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The R_∞-property for nilpotent quotients of Baumslag–Solitar groups

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Abstract. A group G has the R_∞-property if the number $R(\varphi)$ of twisted conjugacy classes is infinite for any automorphism φ of G. For such a group G, the R_∞-nilpotency degree is the least integer c such that $G/\gamma_{c+1}(G)$ still has the R_∞-property. In this paper, we determine the R_∞-nilpotency degree of all Baumslag–Solitar groups.

1 Introduction

Any endomorphism φ of a group G determines an equivalence relation on G by setting $x \sim y \Leftrightarrow$ there exists $z \in G : x = zy\varphi(z)^{-1}$. The equivalence classes of this relation are called Reidemeister classes or twisted conjugacy classes, and their number is denoted by $R(\varphi)$. We are most interested in this number when φ is an automorphism.

For information on the development, historical aspects and the relation of this concept with other topics in mathematics such as fixed-point theory, we refer the reader to the introduction of [3] and its references. An important concept in this context is that of groups having the R_∞-property.

Definition 1.1. A group G is said to have the R_∞-property if, for every automorphism $\varphi\colon G \to G$, the number $R(\varphi)$ is infinite.

A central problem is to decide which groups have the R_∞-property. The study of this problem has been a quite active research topic in recent years. Several families of groups have been studied by many authors. A non-exhaustive list of references is [1–9, 13, 15, 17].

Of particular interest for this paper is the fact that, in [5], it was proved that the Baumslag–Solitar groups $\mathrm{BS}(m,n)$ have the R_∞-property except for $m = n = 1$ (or $m = n = -1$, which is the same group). Recently, in [3], motivated by the

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results of [1], new examples of groups which have the R_∞-property were obtained by looking at quotients of a group which has the R_∞-property by the terms of the lower central series as well the derived central series. So it is natural to ask the same question for the groups $\mathrm{BS}(m,n)$.

Related to this approach, we introduced in [3] the following notion.

Definition 1.2. Let G be a group. The R_∞-nilpotency degree of a group G is the least integer c such that $G/\gamma_{c+1}(G)$ has the R_∞-property. If no such integer exists, then we say that G has R_∞-nilpotency degree infinite.

In this work, we determine the R_∞-nilpotency degree for all the Baumslag–Solitar groups $\mathrm{BS}(m,n)$. The main results of this work are the following two theorems.

Theorem 4.5. *Let m,n be integers with $0 < m \le |n|$ and $\gcd(m,n) = 1$. Let p denote the largest integer such that $2^p \mid 2m + 2$. Then the R_∞-nilpotency degree r of $\mathrm{BS}(m,n)$ is given by the following conditions.*

- *If $n < 0$ and $n \neq -1$, then $r = 2$.*
- *If $n = -1$ (so $m = 1$), then $r = \infty$.*
- *If $n = m$ (so $n = m = 1$), then $r = \infty$.*
- *If $n - m = 1$, then $r = \infty$.*
- *If $n - m = 2$, then $r = p + 2$.*
- *If $n - m \geq 3$, then $r = 2$.*

Theorem 5.4. *Let $0 < m \le |n|$ with $m \neq n$, and take $d = \gcd(m,n)$. Let p denote the largest integer such that $2^p \mid 2\frac{m}{d} + 2$. Then the R_∞-nilpotency degree r of $\mathrm{BS}(m,n)$ is given by the following conditions.*

- *If $n < 0$ and $n \neq -m$, then $r = 2$.*
- *If $n = -m$, then $r = \infty$.*
- *If $n = m$, then $r = \infty$.*
- *If $n - m = d$, then $r = \infty$.*
- *If $n - m = 2d$, then $2 \le r \le p + 2$.*
- *If $n - m \geq 3d$, then $r = 2$.*

At this point, we would also like to mention one interesting family of groups which naturally extends the class of Baumslag–Solitar groups, namely the family of GBS groups, the generalized Baumslag–Solitar groups. In [14], the following strong result about the Reidemeister number of a homomorphism of such groups is proved.

Proposition ([14, Proposition 2.7]). *Let $\alpha\colon G \to G$ be an endomorphism of a non-elementary* GBS *group. If one of the following conditions holds, then $R(\alpha)$ is infinite.*

(1) *α is surjective.*

(2) *α is injective, and G is not unimodular.*

(3) *$G = \mathrm{BS}(m, n)$ with $|m| \neq |n|$, and the image of α is not cyclic.*

Recently, other families of groups, which also naturally extend the class of Baumslag–Solitar groups, were considered. One generalization goes as follows. The class of GBS groups coincides with the class of fundamental groups of graphs all of whose vertex and edge groups are infinite cyclic. So one can generalize this to the class of fundamental groups of graphs where the vertex and edge groups are virtually infinite cyclic. In [11], it was shown by Taback and Wong that any group which is quasi-isometric to a group in this family has the R_∞-property.

Another family was considered by Taback and Whyte in [10], generalizing the solvable Baumslag–Solitar groups $\mathrm{BS}(1, n)$ to another class of groups that are also solvable. These are split extensions fitting into a short exact sequence $1 \to \mathbb{Z}[\frac{1}{n}] \to \Gamma \to \mathbb{Z}^k \to 1$. For this second family, Taback and Wong showed in [12] that any group quasi-isometric to one of these group has the R_∞-property.

As a generalization of the results of this paper, it would be natural to study the R_∞-nilpotency degree for the families of groups above.

This work is divided into three sections besides the introduction. In Section 2, we provide some preliminary results about the description of the terms of the lower central series and the corresponding quotients of $\mathrm{BS}(m, n)$ when the integers (m, n) are coprime. In Section 3, we construct certain specific nilpotent groups in a format which is convenient for our study. Then we identify these groups with the ones that we want to study, namely the quotients $\mathrm{BS}(m, n)/\gamma_{c+1}(\mathrm{BS}(m, n))$. In Section 4, we then show the main result for the $\mathrm{BS}(m, n)$ groups, where m and n are coprime. Finally, in Section 5, we provide a proof for the remaining cases.

