JOURNAL OF DIFFERENTIAL EQUATIONS **135**, 16–40 (1997)
ARTICLE NO. DE963171

Integrability of a System of N Electrons Subjected to Coulombian Interactions[1]

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Received December 13, 1995; revised May 15, 1996

The Liouville integrability of a system of N repelling particles in $\mathbb{R}^n$, for a large class of potentials, is obtained by showing that the asymptotic velocities are smooth first integrals, independent, and in involution. A new proof for the existence of the asymptotic velocities is also presented. © 1997 Academic Press

1. INTRODUCTION

Newton's equations for a system of $N \geqslant 1$ charged particles with masses $m_i > 0$ and charges e_i, $i = 1, ..., N$, read

$$m_i \ddot{\mathbf{x}}_i = \alpha \sum_{j \neq i} (e_i e_j - \beta m_i m_j) \frac{\mathbf{x}_i - \mathbf{x}_j}{|\mathbf{x}_i - \mathbf{x}_j|^3}, \qquad \mathbf{x}_i \neq \mathbf{x}_j, \tag{1.1}$$

where $\mathbf{x}_i(t) \in \mathbb{R}^3$ is the position vector of the ith particle at time t and α and β are positive constants.

The purpose of this work is to show that, if the charges are all of the same sign and such that

$$e_i e_j - \beta m_i m_j > 0, \qquad i, j = 1, ..., N, \tag{1.2}$$

as, for instance, is the case of an isolated system of N electrons, then the system is analitically integrable in the sense of Liouville. In fact we prove that there exist $3N$ analytic functions $F_h: X \times \mathbb{R}^{3N} \to \mathbb{R}$, $h = 1, ..., 3N$, $X = \{(\mathbf{x}_1, ..., \mathbf{x}_N) \in \mathbb{R}^{3N} \mid \mathbf{x}_i \neq \mathbf{x}_j,\ i \neq j\}$ the configuration space, such that

[1] This research was partially supported by C.N.R. (Italy) and Fapesp (Brasil) proc.n.90/3918-5.

0022-0396/97 $18.00

(i) dF_h are linearly independent at each point of $X \times \mathbb{R}^{3N}$

(ii) $\{F_h, H\} = 0$, $h = 1, ..., 3N$,

(iii) $\{F_h, F_k\} = 0$, $h, k = 1 \cdots, 3N$,

where $\{\ ,\ \}$ denotes the standard Poisson bracket and H: $X \times \mathbb{R}^{3N} \to \mathbb{R}$ is the Hamiltonian function corresponding to Eqs. (1.1):

$$H = \frac{1}{2} \sum_i \frac{\mathbf{y}_i \cdot \mathbf{y}_i}{m_i} + \alpha \sum_{j>i} \frac{(e_i e_j - \beta m_i m_j)}{|\mathbf{x}_i - \mathbf{x}_j|}. \tag{1.3}$$

Here . denote the inner product in $\mathbb{R}^3$ and $|\ |$ the norm.

This result is a particular case of the following general theorem that we prove below. Given an integer $K \geqslant 1$ and a number $b \geqslant 0$ we set $\mathbb{R}^K_b = \{r = (r_1, ..., r_K) \mid r_k > b\}$ and if $r \in \mathbb{R}^K$ we denote by $r^{h,k} \in \mathbb{R}^{K-2}$ the vector obtained by r by deleting r_h and r_k.

THEOREM 1.1. *Let* V: $\mathbb{R}_0^{N(N-1)/2} \to \mathbb{R}$ *be a function of class* C^δ, $\delta = l + 1$ $(l \geqslant 1)$, $\delta = \infty$, $\delta = \omega$.

