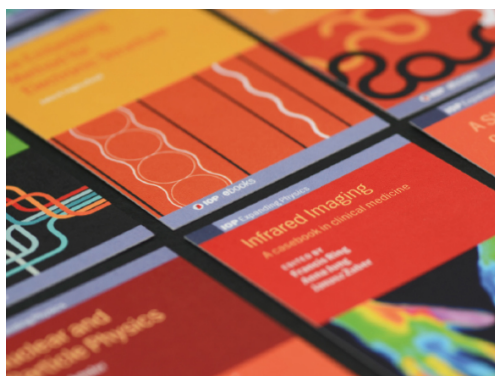SISSA

PAPER

Spherical model with Dzyaloshinskii–Moriya interactions

To cite this article: William de Castilho and S R Salinas *J. Stat. Mech.* (2022) 073106

View the article online for updates and enhancements.

Spherical model with Dzyaloshinskii–Moriya interactions

William de Castilho[*] and S R Salinas

Instituto de Física, Universidade de São Paulo, São Paulo, SP, Brazil

E-mail: william2.castilho@usp.br

Abstract. We analyze the thermodynamic behavior of a ferromagnetic mean-spherical model with three distinct spin components and the addition of Dzyaloshinkii–Moriya interactions. Exact calculations are performed for classical and quantum versions of this lattice model system. We show the onset of space modulated structures at low temperatures.

Contents

[*]Author to whom any correspondence should be addressed.

1. Introduction

The spherical model of ferromagnetism, which was proposed and exactly solved by Berlin and Kac about seventy years ago [1, 2], still remains an excellent laboratory to test some thermodynamic properties of phase transitions and critical phenomena. We then decided to investigate a ferromagnetic mean-spherical model, with three distinct spin components and the addition of Dzyaloshinskii–Moriya (DM) interactions [3, 4]. This model system is still amenable to some exact calculations, both at the classical and at a quantum level, which do indicate the presence of magnetically modulated structures, which are the hallmark of the DM interactions [5, 6].

Spherical models with several spin components, and with ferromagnetic interactions, have been considered in the earlier literature, and have been shown to lead to a simple factorization of the canonical partition function [7]. A mean-spherical model has been recently used by Aqua and Fisher [8] to account for some features of a lattice gas with several components. We were then motivated to revisit these calculations for a mean-spherical model, with the consideration of three spin components, and the addition of DM interactions. The presence of extra couplings between two different sets of degrees of freedom gives rise to a more involved and interesting problem. We were also motivated by the increasing interest in the analysis of systems with DM interactions and the description of the novel skyrmion excitations in the context of condensed matter physics [9–11].

We consider the spin Hamiltonian

$$\mathcal{H} = -J \sum_{(\vec{r}, \vec{r}')} \vec{S}_{\vec{r}} \cdot \vec{S}_{\vec{r}'} - D \sum_{\vec{r}} \left(\vec{S}_{\vec{r}} \times \vec{S}_{\vec{r}+\hat{z}} \right) \cdot \hat{z}, \tag{1}$$

where the classical spin variables $\vec{S}_{\vec{r}} = \left(S_{\vec{r}}^x, S_{\vec{r}}^y, S_{\vec{r}}^z \right)$ are three-component vectors on the sites $\vec{r}$ of a hypercubic lattice of N sites, $J > 0$ is a ferromagnetic exchange interaction, the first sum is over nearest-neighbor pairs of lattice sites, and $\hat{z}$ is a unit vector along an axial direction. This is perhaps the simplest spin Hamiltonian to represent a ferromagnetic model system with the addition of monoaxial DM interactions.

The partition function of this classical mean-spherical model is usually written as

$$\Xi = \prod_{\vec{r}} \left[\int_{-\infty}^{+\infty} \mathrm{d}S_{\vec{r}}^x \int_{-\infty}^{+\infty} \mathrm{d}S_{\vec{r}}^y \int_{-\infty}^{+\infty} \mathrm{d}S_{\vec{r}}^z \right] \exp\left[\overline{\mathcal{H}} \right], \tag{2}$$

with

$$\overline{\mathcal{H}} = -\beta\mathcal{H} - s_1 \sum_{\vec{r}} (S_{\vec{r}}^x)^2 - s_2 \sum_{\vec{r}} (S_{\vec{r}}^y)^2 - s_3 \sum_{\vec{r}} (S_{\vec{r}}^z)^2, \tag{3}$$

where $\beta = 1/k_{\mathrm{B}}T$ is the inverse of temperature, and s_1, s_2, and s_3 are three spherical parameters. In this formulation, we have to take into account three spherical constraints, which are given by

$$\left\langle \sum_{\vec{r}} (S_{\vec{r}}^x)^2 \right\rangle = -\frac{\partial}{\partial s_1} \ln \Xi = N, \tag{4}$$

with similar equations for the y and z spin components,

$$\left\langle \sum_{\vec{r}} (S_{\vec{r}}^y)^2 \right\rangle = -\frac{\partial}{\partial s_2} \ln \Xi = N, \qquad \left\langle \sum_{\vec{r}} (S_{\vec{r}}^z)^2 \right\rangle = -\frac{\partial}{\partial s_3} \ln \Xi = N. \tag{5}$$

In the first section, we analyze the phase diagram of this classical system in terms of temperature T and a parameter $p = D/J$, which describes the strength of the chiral interactions. We perform exact calculations to show the existence of a modulated structure along the $\hat{z}$ direction at sufficiently low temperatures.

