

The Soft Torus: A Variational Analysis of Commutator Norms

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Communicated by the Editors

Received October 28, 1992

The field of C^* -algebras over the interval $[0, 2]$ for which the fibers are the soft tori is shown to be continuous. This result is applied to show that any pair of non-commuting unitary operators can be perturbed (in a weak sense) in a way which decreases the commutator norm. Perturbations in the norm are also considered and a characterization is given for pairs of unitary operators which are local minimum points for the commutator norm in the finite-dimensional case. © 1994 Academic Press, Inc.

1. INTRODUCTION

As in [3], for every ε in the real interval $[0, 2]$ we let A_ε be the universal unital C^* -algebra generated by unitary elements u_ε and v_ε subject to the relation

$$\|u_\varepsilon v_\varepsilon - v_\varepsilon u_\varepsilon\| \leq \varepsilon.$$

Clearly, if $\varepsilon_1 \leq \varepsilon_2$ there is a unique homomorphism $A_{\varepsilon_2} \rightarrow A_{\varepsilon_1}$ sending u_{ε_2} and v_{ε_2} respectively to u_{ε_1} and v_{ε_1} . In the cases $\varepsilon_2 = 2$ and $\varepsilon_1 = \varepsilon$ we denote this map by ϕ_ε , i.e.,

$$\phi_\varepsilon: A_2 \rightarrow A_\varepsilon.$$

One of the main results of the present work (Theorem 3.4) is the fact that there exists a continuous field of C^* -algebras over the interval $[0, 2]$ such that A_ε is the fiber over ε and moreover such that

$$\varepsilon \in [0, 2] \mapsto \phi_\varepsilon(a) \in A_\varepsilon$$

* Partially supported by FAPESP, Brazil.

is a continuous section for every a in A_2 . We refer the reader to [2] for a treatment of the theory of continuous fields of C^* -algebras.

The central point in proving our main result is showing that $\|\phi_\varepsilon(a)\|$ is a continuous function of ε for all a in A_2 . If we let

$$J_\varepsilon = \text{Ker}(\phi_\varepsilon)$$

then we have that $\|\phi_\varepsilon(a)\| = \text{dist}(a, J_\varepsilon)$ and, since the J_ε 's clearly form a decreasing chain of ideals, $\|\phi_\varepsilon(a)\|$ is seen to be an increasing function of ε .

Let us denote by J_ε^+ the closure of the union of all $J_{\varepsilon'}$ for $\varepsilon' > \varepsilon$ and by J_ε^- the intersection of all $J_{\varepsilon'}$ for $\varepsilon' < \varepsilon$. That is,

$$J_\varepsilon^+ = \overline{\bigcup_{\varepsilon' > \varepsilon} J_{\varepsilon'}}$$

and

$$J_\varepsilon^- = \bigcap_{\varepsilon' < \varepsilon} J_{\varepsilon'}$$

1.1. PROPOSITION. *Let ε be in $[0, 2)$ (resp., $(0, 2]$). A sufficient condition for*

$$\varepsilon' \rightarrow \|\phi_{\varepsilon'}(a)\|$$

to be right (resp. left) continuous at ε for all a in A_2 is that $J_\varepsilon = J_\varepsilon^+$ (resp., $J_\varepsilon = J_\varepsilon^-$).

Proof. Note that for a in A_2 we have

$$\text{dist}(a, J_\varepsilon^+) = \inf_{\varepsilon' > \varepsilon} \text{dist}(a, J_{\varepsilon'})$$

and that

$$\text{dist}(a, J_\varepsilon^-) = \sup_{\varepsilon' < \varepsilon} \text{dist}(a, J_{\varepsilon'}).$$

The first identity follows from trivial Banach space facts. On the other hand, the second one cannot be generalized to Banach spaces so let us prove it.