Assume that V *is positive and satisfies*

$$h_1\text{-} \lim_{r_h \to 0} V(r) = \infty \textit{ uniformly for } r_j > 0, \qquad j \neq h,$$

$$h = 1, ..., \frac{N(N-1)}{2},$$

$$h_2 - \frac{\partial V}{\partial r_h}(r) < 0, \qquad h = 1, ..., \frac{N(N-1)}{2},$$

$$h_3\text{-} \lim_{\substack{r_h \to \infty \\ r_k/r_h \to \infty}} \frac{(\partial V/\partial r_k)(r)}{(\partial V/\partial r_h)(r)} = 0, \qquad h, k = 1, ..., \frac{N(N-1)}{2}, \qquad h \neq k,$$

uniformly for $r^{h,k} \in \mathbb{R}_b^{(N(N-1)/2)-2}$, $b > 0$.

h_4*—there exists a number* $\beta > 0$ *such that each one of the functions*

$$(\theta, r) \in (0, \infty) \times \mathbb{R}_0^{N(N-1)/2} \mapsto \theta^{-1/\beta} \frac{\partial V}{\partial r_h} (\theta^{-1/\beta} r_1, ..., \theta^{-1/\beta} r_{N(N-1)/2}) \in \mathbb{R},$$

$$h = 1, ..., \frac{N(N-1)}{2}$$

can be extended to $[0, \infty) \times \mathbb{R}_0^{N(N-1)/2}$ *as a* C^δ $(\delta = l, \delta = \infty, \delta = \omega)$ *function of* (θ, r).

Then the Hamiltonian system defined by the Hamiltonian $H\colon X\times\mathbb{R}^{nN}\to\mathbb{R}$

$$H=\frac{1}{2}\sum_i\frac{\langle \mathbf{y}_i, \mathbf{y}_i\rangle}{m_i}+V(\rho_{12}, \rho_{13}, ..., \rho_{(N-1), N}), \tag{1.4}$$

$\rho_{ij}=|\mathbf{x}_i-\mathbf{x}_j|$ *where* $\mathbf{x}_i, \mathbf{y}_i\in\mathbb{R}^n$ *and* $X=\{(\mathbf{x}_1, ..., \mathbf{x}_N)\mid\rho_{ij}>0\}$, *is* C^δ *integrable in the sense of Liouville.*

The Hamiltonian system defined by (1.4) is equivalent to the N second order equations

$$m_i\ddot{\mathbf{x}}_i=-\operatorname*{grad}_i V(\rho_{12}, ..., \rho_{(N-1), N}), \tag{1.5}$$

which coincide with Eq. (1.1) for $n=3$, and

$$V(\rho_{12}, ..., \rho_{(N-1), N})=\alpha\sum_{j>i}\frac{e_ie_j-\beta m_im_j}{\rho_{ij}}. \tag{1.6}$$

This is a special case of a potential (energy)

$$V=\left(\sum_h\frac{C_h}{r_h^\alpha}\right)^\gamma, \qquad \alpha>0, \quad \gamma>0, \quad C_h>0 \tag{1.7}$$

which satisfies h_1-h_3 and h_4 for $\beta=\gamma\alpha$. The expression (1.7) includes both short range potential $\alpha\gamma>1$ decaying faster than $1/r$ and long range potential $\alpha\gamma<1$ decaying slower than $1/r$. It is easy to check that many other kinds of short range potentials satisfy the assumptions of Theorem 1.1. For instance the very short range potential

$$V=\sum_h C_h\frac{1}{r_h}e^{-c_hr_h}, \qquad C_h, \quad c_h>0 \tag{1.8}$$

is included.

The essential assumption in Theorem 1.1 is h_2 which implies that the interaction between any two particles m_i, m_j, $i\neq j$, of the system, is repelling. In fact, as was proved by Vaserstein [V] and Galperin [G1], this fact, together with the assumption that V is bounded below, implies the existence of the asymptotic velocities (AV)

$$\lim_{t\to\pm\infty}\mathbf{x}_i(t)=\mathbf{v}_i^\pm. \tag{1.9}$$

The nN components of $m_1\mathbf{v}_1^\pm$, ...$m_N\mathbf{v}_N^\pm$ are clearly constants of the motion and therefore it is natural to enquire if they can be identified with the nN first integrals F_h independent and in involution required for Liouville integrability.

