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Generalized Homogeneous Functions and the Two-Body Problem

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GENERALIZED HOMOGENEOUS FUNCTIONS AND THE TWO-BODY PROBLEM

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ABSTRACT. In this article we study a generalization of the homogeneous function concept. An application is done with a solution of the two-body problem.

RESUMO. Neste artigo estudamos uma generalização do conceito de função homogênea. Uma aplicação é feita com uma solução para o problema de dois corpos.

key words and phrases: homogeneous function, Kepler's second law, two-body problem

1. INTRODUCTION

In this paper we generalize the classic concept of homogeneous function of degree α and we study the relation between the homogeneous function concept and the movement of a body that satisfies the Kepler's second law.
As an application of the involved techniques that were used, we present a solution of the two-body problem, giving a way to obtain a time equation for the body that rotates around the other, using the concept of homogeneous function.
We obtained a series like as that was presented in [1].

2. GENERALIZED HOMOGENEOUS FUNCTIONS

Let U be an open subset of $\mathbb{R}^n$ so that if $x \in U$ and λ is a real number, $0 < \lambda < 1$, then $\lambda.x \in U$.

Definition 1. *Let $f : U \to \mathbb{R}$ be a C^r function. We say that f is an homogeneous function of degree α if $f(\lambda.x) = \lambda^\alpha.f(x)$, if $\lambda > 0$.*

We put bellow the well known concept of homogeneous function.

Let θ be a function of class C^r such that $\theta : (0,\infty) \times (0,\infty) \to (0,\infty)$ and

$$(1) \qquad \begin{cases} \theta(1, z) = z, \\ \theta(\lambda_1.\lambda_2, z) = \theta(\lambda_1, \theta(\lambda_2, z)). \end{cases}$$

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We observe that θ is an action of the multiplicative group $(0,\infty)$ to $(0,\infty)$.

Consider the function $\alpha(z) = \dfrac{\partial\theta(1,z)}{\partial\lambda}$, $z \in (0,\infty)$.

Let us generalize the concept of an homogeneous function.

Definition 2. *Let $f : U \to \mathbb{R}$ be a function of class C^r. We say that f is $\theta-$homogeneous if*

$$i)\ f(\lambda . x) = \theta(\lambda, f(x))$$

$$ii)\ \alpha(f(x)) > 0.$$

Lemma 1. *Let θ be an action of $(0,\infty)$.*
Then f is a $\theta-$homogeneous function $\iff \langle \vec{\nabla} f(x),\ x \rangle = \alpha(f(x))$

Proof: $\Rightarrow$) We note that:
$\langle \vec{\nabla} f(\lambda x),\ x \rangle = \dfrac{\partial\theta}{\partial\lambda}(\lambda, f(x))$. Then, for $\lambda = 1$, we have that, $\langle \vec{\nabla} f(x),\ x \rangle = \dfrac{\partial\theta}{\partial\lambda}(1, f(x)) = \alpha(f(x))$.
$\Leftarrow$) We define for each value of x, the functions:
$\varphi(\lambda) = f(\lambda x)$ and $\tilde{\varphi}(\lambda) = \theta(\lambda, f(x))$
We note that $\varphi(1) = f(x) = \tilde{\varphi}(1)$. We will prove that φ and $\tilde{\varphi}$ are solutions of an ordinary differential equation with the same initial condition.
We have that: $\alpha(f(\lambda x)) = \langle \vec{\nabla} f(\lambda x),\ \lambda x \rangle = \lambda \langle \vec{\nabla} f(\lambda x),\ x \rangle = \lambda \varphi'(\lambda)$
Then, $\alpha(\varphi(\lambda)) = \lambda\varphi'(\lambda)$
So φ is a solution of the equation $\varphi' = \dfrac{\alpha}{\lambda}\varphi$.
For the function $\tilde{\varphi}(\lambda)$ we have,
$\lambda\tilde{\varphi}'(\lambda) = \lambda\dfrac{\partial\theta}{\partial\lambda}(t\lambda, f(x))$.
Consider the function $h(t) = \theta(t, \theta(\lambda, f(x))) = \theta(t\lambda, f(x))$.
Then, $h'(t) = \lambda\dfrac{\partial\theta}{\partial\lambda}(t\lambda, f(x))$. So, $h'(1) = \lambda\dfrac{\partial\theta}{\partial\lambda}(\lambda, f(x))$.
By other side, $h'(1) = \dfrac{\partial\theta}{\partial t}(1, \theta(\lambda, f(x)) = \alpha(\theta(\lambda, f(x)))$. Then, $\lambda\tilde{\varphi}'(\lambda) = \alpha(\tilde{\varphi}(\lambda))$. So, $\tilde{\varphi}' = \dfrac{\alpha}{\lambda}\tilde{\varphi}$, and then $\varphi = \tilde{\varphi}$.

