RT-MAE 9734

AN EXCLUSION PROCESS WITH METROPOLIS RATE.

by

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Palavra Chave: Exclusion process, Metropolis Rate, reversibility.

Classificação AMS: 60J99.
(AMS Classification)

- Dezembro de 1997 -

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0. Introduction

The exclusion process was introduced by Spitzer (1970) and its principal characteristic is the conservation of number of particles. Here we deal with an exclusion process with speed change, i.e., when the particles are on any finite set $A \subset I\!N^2$, the rate of jump of each one depends on its position with respect to the others.

We consider the Metropolis dynamics (see (1.3) below) for an exclusion process with nearest neighbors interaction enclosed in a finite torus $\Lambda_N \subset I\!N^2$. This process is reversible with respect to the Gibbs measure that concentrates its mass, on limit where the temperature goes to zero, on configurations that minimize the number of differents neighbors sites.

We start from a configuration with $d = d' \times d'$ particles arranged as a rectangle $d_1 \times d_2$ $d_1 < d_2 \in I\!N$. We analyze the evolution of the system, at low temperatures, until it reaches a configuration where the particles form a square $d' \times d'$ (*square configuration*).

This system presents three differents mechanisms of evolution, namely: creation, rearrange and filling. We estimate the necessary time to each mechanism to be effected and the total time that the process spends to reach a *square configuration.* Even, we present some results about the drift of the process.

The proofs are simple and this is the interesting of this model. The next section presents the formal definitions and the main results and in the second section are the proofs.

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1. Main definitions and results

We start by recalling the main notation and definitions. Consider a simple exclusion process in a finite torus $\Lambda_N = \{1, \ldots N\}^2, N \in \mathbb{N}$, with periodic boundary conditions.

A configuration is a function $\eta : \Lambda_N \to \{-1, +1\}$, that is, $\eta \in \mathcal{X} = \{-1, +1\}^{\Lambda_N}$ (space state). We say that the site $x \in \Lambda_N$ is empty if $\eta(x) = -1$ and it is occupied if $\eta(x) = +1$. We denote by σ_t^η the process at time t with initial configuration η and the value of the site x at time t is $\sigma_t^\eta(x)$.

Definition 1.1. *We define the Hamiltonian of the configuration η as*

$$H(\eta) = -\frac{1}{2} \sum_{\substack{x,y \in \Lambda_N \\ ||x-y||=1}} \eta(x)\eta(y), \tag{1.1}$$

where $||x - y|| = |x_1 - y_1| + |x_2 - y_2|$, and the sum, in the Hamiltonian, runs over the pairs of nearest neighbors sites of Λ_N, counting each pair only once.

Definition 1.2. *Given $\beta > 0$, a* **inverse temperature**, *the Markov jump process's generator may be written as*

$$Lf(\eta) = \sum_{x,y \in \Lambda_N} c(x, y, \eta)[f(\eta^{xy}) - f(\eta)], \tag{1.2}$$

acting on cylinders functions f on $\mathcal{X}$.

In (1.2) η^{xy} is the configuration obtained from η when the contents of sites x and y are interchanged, i.e.,

$$\eta^{xy}(z) = \begin{cases} \eta(z) & \text{if } x \neq z, y \neq z, \\ \eta(x) & \text{if } z = y, \\ \eta(y) & \text{if } z = x. \end{cases}$$

The Metropolis rate $c(x, y, \eta)$ is written as

$$c(x, y, \eta) = \begin{cases} 0, & \text{if } ||x - y|| \neq 1, \\ 1, & \text{if } \Delta_{xy} H(\eta) \leq 0, \\ \exp\{-\beta \Delta_{xy} H(\eta)\} & \text{if } \Delta_{xy} H(\eta) > 0, \end{cases} \tag{1.3}$$

where $\Delta_{xy}H(\eta) = H(\eta^{xy}) - H(\eta)$. Sometimes we use $c(\eta, \zeta)$ to denote $c(x, y, \eta)$ with $\zeta = \eta^{xy}$.

Our dynamics is reversible with respect to the Gibbs measure given by

$$\mu(\eta) = \exp\{-\beta H(\eta)\} Z_{\Lambda_N}^{-1}, \tag{1.4}$$

with the partition function

$$Z_{\Lambda_N} = \sum_{\sigma \in \mathcal{X}} \exp\{-\beta H(\sigma)\},$$

in the sense that the rates satisfy the equalities

$$\mu(\eta) c(x, y, \eta) = \mu(\eta^{xy}) c(x, y, \eta^{xy}), \ \forall x, y \in \Lambda_N.$$

We may construct this process in the following way: at each instant of occurrence of a Poisson Process $\{N(t), t \geq 0\}$ which rate is given by $\lambda = 2N^2$, we choose, uniformly, two neighbors sites $x, y \in \Lambda_N$ and we interchange their values with probability $c(x, y, \eta)$; where η is the configuration at a time of the jump.