Let

$$\phi: A_2 \rightarrow \prod_{\varepsilon' < \varepsilon} A_{\varepsilon'}$$

be given by $\phi(a) = (\phi_{\varepsilon'}(a))_{\varepsilon' < \varepsilon}$ and observe that $\text{Ker}(\phi) = J_\varepsilon^-$ so

$$\text{dist}(a, J_\varepsilon^-) = \|\phi(a)\| = \sup_{\varepsilon' < \varepsilon} \|\phi_{\varepsilon'}(a)\| = \sup_{\varepsilon' < \varepsilon} \text{dist}(a, J_{\varepsilon'}).$$

The conclusion now follows without much trouble. ■

En passant, let us give an example to support our statement that the above fact does not generalize to Banach spaces.

Let $E = C[0, 1]$ and for f in E put

$$\|f\| = |f(0)| + \sup_{t \in [0, 1]} |f(t)|.$$

Let

$$E_n = \{f \in E: f([1/n, 1]) = 0\}$$

and let g be the constant function $g = 1$. Then

$$\text{dist}(g, E_n) = 1$$

for all n while

$$\text{dist}\left(g, \bigcap_{n \geq 1} E_n\right) = \|g\| = 2.$$

1.2. PROPOSITION. For all $\varepsilon \in [0, 2)$ one has $J_\varepsilon^+ = J_\varepsilon$.

Proof. If we denote by u_ε^+ and v_ε^+ the images of u_2 and v_2 in A_2/J_ε^+ then it is clear that

$$\|u_\varepsilon^+ v_\varepsilon^+ - v_\varepsilon^+ u_\varepsilon^+\| \leq \varepsilon,$$

hence by the universal property of A_ε there is a homomorphism

$$A_\varepsilon = A_2/J_\varepsilon \rightarrow A_2/J_\varepsilon^+$$

sending u_ε and v_ε to u_ε^+ and v_ε^+ . Therefore $J_\varepsilon \subseteq J_\varepsilon^+$. But since the reverse inclusion is trivial, our prove is complete. ■

The main technical result in this work, which is proved in the next two sections, is the following.

1.3. THEOREM. For all ε in $(0, 2]$ one has $J_\varepsilon^- = J_\varepsilon$.

Therefore we have the following.

1.4. COROLLARY. For all a in A_2 the function

$$\varepsilon \in [0, 2] \rightarrow \|\phi_\varepsilon(a)\|$$

is continuous.

2. THE CASE $\varepsilon < 2$

Throughout this section we fix a real number ε with $0 < \varepsilon < 2$.

2.1. LEMMA. *Suppose u_0, u_1, \dots, u_n is a finite sequence of unitary elements in a C^* -algebra B such that $\|u_{k-1} - u_k\| \leq \varepsilon$ for all k . Then, for every $\delta > 0$, there are unitaries v_0, v_1, \dots, v_n in B such that*

- (i) $\|v_k - u_k\| \leq \delta$ for $k = 0, \dots, n$
- (ii) $\|v_{k-1} - v_k\| < \varepsilon$ for $k = 1, \dots, n$.

Proof. Since $\|u_{k-1} - u_k\| \leq \varepsilon < 2$, it follows that $\|u_k u_{k-1}^{-1} - 1\| < 2$ so that -1 is not in the spectrum of $u_k u_{k-1}^{-1}$. Therefore we may define

$$h_k = \log(u_k u_{k-1}^{-1})$$

where \log is the principal branch of the logarithm. Each h_k is then a skew adjoint element in B and we have

$$u_k = e^{h_k} u_{k-1} \quad \text{for } k = 1, 2, \dots, n.$$

We choose our sequence v_0, \dots, v_n of the form

$$v_0 = u_0$$

and

$$v_k = e^{t_k h_k} u_{k-1} \quad \text{for } k \geq 1$$

where each t_k is a suitably chosen positive real number approaching 1 from below.

A real function which is useful in our estimates is

$$d(x) = |1 - e^{ix}| = 2 \sin\left(\frac{x}{2}\right) \quad \text{for } x \in \mathbf{R}.$$

For instance, observe that if h is skew adjoint and $\|h\| \leq \pi$ one has

$$\|1 - e^h\| = d(\|h\|)$$

by the spectral theorem. Moreover, for all $k = 1, \dots, n$,

$$d(\|h_k\|) = \|1 - e^{h_k}\| = \|u_{k-1} - u_k\| \leq \varepsilon$$

which implies (since d is increasing in $[0, \pi]$) that

$$\|h_k\| \leq \theta,$$

where $\theta = d^{-1}(\varepsilon)$. Note that $\theta < \pi$ because $\varepsilon < 2$.