2 Baumslag–Solitar groups

Let $\mathrm{BS}(m, n) = \langle a, b \mid a^{-1}b^m a = b^n \rangle$ for m, n integers. It suffices to consider $1 \leq m \leq |n|$. We will use the notation $[x, y] = x^{-1}y^{-1}xy$.

Lemma 2.1. *Consider a Baumslag–Solitar group* $\mathrm{BS}(m,n)$*. For all positive integers k, we have* $b^{(m-n)^k} \in \gamma_{k+1}(\mathrm{BS}(m,n))$.

Proof. Since $a^{-1}b^{-m}a = b^{-n}$, we have $b^{m-n} = [a, b^m] \in \gamma_2(\mathrm{BS}(m,n))$, which proves the lemma for $k = 1$.

Now, we assume that $k \geq 1$ and that $b^{(m-n)^k} \in \gamma_{k+1}(\mathrm{BS}(m,n))$. Then we find

$$\begin{aligned}
&b^{-m(m-n)^k} \in \gamma_{k+1}(\mathrm{BS}(m,n))\\
&\quad\Longrightarrow\ a^{-1}b^{-m(m-n)^k}ab^{m(m-n)^k} \in \gamma_{k+2}(\mathrm{BS}(m,n))\\
&\quad\Longrightarrow\ (a^{-1}b^m a)^{-(m-n)^k}b^{m(m-n)^k} \in \gamma_{k+2}(\mathrm{BS}(m,n))\\
&\quad\Longrightarrow\ b^{-n(m-n)^k}b^{m(m-n)^k} = b^{(m-n)^{k+1}} \in \gamma_{k+2}(\mathrm{BS}(m,n)),
\end{aligned}$$

which proves the lemma, by induction. □

As we will be dealing with nilpotent quotients of the Baumslag–Solitar groups, we introduce the notation

$$\mathrm{BS}_c(m,n) = \frac{\mathrm{BS}(m,n)}{\gamma_{c+1}(\mathrm{BS}(m,n))}.$$

For a nilpotent group N, we use τN to indicate its torsion subgroup.

Lemma 2.2. *Let* $m \neq n$*. For all positive integers c, the nilpotent group* $\mathrm{BS}_c(m,n)$ *has Hirsch length* 1*, and if we denote by* $\bar{b}$ *the natural projection of b in* $\mathrm{BS}_c(m,n)$*, we have* $\tau\mathrm{BS}_c(m,n) = \langle \bar{b}, \gamma_2(\mathrm{BS}_c(m,n))\rangle$.

Proof. We first consider the case $c = 1$. Note that

$$\mathrm{BS}_1(m,n) = \langle \bar{a}, \bar{b} \mid [\bar{a}, \bar{b}] = 1,\ \bar{b}^{m-n} = 1\rangle \cong \mathbb{Z} \oplus \mathbb{Z}_{|m-n|}.$$

So $\tau\mathrm{BS}_1(m,n) = \langle \bar{b} \rangle$.

Now, let $c > 1$. From the case $c = 1$, it follows that

$$\tau\mathrm{BS}_c(m,n) \subseteq \langle \bar{b}, \gamma_2(\mathrm{BS}_c(m,n))\rangle,$$

and hence it suffices to show that $\gamma_2(\mathrm{BS}_c(m,n))$ is a torsion group. To obtain this result, we prove by induction on $i \geq 2$ that

$$\gamma_i(\mathrm{BS}_c(m,n))/\gamma_{i+1}(\mathrm{BS}_c(m,n)) = \gamma_i(\mathrm{BS}(m,n))/\gamma_{i+1}(\mathrm{BS}(m,n))$$

is finite.

The group $\gamma_2(\mathrm{BS}(m,n))/\gamma_3(\mathrm{BS}(m,n))$ is generated by $[a,b]\gamma_3(\mathrm{BS}(m,n))$. By the previous lemma, we know that $b^{m-n} \in \gamma_2(\mathrm{BS}(m,n))$, using this we find

$$[a,b]^{m-n}\gamma_3(\mathrm{BS}(m,n)) = [a,b^{m-n}]\gamma_3(\mathrm{BS}(m,n)) = 1\gamma_3(\mathrm{BS}(m,n)),$$

and so $[a,b]\gamma_3(\mathrm{BS}(m,n))$ is of finite order ($\leq |m-n|$) in

$$\gamma_2(\mathrm{BS}(m,n))/\gamma_3(\mathrm{BS}(m,n)).$$

Now, assume that $\gamma_i(\mathrm{BS}(m,n))/\gamma_{i+1}(\mathrm{BS}(m,n))$ is finite. The group

$$\gamma_{i+1}(\mathrm{BS}(m,n))/\gamma_{i+2}(\mathrm{BS}(m,n))$$

is generated by all elements of the form $[x,y]\gamma_{i+2}(\mathrm{BS}(m,n))$ for $x \in \mathrm{BS}(m,n)$ and $y \in \gamma_i(\mathrm{BS}(m,n))$. By our assumption, there is some $k > 0$ such that we have $y^k \in \gamma_{i+1}(\mathrm{BS}(m,n))$. As before, it then follows that

$$[x,y]^k\gamma_{i+2}(\mathrm{BS}(m,n)) = [x,y^k]\gamma_{i+2}(\mathrm{BS}(m,n)) = 1\gamma_{i+2}(\mathrm{BS}(m,n)),$$

from which we deduce that $\gamma_{i+1}(\mathrm{BS}(m,n))/\gamma_{i+2}(\mathrm{BS}(m,n))$ is finite.