Remark 1. *Let f be a $\theta-$homogeneous function with $\theta(\lambda, z) = \lambda^\alpha z$. Then f is homogeneous of degree α, with $\alpha(z) = \alpha z$.*
In fact, if f is a $\theta-$homogeneous function, then $f(\lambda x) = \theta(\lambda, f(x)) = \lambda^\alpha f(x)$. But $\alpha(z) = \dfrac{\partial\theta}{\partial\lambda}(1, z) = \alpha\lambda^{\alpha-1} z$, for $\lambda = 1$. So, $\alpha(z) = \alpha z$.

Remark 2. *When P_0 is any point, we say that f is a $\theta-homogeneous function$ relative to P_0 if $f(P_0 + \lambda(x - P_0)) = \theta(\lambda, f(x))$.*

As in the proof of Lemma 1 we easily demonstrate that: $\langle \vec{\nabla} f(x),\ x - P_0 \rangle = \alpha(f(x))$.

Theorem 1. *Let θ be an action as in (1), C a curve and $P_0 \notin C$. There exists a $\theta -$ homogeneous function f relative to P_0, so that $f(x) = 1, \forall x \in C$.*

Proof: Define $\psi(\lambda) = \theta(\lambda, 1)$. Suppose $P_0 = 0$. For every $x \in C$, choose y so that $\frac{x}{\psi^{-1}(y)} \in C$. Then we define $f(x) = y$. It is clear that $f(x) = 1$, $\forall x \in C$.

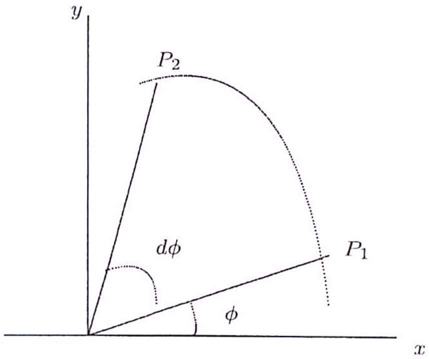
We have that $\frac{\lambda x}{\psi^{-1}(\theta(\lambda, f(x)))} = \frac{\lambda x}{\lambda\psi^{-1}(f(x))} \in C$.

So, $\theta(\lambda\psi^{-1}(f(x)), 1) = \psi(\lambda\psi^{-1}(f(x))) = \psi(\psi^{-1}(\theta(\lambda, f(x)))) = \theta(\lambda, f(x))$.

So, $f(\lambda, x) = \theta(\lambda, f(x))$ and the function f is $\theta -$ *homogeneous.*

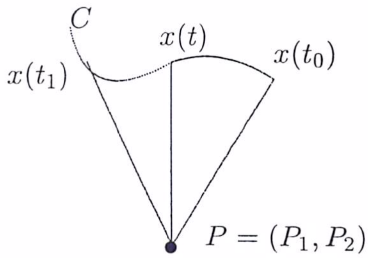
Corollary 1. *If C is any curve, then it is a level curve of a homogeneous function of degree $\alpha > 0$.*

We remember now the Kepler's second law: The Area's Law



Let P_1 and P_2 be two successive positions of a body in a interval of time δt. The element of area in this interval of time is $\delta A = r^2\delta\phi/2$, or, $\frac{\partial A}{\partial t} = \frac{r^2}{2} \cdot \frac{\partial \phi}{\partial t}$ is a constant, that is, the area is proportional to time.