Definition 1.3. *(Contour of a configuration). For $\eta \in \mathcal{X}$, η without rings of particles around Λ_N, trace a line between two neighbors sites whose values are different. The union of this edges results in a set of polygons. The contour of η is the union of the boundaries of these polygons and we denote it by $C(\eta)$. Note that the vertices of these polygons belong to the dual space $\mathbb{Z}^2 + (\frac{1}{2}, \frac{1}{2})$. Define $|C(\eta)|$ the sum of the perimeters of each polygon.*

It follows that

$$H(\eta) = -(N^2 - |C(\eta)|), \ \forall \eta \in \mathcal{X}, \tag{1.5}$$

and then

$$H(\eta^{xy}) - H(\eta) = |C(\eta^{xy})| - |C(\eta)|.$$

From (1.4) and (1.5) we have that the Gibbs measure concentrates its mass on the set of configurations with smaller contour when β goes to infinite. For a system with d particles we denote the set of configurations with minimum energy by $\mathcal{Q}_d$ and if $\xi \in \mathcal{Q}_d$ we call it an *equilibrium* configuration.

For simplicity, we start with $d = d_1 \times d_2$ particles such that $\sqrt{d} = d' \in I\!N$.

A configuration is called *rectangular* if its contour is compound by an unique rectangle.

The goal of our analysis is to study the evolution of this process when it starts from a rectangular configuration $(d_1 \times d_2)$, until it reaches $\mathcal{Q}_d$.

The notation which follows is used at all text.

- $T^{\xi}(A) = \inf\{t \geq 0 : \sigma_t^{\xi} \in A\}, \ \forall A \subset \mathcal{X}, \xi \in \mathcal{X}$.
- $< \eta, \zeta >$ if exist $x, y \in \Lambda_N$ with $||x - y|| = 1$ such that $\zeta = \eta^{xy}$.
- $R_{i,j}$ is a rectangle of dimension $i \times j$ which vertex are on the dual space $\mathbb{Z} + (\frac{1}{2}, \frac{1}{2})$ and we assume that $i \leq j, i, j \in I\!N$.

We namely *protuberance* a particle with only one occupied neighbor, *corner* a particle with two occupied neighbors and *middle* a particle with three occupied neighbors.

Definition 1.4. *We define the slices of a rectangle $R_{i,j}$ as*

$$V_n = \{1, \dots, i\} \times \{n\}, \quad n \in \{1, \dots, j\},$$
$$H_m = \{m\} \times \{1, \dots, j\}, \quad m \in \{1, \dots, i\}.$$
$$R_{i,j} = \bigcup_{m=1}^{i} H_m = \bigcup_{n=1}^{j} V_n.$$

The slices V_1, V_j, H_1 and H_i are called externals. Even $|H_k|$ represents the number of particles in the slice H_k, $\forall k \in \{1, \dots, i\}$.

Now we define two important set of configurations: *semi-solids* $(\mathcal{X}_{\overline{S}})$ and *solids* $(\mathcal{X}_S)$.

Definition 1.5. *Consider η a configuration which contour is compound by only one polygon and $R(\eta)$ the smaller rectangle that contains this polygon. Suppose that $R(\eta) = R_{i,j}$. We say that η is semi-solid, $\eta \in \mathcal{X}_{\overline{S}}$, if i), ii) and iii), below, are satisfied.*

i) $|H_k| \geq 2, \forall k = 2, \ldots, i-1$;

ii) $|V_l| \geq 2, \ \forall l = 2, \ldots, j-1$;

iii) *If* $\eta(x) = 1$ *and* $\eta(x + ke_1) = 1, k \leq j$ *then* $\eta(x + me_1) = 1, \ \forall m = 1, \ldots, k-1$.
If $\eta(x) = 1$ *and* $\eta(x + ke_2) = 1, k \leq i$ *then* $\eta(x + me_2) = 1, \ \forall m = 1, \ldots, k-1$;
where e_1, e_2 *are the vectors* $(1,0)$ *and* $(0,1)$ *respectively.*

Definition 1.6. *Consider* $\eta \in \mathcal{X}_{\overline{S}}$. *We say that* η *is solid,* $\eta \in \mathcal{X}_S$, *if i) and ii), below, are satisfied.*

i) $|H_k| \geq 2, \ \forall k = 1, \ldots, i$.

ii) $|V_k| \geq 2, \ \forall k = 1, \ldots, j$.

Note that

$$\mathcal{X}_S \subset \mathcal{X}_{\overline{S}} \subset \mathcal{X}$$

and if $\eta \in \mathcal{X}_{\overline{S}} \backslash \mathcal{X}_S$ then η has at least one protuberance.

Definition 1.7. *For* $\eta \in \mathcal{X}_{\overline{S}}$, *with* $R(\eta) = R_{i,j}$, *we define envelope of* η, $\mathcal{E}(\eta)$, *as a large rectangle inside of* $R(\eta)$ *such that all its externals slices have at least two particles.*

Note that for $\eta \in \mathcal{X}_S, \mathcal{E}(\eta) = R(\eta)$.