The idea of using asymptotic velocities as first integrals is not new and was recently advanced by Gutkin [GU]. He considered a special class of unidimensional systems and observed that if the potential satisfies a certain cone condition which includes non periodic Toda-lattice and Calogero–Marchioro potential [CM] then asymptotic velocities exist and the system should be integrable. The main point is then the smoothness of the asymptotic velocities with respect to the initial conditions. In fact there exist counterexamples where in spite of the fact that the (cone) potential is C^∞ or even analytic, asymptotic velocities are discontinuous [GZ1].

For the case of non periodic Toda-lattice and Calogero–Marchioro potential, the analytic smoothness of the asymptotic velocities follows from the earlier work of Moser [MO] that showed these systems are integrable by the method of isospectral deformation. Motivated by the work of Gutkin several papers adressed the questions of the smoothness of AV in the context of cone potential.

Oliva and Castilla [OC] Moauro, Negrini, Oliva [MNO] used a geometric method based an a compactification of the phase space, Gorni and Zampieri [GZ1][GZ2][GZ3] used a more analytic approach and also discussed the question of the existence of a scattering operator; see also [H] for related work. For a system of interacting particles the potential cannot satisfy the cone condition of Gutkin unless the particles are constrained to a line and their order cannot change. Therefore the smoothness of AV in the context of the present work requires some new elements. One main point is Theorem 3.1 where we prove that under the assumption h_3 the asymptotic velocities are all distinct and therefore no cluster of particles can form. Another relevant point is the role of the assumption h_4 which, as appears from the proof of Theorem 1.1, is really essential for the smoothness of AV. The plan of the paper is as follows. In Section 2 we present a proof of the existence of AV, in Section 3 we show that the AV are all different under a certain condition which includes h_3. Finally in Section 4 we complete the proof of Theorem 1.1 by using a compactification procedure that allows for the application of classical results from the theory of invariant manifolds.

2. EXISTENCE OF THE ASYMPTOTIC VELOCITIES

In this paragraph we quote a result on the existence of the asymptotic velocities in a setting which is largely sufficient for the proof of Theorem 1.1. The existence of the asymptotic velocities under a very general axiom of repulsivity has been already proved in the work of Vaserstein [V] and Galperin [G1]. Nevertheless in the Appendix we include a proof for making the paper self contained and also because our proof is different from the one in [V] and [G1] and is perhaps more in the spirit of Mechanics.

THEOREM 2.1. *Let* $\mathbf{x}_i(\cdot) \in C^2(\mathbb{R}, \mathbb{R}^n)$; $\mathbf{f}_{ij}(\cdot) \in C^0(\mathbb{R}, \mathbb{R}^n)$, $i, j = 1, ..., N$, $\mathbf{f}_{ii}(\cdot) = 0$ *and let* m_i, $i = 1, ..., N$ *be positive numbers. Assume that* $\mathbf{x}_i(t) \neq \mathbf{x}_j(t)$, *for* $i \neq j$ *and* $t \in \mathbb{R}$.

Assume also

$$\text{(i)} \quad m_i \ddot{\mathbf{x}}_i = \sum_j \mathbf{f}_{ij}$$

$$\text{(ii)} \quad \mathbf{f}_{ij} = -\mathbf{f}_{ji}$$

$$\text{(iii)} \quad \mathbf{f}_{ij} \cdot (\mathbf{x}_i - \mathbf{x}_j) = |\mathbf{f}_{ij}| \, |\mathbf{x}_i - \mathbf{x}_j|.$$

Assume moreover that all the first derivatives $\dot{\mathbf{x}}_i$ *are bounded in* $\mathbb{R}$. *Then there exist vectors* $\mathbf{v}_i^{\pm} \in \mathbb{R}^n$ *such that*

$$\lim_{t \to \pm\infty} \dot{\mathbf{x}}_i(t) = \mathbf{v}_i^{\pm} \tag{2.1}$$

3. PARTICLES CANNOT CLUSTER

In this paragraph we give a sufficient condition ensuring that no two particles have the same asymptotic velocities. We set $\rho_{ij}(t) = |\mathbf{x}_i(t) - \mathbf{x}_j(t)|$.