The fact that $\mathrm{BS}_c(m,n)$ has Hirsch length 1 follows from the fact that

$$\mathrm{BS}_c(m,n)/\gamma_2(\mathrm{BS}_c(m,n)) \cong \mathrm{BS}_1(m,n)$$

has Hirsch length 1 and $\gamma_2(\mathrm{BS}_c(m,n))$ has Hirsch length 0. □

In this paper, the situation where $\gcd(m,n) = 1$ will play a rather crucial role. For these groups, the structure of $\mathrm{BS}_c(m,n)$ is easier to understand than in the general case. For example, we have the following lemma.

Lemma 2.3. *Suppose that* $\gcd(m,n) = 1$ *and* $m \neq n$. *For any* $c > 1$ *and* $k > 1$, *we have* $\gamma_k(\mathrm{BS}_c(m,n)) = \langle \bar{b}^{(m-n)^{k-1}} \rangle$. *Again,* $\bar{b}$ *denotes the projection of* b *in* $\mathrm{BS}_c(m,n)$.

Proof. For sake of simplicity, we will write Γ_i instead of $\gamma_i(\mathrm{BS}_c(m,n))$ in the rest of this proof. We will prove by induction on $k \geq 2$ that $\bar{b}^{(m-n)^{k-1}}\Gamma_{k+1}$ generates Γ_k/Γ_{k+1}.

For $k = 2$, we have that $[\bar{a},\bar{b}]\Gamma_3$ generates Γ_2/Γ_3, and from Lemma 2.1, we know that $[\bar{a},\bar{b}]^{m-n} \in \Gamma_3$; hence the order of $[\bar{a},\bar{b}]\Gamma_3$ in Γ_2/Γ_3 is a divisor of $m-n$. As $\gcd(m,n) = 1$, also $\gcd(m,m-n) = 1$, and therefore also $[\bar{a},\bar{b}]^m\Gamma_3$ is a generator of Γ_2/Γ_3. Now, $[\bar{a},\bar{b}]^m\Gamma_3 = [\bar{a},\bar{b}^m]\Gamma_3 = \bar{b}^{m-n}\Gamma_3$, from which we find that $\bar{b}^{m-n}\Gamma_3$ generates Γ_2/Γ_3.

Now, we assume that $k > 2$ and that Γ_{k-1}/Γ_k is generated by $\bar{b}^{(m-n)^{k-2}}\Gamma_k$. The next quotient Γ_k/Γ_{k+1} is then generated by $[\bar{a}, \bar{b}^{(m-n)^{k-2}}]\Gamma_{k+1}$. Again, by Lemma 2.1, we have

$$[\bar{a}, \bar{b}^{(m-n)^{k-2}}]^{m-n}\Gamma_{k+1} = [\bar{a}, \bar{b}^{(m-n)^{k-1}}]\Gamma_{k+1} = 1\Gamma_{k+1},$$

and so the order of the generator $[\bar{a}, \bar{b}^{(m-n)^{k-2}}]\Gamma_{k+1}$ divides $m-n$. As before, it follows that also $[\bar{a}, \bar{b}^{(m-n)^{k-2}}]^m\Gamma_{k+1}$ generates Γ_k/Γ_{k+1}. In $\mathrm{BS}(m,n)$, we have $[a, b^{km}] = b^{k(m-n)}$, which we now use to obtain

$$[\bar{a}, \bar{b}^{(m-n)^{k-2}}]^m\Gamma_{k+1} = [\bar{a}, \bar{b}^{m(m-n)^{k-2}}]\Gamma_{k+1} = \bar{b}^{(m-n)^{k-1}}\Gamma_{k+1},$$

which finishes the proof. □

Corollary 2.4. *Suppose that* $\gcd(m,n) = 1$ *and* $m \neq n$. *Then, for all* $c \geq 1$, *we have* $\tau\mathrm{BS}_c(m,n) = \langle \bar{b} \rangle$.

3 Some nilpotent quotients of Baumslag–Solitar groups

For the rest of this section, we assume that $m \neq n$. For any positive integer c, we will construct a nilpotent group $G_c(m,n)$ of class $\leq c$ which can be seen as a quotient of $\mathrm{BS}(m,n)$. To construct this group, we fix m, n and c and consider the morphism $\varphi\colon \mathbb{Z}^c \to \mathbb{Z}^c$, which is represented by the matrix

$$\begin{pmatrix} n-m & 0 & 0 & \cdots & 0 & 0 \\ -m & n-m & 0 & \cdots & 0 & 0 \\ 0 & -m & n-m & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & n-m & 0 \\ 0 & 0 & 0 & \cdots & -m & n-m \end{pmatrix}. \tag{3.1}$$

Here we use the convention that elements of $\mathbb{Z}^c$ are written as columns, so also in the matrix above, the image of the i-th standard generator of $\mathbb{Z}^c$ is given by the i-th column of that matrix. We now consider the abelian group

$$A_c(m,n) = \frac{\mathbb{Z}^c}{\operatorname{Im}\varphi}.$$

So $A_c(m,n)$ is a finite group of order $|n-m|^c$.

We consider also the morphism $\psi\colon \mathbb{Z}^c \to \mathbb{Z}^c$, which is represented by

$$M = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 1 \end{pmatrix}.$$

We have $\varphi\psi = \psi\varphi$, and therefore ψ induces an automorphism of $A_c(m,n)$, which we will also denote by the symbol ψ.

Now, we are ready to define the group $G_c(m,n)$, which is given as a semi-direct product $G_c(m,n) = A_c(m,n) \rtimes \langle t\rangle$, where $\langle t\rangle$ is the infinite cyclic group and where the semi-direct product structure is given by the requirement that, for all $a \in A_c(m,n)$, we have $t^{-1}at = \psi(a)$.

For any $z \in \mathbb{Z}^c$, let $\overline{z} = z + \operatorname{Im}\varphi$ denote its natural projection in $A_c(m,n)$. We use $e_1, e_2, \ldots, e_c$ to denote the standard generators of $\mathbb{Z}^c$, so e_i is the column vector having a 1 on the i-th spot and 0's on all other positions. Obviously, we have that $\overline{e_1}, \overline{e_2}, \ldots, \overline{e_c}$ generate $A_c(m,n)$. For sake of simplicity, sometimes, we will write G instead of $G_c(m,n)$.