In search of the correct choice of the t_k 's observe that

$$\|v_0 - v_1\| = \|u_0 - e^{t_1 h_1} u_0\| = d(t_1 \|h_1\|) \leq d(t_1 \theta)$$

and that for $k \geq 1$ we have

$$\begin{aligned} \|v_k - v_{k+1}\| &\leq \|v_k - u_k\| + \|u_k - v_{k+1}\| \\ &= \|e^{t_k h_k} u_{k-1} - e^{h_k} u_{k-1}\| + \|u_k - e^{t_{k+1} h_{k+1}} u_k\| \\ &= d((1 - t_k) \|h_k\|) + d(t_{k+1} \|h_{k+1}\|) \\ &\leq d((1 - t_k) \theta) + d(t_{k+1} \theta). \end{aligned}$$

Note that $d'(x) = \cos(x/2)$ so for $x \in [0, \theta]$ we have $m \leq d'(x) \leq 1$ where $m = \cos(\theta/2)$ is strictly positive since $\theta < \pi$. By the mean value theorem we then have

$$m |t - s| \leq |d(t) - d(s)| \leq |t - s|$$

for all t and s in $[0, \theta]$.

If this last fact is used in our previous computations, we obtain

$$\begin{aligned} \|v_0 - v_1\| &\leq d(t_1 \theta) = \varepsilon - (d(\theta) - d(t_1 \theta)) \\ &\leq \varepsilon - m(\theta - t_1 \theta) = \varepsilon - m\theta(1 - t_1), \end{aligned}$$

while for $k \geq 1$,

$$\begin{aligned} \|v_k - v_{k+1}\| &\leq d((1 - t_k) \theta) + d(t_{k+1} \theta) \\ &\leq (1 - t_k) \theta + \varepsilon - (d(\theta) - d(t_{k+1} \theta)) \\ &\leq (1 - t_k) \theta + \varepsilon - m(\theta - t_{k+1} \theta) \\ &= \varepsilon + (1 - t_k - m(1 - t_{k+1})) \theta. \end{aligned}$$

Therefore the condition that $\|v_{k+1} - v_k\| < \varepsilon$ is fulfilled as long as

$$1 - t_1 > 0$$

and

$$1 - t_{k+1} > \frac{1 - t_k}{m}.$$

If we thus put $t_k = 1 - 2^k \sigma/m^k$ for $\sigma < (m/2)^n$ we have that each t_k is in $(0, 1)$ and $\|v_{k+1} - v_k\| < \varepsilon$.

Clearly, as σ tends to zero, each v_k approaches the corresponding u_k so a suitable choice for σ yields

$$\|v_k - u_k\| \leq \delta$$

for all $k = 0, 1, \dots, n$. ■

Recall from [3] that A_ε is isomorphic to the crossed product

$$A_\varepsilon \simeq B_\varepsilon \times_{\tau} \mathbf{Z},$$

where B_ε is the universal C^* -algebra generated by a sequence $\{u_n^{(\varepsilon)} : n \in \mathbf{Z}\}$ of unitaries satisfying the relations

$$\|u_n^{(\varepsilon)} - u_{n+1}^{(\varepsilon)}\| \leq \varepsilon$$

for all n .

Moreover, τ is the automorphism of B_ε given by

$$\tau(u_n^{(\varepsilon)}) = u_{n+1}^{(\varepsilon)} \quad \text{for } n \in \mathbf{Z}.$$

2.2. PROPOSITION. *There exists a sequence $(\psi_n)_{n \in \mathbf{N}}$ of endomorphisms of B_ε , converging pointwise to the identity map, such that*

$$\sup_{k \in \mathbf{K}} \|\psi_n(u_k^{(\varepsilon)}) - \psi_n(u_{k+1}^{(\varepsilon)})\| < \varepsilon.$$