We then turn to the analysis of the quantum version of this mean-spherical model with DM interactions. According to previous work for the mean-spherical ferromagnet, a quantum version may be obtained by resorting to a standard canonical quantization procedure [12–15]. The first attempt of such quantization was made by Obermair [16], other early and notorious works are by Henkel [17] and Nieuwenhuizen [18]. The problem is then formulated in terms of a set of coupled boson operators, which are duly diagonalized by known techniques of second quantization [19]. Again, we show the persistence of spacial modulated structures in the low-temperature region of the phase diagram.

2. Mean-spherical model with DM interactions

The partition function of the three-component mean-spherical model with DM interactions is given by equation (2) supplemented by the spherical constraints, equations (4) and (5). We now introduce periodic boundary conditions, and write a Fourier representation,

$$S_{\vec{r}}^{\nu} = \frac{1}{\sqrt{N}} \sum_{\vec{q}} \sigma_{\vec{q}}^{\nu} \exp(i\,\vec{q}\cdot\vec{r}), \tag{6}$$

where $\nu = x, y, z$, and the sum is over a symmetric Brillouin zone. The problem is then reduced to the diagonalization of a quadratic form,

$$\begin{aligned}
\overline{\mathcal{H}} = \beta J \sum_{\vec{q}} (\cos q_x + \cos q_y + \cos q_z) \Big(\sigma_{\vec{q}}^x \sigma_{-\vec{q}}^x + \sigma_{\vec{q}}^y \sigma_{-\vec{q}}^y + \sigma_{\vec{q}}^z \sigma_{-\vec{q}}^z \Big) \\
+ -2\beta D i \sum_{\vec{q}} (\sin q_z)\, \sigma_{\vec{q}}^x \sigma_{-\vec{q}}^y - s_1 \sum_{\vec{q}} (\sigma_{\vec{q}}^x \sigma_{-\vec{q}}^x) \\
- s_2 \sum_{\vec{q}} (\sigma_{\vec{q}}^y \sigma_{-\vec{q}}^y) - s_3 \sum_{\vec{q}} (\sigma_{\vec{q}}^z \sigma_{-\vec{q}}^z).
\end{aligned} \tag{7}$$

With the introduction of a standard orthogonal transformation,

$$\sigma_{\vec{q}}^{\alpha} = \frac{1}{\sqrt{2}}\big(R_q^{\alpha} + iI_q^{\alpha}\big), \quad q \neq 0; \quad \sigma_0^{\alpha} = R_0^{\alpha}, \tag{8}$$

where

$$R_{\vec{q}}^{\alpha} = R_{-\vec{q}}^{\alpha}, \qquad I_q^{\alpha} = -I_{-q}^{\alpha}, \tag{9}$$

with $\alpha = x, y, z$, we write the quadratic expression

$$\overline{\mathcal{H}} = \beta J \sum_{\vec{q} \geqslant 0}(\cos q_x + \cos q_y + \cos q_z)\Big[(R_{\vec{q}}^x)^2 + (I_{\vec{q}}^x)^2 + (R_{\vec{q}}^y)^2 + (I_{\vec{q}}^y)^2$$

$$+ (R_{\vec{q}}^z)^2 + (I_{\vec{q}}^z)^2\Big] + 2\beta D \sum_{\vec{q} \geqslant 0}(\sin q_z)\,[R_{\vec{q}}^x I_{\vec{q}}^y - R_{\vec{q}}^y I_{\vec{q}}^x]$$

$$- s_1 \sum_{\vec{q} \geqslant 0}[(R_{\vec{q}}^x)^2 + (I_{\vec{q}}^x)^2] - s_2 \sum_{\vec{q} \geqslant 0}[(R_{\vec{q}}^y)^2 + (I_{\vec{q}}^y)^2] - s_3 \sum_{\vec{q} \geqslant 0}[(R_{\vec{q}}^z)^2 + (I_{\vec{q}}^z)^2]. \tag{10}$$

Due to the DM couplings, this expression still requires a further transformation to be written in a diagonal form.

2.1. Critical ferromagnetic border

In order to carry out the diagonalization of this problem, we consider two distinct quadratic forms. One of these forms, which involves just the z components of the spin variables, is already diagonal,

$$Q_1 = [\beta J(\cos q_x + \cos q_y + \cos q_z) - s_3][(R_{\vec{q}}^z)^2 + (I_{\vec{q}}^z)^2]. \tag{11}$$

Introducing the spherical potential $\mu_3 = s_3/\beta$, we require the inequality

$$\mu_3 > \max_{\vec{q}}[J(\cos q_x + \cos q_y + \cos q_z)], \tag{12}$$

which is the usual condition associated with the existence of a ferromagnetic phase transition in the ferromagnetic spherical model.