Remark. It is easy to see that from the fact that

$$\overline{e_2} = t^{-1}\overline{e_1}t\overline{e_1}^{-1}, \quad \overline{e_3} = t^{-1}\overline{e_2}t\overline{e_2}^{-1}, \quad \ldots$$

follows

$$\begin{aligned} \gamma_2(G) &\subseteq \langle \overline{e_2}, \overline{e_3}, \ldots, \overline{e_c}\rangle \\ \gamma_3(G) &\subseteq \langle \overline{e_3}, \overline{e_4}, \ldots, \overline{e_c}\rangle \\ &\vdots \\ \gamma_c(G) &\subseteq \langle \overline{e_c}\rangle \\ \gamma_{c+1}(G) &= 1. \end{aligned}$$

Hence $G_c(m,n)$ is nilpotent of class $\leq c$.

Lemma 3.1. *There is a surjective morphism of groups $f\colon \mathrm{BS}(n,m) \to G_c(m,n)$ which is determined by $f(a) = t$ and $f(b) = \overline{e_1}$.*

Proof. In order for f to be a morphism, we need to check that f preserves the defining relation of $\mathrm{BS}(n,m)$, that is, the relation $t^{-1}\overline{e_1}^m t = \overline{e_1}^n$ should hold.

This follows from the computation

$$t^{-1}\overline{e_1}^m t = \psi(\overline{e_1}^m) = \overline{e_1}^m \overline{e_2}^m = \overline{e_1}^n(\overline{e_1}^{m-n}\overline{e_2}^m) = \overline{e_1}^n \overline{\varphi(-e_1)} = \overline{e_1}^n.$$

To prove that f is a surjective map, it is enough to show that $\overline{e_1}$ and t generate $G_c(m,n)$. This follows from the fact that

$$\overline{e_2} = t^{-1}\overline{e_1}t\overline{e_1}^{-1}, \quad \overline{e_3} = t^{-1}\overline{e_2}t\overline{e_2}^{-1}, \quad \dots \qquad \square$$

As $G_c(m,n)$ is nilpotent of class $\leq c$, f induces a surjective morphism

$$\mathrm{BS}_c(m,n) = \frac{\mathrm{BS}(m,n)}{\gamma_{c+1}(\mathrm{BS}(m,n))} \to G_c(m,n).$$

For $G_c(m,n)$, we have $\tau G = A$, and so $[\tau G, \tau G] = 1$.

Proposition 3.2. *The morphism* $f\colon \mathrm{BS}(m,n) \to G_c(m,n)$ *induces an isomorphism*

$$\mu\colon \frac{\mathrm{BS}_c(m,n)}{[\tau \mathrm{BS}_c(m,n), \tau \mathrm{BS}_c(m,n)]} \to G_c(m,n).$$

Proof. As already explained, f induces a morphism $\nu\colon \mathrm{BS}_c(m,n) \to G_c(m,n)$. Of course, $\nu(\tau \mathrm{BS}_c(m,n)) \subseteq \tau G$, and so

$$\nu[\tau \mathrm{BS}_c(m,n), \tau \mathrm{BS}_c(m,n)] \subseteq [\tau G, \tau G] = 1.$$

Therefore, there is an induced morphism

$$\mu\colon \frac{\mathrm{BS}_c(m,n)}{[\tau \mathrm{BS}_c(m,n), \tau \mathrm{BS}_c(m,n)]} \to G_c(m,n).$$

As f is surjective, we know that μ is surjective too. In Lemma 2.2, we showed that $\mathrm{BS}_c(m,n)$ has Hirsch length 1. Then also the quotient

$$\mathrm{BS}_c(m,n)/[\tau \mathrm{BS}_c(m,n), \tau \mathrm{BS}_c(m,n)]$$

has Hirsch length 1 since we take the quotient by a finite subgroup. As also, by construction, $G_c(m,n)$ has Hirsch length 1 and μ is surjective, we must have that the kernel of μ has Hirsch length 0, i.e., the kernel of μ has to be finite. For sake of simplicity, we introduce the notation

$$H = \frac{\mathrm{BS}_c(m,n)}{[\tau \mathrm{BS}_c(m,n), \tau \mathrm{BS}_c(m,n)]}.$$

We already know, by Lemma 2.2, that τH is generated by $\bar{b}$ and $\gamma_2(H)$. (Here $\bar{b}$ denotes the image of b in H.) As μ is surjective and has finite kernel (so $\mathrm{Ker}(\mu) \subseteq \tau H$), we know that $\mu(\tau H) = \tau G_c(m,n)$. Therefore, in order to prove that μ is injective, it is enough to show that $\#\tau H \leq \#\tau G_c(m,n) = |m-n|^c$.

To be able to find a bound on $\#\tau H$, we look at the quotients $\gamma_i(H)/\gamma_{i+1}(H)$.