Consider a C^r plane curve C, $r \geq 1$, and a point $P = (P_1, P_2) \notin C$. Suppose C is given by $x = x(t) = (x_1(t), x_2(t))$, so that



(2)

$$det \begin{vmatrix} x(t) - P \\ x'(t) \end{vmatrix} = det \begin{vmatrix} x_1(t) - P_1 & x_2(t) - P_2 \\ x_1'(t) & x_2'(t) \end{vmatrix} > 0$$

We know that the area swept out by a body that moves from $Q_0 = (x_1(t_0), x_2(t_0))$ to $Q_1 = (x_1(t_1), x_2(t_1))$ is

$$A = \frac{1}{2} \int_{t_0}^{t_1} \begin{vmatrix} x(t) - P \\ x'(t) \end{vmatrix} dt$$

So,

$$A'(t) = \frac{1}{2} \begin{vmatrix} x(t) - P \\ x'(t) \end{vmatrix} = c$$

Definition 3. *A curve C satisfies the Kepler's second law relative to the point P if $A'(t) = c$, for some $c > 0$.*

Then, $A(t) = \frac{1}{2}\int_{t_0}^{t} \begin{vmatrix} x(u) - P \\ x'(u) \end{vmatrix} du = \frac{1}{2} 2c\,(t - t_0) = c\,(t - t_0)$.
So, the area is proportional to the time to go from $x(t_0)$ to $x(t)$, and then satisfies the area's Kepler's law.

Remark 3. *If $x(t)$, $t \in (a, b)$ is a parametric curve C that satisfies the Kepler's second law with constant c in relation to P, we have that:*

$$A = \frac{1}{2}\int_{a}^{b} \begin{vmatrix} x(t) - P \\ x'(t) \end{vmatrix} dt = c(b - a) = cp$$

where $p = b - a$.

3. Parametric Representation by Surface Measure

It is often convenient to shift from one parameter representation of a curve, to another, to achieve once a special parametric representation for the Kepler's second law to be satisfied. Let $\tilde{x}(u)$, $u \in (c, d)$, the parameter representation of a curve C. We have that:

$$\tilde{A}(u) = \frac{1}{2}\int_{u_0}^{u} \begin{vmatrix} \tilde{x}(v) - P \\ \tilde{x}'(v) \end{vmatrix} dv$$

So,

$$\tilde{A}'(u) = \frac{1}{2} \begin{vmatrix} \tilde{x}(u) - P \\ \tilde{x}'(u) \end{vmatrix} > 0, \forall u$$

We make the follow change of parameter: $t = \tilde{A}(u)$, $t \in (a, b)$, and $h(t) = u$, and define $x(t) = \tilde{x}(h(t))$.

With this choice of the parametric representation, the curve C satisfies the Kepler's second law in relation to P.
In fact, we have that:

$$\frac{1}{2}\begin{vmatrix} x(t) - P \\ x'(t) \end{vmatrix} = \frac{1}{2}\begin{vmatrix} \tilde{x}(u) - P \\ \tilde{x}'(u).h'(t) \end{vmatrix} = \frac{1}{2}h'(t)\begin{vmatrix} \tilde{x}(u) - P \\ \tilde{x}'(u) \end{vmatrix} = \frac{1}{2}.2\frac{1}{\begin{vmatrix} \tilde{x}(u) - P \\ \tilde{x}'(u) \end{vmatrix}}.\begin{vmatrix} \tilde{x}(u) - P \\ \tilde{x}'(u) \end{vmatrix} = 1$$

So
$\begin{vmatrix} x(t) - P \\ x'(t) \end{vmatrix} = 2$ and the Kepler's second law is satisfied.