The quadratic form (11) contributes to the partition function with a term that depends on s_3. We then write this contribution,

$$\ln \Xi = \cdots - \frac{1}{2}\frac{N}{(2\pi)^3}\int \mathrm{d}^3\vec{q}\,\ln[s_3 - \beta J(\cos q_x + \cos q_y + \cos q_z)] + \cdots. \tag{13}$$

Using equation (4) for the spherical constraint, and taking the maximum value of the spherical potential μ_3, we obtain an integral expression for the ferromagnetic critical border,

$$J\beta_c = \frac{1}{2}\frac{1}{(2\pi)^3}\int \mathrm{d}^3\vec{q}\,\frac{1}{3 - (\cos q_x + \cos q_y + \cos q_z)}, \tag{14}$$

which can be compared with the well-known result for the simple spherical ferromagnet on a cubic lattice [2].

2.2. Critical border of a modulated structure

We now simplify the notation and write the remaining, non diagonal, part of the quadratic form,

$$Q_2 = A\left(x_1^2 + y_1^2\right) + C\left(x_2^2 + y_2^2\right) + B(x_1 y_2 - y_1 x_2), \tag{15}$$

with

$$A = \beta J(\cos q_x + \cos q_y + \cos q_z) - s_1, \tag{16}$$

$$B = 2\beta D \sin q_z, \tag{17}$$

and

$$C = \beta J(\cos q_x + \cos q_y + \cos q_z) - s_2, \tag{18}$$

in which $x_1 = R_q^x$, $y_1 = I_q^x$, $x_2 = R_q^y$, $y_2 = I_q^y$. In order to analyze this quadratic form, it is convenient to introduce a 4×4 matrix,

$$M = \begin{pmatrix} A & 0 & 0 & \frac{1}{2}B \\ 0 & A & -\frac{1}{2}B & 0 \\ 0 & -\frac{1}{2}B & C & 0 \\ \frac{1}{2}B & 0 & 0 & C \end{pmatrix}, \tag{19}$$

which is symmetric, with real elements, and can be easily diagonalized. It is straightforward to write the double-degenerate eigenvalues of this matrix,

$$\Lambda_{1,2} = \beta J(\cos q_x + \cos q_y + \cos q_z) - \frac{1}{2}(s_1 + s_2)$$

$$\pm \frac{1}{2}\left[(s_1 - s_2)^2 + 4\beta^2 D^2 \sin^2 q_z\right]^{1/2}. \tag{20}$$

There are several consequences of the form of this expression for the eigenvalues. Taking into account the symmetry between the exchange of variables s_1 and s_2, and the form of the spherical constraints, given by equations (4) and (5), we can always make

$$s_1 = s_2 = s = \beta\mu, \tag{21}$$

and consider the much simpler expression

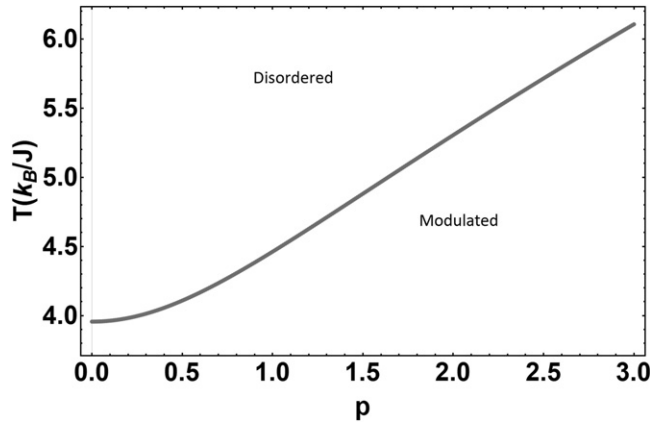$$\Lambda_{1,2} = \beta J(\cos q_x + \cos q_y + \cos q_z) - s \pm \beta D \sin q_z, \tag{22}$$

Figure 1. Transition border in the $T - p$ phase diagram.

from which we obtain a maximum limit for the spherical potential,

$$\mu > \max_{\overrightarrow{q}}\{J(\cos q_x + \cos q_y + \cos q_z) \pm D \sin q_z\}, \tag{23}$$

which can also be written as

$$\mu > \max_{q_z}\{J(2 + \cos q_z) \pm D \sin q_z\}. \tag{24}$$

Therefore, we have a wave solution for the magnetization along the axis of anisotropy, with a wave number q_z given by

$$\tan q_z = \pm\frac{D}{J}, \tag{25}$$

which is a characteristic result associated with monoaxial DM interactions.