- $\gamma_2(H)/\gamma_3(H)$ is generated by $[\bar{a}, \bar{b}]\gamma_3(H)$.
- Then $\gamma_3(H)/\gamma_4(H)$ is generated by $[\bar{a}, [\bar{a}, \bar{b}]]\gamma_4(H)$ and $[\bar{b}, [\bar{a}, \bar{b}]]\gamma_4(H)$. In H, however, we have $[\bar{b}, [\bar{a}, \bar{b}]] = 1$ (since we divide out $[\tau \mathrm{BS}_c(m,n), \tau \mathrm{BS}_c(m,n)]$). So $\gamma_3(H)/\gamma_4(H)$ is generated by $[\bar{a}, [\bar{a}, \bar{b}]]\gamma_4(H)$.
- Continuing by induction, we find that $\gamma_i(H)/\gamma_{i+1}(H)$ is a cyclic group generated by
$$[\bar{a}, [\bar{a}, [\bar{a}, \dots, [\bar{a}, \bar{b}]]]]\gamma_{i+1}(H) \quad \text{(with } i-1 \text{ times } \bar{a}).$$

We already know that $\#\tau H/\gamma_2(H) = |m-n|$ (so $\bar{b}^{m-n}\gamma_2(H) = 1\gamma_2(H)$; see the proof of Lemma 2.2). Let $c_1 = \bar{b}$, and for $i > 1$, we let $c_i = [\bar{a}, [\bar{a}, [\bar{a}, \dots, [\bar{a}, \bar{b}]]]]$ (with $i-1$ times $\bar{a}$). Then $\gamma_i(H)/\gamma_{i+1}(H)$ is generated by $c_i\gamma_{i+1}(H)$ for $i > 1$, and $\tau(H)/\gamma_2(H)$ is generated by $c_1\gamma_2(H)$.

We now show by induction on i that $c_i^{m-n}\gamma_{i+1}(H) = 1\gamma_{i+1}(H)$ and hence $\#\gamma_i(H)/\gamma_{i+1}(H) \leq |m-n|$. We already obtained the case $i = 1$. Now, assume the result holds for c_{i-1} (with $i > 1$). Then $c_i = [\bar{a}, c_{i-1}]$, and we have

$$c_i^{m-n}\gamma_{i+1}(H) = [\bar{a}, c_{i-1}]^{m-n}\gamma_{i+1}(H) = [\bar{a}, c_{i-1}^{m-n}]\gamma_{i+1}(H) = 1\gamma_{i+1}(H).$$

As a conclusion, we find that

$$\#\tau(H) = \#\frac{\tau H}{\gamma_2(H)} \times \#\frac{\gamma_2(H)}{\gamma_3(H)} \times \cdots \times \#\frac{\gamma_c(H)}{\gamma_{c+1}(H)} \leq |m-n|^c = \#\tau G_c(m,n).$$

We can conclude that μ is injective (and hence an isomorphism). □

Corollary 3.3. *If* $\gcd(m,n) = 1$*, the morphism* $f\colon \mathrm{BS}(m,n) \to G_c(m,n)$ *induces an isomorphism* $\mathrm{BS}_c(m,n) \cong G_c(m,n)$.

Proof. It follows from Corollary 2.4 that, in this case,

$$[\tau \mathrm{BS}_c(m,n), \tau \mathrm{BS}_c(m,n)] = 1.$$ □

4 The case where $\gcd(m,n) = 1$

In the next lemma, we will make use of Smith normal form, details about which can be found e.g. in [16]. We remind the reader that Smith normal form is a useful tool in dealing with quotients of free modules (over PIDs).

Lemma 4.1. *Let $a, b \in \mathbb{Z}$ with $\gcd(a,b) = 1$. Then the Smith normal form of the $n \times n$-matrix*

$$A_n = \begin{pmatrix} a & 0 & 0 & 0 & \cdots & 0 & 0 \\ b & a & 0 & 0 & \cdots & 0 & 0 \\ 0 & b & a & 0 & \cdots & 0 & 0 \\ 0 & 0 & b & a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a & 0 \\ 0 & 0 & 0 & 0 & \cdots & b & a \end{pmatrix} \quad \textit{is} \quad \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & a^n \end{pmatrix}.$$

Proof. We will prove a slightly more general version of this lemma and consider for any positive integer k the matrix $A_n(k)$, which is the same matrix as A_n, except that the first entry of $A_n(k)$ (so on the first row and the first column) is a^k instead of a. So $A_n = A_n(1)$. We will now show, by induction on n, that the Smith normal form of $A_n(k)$ is

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & a^{n-1+k} \end{pmatrix}.$$

When $n = 1$, there is nothing to show, so we assume that $n > 1$. As $\gcd(a,b) = 1$, there exist integers $\alpha, \beta \in \mathbb{Z}$ such that $\alpha a^k + \beta b = 1$. Now, consider

$$P = \begin{pmatrix} \alpha & \beta \\ -b & a^k \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z}) \quad \text{and} \quad Q = \begin{pmatrix} 1 & -a\beta \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z}).$$

It is now easy to compute that (with I_{n-2} the $(n-2) \times (n-2)$ identity matrix)

$$\begin{pmatrix} P & 0 \\ 0 & I_{n-2} \end{pmatrix} A_n(k) \begin{pmatrix} Q & 0 \\ 0 & I_{n-2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & A_{n-1}(k+1) \end{pmatrix}.$$

By induction, we know the Smith normal form of $A_{n-1}(k+1)$ and hence also of

$$\begin{pmatrix} 1 & 0 \\ 0 & A_{n-1}(k+1) \end{pmatrix},$$

which is then exactly

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & a^{n-1+k} \end{pmatrix}$$

as claimed. □

Corollary 4.2. *Let m, n be two integers with* $\gcd(m,n) = 1$ *and $m \neq n$. Then*

$$A_c(m,n) \cong \mathbb{Z}_{|m-n|^c}.$$

Proof. Recall that $A_c(m,n) = \frac{\mathbb{Z}^c}{\operatorname{Im}\varphi}$, where $\varphi\colon \mathbb{Z}^c \to \mathbb{Z}^c$ is represented by the matrix (3.1). The lemma above shows that the Smith normal form of this matrix is

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & |n-m|^c \end{pmatrix},$$

from which the result follows. □

For the rest of this section, we will assume that $m \neq n$ and that $\gcd(m,n) = 1$ (so $\mathrm{BS}_c(m,n) \cong G_c(m,n)$; see Corollary 3.3). Moreover, we will use $s = \overline{e_1}$ to denote the canonical projection of the first standard generator of $\mathbb{Z}^c$ in the group $A_c(m,n)$. As $A_c(m,n)$ is a cyclic group and $A_c(m,n)$ is generated as a $\langle t\rangle$-module by s, it follows that s is also a generator of $A_c(m,n)$ as a cyclic group. So

$$\langle s\rangle = A_c(m,n) \cong \mathbb{Z}_{|n-m|^c} \quad \text{and} \quad G_c(m,n) = \langle s\rangle \rtimes \langle t\rangle.$$

Proposition 4.3. *Let $m \neq n$ and* $\gcd(m,n) = 1$. *Then $G_c(m,n) = \langle s\rangle \rtimes \langle t\rangle$, with $t^{-1}st = s^\nu$, where $\nu \in \mathbb{Z}$ is an integer satisfying*

- $\gcd(\nu, n-m) = 1$,
- $\nu m \equiv n \bmod |n-m|^c$.