Proof. By the previous lemma, let for every n ,

$$v_{-n}^{(n)}, v_{-n+1}^{(n)}, \dots, v_0^{(n)}, v_1^{(n)}, \dots, v_n^{(n)}$$

be unitaries in B_ε such that

$$\|v_k^{(n)} - u_k^{(\varepsilon)}\| \leq \frac{1}{n} \quad \text{for } k = -n, \dots, n$$

and

$$\|v_{k-1}^{(n)} - v_k^{(n)}\| < \varepsilon \quad \text{for } k = -n+1, \dots, n.$$

Define $\psi_n: B_\varepsilon \rightarrow B_\varepsilon$ by

$$\psi_n(u_k^{(\varepsilon)}) = \begin{cases} v_{-n}^{(n)} & \text{if } k < -n \\ v_k^{(n)} & \text{if } -n \leq k \leq n \\ v_n^{(n)} & \text{if } k > n. \end{cases}$$

It is then clear that

$$\lim_{n \rightarrow \infty} \psi_n(u_k^{(\varepsilon)}) = u_k^{(\varepsilon)}$$

which implies that ψ_n converges pointwise to the identity.

The condition that

$$\sup_{k \in \mathbf{Z}} \|\psi_n(u_k^{(\varepsilon)}) - \psi_n(u_{k+1}^{(\varepsilon)})\| < \varepsilon$$

is also clearly satisfied. ■

2.3. DEFINITION. Let K_ϵ be the ideal in B_2 given by the kernel of the canonical map

$$\phi_\epsilon: B_2 \rightarrow B_\epsilon.$$

2.4. THEOREM. One has $\bigcap_{\epsilon' < \epsilon} K_{\epsilon'} = K_\epsilon$.

Proof. Let x be in $\bigcap_{\epsilon' < \epsilon} K_{\epsilon'}$ and put $y = \psi_\epsilon(x)$. We have

$$y = \lim_{n \rightarrow \infty} \psi_n(y) = \lim_{n \rightarrow \infty} \psi_n(\phi_\epsilon(x)).$$

Now let

$$\epsilon'_n = \sup_k \|\psi_n \phi_\epsilon(u_k^{(2)}) - \psi_n \phi_\epsilon(u_{k+1}^{(2)})\| = \sup_k \|\psi_n(u_k^{(\epsilon)}) - \psi_n(u_{k+1}^{(\epsilon)})\|$$

which by (2.2) is strictly less than ϵ . So $\psi_n \phi_\epsilon$ factors through $B_{\epsilon'_n}$ and since x is in $K_{\epsilon'_n}$ we have $\psi_n \phi_\epsilon(x) = 0$. So $y = 0$ which implies that x is in K_ϵ . The converse inclusion is trivial. ■

2.5. LEMMA. Let $\phi: A \rightarrow B$ be a C^* -algebra homomorphism and suppose ϕ is equivariant with respect to automorphisms α and β of A and B respectively.

Let $K = \text{Ker}(\phi)$ and $J = \text{Ker}(\bar{\phi})$ where $\bar{\phi}$ is the canonical extension of ϕ to the corresponding crossed products by \mathbf{Z} . Then

$$J = \{x \in A \times_\alpha \mathbf{Z}; E_A(xu^{-n}) \in K \text{ for } n \in \mathbf{Z}\}$$

where

$$E_A: A \times_\alpha \mathbf{Z} \rightarrow A$$

is the associated conditional expectation [5] and u is the unitary implementing α .

Proof. It is clear that $\phi \circ E_A = E_B \circ \bar{\phi}$ where E_B is the conditional expectation for $B \times_\beta \mathbf{Z}$.