The free energy of this system contains terms of the form

$$\ln \Xi = \cdots - \frac{1}{4}\frac{N}{(2\pi)^3}\int \mathrm{d}^3\overrightarrow{q}\ \ln[s - \beta J(\cos q_x + \cos q_y + \cos q_z) + \beta D \sin q_z]$$

$$+ \cdots - \frac{1}{4}\frac{N}{(2\pi)^3}\int \mathrm{d}^3\overrightarrow{q}\ \ln[s - \beta J(\cos q_x + \cos q_y + \cos q_z) - \beta D \sin q_z]$$

$$+ \cdots, \tag{26}$$

from which we have

$$\beta = \frac{1}{4}\frac{1}{(2\pi)^3}\int \mathrm{d}^3\overrightarrow{q}\ \left[\frac{1}{\mu - J(\cos q_x + \cos q_y + \cos q_z) + D \sin q_z}\right.$$

$$\left. + \frac{1}{\mu - J(\cos q_x + \cos q_y + \cos q_z) - D \sin q_z}\right]. \tag{27}$$

The critical border of the modulated transition is given by this expression, with the largest values of the spherical potential, according to equations (24) and (25). In figure 1,

we draw this border in terms of temperature T and the parameter of chiral competition, $p = D/J$. This is a typical result for a simple ferromagnetic system with monoaxial DM interactions [20].

3. Quantum spherical model with DM interactions

In a quantum version of the classical mean-spherical model, the spin variable $S_{\vec{r}}^a$ becomes a position operator at lattice site $\vec{r}$, which is canonically conjugated to a momentum operator $P_{\vec{r}}^\alpha$. We then introduce a set of momentum operators, $P_{\vec{r}}^\alpha$, with $\alpha = x, y, z$, and write the commutation relations

$$[S_{\vec{r}}^a, S_{\vec{r}'}^{\alpha'}] = 0, \qquad [P_{\vec{r}}^\alpha, P_{\vec{r}'}^{\alpha'}] = 0, \qquad [S_{\vec{r}}^\alpha, P_{\vec{r}'}^{\alpha'}] = i\delta_{\vec{r},\vec{r}'}\delta_{\alpha,\alpha'}, \tag{28}$$

where α and α' are the three Cartesian coordinates. With the introduction of a kinetic energy term, we have the quantum Hamiltonian of this system,

$$\mathcal{H}_q = \mathcal{H}_1 + \mathcal{H}_2 + \mathcal{H}_3, \tag{29}$$

with

$$\mathcal{H}_1 = \frac{1}{2}g\sum_{\vec{r}}\vec{P}_{\vec{r}}^2 + \mu_1\sum_{\vec{r}}(S_{\vec{r}}^x)^2 + \mu_2\sum_{\vec{r}}(S_{\vec{r}}^y)^2 + \mu_3\sum_{\vec{r}}(S_{\vec{r}}^z)^2, \tag{30}$$

$$\mathcal{H}_2 = -J\sum_{(\vec{r},\vec{r}')}\vec{S}_{\vec{r}} \cdot \vec{S}_{\vec{r}'}, \tag{31}$$

and

$$\mathcal{H}_3 = -D\sum_{\vec{r}}\left[S_{\vec{r}}^x S_{\vec{r}+\hat{z}}^y - S_{\vec{r}}^y S_{\vec{r}+\hat{z}}^x\right], \tag{32}$$

where we adopt the same quantum parameter $g > 0$ for the three directions, but assume three distinct spherical potentials. Although we use the same notation as in equation (1), spin and momenta are standard quantum operators.

We now introduce bosonic operators of creation, $(a_{\vec{r}}^\alpha)^\dagger$, and of annihilation, $a_{\vec{r}}^\alpha$, and change to a language of second quantization. We then write

$$S_{\vec{r}}^\alpha = \frac{1}{\sqrt{2}}\left(\frac{g}{2\mu_\alpha}\right)^{1/4}[a_{\vec{r}}^\alpha + (a_{\vec{r}}^\alpha)^\dagger] \tag{33}$$

and

$$P_{\vec{r}}^\alpha = -\frac{i}{\sqrt{2}}\left(\frac{2\mu_\alpha}{g}\right)^{1/4}[a_{\vec{r}}^\alpha - (a_{\vec{r}}^\alpha)^\dagger], \tag{34}$$

for $\alpha = x, y, z$. In the next step toward the diagonalization of the Hamiltonian, we adopt periodic boundary conditions, and use a Fourier representation,

$$a_{\vec{r}}^{\alpha} = \frac{1}{\sqrt{N}} \sum_{\vec{q}} \eta_{\vec{q}}^{\alpha} \exp(i\vec{q}.\vec{r}), \tag{35}$$

in which the sum is restricted to the first, symmetric, Brillouin zone, and the new bosonic operators, $\left\{ \eta_{\vec{q}}^{\alpha} \right\}$, obey canonical commutation relations,

$$[\eta_{\vec{q}}^{\alpha}, \eta_{\vec{q}'}^{\gamma}] = 0, \qquad [(\eta_{\vec{q}}^{\alpha})^{\dagger}, (\eta_{\vec{q}'}^{\gamma})^{\dagger}] = 0, \qquad [\eta_{\vec{q}}^{\alpha}, (\eta_{\vec{q}'}^{\gamma})^{\dagger}] = \delta_{\vec{q}, \vec{q}'} \delta_{\alpha, \gamma}. \tag{36}$$