Proof. We already explained that $G_c(m,n) = \langle s\rangle \rtimes \langle t\rangle$. As s is a generator of the cyclic group of order $|n-m|^c$, we must have that also $t^{-1}st$ is a generator of $\langle s\rangle$, which implies that $t^{-1}st = s^\nu$ for some integer ν with $\gcd(\nu, n-m) = 1$.

As we already saw in the proof of Lemma 3.1 (recall $s = \overline{e_1}$), we also have $t^{-1}s^m t = s^n$. As $t^{-1}s^m t = s^{\nu m}$, it follows that $s^{\nu m} = s^n$; hence

$$\nu m \equiv n \bmod |n-m|^c. \qquad \square$$

Let $\varphi: G_c(m,n) \to G_c(m,n)$ be an automorphism. Since $\langle s \rangle = \tau G_c(m,n)$, φ induces an automorphism

$$\bar{\varphi}: G_c(m,n)/\langle s \rangle = \langle t \rangle \cong \mathbb{Z} \to G_c(m,n)/\langle s \rangle = \langle t \rangle \cong \mathbb{Z}.$$

So $\bar{\varphi}(t) = t^{\pm 1}$. The following lemma is easy to check.

Lemma 4.4. *With the notation above, we have $R(\varphi) < \infty \Leftrightarrow \bar{\varphi}(t) = t^{-1}$.*

It follows that $G_c(m,n)$ does not have the R_∞-property if and only if there exists an automorphism φ of $G_c(m,n)$ such that $\bar{\varphi}(t) = t^{-1}$. We are now ready to prove the main theorem of this section which gives us the R_∞-nilpotency degree of any Baumslag–Solitar group which is determined by coprime parameters m and n.

Theorem 4.5. *Let m,n be integers with $0 < m \le |n|$ and $\gcd(m,n) = 1$. Let p denote the largest integer such that $2^p \mid 2m+2$. Then the R_∞-nilpotency degree r of $\mathrm{BS}(m,n)$ is given by the following conditions.*

- *If $n < -1$, then $r = 2$.*
- *If $n = -1$ (so $m = 1$), then $r = \infty$.*
- *If $n = m$ (so $n = m = 1$), then $r = \infty$.*
- *If $n - m = 1$, then $r = \infty$.*
- *If $n - m = 2$, then $r = p + 2$.*
- *If $n - m \ge 3$, then $r = 2$.*

Proof. Let m and n be as in the statement of the theorem.

Let $m = n$. Then the fact that $\gcd(m,n) = 1$ and $m > 0$ implies that $m = n = 1$. We have that $\mathrm{BS}(1,1) = \mathbb{Z}^2 = \mathrm{BS}_c(m,n)$ (for all c) does not have the R_∞-property, from which it follows that, in this case, the R_∞-nilpotency index is ∞.

So, from now onwards, we assume that $m \neq n$. We have to examine for which c the group $G_c(m,n)$ has the R_∞-property. So we have to investigate when $G_c(m,n)$ admits an automorphism φ with $\bar{\varphi}(t) = t^{-1}$ (Lemma 4.4). Such a morphism φ satisfies

$$\varphi(s) = s^\mu \quad \text{and} \quad \varphi(t) = s^\beta t^{-1} \quad \text{for some } \mu, \beta \in \mathbb{Z}. \tag{4.1}$$

In fact, given $\mu, \beta \in \mathbb{Z}$, the expressions of (4.1) above determine an endomorphism of $G_c(m,n)$ if and only if the relation $t^{-1}st = s^\nu$ (where ν is as in Proposition 4.3) is preserved, i.e., it must hold that

$$\begin{aligned} \varphi(t)^{-1}\varphi(s)\varphi(t) &= \varphi(s)^\nu \\ &\Updownarrow \\ ts^{-\beta}s^\mu s^\beta t^{-1} &= s^{\mu\nu} \\ &\Updownarrow \\ s^\mu &= t^{-1}s^{\mu\nu}t = s^{\mu\nu^2}. \end{aligned}$$

Moreover, such a φ is an automorphism if s^μ is a generator of $\langle s\rangle$, i.e. when $\gcd(\mu, |n-m|) = 1$. In this case, the last condition is equivalent to

$$\nu^2 \equiv 1 \bmod |n-m|^c.$$

Moreover, as we also have $\gcd(m, |n-m|) = 1$, this is also equivalent to the requirement that

$$\nu^2 m^2 \equiv m^2 \bmod |n-m|^c.$$

Finally, using Proposition 4.3, which says that $\nu m \equiv n \bmod |n-m|^c$, we find that

$$\begin{gathered} G_c(m,n) \text{ does not have the } R_\infty\text{-property} \\ \Updownarrow \\ n^2 \equiv m^2 \bmod |n-m|^c \\ \Updownarrow \\ n+m \equiv 0 \bmod |n-m|^{c-1} \end{gathered}$$

So, from now on, we have to examine when the condition

$$n+m \equiv 0 \bmod |n-m|^{c-1} \tag{4.2}$$

is satisfied.

When $c = 1$, the equation is always satisfied (reflecting the fact that finitely generated abelian groups do not have the R_∞-property). So, from now onwards, we consider the case $c > 1$.