The next two lemmas will be used many times in this paper and to prove them we introduce a dynamics restricted to the connected set $\mathcal{S}$. We say that a set $\mathcal{S}$ of configurations is connected if for any pair of configurations $\eta, \zeta \in \mathcal{S}$ exist a sequence of configurations $\xi_0 = \eta, \ \xi_1, \ldots, \ \xi_k = \zeta$, for some $k \in I\!N$, with $< \xi_i, \xi_{i+1} >, \ \xi_i \in \mathcal{S}$, for $i = 1, \ldots, k-1$. We present the idea in a general fashion.

The dynamics restricted to $\mathcal{S}$ is defined by the rates

$$\tilde{c}(x, y, \eta) = \begin{cases} c(x, y, \eta) & \text{if } \eta, \eta^{xy} \in \mathcal{S}, \\ 0 & \text{otherwise} \end{cases} .$$

Denote the restricted process by $\tilde{\sigma}_t$, and consider $\tilde{\mu}$ the measure defined by

$$\tilde{\mu}(\eta) = \begin{cases} \frac{\mu(\eta)}{\sum\limits_{\zeta\in\mathcal{S}} \mu(\zeta)} & \text{if } \eta \in \mathcal{S}, \\ 0 & \text{otherwise} \ ; \end{cases}$$

where μ is the Gibbs measure associated to the process. We know that $\tilde{\sigma}_t$ is reversible with respect to $\tilde{\mu}$ and we remark that $\mathbb{P}(\tilde{\sigma}_0^{\tilde{\mu}} = \eta) = \tilde{\mu}(\eta), \ \forall \eta \in \mathcal{X}$.

We use two types of coupling:

- *Coupling A:* The process $\{\tilde{\sigma}_t^\eta\}$ and $\{\sigma_t^\eta\}$ jump together until the latter escapes from $\mathcal{S}$; at this moment the former process stays still and afterwards they evolve independently.
- *Coupling B:* The process $\{\tilde{\sigma}_t^\eta\}$ and $\{\tilde{\sigma}_t^{\tilde{\mu}}\}$ evolve independently until they meet and afterwards they jump together.

For $\tilde{T}^\eta(\xi) = \inf\{t \geq 0 : \tilde{\sigma}_t^\eta = \xi\}, \ \forall \eta, \xi \in \mathcal{X}$, and $\tilde{T}^{\tilde{\mu}}(\xi) = \inf\{t \geq 0 : \tilde{\sigma}_t^{\tilde{\mu}} = \xi\}$, we have that

Lemma 1.8.([NS]) *Let $\eta \in \mathcal{S}$, $\mathcal{S}$ a connected set such that for all $\xi \in \mathcal{S}\backslash\{\eta\}$ we have that $H(\xi) > H(\eta)$. Then, for all $\xi \in \mathcal{S}$ and $\epsilon > 0$,*

$$\lim_{\beta\to\infty} \mathbb{P}\big(\tilde{T}^\eta(\xi) < \exp\{\beta(H(\xi) - H(\eta) - \epsilon)\}\big) = 0. \tag{1.6}$$

Proof: Using coupling B to $\tilde{\sigma}_t^\eta$ and $\tilde{\sigma}_t^{\tilde{\mu}}$ on $\mathcal{S}$ and stationarity of the process $\tilde{\sigma}_t^{\tilde{\mu}}$ we have that

$$
\begin{aligned}
I\!P\big(\tilde T^{\eta}(\xi) < \exp\{\beta(H(\xi)-H(\eta)-\epsilon)\}\big) &= I\!P\big(\tilde T^{\eta}(\xi) < \exp\{\beta(H(\xi)-H(\eta)-\epsilon)\}, \tilde\sigma_0^{\eta} \neq \tilde\sigma_0^{\tilde\mu}\big) \\
&+ I\!P\big(\tilde T^{\eta}(\xi) < \exp\{\beta(H(\xi)-H(\eta)-\epsilon)\}, \tilde\sigma_0^{\eta} = \tilde\sigma_0^{\tilde\mu}\big) \\
&\leq I\!P\big(\tilde\sigma_0^{\eta} \neq \tilde\sigma_0^{\tilde\mu}\big) \\
&+ I\!P\big(\tilde\sigma_t^{\tilde\mu} = \xi \text{ for some } t \in \{\beta^{-1}, 2\beta^{-1}, \ldots, \lceil \beta \exp\{\beta(H(\xi)-H(\eta)-\epsilon)\}\rceil \beta^{-1}\}\big) \\
&+ I\!P\big(\tilde\sigma_t^{\tilde\mu} \text{ jumps between the times } \tilde T^{\tilde\mu}(\xi) \text{ and } \tilde T^{\tilde\mu}(\xi) + \beta^{-1}\big) \\
&\leq \big(1 - \tilde\mu(\eta)\big) + \big(\beta \exp\{\beta(H(\xi)-H(\eta)-\epsilon)\} + 1\big)\tilde\mu(\xi) \\
&+ \big(1 - \exp\{-\beta^{-1} 2N^2\}\big),
\end{aligned}
$$

where $\lceil k \rceil$ is the smaller integer large or equal to k.