In the Fourier space, we finally have

$$\mathcal{H}_1 = \sum_{\alpha} \frac{1}{2} N (g\mu_{\alpha})^{1/2} + \sum_{\alpha} \sum_{\vec{q} \geqslant 0} (2g\mu_{\alpha})^{1/2} \left[(\eta_{\vec{q}}^{\alpha})^{\dagger} \eta_{\vec{q}}^{\alpha} + (\eta_{-\vec{q}}^{\alpha})^{\dagger} \eta_{-\vec{q}}^{\alpha} \right], \tag{37}$$

$$\mathcal{H}_2 = \sum_{\alpha} \left\{ -\frac{1}{2} \left(\frac{g}{2\mu_{\alpha}} \right)^{1/2} \sum_{\vec{q} \geqslant 0} \widehat{J}(\vec{q}) \left[\eta_{\vec{q}}^{\alpha} \eta_{-\vec{q}}^{\alpha} + (\eta_{\vec{q}}^{\alpha})^{\dagger} (\eta_{-\vec{q}}^{\alpha})^{\dagger} \right] \right.$$

$$\left. -\frac{1}{2} \left(\frac{g}{2\mu_{\alpha}} \right)^{1/2} \sum_{\vec{q} \geqslant 0} \widehat{J}(\vec{q}) \left[(\eta_{\vec{q}}^{\alpha})^{\dagger} \eta_{\vec{q}}^{\alpha} + (\eta_{-\vec{q}}^{\alpha})^{\dagger} \eta_{-\vec{q}}^{\alpha} \right] \right\}, \tag{38}$$

and

$$\mathcal{H}_3 = -D \left(\frac{g^2}{4\mu_x \mu_y} \right)^{1/4} \sum_{\vec{q} \geqslant 0} i \sin q_z \left\{ \eta_{\vec{q}}^{x} \eta_{-\vec{q}}^{y} - (\eta_{\vec{q}}^{x})^{\dagger} \left(\eta_{-\vec{q}}^{y} \right)^{\dagger} - \eta_{-\vec{q}}^{x} \eta_{\vec{q}}^{y} \right.$$

$$\left. + (\eta_{-\vec{q}}^{x})^{\dagger} \left(\eta_{\vec{q}}^{y} \right)^{\dagger} + \eta_{\vec{q}}^{x} \left(\eta_{\vec{q}}^{y} \right)^{\dagger} - (\eta_{\vec{q}}^{x})^{\dagger} \eta_{\vec{q}}^{y} - \eta_{-\vec{q}}^{x} \left(\eta_{-\vec{q}}^{y} \right)^{\dagger} + (\eta_{-\vec{q}}^{x})^{\dagger} \eta_{-\vec{q}}^{y} \right\}, \tag{39}$$

with the definition

$$\widehat{J}(\vec{q}) = 2J(\cos q_x + \cos q_y + \cos q_z). \tag{40}$$

Taking into account the couplings between terms dependent on $+\vec{q}$ and of $-\vec{q}$, we have to consider the quadratic form

$$Q = \sum_{\alpha} \left\{ (2g\mu_{\alpha})^{1/2} \left[(\eta_{\vec{q}}^{\alpha})^{\dagger} \eta_{\vec{q}}^{\alpha} + (\eta_{-\vec{q}}^{\alpha})^{\dagger} \eta_{-\vec{q}}^{\alpha} \right] - \left(\frac{g}{2\mu_{\alpha}} \right)^{1/2} \right.$$

$$\left. \times \frac{1}{2} \widehat{J}(\vec{q}) \left[\eta_{\vec{q}}^{\alpha} \eta_{-\vec{q}}^{\alpha} + (\eta_{\vec{q}}^{\alpha})^{\dagger} (\eta_{-\vec{q}}^{\alpha})^{\dagger} + (\eta_{\vec{q}}^{\alpha})^{\dagger} \eta_{\vec{q}}^{\alpha} + (\eta_{-\vec{q}}^{\alpha})^{\dagger} \eta_{-\vec{q}}^{\alpha} \right] \right\}$$

$$- D \left(\frac{g^2}{4\mu_x \mu_y} \right)^{1/4} (i \sin q_z) \left[\eta_{\vec{q}}^{x} \eta_{-\vec{q}}^{y} - (\eta_{\vec{q}}^{x})^{\dagger} \left(\eta_{-\vec{q}}^{y} \right)^{\dagger} - \eta_{-\vec{q}}^{x} \eta_{\vec{q}}^{y} \right.$$

$$\left. + (\eta_{-\vec{q}}^{x})^{\dagger} \left(\eta_{\vec{q}}^{y} \right)^{\dagger} + \eta_{\vec{q}}^{x} \left(\eta_{\vec{q}}^{y} \right)^{\dagger} - (\eta_{\vec{q}}^{x})^{\dagger} \eta_{\vec{q}}^{y} - \eta_{-\vec{q}}^{x} \left(\eta_{-\vec{q}}^{y} \right)^{\dagger} + (\eta_{-\vec{q}}^{x})^{\dagger} \eta_{-\vec{q}}^{y} \right]. \tag{41}$$