Let $n = -m$. In this case, $n = -1$ and $m = 1$ since $\gcd(m, n) = 1$; then equation (4.2) is always satisfied. This shows that $\mathrm{BS}(-1, 1)$ (which is the fundamental group of the Klein bottle) has an infinite R_∞-nilpotency degree (although $\mathrm{BS}(-1, 1)$ does have the R_∞-property [9, Theorem 2.2]).

Let $n < -1$. Then $|n-m|^{c-1} = (|n| + m)^{c-1} > |n+m| \neq 0$. This implies that equation (4.2) is never satisfied. This means that, in this case, the R_∞-nilpotency degree of $\mathrm{BS}(n, m)$ is 2.

Now, we consider the case of positive n, where we already treated the case when $n = m$. So we have $n = m + k$ for $k > 0$. Moreover, as $\gcd(n, m) = 1$, we also have $\gcd(k, m) = 1$. If equation (4.2) is satisfied, then $|n - m| = k$ divides $n + m = 2m + k$, so $k \mid 2m$, and as $\gcd(k, m) = 1$, we must have $k \mid 2$, so $k = 1$ or $k = 2$.

Let $n = m + k$ for $k \geq 3$. From the considerations of the paragraph above, we have that the R_∞-nilpotency degree of $\mathrm{BS}(m + k, k)$ is 2.

Let $n = m + 1$. In this case, equation (4.2) is again satisfied for all c, and hence the R_∞-nilpotency degree of $\mathrm{BS}(m + 1, m)$ is ∞.

Let $n = m + 2$. Then equation (4.2) is of the form $2m + 2 \equiv 0 \bmod 2^{c-1}$. This equation is satisfied exactly when $c \leq p + 1$. It follows that the R_∞-degree of $\mathrm{BS}(m + 2, m)$ (where m is odd) is $p + 2$. This finishes the proof. □

5 The case where $\gcd(m, n) \neq 1$

Lemma 5.1. *Let m, n be non-zero integers with $m \neq n$. If $d = \gcd(m, n)$, then*

$$A_c(m, n) \cong \mathbb{Z}_d^c \oplus \mathbb{Z}_{|\frac{n-m}{d}|^c}.$$

Proof. Note that the matrix (3.1) is

$$d \begin{pmatrix} \frac{n-m}{d} & 0 & 0 & \cdots & 0 & 0 \\ -\frac{m}{d} & \frac{n-m}{d} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{m}{d} & \frac{n-m}{d} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{n-m}{m} & 0 \\ 0 & 0 & 0 & \cdots & -\frac{m}{d} & \frac{n-m}{d} \end{pmatrix}.$$

with $\gcd\left(\frac{m}{d}, \frac{n-m}{d}\right) = 1$. It follows that the Smith normal form of (3.1) is

$$\begin{pmatrix} d & 0 & \cdots & 0 & 0 \\ 0 & d & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & d & 0 \\ 0 & 0 & \cdots & 0 & d\left|\frac{n-m}{d}\right|^c \end{pmatrix},$$

from which the result follows. □

It follows that $dA_c(m, n) \cong \mathbb{Z}_{|\frac{n-m}{d}|^c}$ is a cyclic subgroup of $A_c(m, n)$, and since this subgroup is invariant under the action of $\langle t \rangle$, the semi-direct product $(dA_c(m, n)) \rtimes \langle t \rangle$ is a subgroup of $G_c(m, n)$.

Lemma 5.2. *Let* $0 < m \leq |n|$ *with* $m \neq n$*, and take* $d = \gcd(m,n)$*. Then we have that* $(dA_c(m,n)) \rtimes \langle t \rangle$ *is a subgroup of* $G_c(m,n)$ *and*

$$(dA_c(m,n)) \rtimes \langle t \rangle \cong G_c\left(\frac{m}{d}, \frac{n}{d}\right).$$

Proof. The fact that $(dA_c(m,n)) \rtimes \langle t \rangle$ is a subgroup of $G_c(m,n)$ was already discussed before the statement of the lemma. Let $\varphi' \colon \mathbb{Z}^c \to \mathbb{Z}^c$ be the morphism represented by the matrix

$$\begin{pmatrix} \frac{n-m}{d} & 0 & 0 & \cdots & 0 & 0 \\ -\frac{m}{d} & \frac{n-m}{d} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{m}{d} & \frac{n-m}{d} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{n-m}{m} & 0 \\ 0 & 0 & 0 & \cdots & -\frac{m}{d} & \frac{n-m}{d} \end{pmatrix};$$

then $\varphi = d\varphi'$. We have

$$dA_c(m,n) = d\frac{\mathbb{Z}^c}{\operatorname{Im}\varphi} = \frac{(d\mathbb{Z})^c}{d\operatorname{Im}\varphi'} \cong \frac{\mathbb{Z}^c}{\operatorname{Im}\varphi'} = A_c\left(\frac{m}{d}, \frac{n}{d}\right).$$

It is now easy to see that, under the identification $dA_c(m,n) \cong A_c\left(\frac{m}{d}, \frac{n}{d}\right)$, the action of t is still the same as what we had before, and so

$$(dA_c(m,n)) \rtimes \langle t \rangle \cong G_c\left(\frac{m}{d}, \frac{n}{d}\right). \qquad \square$$

Lemma 5.3. *Let* $0 < m \leq |n|$ *with* $m \neq n$*, and take* $d = \gcd(m,n)$*. If* $G_c\left(\frac{m}{d}, \frac{n}{d}\right)$ *has property* R_∞*, then also* $G_c(m,n)$ *has property* R_∞*.*

Proof. First let us remark that, for any $\alpha \in A_c(m,n)$, there is an automorphism ψ_α of $G_c(m,n) = A_c(m,n) \rtimes \langle t \rangle$ with

$$\psi_\alpha(a) = a \quad \text{for all } a \in A_c(m,n) \quad \text{and} \quad \psi_\alpha(t) = t\alpha.$$