Taking the limit when $\beta \to \infty$ we get the result.

■

Next we define, to *semi-solids* configurations, what we call of *movement of external slice*.

Let $\eta \in \mathcal{X}_{\overline{S}}$ and consider P_1, P_2, P_3, P_4 the vertex of $R(\eta)$ and $l_n, n \geq 4$ the sides of $C(\eta)$ contained in $\overline{P_1P_2}, \overline{P_2P_3}, \overline{P_3P_4}, \overline{P_4P_1}$ respectively.

A *movement of external slice* is characterized by a shift of lenght 1 of some l_n via jumps of rate large or equal to $\exp\{-2\beta\}$.

Formally, for $\eta \in \mathcal{X}_{\overline{S}}$ consider

$$
\mathcal{X}_{\eta^*} = \left\{ \begin{array}{c} \text{all configurations } \xi \in \mathcal{X}_{\overline{S}},\ H(\xi) = H(\eta), \text{ obtained} \\ \text{from } \eta \text{ via jumps of rate large or equal to} \\ \exp\{-2\beta\} \text{ with one movement of external slice} \end{array} \right\}. \tag{1.7}
$$

Example:

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```

η $\qquad\qquad\qquad$ $\zeta \in \mathcal{X}_{\eta^*}$

Remark: For $\eta \in \mathcal{X}_{\overline{S}}$ a rectangular configuration we have that $\mathcal{X}_{\eta^*} = \emptyset$.

Lemma 1.9. *For $\eta \in \mathcal{X}_{\overline{S}}$, $\mathcal{X}_{\eta^*} \neq \emptyset$ and $\forall \delta > 0$ we have that*

$$\lim_{\beta \to \infty} I\!P\big(T^\eta(\mathcal{X}_{\eta^*}) < \exp\{\beta(2+\delta)\}\big) = 1, \tag{1.8}$$

where $T^\eta(\mathcal{X}_{\eta^*}) = \inf\{t \geq 0 : \tilde{\sigma}_t^\eta \in \mathcal{X}_{\eta^*}\}$.

Proof: First note that if an external slice has only one particle, this jumps to site by side with rate 1 and thus the result is immediate.

Now fix $\eta \in \mathcal{X}_S$ with $\mathcal{X}_{\eta^*} \neq \emptyset$ and consider S_η a connected set of configurations such that

- $\mathcal{X}_{\eta^*} \subset \mathcal{X}_\eta$.
- for $\xi \in \mathcal{X}_{\eta^*}, \exists \eta = \eta_0, \eta_1, \ldots, \eta_l = \xi, l \in I\!N, \eta_i \in \mathcal{X}_\eta$,
  $< \eta_i, \eta_{i+1} >$, $c(\eta_i, \eta_{i+1}) \geq e^{-2\beta}$, and $c(\eta_{i+1}, \eta_i) \geq e^{-2\beta}, \forall i \in \{1, \ldots, l-1\}$.

For $\tilde{\sigma}_t^\eta$ the process restrict to S_η and using coupling A,

$$\lim_{\beta \to \infty} I\!P\big(\tilde{\sigma}_t^\eta \neq \sigma_t^\eta, \text{ for some } t < e^{\beta(4-\epsilon)}\big) = 0, \ \forall \epsilon > 0. \tag{1.9}$$

On the another hand,

$$\lim_{\beta\to\infty} \mathbb{P}(\tilde{\sigma}_t^\eta \in \mathcal{X}_{\eta^*}, \text{ for some } t < e^{\beta(2+\delta)}) = 1, \ \forall\delta > 0; \tag{1.10}$$

because $\{\tilde{\sigma}_t^\eta\}_{t\geq 0}$ is a finite Markov chain with $\min\limits_{x,y,\eta} c(x,y,\eta) = e^{-2\beta}$.

By (1.9) and (1.10) the result follows.

■

The process presents three differents mechanisms: *Rearrange, Creation* and *Filling.* To explain them we decompose $\mathcal{X}_S$ into $\mathcal{X}_{S_r}, \mathcal{X}_{S_c}$ and $\mathcal{X}_{S_f}$.

Rearrange

This mechanism transfers particles of an external slice to another one with jumps of rate large or equal to $e^{-2\beta}$.

