ON THE COMPLETION OF THE SPACE OF KURZWEIL-HENSTOCK-DENJOY-PERRON INTEGRABLE FUNCTIONS

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ABSTRACT.

We prove that the space $K([a,b])$ of the functions $f : [a,b] \to \mathbf{R}$ that are integrable in the sense of Kurzweil-Henstock is not complete in its natural norm; its completion is the space of the distributions that are derivatives of the continuous functions on $[a,b]$. We conjecture that there exists no "natural" Banach space norm on $K([a,b])$.

0. Introduction.

The integral we deal with was defined by Kurzweil (see [10], [11] and [14]) and, independently, by Henstock (see [7] and the references given there). It coincides with the (special or restricted) Denjoy integral (see [6] for an elementary proof of this fact) and with the Perron integral, that were already defined at the beginning of the century (see [13] for information about these integrals, in particular Chap. VIII, par. 3 for a proof of their equivalence). This integral extends the one of Lebesgue (see [4] for an elementary proof of this fact) and is also known as the gauge integral,, the Riemann complete integral, the generalized Riemann integral etc. (see, for instance, [12]). The existence of these three different characterizations of the same integral suggests that we are dealing with a very rich theory. Many properties that are trivial or easy to prove in one of the formulations may have a difficult proof in the others. On the other hand a serious handicap of these integrals lies in the fact that we know of no functional analytic characterization of $K([a,b])$ (while for the Lebesgue integral, $L_1([a,b])$ is the completion of $C([a,b])$ and of the space of step-functions with respect to the norm $\| \|_1$, a fact that makes easy or automatic the proofs of many properties of the Lebesgue integral).

1 Basic definitions and properties.

We say that a function $f : [a,b] \to \mathbf{R}$ is integrable in the sense of Kurzweil-Henstock (we write $f \in \mathcal{K}([a,b])$) and that the real number I is its Kurzweil-Henstock integral (we write $(K)\int_a^b f(t)dt = I$) if for every $\varepsilon > 0$ there exists a function $\delta : [a,b] \to]0,\infty[$ (called *gauge* function) such that for every pointed division of $[a,b]$ that is δ–fine (i.e., a division $t_0 = a < t_1 < t_2 \cdots < t_n = b$ of $[a,b]$ with points $\xi_1, \cdots, \xi_n$ such that $\xi_i \in [t_{i-1}, t_i] \subset]\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i)[$) we have

$$|I - \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1})| < \varepsilon.$$

It can be proved that for any gauge δ there exist always δ–fine pointed divisions of $[a,b]$ and that I is unique.

We denote by $\mathcal{L}_1([a,b])[L_1([a,b])]$ the space of [equivalence classes of] all functions $f : [a,b] \to \mathbf{R}$ that are Lebesgue integrable. The Kurzweil-Henstock integral encompasses the Lebesgue integral·

1.1- $\mathcal{L}_1([a,b]) \subset \mathcal{K}([a,b])$; $f \in \mathcal{L}_1([a,b]) \iff f, |f| \in \mathcal{K}([a,b])$.

Proof: see [13] where the proofs are given for the Denjoy integral (Theorem VII.1.1) and the Perron integral (VI, par. 6 and Theorem VI.6.5).

For $f \in \mathcal{K}([a,b])$ we define $\tilde{f}(t) = (K)\int_a^t f(s)ds, a \le t \le b$. We denote by $\mathcal{C}_{\{a\}}([a,b])$ the space of continuous functions $\varphi : [a,b] \to \mathbf{R}$ with $\varphi(a) = 0$.

1.2- $f \in \mathcal{K}([a,b]) \Rightarrow \tilde{f} \in \mathcal{C}_{\{a\}}([a,b])$.

The proof is easy if we apply the Cauchy criterion for the existence of the Kurzweil-Henstock integral.

1.3- For $f \in \mathcal{K}([a,b])$ we have: $\tilde{f} = 0 \iff f = 0$ a.e. (almost everywhere).

This result is immediate by the properties of the Denjoy integral.

We denote by $K([a,b])$ the vector space of all equivalence classes (with respect to the relation "$f = 0$ a.e.") of the functions of $\mathcal{K}([a,b])$.

The Kurzweil-Henstock integral is the ideal tool to recover primitives, in their most general definition, as integrals. We recall that a function $g \in C([a,b])$ is a *primitive* of $f : [a,b] \to \mathbf{R}$ if at the complement of a countable subset of $[a,b]$ there exists $g'(t) = f(t)$.

1.4.a- If $g \in C([a,b])$ is a primitive of $f : [a,b] \to \mathbf{R}$ then $f \in \mathcal{K}([a,b])$ and for every $c \in [a,b]$ we have

$$g(c) = g(a) + (K)\int_a^c f(t)dt.$$

b- If $f \in \mathcal{K}([a,b])$ then $\tilde{f}$ is differentiable a.e. and we have $\tilde{f}' = f$ a.e.