Now, suppose that $\psi \in \operatorname{Aut}(G_c(m,n))$ is an automorphism with $R(\psi) < \infty$. This means that $\psi(t) = \alpha t^{-1}$ for some $\alpha \in A_c(m,n)$. After composing ψ with ψ_α, we may assume that $\psi(t) = t^{-1}$. Since we also have $\psi(dA_c(m,n)) = dA_c(m,n)$ (since $A_c(m,n)$ is a characteristic subgroup of $G_c(m,n)$), we have that ψ restricts to an automorphism of $(dA_c(m,n)) \rtimes \langle t \rangle \cong G_c\left(\frac{m}{d}, \frac{n}{d}\right)$ with finite Reidemeister number. This shows that if $G_c(m,n)$ does not have property R_∞, then also $G_c\left(\frac{m}{d}, \frac{n}{d}\right)$ does not have this property. $\square$

The main result of this section is the following result.

Theorem 5.4. *Let* $0 < m \leq |n|$, *and take* $d = \gcd(m, n)$. *Let* p *denote the largest integer such that* $2^p \mid 2\frac{m}{d} + 2$. *Then the* R_∞*-nilpotency degree* r *of* $\mathrm{BS}(m, n)$ *is given by the following conditions.*

- *If* $n < 0$ *and* $n \neq -m$, *then* $r = 2$.
- *If* $n = -m$, *then* $r = \infty$.
- *If* $n = m$, *then* $r = \infty$.
- *If* $n - m = d$, *then* $r = \infty$.
- *If* $n - m = 2d$, *then* $2 \leq r \leq p + 2$.
- *If* $n - m \geq 3d$, *then* $r = 2$.

Remark. As $d = \gcd(m, n)$, the difference $n - m$ is a multiple of d, so the theorem above does treat all possible cases.

Proof. We will first deal with a few special cases and then treat the general case.

Let $n = m$. In this case, there is an automorphism ψ of $\mathrm{BS}(m, m)$ mapping a to a^{-1} and b to b^{-1}. This automorphism induces minus the identity map on $\mathrm{BS}_1(m, m) = \frac{\gamma_1(\mathrm{BS}(m,m))}{\gamma_2(\mathrm{BS}(m,m))} \cong \mathbb{Z}^2$. It now follows that the induced map $\bar{\psi}$ on any quotient $\mathrm{BS}_c(m, m)$ has -1 as an eigenvalue, and hence $R(\bar{\psi}) < \infty$ (see [3]). It follows that the R_∞-nilpotency index of $\mathrm{BS}(m, m)$ is ∞.

Let $n = -m$. Now, consider the automorphism ψ of $\mathrm{BS}(m, -m)$ mapping a to a^{-1} and b to b. Then ψ induces a map on $\mathrm{BS}_1(m, -m) \cong \mathbb{Z} \oplus \mathbb{Z}_{2m}$, which is minus the identity on the $\mathbb{Z}$-factor and the identity on the $\mathbb{Z}_{2m}$ factor. The same argument as in the previous case now allows us to conclude that the R_∞-nilpotency index of $\mathrm{BS}(m, -m)$ is ∞.

Let $n = m + d$. So $m = kd$ and $n = (k + 1)d$ for some positive integers k and d. We claim that $b^d \in \gamma_c(\mathrm{BS}(kd, (k + 1)d))$ for all $c \geq 1$. This claim is certainly correct for $c = 1$. Now, fix $c \geq 1$, and assume that

$$b^d \in \gamma_c(\mathrm{BS}(kd, (k + 1)d)).$$

Then also $b^{kd} \in \gamma_c(\mathrm{BS}(kd, (k + 1)d))$, and hence

$$[a, b^{kd}] \in \gamma_{c+1}(\mathrm{BS}(kd, (k + 1)d)).$$

But as $a^{-1} b^{kd} a = b^{(k+1)d}$, we have

$$[a, b^{kd}] = a^{-1} b^{-kd} a b^{kd} = b^{-d} \in \gamma_{c+1}(\mathrm{BS}(kd, (k + 1)d)).$$

By induction, this finishes the proof of the claim.

It follows that the group $\mathrm{BS}(kd, (k+1)d)$ and the group

$$C(d) = \langle a, b \mid a^{-1}b^{kd}a = b^{(k+1)d}, b^d \rangle = \langle a, b \mid b^d \rangle$$

have isomorphic nilpotent quotients, i.e.,

$$\frac{\mathrm{BS}(kd, (k+1)d)}{\gamma_{c+1}(\mathrm{BS}(kd, (k+1)d))} \cong \frac{C(d)}{\gamma_{c+1}(C(d))}.$$

Now, it is easy to see that $C(d)$ has an automorphism ψ mapping a to a^{-1} and b to b such that ψ induces an automorphism $\bar{\psi}$ on $\frac{C(d)}{\gamma_{c+1}(C(d))}$ with finite Reidemeister number. It follows that the R_∞-nilpotency degree of $\mathrm{BS}(kd, (k+1)d)$ is infinite.

All the other cases. As a finitely generated abelian group never has the R_∞-property, we have $r \geq 2$. Now, assume that ψ is an automorphism of $\mathrm{BS}_c(m, n)$ with $R(\psi) < \infty$. Then ψ induces an automorphism $\bar{\psi}$ of $G_c(m, n)$ (since we divide out a characteristic subgroup to go from $\mathrm{BS}_c(m, n)$ to $G_c(m, n)$ by Proposition 3.2) with $R(\bar{\psi}) < \infty$. It follows that the R_∞-nilpotency degree is bounded above by the smallest c for which $G_c(m, n)$ has property R_∞. In turn, this number is bounded above by the smallest c such that $G_c\left(\frac{m}{d}, \frac{n}{d}\right)$ has the R_∞-property. This is exactly what we determined in the proof of Theorem 4.5, which finishes the proof. □

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