Let $\eta \in \mathcal{X}_S$ with $\mathcal{E}(\eta) = R_{i,j}, i,j \in \mathbb{N}$. We say that a rearrange occurred if the process starting from η reached a new configuration $\zeta \in \mathcal{X}_S$ via successives movements of external slices plus jumps of protuberances and $\mathcal{E}(\zeta)$ has one dimension (i or j) reduced and $H(\zeta) \leq H(\eta)$. If $H(\zeta) < H(\eta)$ we say that a rearrange is successful.

Definition 1.10. *Consider $\eta \in \mathcal{X}_S$ with $\mathcal{E}(\eta) = R_{i,j}, i,j \in \mathbb{N}$. We say that $\eta \in \mathcal{X}_{S_r}$ if*

i) $d \leq i(j-1)$,

ii) $\exists$ *a sequence of configurations* $\eta_0 = \eta, \eta_1, \dots, \eta_k$ *for some* $k \in \mathbb{N}$ *with* $<\eta_i, \eta_{i+1}>$, $c(\eta_i,\eta_{i+1}) \geq e^{-2\beta}$, $\forall i \in \{0,\dots,k-1\}, \eta_k \in \mathcal{X}_S$ *and* $\mathcal{E}(\eta_k) = R_{n_1,n_2}$ *where* $n_1 \leq i-1$ *or* $n_2 \leq j-1$.

Theorem 1. *Suppose that $\eta \in \mathcal{X}_{S_r}$ and $\mathcal{E}(\eta) = R_{i,j}, i,j \in \mathbb{N}$. For*

$$T_1 = \inf\{t \geq 0 : \sigma_t^\eta \in \mathcal{X}_S \text{ and } \mathcal{E}(\sigma_t^\eta) = R_{k,l} \text{ for } k \leq i-1 \text{ or } l \leq j-1\},$$

we have that

$$\lim_{\beta\to\infty} I\!P\left(e^{\beta(2-\delta)} \le T_1 \le e^{\beta(2+\delta)}\right) = 1, \ \forall \delta > 0.$$

Creation

We say that a new slice was created if the process, starting from $\eta \in \mathcal{X}_S$, with $\mathcal{E}(\eta) = R_{i,j}, i,j \in I\!N$, reached a configuration $\zeta \in \mathcal{X}_S$ such that $\mathcal{E}(\zeta)$ has one dimension (i or j) increased. In this manner, it is need that two particles to take place in a new external slice.

Definition 1.11. *Consider $\eta \in \mathcal{X}_S$ with $\mathcal{E}(\eta) = R_{i,j}, i,j \in I\!N$. We say that $\eta \in \mathcal{X}_{S_c}$ if $d > i(j-1)$.*

For a simple example consider a rectangular configuration.

Theorem 2. *Suppose that $\eta \in \mathcal{X}_{S_c}$ and $\mathcal{E}(\eta) = R_{i,j}, i,j \in I\!N$. For*

$$T_2 = \inf\{t \ge 0 : \sigma_t^\eta \in \mathcal{X}_S, \ \textit{and} \ \mathcal{E}(\sigma_t^\eta) = R_{k,l} \ \textit{for} \ k \ge i+1 \ \textit{or} \ l \ge j+1\},$$

we have that

$$\lim_{\beta\to\infty} I\!P\left(e^{\beta(4-\delta)} \le T_2 \le e^{\beta(4+\delta)}\right) = 1, \forall \delta > 0.$$

Filling

Definition 1.12. *Consider $\eta \in \mathcal{X}_S$ with $\mathcal{E}(\eta) = R_{i,j}, i,j \in I\!N$. We say that $\eta \in \mathcal{X}_{S_f}$ if $\eta \in \mathcal{X}_S \backslash \{\mathcal{X}_{S_c} \cup \mathcal{X}_{S_r}\}$.*

In fact, if $\eta \in \mathcal{X}_{S_f}$ with $\mathcal{E}(\eta) = R_{i,j}, i,j \in I\!N$, we have that $d < i(j-1)$ but it is not possible to change its envelope with jumps of rate large or equal to $e^{-2\beta}$. For these configurations the process has two alternatives: to fill the empty sites of $R(.)$ with corner particles or, if possible, to create a new slice.

Example:

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$\eta \in \mathcal{X}_{S_f}$

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```

$\zeta \in \mathcal{X}_{S_f}$

Theorem 3. *Consider* $\eta \in \mathcal{X}_{S_f}$ and $\mathcal{E}(\eta) = R_{i,j}, i,j \in I\!N$. *For*

$$T_3 = \inf\{t \geq 0 : \sigma_t^\eta \in \mathcal{X}_S \text{ and } \mathcal{E}(\sigma_t^\eta) \neq R_{i,j}\},$$

we have that

$$\lim_{\beta \to \infty} I\!P\big(e^{\beta(4-\delta)} \leq T_3 \leq e^{\beta(4+\delta)}\big) = 1, \ \forall \delta > 0.$$

Now we present two results about the drift of the process.

Consider $\mathcal{C}(\mathcal{E}_{i,j}) \subset \mathcal{X}_S$ the class of configurations whose envelope are rectangles $R_{i,j}$, $i,j \in I\!N$.

