K_d satisfy h_3). Therefore Theorem-1 implies that, for each $1 \le d < n$, there is a T-invariant d-dimensional subspace $W_1^d \subset \{0\} \cup \text{int}\, K_d$ and a T-invariant (n-d)-dimensional subspace W_2^d, $W_2^d \cap K_d = \{0\}$ with corresponding spectral gap. Clearly (with $W_1^n =: W_2^o =: \mathbb{R}^n$) we have $W_1^d \subset W_1^{d+1}$, $W_2^d \subset W_2^{d-1}$, $1 \le d < n$. For each $1 \le d \le n$ let $V_d = : W_1^d \cap W_2^{d-1}$. V_d is a T-invariant subspace and $\dim V_d = 1$ because $W_1^d \cap W_2^d = \{0\}$. It is also clear that span $\{V_1,\ldots,V_n\} = \mathbb{R}^n$, therefore the spectrum of T is the set of the (real) eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$ corresponding to $V_1,\ldots,V_n$ and $\lambda_1 > \lambda_2 > \ldots > \lambda_n$ by the spectral gap. Let $v_d \in V_d$ be a non zero vector, then $v_d \in \text{int}\, K_d$ and therefore $N_M(v_d) \le d$. Also $v_d \notin K_{d-1}$ and therefore $N_m(v_d) > d-1$. It follows that $v_d \in N$ and $N(v_d) = d$. The last statement is proved in a similar way by observing that $v = \sum_{j=h}^{k} \alpha_j v_j \ne 0$ belongs to the subspaces $W_1^k \cap W_2^{h-1}$. □

In a similar way one can derive from Theorem-1 detailed information on spectrum and eigenvectors of matrices A of the type

$$A = \begin{bmatrix} a_1 & b_1 & 0 & \cdots & c_1 \\ c_2 & a_2 & b_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \\ 0 & \cdots & c_{n-1} & a_{n-1} & b_{n-1} \\ b_n & 0 & \cdots & c_n & a_n \end{bmatrix}$$

with $b_i, c_i \geq 0$ and $\pi_i b_i + \pi_i c_i > 0$, $i=1,\ldots,n$. We discuss this interesting class of matrices in [F,O 2].

Our second application of Theorem-1 is the following theorem which concerns matrices H of the type

14) $$H = \begin{bmatrix} a_1 & b_1 & \cdots & b_{n-1} \\ c_1 & a_2 & & \bigcirc \\ \vdots & \bigcirc & \ddots & \\ c_{n-1} & & & a_n \end{bmatrix}$$

satisfying the condition $b_i c_i > 0$, $i=1,\ldots,n-1$. Any such matrix is similar to a matrix satisfying the more restrictive condition,

15) $$b_i > 0 \ , \ c_i > 0 \ , \ i=1,\ldots,n-1 .$$

Therefore we only consider this special case.

Theorem-3. Let H *be a matrix of type* 14) *satisfying condition* 15). *Then*

(i) *The eigenvalues of* H *are real and* H *has* n *linearly independent eigenvectors.*

(ii) *The largest and the smallest eigenvalues* λ_1, λ_n *are simple :*

$$\lambda_1 > \lambda_2 \geq \ldots \geq \lambda_{n-1} > \lambda_n .$$

(iii) *If* $v_1, \ldots, v_n$ *are eigenvectors corresponding to* $\lambda_1, \ldots, \lambda_n$ *then*

$$v_1 \in \{x \in \mathbb{R}^n | x_1 x_{i-1} > 0,\ 2 \leq i \leq n\},$$

$$v_2, \ldots, v_{n-1} \in \{x \in \mathbb{R}^n | \exists\ i, j \geq 2 \text{ such that } x_i x_j < 0\},$$

$$v_n \in \{x \in \mathbb{R}^n | x_1 x_i < 0,\ i=2,\ldots,n\}.$$

We don't present a detailed proof of this theorem. The first statment follows from the fact that H is similar to a symmetric matrix. The proof of (ii) and (iii) is analogous to the proof of Theorem-2 and it is based on the fact that by using the differential equation $\dot{x} = Hx$ it is seen that the exponential $T = e^{tH}$ and the cones K_1, K_{n-1} defined by

$$K_1 = \text{Closure } \{x | x_1 x_{i-1} > 0,\ 2 \leq i \leq n\},$$

$$K_{n-1} = R^n \backslash \{x \mid x_1 x_i < 0, \ i=2,\ldots,n\},$$

satisfy the hypothesis of theorem-1 for d=1, d=n-1. □

Our next example is about matrices obtained by discretizing the Laplacian operator Δ in a bounded domain Ω with Direchlet or Newmann boundary conditions. We consider the simple case when Ω is a cube $C \subset R^k$, $k \geq 1$, divided in $n=m^k$ equal cubes $c_1,\ldots,c_n$ of size(1/m)th of the size of C. Then, if we let I_i be the set of indices $j \neq i$ such that $j \in I_i \Leftrightarrow c_i$ and c_j have a commun face, the n×n matrix $L = (\ell_{ij})$, which is the standard discretization of Δ corresponding to the above partition of C, is symmetric and satisfies

$$\ell_{ij} > 0 \text{ if } i \neq j \text{ and } j \in I_i,$$

$$\ell_{ij} = 0 \text{ if } i \neq j \text{ and } j \notin I_i.$$

By using these properties of L and the differential equation $\dot{x} = Lx$ one can show that $T = e^L$ and the cones

$$K_1 = \text{Closure}\{x \in R^n \mid x_i x_{i-1} > 0, \ i=2,\ldots,n\},$$

$$K_{n-1} = R^n \backslash \{x \in R^n \mid x_i x_j < 0, \ j \in I_i, \ i=1,\ldots,n\},$$

satisfy the hypothesis of Theorem-1 with d=1, d=n-1. Therefore one can prove the following theorem

Theorem-4. If $\lambda_1 \geq \dots \geq \lambda_n$ are the eigenvalues of L *and* $v_1,\dots,v_n$ *are corresponding eigenvectors, then*

(i) $\lambda_1 > \lambda_2 \geq \dots \geq \lambda_{n-1} > \lambda_n$

(ii) $v_1 \in \{x | x_i x_{i-1} > 0,\ i=2,\dots,n\}$,

$v_2,\dots,v_{n-1} \in \mathrm{int}(K_{n-1} \setminus K_1)$,

$v_n \in \{x | x_i x_j < 0,\ j \in I_i,\ i=1,\dots,n\}$.

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