Definition 4. *If a curve C satisfies the Kepler's second law with constant $c = 1$ in relation to P we say that the curve C has a parametric representation by surface measure.*

Then, above we prove that any curve can have a parametric representation by surface measure, what is analogous that we know by the parametric representation of one curve by arc length. [2]

So we prove that if $x(t)$ is a parametric representation of a curve by surface measure and the time to go from a point Q_1 to Q_2 is T, then the area swept out is T.

4. Generalized Homogeneity and the Kepler's Second Law

Let U be an open subset of $\mathbb{R}^2$. Let $f\colon U \to \mathbb{R}$ be a C^1 function whose derivative at a point x is denoted by $f'(x)$. There exists a unique vector $g(x) \in \mathbb{R}^2$ such that $f'(x)\cdot v = \langle g(x), v\rangle$, for all $v \in \mathbb{R}^2$, where $\langle\,,\,\rangle$ denotes the inner product in $\mathbb{R}^2$. Let $u(x)$ be the hamiltonian field obtained by rotating $g(x)$ by an angle of $\frac{\pi}{2}$ radians. Observe that the vectors $u(x)$ are tangent to the level curves of f, so the vectors $g(x)$ are orthogonal to the level curves of f.

Theorem 2. *Let f be a θ – homogeneous function and $x(t)$ a solution of the initial value problem*

$$\begin{cases} \dot{x} = u(x) \\ x(t_0) = x_0 \end{cases} \tag{3}$$

where u is defined above. Then $x(t)$ satisfies the Kepler's second law in relation to the origin.

Proof We note that because $x(t)$ is a solution of (3), then it is a parametric function of a part of the level curve $f^{-1}(f(x_0))$. This fact and lemma 1 imply that

$$A'(t) = \begin{vmatrix} x_1(t) & x_2(t) \\ x_1{}'(t) & x_2{}'(t) \end{vmatrix} = \langle \nabla f(x), x\rangle = \alpha(f(x)) = \alpha f(x_0))$$

which is constant.

Corollary 2. *Let f be homogeneous of degree $\alpha > 0$. Then the solution of the equation $\dot{x} = u(x)$ satisfies the Kepler's second law with constant $c = \dfrac{\alpha}{2}$.*

Proof The function f is homogeneous of degree α, then by Remark 1, $\alpha(z) = \alpha z$. Let $x(t)$ be the solution of $\dot{x} = u(x)$ so that $f(x(t)) = 1$. Then by Theorem 2,
$2c = A'(t) = \alpha.f(x_0) = \alpha.1$, and so $c = \dfrac{\alpha}{2}$.

Lemma 2. *Let $x(t)$, $t \in (a,b)$ a parameter representation of a curve of a curve C that satisfies the second Kepler's law with constant c in relation to P. Then to obtain another parametric representation $x_1(s)$, $s \in (c,d)$ that satisfies the Kepler's second law with constant c_1 it is suficient to take $x_1(s) = x(s\frac{c}{c_1})$.*

Proof Let $h : (c,d) \to (a,b)$ be a function so that $t = h(s)$ and $x(t) = x_1(h(s))$. Then,

$$2c = \begin{vmatrix} x(t) - P \\ x'(t) \end{vmatrix} = \begin{vmatrix} x_1(h(s)) - P \\ h'(s)x_1{}'(h(s)) \end{vmatrix} = h'(s)\begin{vmatrix} x_1(h(s)) - P \\ x_1{}'(h(s)) \end{vmatrix} = h'(s)2c_1$$

Then $h'(s) = \frac{c}{c_1}$ which implies that $t = h(s) = \frac{c}{c_1}s + t_0$,
So, $x_1(s) = x(s\frac{c}{c_1} + t_0)$.