Proposition 1.13. *For* $\eta \notin \mathcal{Q}_d$ *a rectangular configuration* $d_1 \times d_2$ *with* $3 < d_1 < d_2$ *we have that*

$$\lim_{\beta \to \infty} I\!P\big(T^\eta(\mathcal{C}(\mathcal{E}_{k_1,k_2})) < e^{\beta(4+\delta)}\big) = 1, \ \forall \delta > 0, \tag{1.11}$$

for $k_1 = d_1 + 1$ and $k_2 = d_2 - [\frac{d_2}{d_1+1}]$ *where* $[k] =$ *the bigger integer smaller or equal to* k.

Even

$$\lim_{\beta \to \infty} I\!P\big(T^\eta(\mathcal{C}(\mathcal{E}_{d_1,(d_2+2)})) > e^{\beta(6-\delta)}\big) = 1, \ \forall \delta > 0. \tag{1.12}$$

Proposition 1.14. *Consider $\eta \in \mathcal{X}_{S_c} \setminus \mathcal{Q}_d$ with $\mathcal{E}(\eta) = R_{i,j}, i,j \in I\!N$; $\tau_1 = \inf\{t \geq 0 : \mathcal{E}(\sigma_t^\eta) \neq R_{i,j}\}$ and $\tau_2 = \inf\{t \geq \tau_1 : \sigma_t^\eta \in \mathcal{X}_{S_c}\}$. If $\mathcal{E}(\sigma_{\tau_2}^\eta) = R_{k_1,k_2}$ and $k_2 \leq j-2$ then*

$$\lim_{\beta\to\infty} I\!P(T^{\sigma_{\tau_2}^\eta}(\mathcal{C}(\mathcal{E}_{i,j})) > e^{\beta(6-\delta)}) = 1, \ \forall \delta > 0.$$

2. Proofs

2.1 *Proof of Theorem 1.*

Fix $\eta \in \mathcal{X}_{S_r}$ with $\mathcal{E}(\eta) = R_{i,j}, i,j \in I\!N$. We consider the following subset of configurations

$$\mathcal{S}_\eta = \eta \cup \left\{ \begin{array}{c} \text{all configurations } \xi \text{ obtained from } \eta \text{ via jumps} \\ \text{of rates large or equal to } \exp\{-2\beta\} \text{ with } H(\xi) \leq H(\eta) + 2 \end{array} \right\}$$

Note that, by definition 1.11 and by (1.8) it follows that

$$\mathcal{X}_{\eta^*} \neq \emptyset \subset \mathcal{S}_\eta.$$

Consider $\{\tilde{\sigma}_t^\eta\}$ the process restricted to $\mathcal{S}_\eta$ with rates

$$\tilde{c}(x,y,\eta) = \begin{cases} c(x,y,\eta) & \text{if } c(x,y,\eta) \geq e^{-2\beta}, \\ 0, & \text{otherwise.} \end{cases}$$

Using coupling A for $\{\tilde{\sigma}_t^\eta\}$ and $\{\sigma_t^\eta\}$ we have that

$$\lim_{\beta\to\infty} I\!P(\tilde{\sigma}_t^\eta \neq \sigma_t^\eta, \text{ for some } t < e^{\beta(4-\delta)}) = 0, \ \forall \delta > 0.$$

Define

$$\tau_0 = 0,$$
$$\tau_i = \inf\{t \geq \tau_{i-1} : \tilde{\sigma}_t^\eta \neq \tilde{\sigma}_{t-}^\eta\}, \quad \text{for } i \geq 1;$$

and consider the Markov chain $\{\tilde{\sigma}^{\eta}_{\tau_i}\}_{i\geq 0}$. Using Lemma 1.9 it follows that

$$\lim_{\beta\to\infty} I\!P\big(\tau_i - \tau_{i-1} > e^{\beta(2+\epsilon)} \;/\; T_1 > \tau_{i-1}\big) = 0,\ \forall \epsilon > 0.$$

As $\tilde{\sigma}^{\eta}_t$ is a finite Markov chain on S_η we have the result.

■

2.2 *Proof of Theorem 2.*

First we consider configurations of $\mathcal{X}_{\overline{S}}\backslash\mathcal{X}_S$ with one protuberance particle. We denote this set by $\mathcal{X}_{\overline{S}_1}$. It follows that

Lemma 2.1. *For $\eta \in \mathcal{X}_{S_c}$ we have that*

i) $\lim_{\beta\to\infty} I\!P\big(T^{\eta}(\mathcal{X}_{\overline{S}_1}) < e^{\beta(4+\delta)}\big) = 1, \forall \delta > 0.$

ii) $\lim_{\beta\to\infty} I\!P\big(T^{\eta}(\mathcal{X}_{\overline{S}_1}) < T_2\big) = 1.$

Proof: *i)* It is enough a corner carry out a random walk on Λ_N and after some jumps becomes a protuberance for i) to happen.

ii) Starting from $\eta \in \mathcal{X}_{S_c}$, to create a new slice without to visit $\mathcal{X}_{\overline{S}_1}$ the process needs that two particles, in the same time, carry out a random walk (with rate 1) on the torus. But this increase by at least 6 the energy with respect to η. In this manner, when β goes to infinite we have the result.