Let v be the implementing unitary for $B \times_\beta \mathbf{Z}$. Given $x \in A \times_\alpha \mathbf{Z}$ we have that x is in J if and only if $\bar{\phi}(x) = 0$ which is equivalent to the fact that $E_B(\bar{\phi}(x)v^{-n}) = 0$ for all n , or that $\phi(E_A(xu^{-n})) = 0$. But this is to say that $E_A(xu^{-n})$ is in K . ■

2.6. THEOREM. For every $\epsilon \in (0, 2)$ one has $J_\epsilon^- = J_\epsilon$.

Proof. Let $E: A_2 \rightarrow B_2$ be the conditional expectation induced by the isomorphism

$$A_2 \simeq B_2 \times_\tau \mathbf{Z}.$$

Given x in J_ε^- we have that $E(xu^{-n})$ is in $K_{\varepsilon'}$ for all n and $\varepsilon' < \varepsilon$ so

$$E(xu^{-n}) \in \bigcap_{\varepsilon' < \varepsilon} K_{\varepsilon'} = K_\varepsilon$$

for all n , which shows that $x \in J_\varepsilon$. The converse inclusion is trivial. ■

3. THE CASE $\varepsilon = 2$

The purpose of this section is to prove that $J_2^- = J_2$ or, since $J_2 = (0)$, that

$$\bigcap_{\varepsilon < 2} J_\varepsilon = (0).$$

Note that the techniques employed in the previous section do not work here because one could not take logarithms in the proof of (2.1) if $\varepsilon = 2$. A different approach is thus necessary.

3.1. LEMMA. *Let w_1 and w_2 be $n \times n$ unitary matrices. Then $\|w_1 - w_2\| = 2$ if and only if $\det(w_1 + w_2) = 0$.*

Proof. We have that $\|w_1 - w_2\| = 2$ if and only if $\|w_1 w_2^{-1} - 1\| = 2$ which is equivalent to -1 being in the spectrum of $w_1 w_2^{-1}$, which is to say that $\det(w_1 w_2^{-1} + 1) = 0$ or that $\det(w_1 + w_2) = 0$. ■

3.2. PROPOSITION. *Given unitary $n \times n$ matrices u and v such that $\|uv - vu\| = 2$ there is, for every $\delta > 0$, a unitary u' with*

$$\|u' - u\| < \delta$$

and

$$\|u'v - vu'\| < 2.$$

Proof. Write $u = e^h$ for some skew adjoint h and let $u(t) = ue^{-th}$ for all real t . Put

$$f(t) = \det(u(t)v + vu(t))$$

and observe that

$$f(1) = \det(2v) \neq 0,$$

while

$$f(0) = \det(uv + vu) = 0$$

by (3.1). Therefore f is not a constant function and since it is analytic, its zeros are isolated. So there are arbitrarily small values of t for which $f(t) \neq 0$, which is to say, by (3.1) again, that

$$\|u(t)v - vu(t)\| < 2.$$

Taking t sufficiently small also ensures that $\|u(t) - u\| < \delta$. ■

3.3. THEOREM. *One has $J_2^- = J_2$.*

Proof. Assume by way of contradiction that $a \in J_2^-$ is non-zero.

Note that A_2 is isomorphic to the full C^* -algebra of the free group on two generators so that by [1] A_2 has a separating family of finite dimensional representations. Let therefore

$$\pi: A_2 \rightarrow M_n(\mathbf{C})$$

be a representation such that $\pi(a) \neq 0$.

Let $u = \pi(u_2)$ and $v = \pi(v_2)$ and write

$$u = \lim_{i \rightarrow \infty} u'_i$$

where $\|u'_i v - v u'_i\| < 2$ by (3.2). For each i let π_i be the representation of A_2 such that

$$\pi_i(u_2) = u'_i \quad \text{and} \quad \pi_i(v_2) = v.$$

Then, if $\varepsilon_i = \|u'_i v - v u'_i\|$, we have that π_i vanishes on J_{ε_i} and thus also on J_2^- so $\pi_i(a) = 0$.

On the other hand, it is clear that π_i converges pointwise to π so $\pi(a) = 0$ which is a contradiction. ■

3.4. THEOREM. *There exists a continuous field of C^* -algebras over the interval $[0, 2]$ such that A_ε is the fiber over ε and such that*

$$\varepsilon \in [0, 2] \mapsto \phi_\varepsilon(a) \in A_\varepsilon$$

is a continuous section for every a in A_2 .