Lemma 3. *Let $x(t)$, $t \in J$ and $\tilde{x}(s)$, $s \in J_1$, parametric representations of a curve C that satisfies the Kepler's second law with the same constant c in relation to P. Then $t = s + d$, with d constant.*

Proof Because $x(t)$ and $\tilde{x}(s)$ parameterize the same curve C we have that $\tilde{x}(s) = x(t) = x(h(s))$, $t = h(s)$, and then $\tilde{x}'(s) = h'(s)x'(h(s)) = h'(s)x'(t)$. So,

$$2c = \begin{vmatrix} \tilde{x}(s) - P \\ \tilde{x}'(s) \end{vmatrix} = h'(s) \begin{vmatrix} x(t) - P \\ x'(t) \end{vmatrix} = h'(s)2c$$

Then, $h'(s) = 1$ and so $h(s) = s + d$.

Theorem 3. *Let $x(t)$ be a parametric representation of a curve C that satisfies the Kepler's second law with a constant $\alpha = c$ in relation to origin. Then there exists a homogeneous function of degree $\alpha = 2c$ so that $x(t)$ is a solution of (3).*

Proof By Corollary 1 we have that $C = \{x(t), t \in J\}$ is a level curve of a homogeneous function f of degree $\alpha = 2c$, that is, $f(x(t)) = 1$, $x(t_0) = x_0$, $f(x_0) = 1$.
Consider the equation (3) and let $\tilde{x}(s)$ be a solution so that $\tilde{x}(t_0) = x_0$. Then, $f(\tilde{x}(s)) = 1$, and because $\tilde{x}(s)$ satisfies the Kepler's second law with the same constant c and $\tilde{x}(t_0) = x_0$, then $\tilde{x}(t) = x(t)$.

We observe that if we change the point P and consider the same parameter, the relation between the areas swept out by a point that moves from point P to a point P_1 is given by:

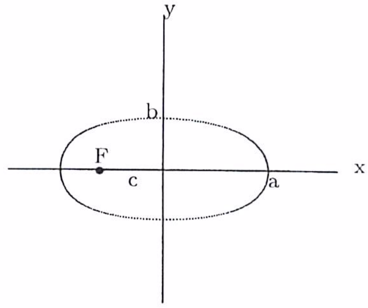
$$A = \int_a^b \frac{1}{2} \begin{vmatrix} x(t) - P \\ x'(t) \end{vmatrix} dt = \int_a^b \frac{1}{2} \begin{vmatrix} x(t) - P_1 + P_1 - P \\ x'(t) \end{vmatrix} dt \tag{4}$$

$$= \int_a^b \frac{1}{2} \begin{vmatrix} x(t) - P_1 \\ x'(t) \end{vmatrix} dt + \int_a^b \frac{1}{2} \begin{vmatrix} P_1 - P \\ \int_a^b x'(t) \end{vmatrix} dt = A_1 + \frac{1}{2} \begin{vmatrix} P_1 - P \\ x(t) - x(a) \end{vmatrix}$$

If f is θ homogeneous and $x(t)$ is a solution of (3) for $t \in (t_0, t_1)$, we have that $\begin{vmatrix} x(t) - P \\ x'(t) \end{vmatrix} = \begin{vmatrix} x(t) - P \\ u(x) \end{vmatrix} = \alpha(f(x))$
Then, $A = \frac{1}{2} \int_{t_0}^{t_1} \alpha(f(x(t)))\, dt = \frac{1}{2} \int_{t_0}^{t_1} \alpha(z_0)\, dt = \frac{1}{2}\alpha(z_0)(t_1 - t_0)$.

5. An Application: The Two-Body Problem

Consider the classical problem in which an object of mass m orbits another object of a much larger mass M. Let F be the center of mass between m and M, and suppose that the movement is elliptical, that is, the object of mass m describes an ellipse whose focus is F.
The orbital period P is knew in terms of the masses m and M, that is: $P = \dfrac{4\pi^2 a^3}{G(m+M)}$, where G is the gravitational constant.