THEOREM 3.1. *Let the functions* $\mathbf{x}_i(\cdot)$; $\mathbf{f}_{ij}(\cdot)$, $i, j = 1, ..., N$ *be as in Theorem* 2.1 *and assume moreover that*

$$\text{(iv)} \quad |\mathbf{f}_{ij}(t)| \, \rho_{ij}(t) > 0, \; i \neq j,$$

$$\text{(v)} \quad \begin{cases} \liminf_{t \to \infty} |\mathbf{f}_{ij}(t)| > 0, \text{ whenever } i \neq j \text{ are such that} \\ \qquad \limsup_{t \to \infty} \rho_{ij}(t) < \infty \\ \liminf_{t \to -\infty} |\mathbf{f}_{ij}(t)| > 0, \text{ whenever } i \neq j \text{ are such that} \\ \qquad \limsup_{t \to -\infty} \rho_{ij}(t) < \infty \end{cases}$$

$$\text{(vi)} \quad \lim_{t \to \infty} \frac{|\mathbf{f}_{ij}(t)|}{|\mathbf{f}_{hk}(t)|} = 0 \text{ whenever } i \neq j \text{ and } h \neq k \text{ are}$$

$$\text{such that } \lim_{t \to \infty} \frac{\rho_{ij}(t)}{\rho_{hk}(t)} = \infty.$$

Then the asymptotic velocities $\mathbf{v}_i^{\pm}$, $i = 1, ..., N$ *satisfy*

$$\mathbf{v}_i^{\pm} \neq \mathbf{v}_j^{\pm}, \qquad i \neq j. \tag{3.1}$$

Proof. We only prove (3.1) with the + sign and drop the superscript. The other case is proved in the same way. Let $I_0 \subset \{1, ..., N\}$ be a maximal set such that $\mathbf{v}_i = \mathbf{v}_j$, $i, j \in I_0$ and let $m_0 = \sum_{i \in I_0} m_i$. Define

$$\mathbf{x}_0(\cdot) = \frac{1}{m_0} \sum_{i \in I_0} m_i \mathbf{x}_i(\cdot), \tag{3.2}$$

$$\mathbf{z}_i(\cdot) = \mathbf{x}_i(\cdot) - \mathbf{x}_0(\cdot). \tag{3.3}$$

$\mathbf{x}_0(t)$ is the center of mass of the masses m_i, $i \in I_0$ and $\mathbf{z}_i(\cdot)$ describes the motion of m_i in a frame attached to $\mathbf{x}_0(t)$.

We set

$$\begin{aligned} r_i &= |\mathbf{z}_i|, \qquad && i \in I_0, \\ \mathbf{u}_i &= \frac{\mathbf{z}_i}{|\mathbf{z}_i|}, \qquad && i \in I_0, \quad r_i \neq 0. \end{aligned} \tag{3.4}$$

Then, for any t such that $r_i > 0$,

$$\ddot{\mathbf{z}}_i = \ddot{r}_i \mathbf{u}_i + \frac{1}{r_i} (r_i^2 \dot{\mathbf{u}}_i)^{\cdot}.$$

From this and the identity

$$(r_i^2 \dot{\mathbf{u}}_i)^{\cdot} \cdot \mathbf{u}_i = -r_i^2 \, |\dot{\mathbf{u}}_i|^2,$$

it follows

$$\ddot{\mathbf{z}}_i \cdot \mathbf{u}_i = \ddot{r}_i - r_i \, |\dot{\mathbf{u}}_i|^2,$$

which implies

$$\ddot{r}_i \geqslant \ddot{\mathbf{z}}_i \cdot \mathbf{u}_i \tag{3.5}$$

for any t such that $r_i > 0$.