Proof of a: Let $A = \{u_k \mid k \in \mathbf{N}\}$ be the subset of $[a,b]$ such that for $t \notin A$ there exists $g'(t) = f(t)$. We will prove the result for $c = b$. Given an $\varepsilon > 0$ we define the following gauge δ:

1) for $u_k \in A$ we take $\delta(u_k) > 0$ such that for $x, y \in]u_k - \delta(u_k), u_k + \delta(u_k)[\cap[a,b]$ we have

$$|g(y) - g(x)| + |f(u_k)(y - x)| < \frac{\varepsilon}{2^k}.$$

2) for $u \in [a,b] \cap A$ there exists $g'(u) = f(u)$ hence we may take $\delta(u) > 0$ such that for $z \in]u - \delta(u), u + \delta(u)[\cap[a,b]$ we have

$$|g(z) - g(u) - f(u)(z - u)| \leq \varepsilon|z - u|$$

hence for $u \in [x,y] \subset]u - \delta(u), u + \delta(u)[\cap[a,b]$ we have

$$|g(y) - g(x) - f(u)(y - x)| \leq \varepsilon(y - x).$$

Now, given a pointed division of $[a,b]$ that is δ-fine,

$$\xi_i \in [t_{i-1}, t_i] \subset]\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i)[\quad i = 1, 2, \cdots, n.$$

from 1) and 2) follow respectively 1') and 2'):

1') $|g(t_i) - g(t_{i-1}) - f(\xi_i)(t_i - t_{i-1})| < \frac{\varepsilon}{2^k}$ if $\xi_i = u_k \in A$.

2') $|g(t_i) - g(t_{i-1}) - f(\xi_i)(t_i - t_{i-1})| \leq \varepsilon(t_i - t_{i-1})$ if $\xi_i \in [a,b] \cap CA$.

Hence $|g(b) - g(a) - \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1})| = |\sum_{i=1}^{n}[g(t_i) - g(t_{i-1}) - f(\xi_i)(t_i - t_{i-1})]| < \varepsilon \sum_{k=1}^{\infty} \frac{1}{2^k} + \varepsilon \sum_{i=1}^{n}(t_i - t_{i-1}) = \varepsilon(1 + b - a)$.

See also [3].

The proof of b follows from the characterization of the Perron integral (see [13], Theorem VI.6.1) or from the definition of the Denjoy integral.

For $f \in \mathcal{K}([a,b])$ and $\varphi \in C([a,b])$ it does not necessarely follow that $f\varphi \in \mathcal{K}([a,b])$. Example: in $[a,b] = [0,1]$ we take $f(t) = \frac{d}{dt}(t^2 \sin t^{-3})$ and $\varphi(t) = t cos t^{-3}$. ($f \in \mathcal{K}([0,1])$ by 1.4.a; from 1.1 it follows that $f\varphi \notin \mathcal{K}([0,1])$).

We denote by $BV([a,b])$ the space of all functions $\alpha : [a,b] \to \mathbf{R}$ that are of bounded variation. If $\varphi \in C([a,b])$ we denote by $\int_a^b \varphi(t)d\alpha(t)$ the corresponding Riemann- Stieltjes integral; see [8] or [9], exerc. I.7.5 and I.8.6.

1.5- For $f \in \mathcal{K}([a,b])$ and $\alpha \in BV([a,b])$ we have $f\alpha \in \mathcal{K}([a,b])$ and the integration by parts formula:

$$(K)\int_a^b f(t)\alpha(t)dt + \int_a^b \tilde{f}(t)d\alpha(t) = \tilde{f}(b)\alpha(b) - \tilde{f}(a)\alpha(a).$$

If α is absolutely continuous we have $\int_a^b \tilde{f}(t)d\alpha(t) = L\int_a^b \tilde{f}(t)\alpha'(t)dt$, where $L\int$ denotes the Lebesgue integral.

Proof: see [13], Theorem VIII.2.5 where the Denjoy integral is used. For a proof using the Perron integral see [6] and [2], Theorem 4.

2. *A natural norm on* $K([a,b])$.

For every $f \in K([a,b])$ we define $\|f\|_A = \sup_{a \leq t \leq b} \left|(K)\int_a^t f(s)ds\right|$; see [1] and [5].

It is immediate:

2.1- The mapping $f \in K([a,b]) \mapsto \|f\|_A \in \mathbb{R}_+$ is a norm and we have $\|f\|_A = \|\tilde{f}\|$, where $\|\varphi\| = \sup_{a \leq t \leq b} |\varphi(t)|$.