Proof. Let S be the set of sections

$$\varepsilon \in [0, 2] \mapsto \phi_\varepsilon(a) \in A_\varepsilon$$

for a in A_2 . According to [2, Props. 10.2.3 and 10.3.2] all one needs to check is that S is a $*$ -subalgebra of the algebra of all sections, that the set

of all $s(\varepsilon)$ as s runs through S is dense in A_ε and that $\|s(\varepsilon)\|$ is continuous as a function of ε for all s in S .

The first two properties are trivial while the last one follows from (1.1), (1.2), (2.6), and (3.3). ■

4. LOCAL MINIMA FOR COMMUTATOR NORMS

As clearly indicated by the results obtained above, the phenomenon under consideration is related to the following question.

4.1. QUESTION. *Given unitary operators u and v which do not commute, when is it possible to perturb u and v in order to obtain a new pair u' and v' such that*

$$\|u'v' - v'u'\| < \|uv - vu\| ?$$

In other words, (4.1) calls for a characterization of pairs of unitary operators which are not local minimum points for the commutator norm. Proposition 3.2 is clearly a partial answer to (4.1) and it says that when $\|uv - vu\| = 2$ in finite dimensions then (u, v) is never such a local minimum point.

In formulating the question above we chose not to specify the precise meaning of “to perturb” in order to allow for different points of view.

The following is a complete answer to (4.1) under quite a loose type of perturbation which we might call **-strong dilated perturbation*.

4.2. THEOREM. *Let u and v be non-commuting unitary operators on a Hilbert space H . Then there are nets $(u_i)_i$ and $(v_i)_i$ of unitary operators on H^∞ (the direct sum of infinitely many copies of H) satisfying*

$$\|u_i v_i - v_i u_i\| < \|uv - vu\|$$

and such that the compressions

$$\text{proj}_{H^\infty} \circ u_i|_H \quad \text{and} \quad \text{proj}_{H^\infty} \circ v_i|_H$$

of u_i and v_i to H converge *-strongly to u and v respectively.

Proof. After Theorems 2.6 and 3.3 this basically becomes a consequence of [4] and some well known results on representation theory. We therefore restrict ourselves to a sketch of the proof.

Let $\varepsilon = \|uv - vu\|$ and consider the set N of states on A_2 which vanish on some $J_{\varepsilon'}$ for $\varepsilon' < \varepsilon$. Since $\bigcap_{\varepsilon' < \varepsilon} J_{\varepsilon'} = J_\varepsilon$ it can be proved that N is weakly dense in the set of states of A_2 that vanish on J_ε .

Given u and v let π be the representation of A_2 on H such that $\pi(u_2) = u$ and $\pi(v_2) = v$. Assume without loss of generality that π is cyclic with cyclic vector ξ and put

$$f: a \in A_2 \rightarrow \langle \pi(a) \xi, \xi \rangle.$$

Since π factors through A_ε we have that f vanish on J_ε so there exists a net $(f_i)_{i \in I}$ in N converging weakly to f . Let, for every i , π_i be the GNS representation of A_2 corresponding to f_i . Since each f_i vanishes on some $J_{\varepsilon'_i}$ with $\varepsilon'_i < \varepsilon$ the same is true for π_i , hence

$$\|\pi_i(u_2) \pi_i(v_2) - \pi_i(v_2) \pi_i(u_2)\| \leq \varepsilon'_i < \varepsilon.$$

Using the methods of [4] one may assume that the space H_i where π_i acts is a subspace of H^∞ and that the conclusion holds with

$$u_i = \pi_i(u_2) + 1 - p_i$$

and

$$v_i = \pi_i(v_2) + 1 - p_i,$$

where p_i is the projection onto H_i . ■

We therefore see that there are no local minimum points for the commutator norm other than the commuting pairs, as long as we consider *-strong dilated perturbations.

The situation is quite different if norm perturbations are considered, as we see in the next section

5. NORM PERTURBATIONS IN FINITE DIMENSIONS

Let us now study Question 4.1 for pairs of unitary operators on a finite dimensional Hilbert space. From now on we only consider norm perturbations.