We observe that this is a movement that satisfies the Kepler's second law and then we know that:

$$\frac{1}{2}\begin{vmatrix} x(t) - F \\ x'(t) \end{vmatrix} = c \qquad \text{and} \qquad A = \pi ab$$

Then, $\pi ab = \displaystyle\int_{t_0}^{t_1} \frac{1}{2}\begin{vmatrix} x(t) - F \\ x'(t) \end{vmatrix} dt = (t_1 - t_0)c = Pc.$

So, $c = \dfrac{\pi ab}{P}$.
Our objective is to obtain the equation of the movement $x(t)$ of the body of mass m. (with Kepler's constant c relatively to F).
Given $f = f(x_1, x_2) = \dfrac{{x_1}^2}{a^2} + \dfrac{{x_2}^2}{b^2} - 1$, the movement's orbit is given by the level curve one of f, that is, $\dfrac{{x_1}^2}{a^2} + \dfrac{{x_2}^2}{b^2} = 1$.
We observe that f is homogeneous of degree 2, $\vec{\nabla} f(x) = (\dfrac{2x_1}{a^2}, \dfrac{2x_2}{b^2})$, and $u(x) = (\dfrac{-2x_2}{b^2}, \dfrac{2x_1}{a^2})$.
The ordinary differential equation is $\dot{x} = (\dfrac{-2x_2}{b^2}, \dfrac{2x_1}{a^2})$.

The solution for this equation is $\tilde{x}(s) = \left(a \cos\dfrac{2s}{ab},\ b \sin\dfrac{2s}{ab}\right)$. For this solution the constant in relation to $P = 0$ is:

$$\frac{1}{2}\begin{vmatrix} x(\tilde{t}) - P \\ x'(\tilde{t}) \end{vmatrix} = \frac{1}{2}\begin{vmatrix} a\cos\frac{2\tilde{t}}{ab} & b\sin\frac{2\tilde{t}}{ab} \\ \frac{-2}{b}\sin\frac{2\tilde{t}}{ab} & \frac{2}{a}\cos\frac{2\tilde{t}}{ab} \end{vmatrix} = 1$$

Let $\bar{x}(\tilde{t})$ be the parametric representation with constant $c = 1$ in relation to F. We know that any curve can have a parametric representation for to satisfy the Kepler's

second law.
So, $\tilde{t} = \tilde{A}(s) = \frac{1}{2}\int_{s_0}^{s} \begin{vmatrix} \tilde{x}(v) - F \\ \tilde{x}'(v) \end{vmatrix} dv = \frac{1}{2}\int_{s_0}^{s} \begin{vmatrix} \tilde{x}(v) \\ \tilde{x}'(v) \end{vmatrix} dv - \frac{1}{2}\int_{s_0}^{s} \begin{vmatrix} F \\ \tilde{x}'(v) \end{vmatrix} dv =$

$$s - s_0 - \frac{1}{2}\begin{vmatrix} F \\ \tilde{x}(s) - \tilde{x}(s_0) \end{vmatrix}$$

Since $F = -\sqrt{a^2 - b^2}$ and taking $s_0 = 0$, it follows that:

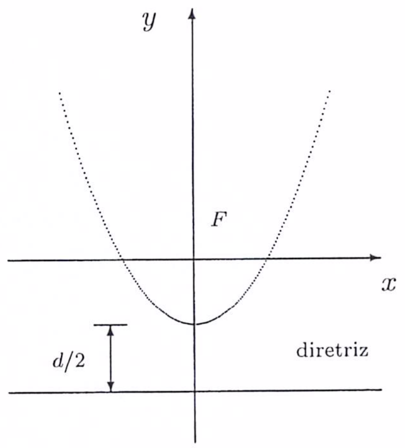
$$\tilde{t} = \tilde{A}(s) = s - s_0 + \frac{1}{2}\begin{vmatrix} (-\sqrt{a^2 - b^2}, 0) \\ \tilde{x}(s) - \tilde{x}(0) \end{vmatrix} = s + \frac{1}{2}\begin{vmatrix} -\sqrt{a^2 - b^2} & 0 \\ a(\cos\frac{2s}{ab} - 1) & b\sin\frac{2s}{ab} \end{vmatrix}.$$