■

Now, observe that starting from $\mathcal{X}_{\overline{S}_1}$, it is enough to put a second particle beside of the protuberance to change the dimension of the envelope. At this point, the question is *Where is the second particle from?* The answer is not difficult and we will see that this is possible by jumps of rate large or equal to $e^{-2\beta}$.

Consider $\eta, \zeta \in \mathcal{X}_{\overline{S}_1}$ below

```
- - - - - - - -        - - - - - - - - -
- - - - + - - -        - - - + - - - - -
- + + + + + - -        - + + + + + - - -
- + + + + + + -        - + + + + + + - -
- + + + + + + -        - + + + + + + + -
- - - + + + - -        - - + + + + + + -
- - - - - - - -        - - - - - - - - -
```

η ζ

Note that it is possible to *create* the second particle, starting from η, with jumps of rate large or equal to $e^{-2\beta}$ (movement of external slice) but this is not possible starting from another configuration ζ.

In this manner, we call η (above) a *good configuration* of $\mathcal{X}_{\overline{S}_1}$, i.e., starting from a *good configuration* the process may change its envelope by jumps of rate large or equal to $e^{-2\beta}$.

Even, note that the creation of a new slice without to reach a *good configuration* spends a time of order large or equal to $e^{\beta(6-\delta)}, \forall \delta > 0$, with probability 1 when $\beta \to \infty$ (using Lemma 1.8).

Lemma 2.2. *Consider* $\eta \in \mathcal{X}_{S_c}$ *and* $T_g^\eta = \inf\{t \geq 0 : \sigma_t^\eta$ *is a good configuration of* $\mathcal{X}_{\overline{S}_1}\}$. *We have that*

$$\lim_{\beta \to \infty} \mathbb{P}\big(T_g^\eta < e^{\beta(4+\delta)}\big) = 1, \forall \delta > 0.$$

Proof: Observe that every time that a corner particle leaves $R(\eta)$ (with rate $e^{-4\beta}$) it returns (with rate 1) as a protuberance or again as a corner (but not necessarily the same corner). In some sense, starting from $\eta \in \mathcal{X}_{S_c}$ the process realizes a random walk on the configurations with the same envelope of η until the new slice to be created.

Thus, without to reach a *good configuration* the process would spend a time of order $e^{6\beta}$ to create a new slice and then the result follows. ■

Now observe that when the process starts from a *good configuration* at a time of order $e^{\beta(2+\epsilon)}, \forall \epsilon > 0$, the process may

- increases the dimension of the envelope,
- returns to $\mathcal{X}_{S_c}$,
- reaches another configuration of $\mathcal{X}_{\overline{S}_1}$.

As $\delta > 0$ in Theorem 1 and we have independent attempts of realize the above itens and then the result follows. ■

2.3 *Proof of Theorem 3.*

To prove Theorem 3 it is enough to observe that to fill the holes of $R(.)$ the corner particles carry out a random walk on the torus and it returns filling a hole that spends a time of order $e^{4\beta}$ and by Theorem 2 the process spends the same time to create a new slice. ■

2.4 *Proof of Proposition 1.13.*

First, we show (1.12) and for this it is enough to use Lemma 1.8 for S a suitable connect set of configurations.

The construction of $\mathcal{S}$ needs a remark: for the process to reach $\mathcal{C}(\mathcal{E}_{d_1(d_2+2)})$ it first reaches $\mathcal{C}(\mathcal{E}_{d_1(d_2+1)})$ with probability close to one when β goes to infinite.

Formally,

$$\mathcal{S} = \eta \bigcup \mathcal{C}'(\mathcal{E}_{d_1 d_2}) \bigcup \mathcal{C}(\mathcal{E}_{d_1(d_2+1)}) \bigcup \mathcal{C}'(\mathcal{E}_{d_1(d_2+1)}), \tag{2.1}$$

where $\zeta \in C'(\mathcal{E}_{k_1 k_2})$ is a configuration obtained from η such that exists at most one site $x \notin \mathcal{E}(\eta)$ such that $\zeta(x) = +1$.

Note that $H(\eta) < H(\xi)$, $\forall \xi \in \mathcal{S}\backslash\{\eta\}$. As $H(\xi) - H(\eta) \geq 6$, $\forall \xi \in C'(\mathcal{E}_{d_1(d_2+1)})$, we have by Lemma 1.8 that

$$\lim_{\beta \to \infty} I\!P\big(\tilde{T}^{\eta}(C'(\mathcal{E}_{d_1(d_2+1)})) < e^{\beta(H(\xi)-H(\eta)-\epsilon)}\big) = 0.$$

Also the estimate above is true for the original process and therefore we get (1.12).