For $n \geq 3$ denote by Ω_n and S_n the $n \times n$ Voiculescu unitary matrices (see [6])

$$\Omega_n = \begin{pmatrix} \omega & & & & \\ & \omega^2 & & & \\ & & \omega^3 & & \\ & & & \ddots & \\ & & & & \omega^n \end{pmatrix} \quad \text{and} \quad S_n = \begin{pmatrix} 0 & & & & 1 \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & & \\ & & & 1 & 0 \end{pmatrix}$$

where $\omega = e^{2\pi i/n}$.

5.1. THEOREM. For $n \geq 3$ there exists a neighborhood V of the pair (Ω_n, S_n) in $U(n) \times U(n)$ so that

$$\|uv - vu\| \geq \|\Omega_n S_n - S_n \Omega_n\|$$

for all pairs (u, v) in V .

Proof. Note that $\Omega_n S_n \Omega_n^{-1} S_n^{-1} = \omega I_n$ so that if (u, v) is close enough to (Ω_n, S_n) then the spectrum of $uvu^{-1}v^{-1}$ is in a small neighborhood of ω in the complex plane.

On the other hand, note that

$$\det(uvu^{-1}v^{-1}) = 1$$

so that if the spectrum of $uvu^{-1}v^{-1}$ is the set $\{e^{i\theta_1}, \dots, e^{i\theta_n}\}$ with $-\pi < \theta_i < \pi$ one has that $\sum_{k=1}^n \theta_k$ is in $2\pi\mathbf{Z}$. So, by continuity,

$$\sum_{k=1}^n \theta_k = 2\pi.$$

Therefore, for some k_0 we must have $\theta_{k_0} \geq 2\pi/n$ and it follows that

$$\|uv - vu\| \geq |e^{i\theta_{k_0}} - 1| \geq |\omega - 1| = \|\Omega_n S_n - S_n \Omega_n\|. \blacksquare$$

In other words, the pair (Ω_n, S_n) is a local minimum for the commutator norm. This shows that the situation is quite different from what we saw when we considered *-strong dilated perturbations in Section 4. This result should also be compared with [6].

Clearly the method used in Theorem 5.1 above applies to show that the conclusion is also true for any pair of unitary matrices whose multiplicative commutator is a scalar multiple of the identity, but not equal to $-I$ (see Proposition 3.2).

In fact, among irreducible pairs there are no other examples, as we prove shortly. We say that a pair of unitary operators is irreducible when there is no proper invariant subspace for both elements of the pair.

Denote by γ the map

$$\gamma: (u, v) \in U(n) \times U(n) \mapsto uvu^{-1}v^{-1} \in SU(n).$$

5.2. LEMMA. A point $(u, v) \in U(n) \times U(n)$ is regular for γ (in the sense that γ is a submersion at (u, v)) if and only if (u, v) is an irreducible pair.

Proof. If h, k are in the Lie algebra $u(n)$ of $U(n)$, a simple computation shows that

$$d\gamma_{(u,v)}(uh, vk) = uv(v^{-1}hv - h + k - u^{-1}ku)u^{-1}v^{-1}.$$

So γ is a submersion at (u, v) if and only if the map

$$L: (h, k) \in \mathfrak{u}(n) \times \mathfrak{u}(n) \rightarrow v^{-1}hv - h + k - u^{-1}ku \in \mathfrak{su}(n)$$

is onto the Lie algebra $\mathfrak{su}(n)$ of $SU(n)$.

Under the inner product on $\mathfrak{su}(n)$ defined by

$$\langle x, y \rangle = \text{Trace}(xy^*),$$

the orthogonal space to the image of L can easily be seen to be the set

$$\{x \in \mathfrak{su}(n) : xu = ux \text{ and } xv = vx\}.$$

Now, by Schur's lemma, irreducibility of (u, v) can be characterized by the fact that only scalars commute with both u and v . Since $\mathfrak{su}(n)$ contains no scalar matrices the result follows. ■

5.3. THEOREM. *If (u, v) is an irreducible pair in $U(n) \times U(n)$ and at the same time a local minimum for the commutator norm then $uvu^{-1}v^{-1}$ is a scalar.*