Then, $\tilde{t} = s - \frac{1}{2}b\sqrt{a^2 - b^2}\sin\frac{2s}{ab} = s - \frac{1}{2}b\sqrt{a^2 - b^2}\,[\frac{2s}{ab} - (\frac{2s}{ab})^3\frac{1}{3!} + (\frac{2s}{ab})^5\frac{1}{5!} + \ldots]$

But $\tilde{t} = \tilde{A}(s)$, $s = h(\tilde{t})$, then $\bar{x}(\tilde{t}) = \tilde{x}(h(\tilde{t}))$, and we remember that in this manner $\bar{x}(\tilde{t})$ satisfies the Kepler's second law with constant $c = 1$.

Taking $x(t) = \bar{x}(ct)$, where $\tilde{t} = ct$, the Kepler's second law is satisfied with constant $c = \dfrac{\pi ab}{P}$.
This is the parametric representation of the planetary movement of two-body problem whose Kepler's constant is given as a function of the period.

Let us now consider the case when the orbit is a parabola, $y = \frac{1}{2d}(x^2 - d^2)$.



The parametric representation $\begin{cases} x = u \\ y = \frac{1}{2d}(u^2 - d^2) \end{cases}$

cannot satisfies the Kepler's second law, and then we make a parametric representation by surface measure by making:

$$\frac{1}{2}\begin{vmatrix} u & \frac{1}{2d}(u^2-d^2) \\ 1 & \frac{u}{d} \end{vmatrix} = \frac{1}{2}\left(\frac{u^2}{d} - \frac{1}{2d}(u^2-d^2)\right) = \frac{1}{2}\left(\frac{u^2+d^2}{2d}\right) = \frac{1}{4d}(u^2+d^2)$$

Then, $\tilde{t} = \frac{1}{2}\int_0^u \frac{v^2+d^2}{2d}\,dv = \frac{1}{2}\left(\frac{1}{6d}u^3 + \frac{d}{2}u\right) = \frac{1}{12d}u^3 + \frac{d}{4}u = \theta(u)$ and so, $u = \theta^{-1}(\tilde{t})$

By making a change of parameters, we obtain a new parametric representation that satisfies the Kepler's second law with constant 1 in relation to F. Let $(\tilde{x}(\tilde{t}), \tilde{y}(\tilde{t}))$ this parametric representation. So, $(\tilde{x}(ct), \tilde{y}(ct))$ will satisfies the Kepler's second law with constant c determined by the velocity v_0 at $t = 0$.
But, $(\tilde{x}(ct), \tilde{y}(ct)) = ((u, \frac{1}{2d}(u^2-d^2)) = (\theta^{-1}(ct), \frac{1}{2d}(\theta^{-1}(ct))^2 - d^2)$ and $\left(\tilde{x}'(ct), \tilde{y}'(ct)\right)_{t=0} = \left(\frac{1}{\theta'(u)}, \frac{2}{2d}\frac{1}{\theta'(u)}u\right)_{u=0}$, where $\theta'(u) = \left(\frac{3u^2+3d^2}{12d}\right)_{u=0} = \frac{1}{4}d$ and we observe that in this example the constant c was not determined in function of the period because we are treating of the parabolic case; it is done in function of the initial velocity.

Put now $x(t) = \tilde{x}(ct)$ and $y(t) = \tilde{y}(ct)$. Then, $(x'(0), y'(0))\frac{4}{d} = (1, 0)$ and then $\vec{v_0} = (c\frac{4}{d}, 0)$ and $v_0 = |\vec{v_0}| = c\frac{4}{d}$ and then, $c = \frac{dv_0}{4}$. So, we have that $(x(t), y(t))$ is a parametric representation that satisfies the Kepler's second law with constant c in relation to F, where c is given above.

We observe that in this case we obtain u in function of $\tilde{t}$ by resolving a cubic equation.

In the elliptic case the equation is transcendent.

In a analogous way we can describe the movement in the case that the orbit is a hyperbole. In this case the parametric representation involves hyperbolic functions.

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