3.1. Ferromagnetic sector

The quadratic form (41) has two different sectors, so that operators associated with the z direction can be treated separately. We then write

$$Q = Q_z + Q_{xy}, \tag{42}$$

with

$$Q_z = A_z \left[\left(\eta^z_{\vec{q}} \right)^\dagger \eta^z_{\vec{q}} + \left(\eta^z_{-\vec{q}} \right)^\dagger \eta^z_{-\vec{q}} \right] + B_z \left[\eta^z_{\vec{q}} \eta^z_{-\vec{q}} + \left(\eta^z_{\vec{q}} \right)^\dagger \left(\eta^z_{-\vec{q}} \right)^\dagger \right], \tag{43}$$

where

$$A_z = (2g\mu_3)^{1/2} \left[1 - \frac{1}{4\mu_3} \widehat{J}(\vec{q}) \right] \tag{44}$$

and

$$B_z = -(2g\mu_3)^{1/2} \frac{1}{4\mu_3} \widehat{J}(\vec{q}). \tag{45}$$

It is immediate to use a standard Bogoliubov transformation [19] to write Q_z in a diagonal form in terms of a new set of bosonic operators. According to this well-known procedure, and discarding all the constant terms, we write

$$Q_z = \lambda_{\vec{q}} \left[\left(\alpha^z_{\vec{q}} \right)^\dagger \alpha^z_{\vec{q}} + \left(\alpha^z_{-\vec{q}} \right)^\dagger \alpha^z_{-\vec{q}} \right], \tag{46}$$

where $\left\{ \alpha^z_{\vec{q}} \right\}$ is a set of transformed boson operators, associated with the energy spectrum

$$\lambda_{\vec{q}} = \left\{ 2g \left[\mu_3 - \frac{1}{2} \widehat{J}(\vec{q}) \right] \right\}^{1/2}, \tag{47}$$

which is the well known result for the quantum ferromagnetic mean-spherical model [12–14].

We now use the energy spectrum, given by equation (47), and take care of the proper constant terms of the spin Hamiltonian. We then have the diagonal form

$$\mathcal{H}_z = \sum_{\vec{q}} \lambda_{\vec{q}} \left[\left(\alpha^z_{\vec{q}} \right)^\dagger \alpha^z_{\vec{q}} + \frac{1}{2} \right], \tag{48}$$

from which we obtain the partition function

$$Z_z = \prod_{\vec{q}} \left\{ \sum_{n=0}^{\infty} \exp \left[-\beta \lambda_{\vec{q}} \left(n + \frac{1}{2} \right) \right] \right\}. \tag{49}$$

Taking into account the form of the spectrum of energy in this ferromagnetic sector, we have

$$N = -\frac{1}{\beta}\frac{\partial}{\partial \mu}\ln Z_z = \sum_{\vec{q}}\frac{g}{2\lambda_{\vec{q}}}\coth\left(\frac{1}{2}\beta\lambda_{\vec{q}}\right). \tag{50}$$

In the classical limit, $g \to 0$, we obtain a much simpler expression,

$$N \to \frac{1}{2\beta}\sum_{\vec{q}}\frac{1}{\mu - \frac{1}{2}\widehat{J}(\vec{q})}, \tag{51}$$

which can be shown to lead to the same equation (14) for the ferromagnetic border as we have obtained in the preceding section.

3.2. Modulated sector

We now sketch some calculations for the other sector of the quadratic form. As in the classical case, there is a symmetry that relates the spherical potential variables. We then make $\mu_1 = \mu_2 = \mu$. Also, we simplify the notation to emphasize the couplings involving four bosonic modes, and discard the vector symbols. We then write

$$\begin{aligned}
Q_{xy} = {}&A_\perp\left\{a_q^\dagger a_q + a_{-q}^\dagger a_{-q} + b_q^\dagger b_q + b_{-q}^\dagger b_{-q}\right\}\\
&+ B_\perp\left\{a_q a_{-q} + a_q^\dagger a_{-q}^\dagger + b_q b_{-q} + b_q^\dagger b_{-q}^\dagger\right\}\\
&+ C_\perp\left\{a_q b_{-q} - a_q^\dagger b_{-q}^\dagger - a_{-q}b_q + a_{-q}^\dagger b_q^\dagger + a_q b_q^\dagger - a_q^\dagger b_q - a_{-q}b_{-q}^\dagger + a_{-q}^\dagger b_{-q}\right\},
\end{aligned} \tag{52}$$