The extension of the Kurzweil-Henstock integral to complex-valued functions is immediate. If we take the sequence of functions $f_n(t) = e^{2\pi i n t}, a \leq t \leq b$, it follows from the Riemann-Lebesgue Lemma that $\lim_{n \to \infty} \|f_n\|_A = 0$. However the sequence f_n contains no subsequence that converges to 0 at some point t (since $|f_n(t)| = 1$).

Every $f \in K([a,b])$ defines a distribution $T_f \in \mathrm{D}'(]a,b[)$ (in the sense of Laurent Schwartz, [15]) where for every $\varphi \in \mathrm{D}(]a,b[)$ we take

$$< T_f, \varphi >= (K) \int_a^b f(t)\varphi(t)dt \text{ (Cf. 1.5).}$$

For every distribution T we denote by T' its derivative in the sense of distributions: $< T', \varphi >= - < T, \varphi' >$ for every $\varphi \in \mathrm{D}(]a,b[)$.

2.2- For every $f \in K([a,b])$ we have $T_f = T'_{\tilde{f}}$, i.e., the following diagram is commutative

$$\begin{array}{ccc} f \in K([a,b]) & \longrightarrow & T_f \in \mathrm{D}'(]a,b[) \\ \big\downarrow & & \big\uparrow \frac{d}{dt} \\ \tilde{f} \in C_{\{a\}}([a,b]) & \longrightarrow & T_{\tilde{f}} \in \mathrm{D}'(]a,b[) \end{array}$$

Proof: By 1.5 and from the definition of the derivative of a distribution it follows that for every $\varphi \in \mathrm{D}(]a,b[)$ we have

$$< T_f, \varphi >= (K) \int_a^b f(t)\varphi(t)dt = - \int_a^b \tilde{f}(t)\varphi'(t)dt = - < T_{\tilde{f}}, \varphi' >=< T'_{\tilde{f}}, \varphi > .$$

The horizontal mappings are immersions since, by 2.1, $f \neq 0$ implies that $\tilde{f} \dot{\neq} 0$, hence $T_{\tilde{f}} \neq 0$ and $T_{\tilde{f}}$ is not a constant, hence $T'_{\tilde{f}} \neq 0$, i.e., $T_f \neq 0$.

In what follows we consider the commutative diagram above with the corresponding natural immersion of $K([a,b])$ in $\mathrm{D}'(]a,b[)$:

Theorem 2.3- The completion of $K([a,b])$ with respect to the norm $\| \|_A$ is the subspace of all distributions of $D'(]a,b[)$ that are derivatives, in the sense of distributions, of the functions of $C_{\{a\}}([a,b])$.

Proof: let $f_n \in K([a,b])$ be a $\| \|_A$-Cauchy sequence. By 2.1 it follows that $\tilde{f}_n \in C_{\{a\}}([a,b])$ is a $\| \|$-Cauchy sequence, hence converges uniformly to a function $g \in C_{\{a\}}([a,b])$ and therefore $T_{\tilde{f}_n} \to T_g$ (i.e., the distributions $T_{\tilde{f}_n}$ converge to T_g in the sense of distributions: for every $\varphi \in D(]a,b[)$ we have $< T_{\tilde{f}_n}, \varphi > \to < T_g, \varphi >$) hence we have also $T'_{\tilde{f}_n} \to T'_g$. But by 2.2 we have $T'_{\tilde{f}_n} = T_{f_n}$, hence $T_{f_n} \to T'_g$.

Reciprocally if $T \in D'(]a,b[)$ is of the form $T = T'_g$ with $g \in C_{\{a\}}([a,b])$ we take a sequence $p_n \in C^{(1)}_{\{a\}}([a,b])$ (the space of the elements of $C_{\{a\}}([a,b])$ that have a continuous derivative) that converges uniformly to g. By 2.1 $p'_n \in K([a,b])$ is a $\| \|_A$- Cauchy sequence; its image in $D'(]a,b[)$ converges to T : $\|p_n - g\| \to 0$ implies $T_{p_n} \to T_g$, hence $T'_{p_n} \to T'_g = T$; since $p_n = \widetilde{p'_n}$ the assertion follows from 2.2 ($T_{p'_n} = T'_{\widetilde{p'_n}} = T'_{p_n} \to T'_g = T$).

Remark 2.4- From the proof of the preceeding theorem we see that $C([a,b])$ is $\| \|_A$-dense in $K([a,b])$.

2.5- $K([a,b])$ is not complete in the norm $\| \|_A$.

Proof: with the notation of Theorem 2.3 we take a $g \in C_{\{a\}}([a,b])$ that is nowhere differentiable and $p_n \in C^{(1)}_{\{a\}}([a,b])$ such that $\|p_n - g\| \to 0$. The $\| \|_A$-Cauchy sequence p'_n does not $\| \|_A$-converge in $K([a,b])$ since from $\|p'_n - h\|_A \to 0$ it follows that $T_{p'_n} \to T_h$, hence $T_h = T'_g$. By 2.2 we have $T_h = T'_{\tilde{h}}$ hence $T'_{\tilde{h}} = T'_g$ and therefore, by a well known result of the theory of distributions, we have $\tilde{h} = g$. But by 1.4.b we have $\tilde{h}' = h$ a.e. and therefore g would be almost everywhere differentiable.

