THE GROUP OF UNITS OF $Z D_4$

by

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Recently, Hughes and Pearson [4] studied the group of units of the integral group ring $\mathbb{Z}S_3$, where S_3 is the symmetric group on three symbols. Using similar methods we have studied the units of the integral group ring of the dihedral group of eight elements. i.e. the group D_4 with two generatous a and b and relations:

$$a^4 = b^2 = baba = 1.$$

For an arbitrary group G we introduce the following notation: $U(\mathbb{Z}G)$ will stand for the group of units of $\mathbb{Z}G$. The elements $\pm g$, with g in G are called the trivial units of $\mathbb{Z}G$.

The homomorphism $\epsilon: \mathbb{Z}G \to \mathbb{Z}$ such that $\epsilon(g) = 1$ for all g in G is called the augmentation function. $V(\mathbb{Z}G)$ will denote the normal subgroup of units $u \in \mathbb{Z}G$ such that $\epsilon(u) = 1$. An element in $V(\mathbb{Z}G)$ is called a normalized unit. Finally an automorphism θ of $\mathbb{Z}G$ is said to be normalized if $\epsilon \circ \theta(g) = 1$ for all g in G.

The following questions were raised in [4].

(a) Is every unit of finite order in $\mathbb{Z}G$ conjugate to a trivial unit?

(b) What are the maximal finite subgroups of $U(\mathbb{Z}G)$?

(c) Is every normalized automorphism of $\mathbb{Z}G$ the product of an inner automorphism and an automorphism of G ?

We answer these questions for $G = D_4$. A detailed account will appear in [6].

1. <u>The group of units</u>

Let $M_2(Q)$ denote the full ring of 2×2 matrices over the rational field Q. It is well known that there exists an isomorphism:

(1) $$\theta : QD_4 \to Q \oplus Q \oplus Q \oplus Q \oplus M_2(Q)$$

such that:

(2)
$$\phi(a) = (1,1,-1,-1, \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix})$$
$$\phi(b) = (1,-1,1,-1, \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix})$$

Take D_4 as a Q-basis of QD_4 and the canonical basis of the direct sum. Considered as a Q-isomorphism, ϕ is expressed by a matrix A with respect to this basis, whose inverse is:

$$A^{-1} = \frac{1}{8} \begin{vmatrix} 1 & 1 & 1 & 1 & 2 & 0 & 0 & 2 \\ 1 & 1 & -1 & -1 & 0 & -2 & 2 & 0 \\ 1 & 1 & 1 & 1 & -2 & 0 & 0 & -2 \\ 1 & 1 & -1 & -1 & 0 & 2 & -2 & 0 \\ 1 & -1 & 1 & -1 & 0 & 2 & 2 & 0 \\ 1 & -1 & -1 & 1 & -2 & 0 & 0 & 2 \\ 1 & -1 & 1 & -1 & 0 & -2 & -2 & 0 \\ 1 & -1 & -1 & 1 & 2 & 0 & 0 & -2 \end{vmatrix}$$

We now see that an element

$$X = (x_1, x_2, x_3, x_4, \begin{vmatrix} x_5 & x_6 \\ x_7 & x_8 \end{vmatrix}) \in \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus M_2(Z)$$

belongs to $\phi(ZD_4)$ if and only if:

(3) $\quad x_1 + x_2 + x_3 + x_4 + 2x_5 + 2x_8 \equiv 0 \pmod 8$

and seven other congruence equations obtained from the rows of A^{-1} are satisfied.

It can be shown that X belongs to $\phi\ (U(\mathbb{Z}D_4))$ if and only if one of the following conditions also holds (see [6]):

(4) (i) $x_8 \equiv 1 \pmod 2$; $x_5+x_6+x_7-x_8 \equiv 0 \pmod 4$;

$x_5+x_8 \equiv 2 \pmod 4$

(ii) $x_8 \equiv 1 \pmod 2$; $x_5+x_6+x_7-x_8 \equiv 2 \pmod 4$;

$x_5+x_8 \equiv 2 \pmod 4$

(iii) $x_8 \equiv 0 \pmod 2$; $x_5+x_8 \equiv 0 \pmod 4$

If we denote by Ω the subgroup of matrices $X = \begin{vmatrix} x_5 & x_6 \\ x_7 & x_8 \end{vmatrix}$ in $GL(2,\mathbb{Z})$ verifying any one of the conditions in (4) and which is part of a solution (x_1,x_2,x_3,x_4,X) of the system in (3), it can be shown that:

(5) $$V(ZD_4) \cong \Omega \quad \text{and} \quad |GL(2,\mathbb{Z}):\Omega| = 6$$

Actually:

$$w_1 = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \quad w_2 = \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} \quad w_3 = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}$$

$$w_4 = \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} \quad w_5 = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} \quad w_6 = \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}$$

is a complete set of representatives of the left cosets of Ω in $GL(2,\mathbb{Z})$.

To give an answer to the first question, we have shown that there are five conjugacy classes of elements of order 2 in Ω.

The following are representatives of these classes:

$-I$; $X_1 = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}$; $Y_1 = \begin{vmatrix} 1 & 4 \\ 0 & -1 \end{vmatrix}$; $X_2 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$; $Y_2 = \begin{vmatrix} -2 & -3 \\ 1 & 3 \end{vmatrix}$

It is easy to see that X_i is conjugate to Y_i, $i = 1,2$, in $GL(2,\mathbb{Z})$.

The elements of order 2 in D_4 are a^2, b, a^2b, ab, a^3b. If π stands for the natural projection of the direct sum onto $M_2(Q)$ we see that: $\pi\circ\phi(a^2) = -I$; $\pi\circ\phi(b) \in X_1\,\Omega$; $\pi\circ\phi(a^2b) \in X_1\,\Omega$; $\pi\circ\phi(a^3b) \in X_2\,\Omega$.

So the answer to the first question is negative.

The following Proposition answers the second question:

<u>Proposition</u> - A maximal finite subgroup of Ω is conjugate to one of the following subgroups: $D_4^* = \langle A,B\rangle$, $D_4' = \langle A',Y_2\rangle$ where

$$A = \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix}; \quad B = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}; \quad A' = \begin{vmatrix} -2 & -5 \\ 2 & 2 \end{vmatrix}.$$

Finally, the answer to question (c) is also negative.

The function $\psi : \mathbb{Z}D_4 \to \mathbb{Z}D_4$ such that

$$\psi(a) = 2a - a^3 - b + ab + a^2b - a^3b$$

$$\psi(b) = a - a^3 + ab + a^2b - a^3b$$

provides a counter example.

References

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