with

$$A_\perp = (2g\mu)^{1/2}\left[1 - \frac{1}{4\mu}\widehat{J}(\vec{q})\right], \tag{53}$$

$$B_\perp = -(2g\mu)^{1/2}\frac{1}{4\mu}\widehat{J}(\vec{q}), \tag{54}$$

and

$$C_\perp = iD\left(\frac{g}{2\mu}\right)^{1/2}(\sin q_z). \tag{55}$$

In order to analyze this four-component system, it is easier to write a set of transformed bosonic operators,

$$\gamma_q = w a_q + x b_q + y a_{-q}^\dagger + z b_{-q}^\dagger, \tag{56}$$

so that

$$[\gamma_q, Q_{xy}] = E_q\,\gamma_q, \tag{57}$$

where E_q is a bosonic energy spectrum [19, 21]. We then write

$$[a_q, Q_{xy}] = A_\perp\,a_q + B_\perp\,a_{-q}^\dagger + C_\perp\,(-b_q - b_{-q}^\dagger), \tag{58}$$

$$[b_q, Q_{xy}] = A_\perp \, b_q + B_\perp \, b_{-q}^\dagger + C_\perp \, (a_q + a_{-q}^\dagger), \tag{59}$$

$$\left[a_{-q}^\dagger, Q_{xy}\right] = -A_\perp \, a_{-q}^\dagger - B_\perp \, a_q + C_\perp \, (b_q + b_{-q}^\dagger), \tag{60}$$

and

$$\left[b_{-q}^\dagger, Q_{xy}\right] = -A_\perp \, b_{-k}^\dagger - B_\perp \, b_q + C_\perp \, (-a_q - a_{-q}^\dagger), \tag{61}$$

from which we have

$$[A_\perp w - B_\perp y + C_\perp(x - z)]a_q + [A_\perp x - B_\perp z + C_\perp(y - w)]b_q$$
$$+ [B_\perp w - A_\perp y + C_\perp(x - z)]a_{-q}^\dagger + [B_\perp x - A_\perp z + C_\perp(y - w)]b_{-q}^\dagger$$
$$= E_q\Big[wa_q + xb_q + ya_{-q}^\dagger + zb_{-q}^\dagger\Big]. \tag{62}$$

This equation may be written in a matrix form,

$$\mathbf{M}_q \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = E_q \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}, \tag{63}$$

with

$$\mathbf{M}_q = \begin{pmatrix} A_\perp & C_\perp & -B_\perp & -C_\perp \\ -C_\perp & A_\perp & C_\perp & -B_\perp \\ B_\perp & C_\perp & -A_\perp & -C_\perp \\ -C_\perp & B_\perp & C_\perp & -A_\perp \end{pmatrix}. \tag{64}$$

From the eigenvalues of this matrix, we obtain the two branches of the spectrum of energy of the diagonal bosonic system,

$$\lambda^2 = A_\perp^2 - B_\perp^2 \pm 2\sqrt{C_\perp^2(-A_\perp^2 + 2A_\perp B_\perp - B_\perp^2)}. \tag{65}$$

Taking into account that $C_\perp$ is a pure imaginary number, we can also write

$$\lambda^2 = A_\perp^2 - B_\perp^2 \pm 2|C_\perp|(A_\perp - B_\perp). \tag{66}$$

We now take into account that $A_\perp$, $B_\perp$, and $C_\perp$, are given by equations (53), (54) and (55), and write the expressions

$$-C_\perp^2 = D^2 \frac{g}{2\mu} \sin^2 q_z, \tag{67}$$

$$A_\perp - B_\perp = (2g\mu)^{1/2}\left[1 - \frac{1}{4\mu}\widehat{J}(\vec{q})\right] + (2g\mu)^{1/2}\frac{1}{4\mu}\widehat{J}(\vec{q}) = (2g\mu)^{1/2}, \tag{68}$$
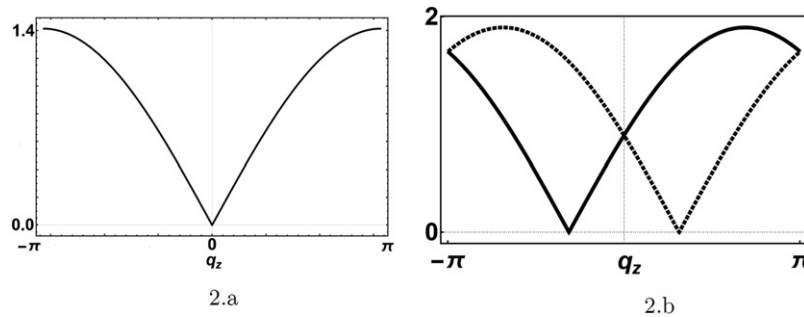
Figure 2. (a) A plot of the spectrum of energy in the ferromagnetic sector. (b) W plot the spectrum in the modulated sector (solid and dashed lines represent positive and negative values of the ratio D/J).

