Nº 01

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Janeiro 1987

TOTAL CURVATURE OF ORTHOGONAL VECTOR FIELDS ON THREE-MANIFOLDS

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In [LL], the authors proved that if F_1 and F_2 are ortho-orthogonal orientable foliations of S^2, then

$$(1) \qquad \int_{S^2} (|k_1| + |k_2|) > 4\pi ,$$

where k_i is the geodesic curvature of F_i and, F_i's are assumed to have a finite number of singularities.

The analogous result for S^3 cannot be obtained since S^3 is a compact Lie group with bi-invariant metric, so it admits three orthogonal geodesic vector fields X_1, X_2, X_3. If k_i is the geodesic curvature of X_i then

$$\int_{S^3} (|k_1| + |k_2| + |k_3|) = 0.$$

However, one can easily see that the normal bundles D_i of X_i are not integrable. In this note, we prove that integrability of one of D_i's implies an inequality analogous to (1).

<u>Theorem</u>. Let M be a closed, oriented three-dimensional Riemannian manifold equipped with three unit mutually orthogonal vector fields X_1, X_2, X_3. Denote by k_i the geodesic curvature of X_i and assume that one of the normal bundles D_i of X_i is integrable. Then

$$(2) \qquad \int_M (k_1^2 + k_2^2 + k_3^2) > \int_M s ,$$

where s is the scalar curvature of M. Moreover, if the equality holds in (2) then the foliations determined by all integrable bundles D_i are totally umbilical.

<u>Proof</u>. Denote by A_i, T_i and H_i respectively the second fundamental tensor, the integrability tensor and the mean curva-

ture vector of D_i. Then

$$A_i, T_i : D_i \otimes D_i \to \{X_i\},$$

$$A_i(Y,Z) = \frac{1}{2} < \nabla_x Y + \nabla_y X, X_i > : X_i$$

$$T_i(Y,Z) = \frac{1}{2} < \nabla_x Y - \nabla_y X, X_i > . X_i$$

and

$$H_i = \text{trace } A_i,$$

where Y and Z are sections of D_i, $\{X_i\}$ is the 1-dimensional bundle spanned by X_i and ∇ is the Levi-Civita connection on M.

With these notation we have

$$(3) \qquad \int_M [\,\mathrm{Ric}\,(X_i) + |A_i|^2 - |H_i|^2 - |T_i|^2\,] = 0 \qquad (i=1,2,3)$$

(see [W], and [B] in the case of an integrable distribution).

Writting Γ_{ij}^k for $< \nabla_{x_k} X_i, X_j >$ we may express formulae ( ) in the form

$$(4) \qquad \int_M [\,\mathrm{Ric}\,(X_1) - 2\Gamma_{21}^2 \Gamma_{31}^3 + 2\Gamma_{21}^3 \Gamma_{31}^2\,] = 0,$$

$$(5) \qquad \int_M [\,\mathrm{Ric}\,(X_2) - 2\Gamma_{12}^1 \Gamma_{32}^3 + 2\Gamma_{12}^3 \Gamma_{32}^1\,] = 0,$$

and

$$(6) \qquad \int_M [\,\mathrm{Ric}\,(X_3) - 2\Gamma_{13}^1 \Gamma_{23}^2 + 2\Gamma_{23}^1 \Gamma_{13}^2\,] = 0.$$

Summing up formulae (4) - (6) we get

$$(7) \qquad 2\int_M (\Gamma_{13}^1 \Gamma_{23}^2 + \Gamma_{12}^1 \Gamma_{32}^3 + \Gamma_{21}^2 \Gamma_{31}^3) = \int_M [\,s + 2(\Gamma_{21}^3 \Gamma_{31}^2 + \Gamma_{32}^1 \Gamma_{12}^3 + \Gamma_{13}^2 \Gamma_{23}^1)\,]$$

Also,

$$k_1^2 + k_2^2 + k_3^2 = (\Gamma_{21}^2)^2 + (\Gamma_{31}^3)^2 + (\Gamma_{12}^1)^2 + (\Gamma_{32}^3)^2 + (\Gamma_{13}^1)^2 + (\Gamma_{23}^2)^2 >$$

(8)

$$> 2(\Gamma_{13}^1 \Gamma_{23}^2 + \Gamma_{12}^1 \Gamma_{32}^3 + \Gamma_{21}^2 \Gamma_{31}^3).$$

Now, assume that D_1 is integrable. In this case, we have

(9) $\Gamma^2_{31} = \Gamma^3_{21}$

and

(10) $\Gamma^1_{32}\Gamma^3_{12} + \Gamma^1_{23}\Gamma^2_{13} = \Gamma^1_{32}(\Gamma^2_{31} - \Gamma^3_{21}) = 0.$

Therefore, the left side of (7) is not biger than the left side of (2), while the right side of (7) is biger or equal to the right side of (2). This proves the inequality (2).

Finally, equality in (2) holds if and only if

$$\Gamma^2_{31} = \Gamma^3_{21} = 0,$$

$$\Gamma^2_{21} = \Gamma^3_{31}, \; \Gamma^1_{12} = \Gamma^3_{32} \text{ and } \Gamma^2_{21} = \Gamma^3_{31}.$$

In this case, the foliation determined by D_1 is umbilical. ∎

FINAL REMARKS

1). Every orientable foliation of S^3 is one example of the situation described in the theorem because it is always possible to find two mutually orthogonal vector fields of S^3 tangent to a given foliation. Moreover it is known that S^3 doesn't admit totally umbilic foliations so equality will never hold in (2).

2) The theorem can be generalized to dimension n in the following way. Let M be a closed oriented n-dimensional Riemannian manifold and $X_1, X_2, \ldots, X_n$ be n mutually orthogonal flows in M. Denote k_i the geodesic curvature of X_i. Assume that all but two of the normal bundles D_i of X_i are integrable. Then,

$$\int_M \sum_{i=1}^{n} k_i^2 > \int_M S$$, S being the scalar curvature of M.

We don't know if the assumption of integrability on (n - 2) normal bundles is really necessary and it would be nice to get a result with weaker integrability assumption.

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