To show (1.11) we consider the random times

$$\tau_1 = \inf\{t \geq 0 : \mathcal{E}(\sigma_t^{\eta}) \neq R_{d_1 d_2}\}$$

and

$$\tau_2 = \inf\{t \geq \tau_1 : \mathcal{E}(\sigma_t^{\eta}) \neq \mathcal{E}(\sigma_{\tau_1}^{\eta})\}.$$

Note that $\sigma_{\tau_1}^{\eta} \in \mathcal{X}_{S_r}$ with probability close to one when β goes to infinite. Using Theorem 1 we have that

- If $\mathcal{E}(\sigma_{\tau_1}^{\eta}) = R_{d_1,(d_2+1)}$ then

$$\lim_{\beta \to \infty} I\!P\big(\mathcal{E}(\sigma_{\tau_2}^{\eta}) = R_{d_1,d_2};\ \tau_2 - \tau_1 < e^{\beta(2+\epsilon)}\big) = 1,\ \forall \epsilon > 0.$$

- If $\mathcal{E}(\sigma_{\tau_1}^{\eta}) = R_{(d_1+1),d_2}$ then

$$\lim_{\beta \to \infty} I\!P\big(\mathcal{E}(\sigma_{\tau_2}^{\eta}) = R_{(d_1+1),(d_2-1)};\ \tau_2 - \tau_1 < e^{\beta(2+\epsilon_1)}\big) = c_1$$

and

$$\lim_{\beta \to \infty} I\!P(\sigma_{\tau_2}^{\eta} \in \mathcal{X}_{S_r}) = 1$$

or

$$\lim_{\beta \to \infty} I\!P\big(\mathcal{E}(\sigma_{\tau_2}^{\eta}) = R_{d,1d_2};\ \tau_2 - \tau_1 < e^{\beta(2+\epsilon_2)}\big) = c_2,\ \forall \epsilon_1, \epsilon_2 > 0,$$

where c_1, c_2 are positive constants.

In this way, or the process starts from the same point or it reaches the set $\mathcal{X}_{S_r}$ and by rearrange it reaches a configuration which envelope is R_{k_1,k_2}, k_1 and k_2 as state in Proposition 1.13.

■

2.5 *Proof of Proposition 1.14*

If $\sigma^{\eta}_{\tau_2}$ is a rectangular configuration the result is proved by Proposition 1.13.

For the process, starting from $\sigma^{\eta}_{\tau_2}$, to reach a configuration with envelope $R_{i,j}$ it needs to visit before some configuration $\zeta \in \mathcal{X}_{\overline{S}}$ with $R(\zeta) = R_{m,(j-1)}$ for $m = i$ or $m = i+1$.

Consider

$$\tau_1 = \inf\{t > 0 : \sigma_t^{\sigma^{\eta}_{\tau_2}} \in \mathcal{X}_{\overline{S}} \text{ and } R(\sigma_t^{\sigma^{\eta}_{\tau_2}}) = R_{m,(j-1)}\}, \quad \text{for } m = i \text{ or } m = i+1.$$

By definition for $\mathcal{X}_{S_c}$ we have that

$$d > i(j-1) \text{ and } d > (i+1)(j-3). \tag{2.2}$$

But if $R(\sigma^{\eta}_{\tau_1}) = R_{i,(j-1)}$ then $d \leq i(j-1)$ what is contrary to (2.2).

Observing that, starting from $\sigma^{\eta}_{\tau_2}$, to reach $\mathcal{C}(\mathcal{E}_{i,j})$ the process needs to visit some configuration which contour is formed by a square of side 1 plus a polygon with perimeter $(i+1) \times (j-1)$, what spends a time of order large than $e^{\beta(6-\delta)}, \forall \delta > 0$ when β goes to infinite, and this finishes the proof.

■

2.6 *Proof of Theorem 4*

Definition 2.1. *A block of particles is formed by at least 4 particles where each one has at least two occupied neighbour.*

Consider $B_i, i \in I\!N$, the set of configurations with i blocks of particles. We have the following result

Proposition 2.1. *For $\eta \in \mathcal{X}_S$ and $\delta > 0$ we have that*

$$\lim_{\beta \to \infty} I\!P\big(T^{\eta}(B_2) > e^{\beta(6-\delta)}\big) = 1.$$

Proof: It is enough to apply Lemma 1.8 for initial configurations belong to $\mathcal{X}_S$.

■

In this way, using Proposition 2.1 and the precedings results we finish the proof of Theorem 4.

■

Acknowledgements. I want to acknowledge E. J. Neves who suggested the theme, A. Simonis by careful reading and comments and the friendly hospitality of IMPA and Università Tor Vergata (Roma II) where part of this work was done. This work was partially supported by Fapesp 96/8245-5.

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