Proof. By the open mapping theorem the image under γ of a neighborhood of (u, v) is a neighborhood of $\gamma(u, v)$. But since

$$\|uv - vu\| = \|\gamma(u, v) - 1\|,$$

it follows that $\gamma(u, v)$ is a local minimum for the map

$$w \in SU(n) \rightarrow \|w - 1\|$$

and this implies, as a moment of thought will reveal, that $\gamma(u, v)$ is a scalar. ■

This completes the classification of local minima for irreducible pairs. So let us now consider a reducible pair (u, v) of unitary $n \times n$ matrices. As usual write

$$u = \bigoplus u_j \quad \text{and} \quad v = \bigoplus v_j$$

where each pair (u_j, v_j) is irreducible and observe that

$$\|uv - vu\| = \max_j \|u_j v_j - v_j u_j\|.$$

5.4. THEOREM. *Let $u = \bigoplus u_j$ and $v = \bigoplus v_j$ be as above and suppose that the pair (u, v) is a local minimum for the commutator norm. Then for some value of j for which*

$$\|u_j v_j - v_j u_j\| = \|uv - vu\|$$

one has that $u_j v_j u_j^{-1} v_j^{-1}$ is a scalar.

Proof. If this is not so then for all such j the pair (u_j, v_j) admits by (5.3) a small perturbation decreasing the commutator norm. Together, these perturbations yield a contradiction to the hypothesis. ■

A natural question which one could ask is, of course, whether the converse to Theorem 5.4 is also true. A good test case is given by the pair

$$(\Omega_n \oplus I_m, S_n \oplus I_m),$$

that is, the direct sum of Voiculescu's unitaries with the $m \times m$ identity matrix.

This pair clearly satisfies the conclusion of Theorem 5.4 and so it is natural to ask whether or not it is a local minimum for the commutator norm.

Despite strong favorable evidence given by some partial positive results and a large amount of computer simulation supporting this thesis, we were unable to establish a proof for this fact. In fact, we do not even know whether the above pair is a local minimum for $n=3$ and $m=1$. Nevertheless, we conjecture the following.

5.5. CONJECTURE. *The converse of Theorem 5.4 is also true.*

6. AN EXAMPLE

Considering the apparent discrepancy between (4.2) and (5.1) it is perhaps interesting to see a concrete example of the nets $(u_i)_i$ and $(v_i)_i$, whose existence is guaranteed by (4.2), in case the given unitaries are taken to be Voiculescu's unitaries, i.e., $u = \Omega_n$ and $v = S_n$.

For that purpose it is enough to find, for all $\delta > 0$, unitary operators u' and v' on a separable, infinite dimensional Hilbert space H , and an orthonormal set $\{\xi_k : k \in \mathbf{Z}/n\mathbf{Z}\}$ of vectors in H such that

- (i) $\|u'v' - v'u'\| < \|\Omega_n S_n - S_n \Omega_n\|$
- (ii) $\|u'(\xi_k) - \omega^k \xi_k\| < \delta$, and
- (iii) $\|v'(\xi_k) - \xi_{k+1}\| < \delta$

for all k in $\mathbf{Z}/n\mathbf{Z}$ where $\omega = e^{2\pi i/n}$.

Let $H = L_2(\mathbf{S}^1)$ and let u' be the unitary operator on H defined by

$$u'(\xi)|_z = z\xi(z) \quad \text{for } \xi \in H, \quad z \in \mathbf{S}^1,$$

and, for $\theta < 2\pi/n$, let v' be defined by

$$v'(\xi)|_z = \xi(e^{-i\theta}z) \quad \text{for } \xi \in H, \quad z \in \mathbf{S}^1.$$

Let ξ_0 be a unit vector in H represented by a function f on \mathbf{S}^1 supported in a neighborhood V of $z = 1$ which is small enough so that V is disjoint from $e^{ik\theta}V$ for $k = 1, 2, \dots, n-1$. For all such k let $\xi_k = v'^k(\xi_0)$. The reader may now check that (i), (ii), and (iii) hold as long as θ is close to $2\pi/n$ and the diameter of V is small enough.

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