We say that a norm p on $K([a,b])$ is *natural* if $p(f_n) \to 0$ implies that $(K) \int_a^t f_n(s)ds \to 0$ for every $t \in [a,b]$.

2.6- If p is a natural Banach space norm on $K([a,b])$ then p is finer than the norm $\| \|_A$ (i.e., there exists $M > 0$ such that for every $f \in K([a,b])$ we have $\|f\|_A \leq Mp(f)$).

Proof: for every $t \in [a,b]$ and $f \in K([a,b])$ we define $F_t(f) = \tilde{f}(t)$. By our hypothesis we have $F_t \in K_p([a,b])'$ (the topological dual of $K([a,b])$ endowed with the norm p). For every $f \in K([a,b])$ we have

$$\sup_{a\leq t\leq b} |F_t(f)| = \sup_{a\leq t\leq b} |\tilde{f}(t)| = \|\tilde{f}\| = \|f\|_A.$$

From the uniform boundedness principle (see [8] or [9]) it follows that

$$\sup_{a\leq t\leq b} p(F_t) = M < \infty$$

where $p(F_t) = \sup\{|F_t(f)| \mid f \in K([a,b]), p(f) \leq 1\}$.But

$$\sup_{a\leq t\leq b} p(F_t) = \sup_{a\leq t\leq b} [\sup_{p(f)\leq 1} |F_t(f)|] = \sup_{p(f)\leq 1} [\sup_{a\leq t\leq b} |F_t(f)|]$$
$$= \sup_{p(f)\leq 1} \|\tilde{f}\| = \sup_{p(f)\leq 1} \|f\|_A$$

hence $p(f) \leq 1$ implies that $\|f\|_A \leq M$.

An open problem: we conjecture that there exists no natural norm on $K([a,b])$ that makes it a Banach space. We believe even that there exists no Banach space norm p on $K([a,b])$ such that $p(f_n) \to 0$ implies $T_{f_n} \to 0$.

REFERENCES

[1] A. Alexiewicz, "Linear functionals on Denjoy integrable functions", Colloquium Math. 1(1948), 289-293. (Math. Reviews 10(1949), 717).

[2] P.S. Bullen, "A survey of integration by parts for Perron integrals", J. Austral. Math. Soc. 40(1986), 343-363.

[3] G. Cross and O. Shisha, "A new approach to integration", J. of Math. An. and Appl. 114(1986), 289-294.

[4] R.O. Davis and Z. Schuss, "A proof that Henstock's integral includes Lebesgue's", J. London Math. Soc. (2), 2(1970), 561-562.

[5] A.G. Džarševsvili, "On the normed space of D^*-integrable functions", Akad. Nauk Gruzin S.S.R. Trudy Tbiliss Mat. Inst. Razmadze 19(1953), 153-162. (Math. Reviews 16(1955), 490).

[6] L. Gordon and S. Lasher, "An elementary proof of integration by parts for the Perron integral", Proc. Amer. Math. Soc. 18(1967), 394-398.

[7] R. Henstock, "A Riemann-type integral of Lebesgue power", Canad. J. Math. 20(1968), 79-87.

[8] C.S. Hönig, "Análise Funcional e Aplicações", 2 Vol., Publicações do Instituto de Matemática e Estatística da USP, 1970.

[9] C.S. Hönig, "Aplicações da Topologia à Análise", Projeto Euclides, IMPA do CNPq, 1976.

[10] J. Kurzweil, "Generalized Ordinary Differential Equations and continuous dependence on a parameter", Czech. Math. J. 7(82), (1957), 418-446.

[11] J. Kurzweil, "Nichtabsolut konvergente Integrale", Teubner-Texte zur Mathematik, Band 26. Leipzig, 1980.

[12] E.J. McShane, "Unified Integration", Academic Press, 1983.

[13] S. Saks, "Theory of the integral", Hafner Publ. Co., New York, 1937.

[14] Št. Schwabik, "Generalized Differential Equations. Fundamental Results", Rozpravy Československé Akademie Věd-Řada Matematických a Přírodních Věd. Praha 1985.

[15] L. Schwartz, Méthods mathématiques pour les sciences physiques", Hermann, Paris, 1961 - English translation: "Mathematics for physical sciences", Addison-Wesley, 1966.

[16] L.P Yee and W. Naak-In, "A direct proof that Henstock and Denjoy integrals are equivalent". Bull. Malayasian Math. Soc.(2)5(1982), 43-47.

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