and

$$A_\perp^2 - B_\perp^2 = (2g\mu)\left[1 - \frac{1}{4\mu}\widehat{J}(\overrightarrow{q})\right]^2 - (2g\mu)\frac{1}{16\mu^2}\left[\widehat{J}(\overrightarrow{q})\right]^2 = 2g\mu - g\widehat{J}(\overrightarrow{q}),$$
(69)

in which the form of $C_\perp$ already gives an indication of the oscillation along the axis. Using these expressions, we finally have the energy spectrum,

$$[\Lambda(\overrightarrow{q})]^2 = 2g\left[\mu - \frac{1}{2}\widehat{J}(\overrightarrow{q}) \pm D \sin q_z\right],$$
(70)

which is a characteristic result for an oscillating alignment along the z axis. It is worth pointing out the similarities with the energy spectrum of ferromagnetic sector, given by equation (47). Also, it is easy to perform calculations to recover the classical spectrum.

With $D = 0$, we recover the well-known quantum ferromagnetic case [13],

$$[\Lambda_{\text{Ferro}}(\overrightarrow{q})]^2 = 2g\left[\mu - \frac{1}{2}\widehat{J}(\overrightarrow{q})\right].$$
(71)

If we restrict the problem to a linear chain, along the z direction, equation (70) can be written as

$$[\Lambda_{\text{chain}}(\overrightarrow{q})]^2 = 2g[\mu - J \cos q_z \pm D \sin q_z],$$
(72)

which indicates the oscillations along the z direction (and which is similar to the exact calculations for the energy spectrum of an XY chain with DM interactions [22]).

In figure 2(a) we plot the spectrum of energy in the ferromagnetic sector for the maximum value of the spherical potential. In figure 2(b), we show a similar plot, for the modulated sector, with the maximum value of the associated spherical potential. In this second figure, the minimum of energy is shifted to $q_z \neq 0$, depending on the sign of $d = D/J$.

4. Conclusions

We performed some exact calculations to investigate the phase transitions in a ferromagnetic mean-spherical model, with the consideration of three distinct spin components, and the addition of monoaxial DM interactions.

In the classical case, we show the existence of a modulated structure along the $\widehat{z}$ direction at sufficiently low temperatures. We then define a quantum version of this model system, which can be analyzed by standard techniques of second quantization. We obtain the quantum energy spectrum, and give arguments to show the persistence of spacial modulated structures in the low-temperature region of the phase diagram.

Acknowledgments

References

[1] Berlin T H and Kac M 1952 *Phys. Rev.* **86** 821
[2] Joyce G S 1972 Critical properties of the spherical model *Phase Transitions and Critical Phenomena* vol 2 ed C Domb and M S Green (New York: Academic)
[3] Dzyaloshinskii I E 1958 *J. Phys. Chem. Solids* **4** 24
[4] Moriya T 1960 *Phys. Rev.* **120** 91
[5] Izyumov Y A 1984 *Sov. Phys.-Usp.* **27** 245
[6] Togawa Y, Kousaka Y, Inoue K and Kishine J-i 2016 *J. Phys. Soc. Japan* **85** 112001
[7] Bettoney G V and Mazo R M 1970 *J. Math. Phys.* **11** 1147
[8] Aqua J-N and Fisher M E 2009 *Phys. Rev.* E **79** 011118
[9] Fert A, Reyren N and Cros V 2017 *Nat. Rev. Mater.* **2** 17031
[10] Nagaosa N and Tokura Y 2013 *Nat. Nanotechnol.* **8** 899
[11] Rössler U K, Bogdanov A N and Pfleiderer C 2006 *Nature* **442** 797
[12] Vojta T 1996 *Phys. Rev.* B **53** 710
[13] Oliveira M H, Raposo E P and Coutinho-Filho M D 2006 *Phys. Rev.* B **74** 184101
[14] Bienzobaz P F and Salinas S R 2012 *Physica* A **391** 6399–408
[15] Wald S and Henkel M 2015 *J. Stat. Mech.* P07006
[16] Obermair G 1972 *Dynamical Aspects of Critical Phenomena* ed J I Budnick and M P Kawatra (London: Gordon and Breach) p 137
[17] Henkel M and Hoeger C 1984 *Z. Phys.* B **55** 67–73
[18] Nieuwenhuizen T M 1995 *Phys. Rev. Lett.* **74** 4293
[19] Bogoliubov N N and Bogoliubov N N Jr 1982 *Introduction to Quantum Statistical Mechanics* (Singapore: World Scientific)
[20] Bak P and Jensen M H 1980 *J. Phys. C: Solid State Phys.* **13** L881
[21] Hopfield J J 1958 *Phys. Rev.* **112** 1555
[22] Ming Z, Hui X, Xiao-Xian L and Pei-Qing T 2013 *Chin. Phys.* B **22** 090313