Let $r(t)$ be defined by

$$r(t) = \max_{i \in I_0} r_i(t). \tag{3.6}$$

Assumption (iv) implies that, unless I_0 is a singleton,

$$r(t) > 0, \qquad t \in \mathbb{R}. \tag{3.7}$$

From (i), (ii) and (3.2), (3.3) it follows

$$\ddot{\mathbf{x}}_0 = \frac{1}{m_0} \sum_{i \in I_0} \sum_{j \notin I_0} \mathbf{f}_{ij},$$

$$\ddot{\mathbf{z}}_i = \frac{1}{m_i} \left(\sum_{j \in I_0} \mathbf{f}_{ij} + \sum_{j \notin I_0} \mathbf{f}_{ij} \right) - \ddot{\mathbf{x}}_0$$

$$= \frac{1}{m_i} \left(\sum_{j \in I_0} \mathbf{f}_{ij} + \sum_{j \notin I_0} \mathbf{f}_{ij} \right) - \frac{1}{m_0} \sum_{h \in I_0} \sum_{j \notin I_0} \mathbf{f}_{hj}, \tag{3.8}$$

and therefore

$$\ddot{\mathbf{z}}_i \cdot \mathbf{u}_i \geqslant \varphi + \psi \tag{3.9}$$

where we have set

$$\varphi = \frac{1}{m_i} \sum_{j \in I_0} \mathbf{f}_{ij} \cdot \mathbf{u}_i, \tag{3.10}$$

$$\psi = \frac{1}{m_i} \sum_{j \notin I_0} \mathbf{f}_{ij} \cdot \mathbf{u}_i - \frac{1}{m_0} \sum_{i \in I_0} \sum_{j \notin I_0} \mathbf{f}_{ij} \cdot \mathbf{u}_i. \tag{3.11}$$

By definition of I_0, $h, k \in I_0$ implies $\mathbf{v}_h = \mathbf{v}_k$ and therefore that given $\varepsilon > 0$ there is t_ε such that

$$\rho_{hk}(t) < \varepsilon t, \qquad t > t_\varepsilon, \qquad h, k \in I_0. \tag{3.12}$$

On the other hand, if $\alpha \in I_0$ and $\beta \neq I_0$ we have $\mathbf{v}_\alpha \neq \mathbf{v}_\beta$ which implies the existence of constants $c > 0$, $\bar{t}$ such that

$$\rho_{\alpha\beta}(t) > ct, \qquad t > \bar{t}. \tag{3.13}$$

It follows that

$$\lim_{t \to \infty} \frac{\rho_{\alpha\beta}(t)}{\rho_{hk}(t)} = \infty \qquad \forall h, k, \alpha \in I_0, \qquad \beta \notin I_0. \tag{3.14}$$

From this and assumption (vi) it follows that

$$\lim_{t \to \infty} \frac{\psi}{m_h |\mathbf{f}_{hk}|} = 0 \qquad \forall h, k \in I_0. \tag{3.15}$$

Now choose i such that $r_i(t) = r(t)$.

Eq. (3.13) and assumptions (i) and (iv) imply

$$\varphi = \frac{1}{m_i} \sum_{j \in I_0} |f_{ij}| \, \mathbf{u}_{ij} \cdot \mathbf{u}_i > 0. \tag{3.16}$$

Moreover the definition of $\mathbf{x}_0$ implies $\sum_{h \in I_0} m_h \mathbf{z}_h \cdot \mathbf{u}_i = 0$ and therefore since $m_i \mathbf{z}_i \cdot \mathbf{u}_i = m_i r > 0$ it follows the existence of $j \in I_0$ such that $m_j \mathbf{z}_j \cdot \mathbf{u}_i < 0$. This implies

$$\mathbf{u}_{ij} \cdot \mathbf{u}_i > \frac{1}{\sqrt{2}}$$

and therefore from (3.16) we get

$$\varphi > \frac{1}{\sqrt{2m_i}} |\mathbf{f}_{ij}|. \tag{3.17}$$

Equations (3.15), (3.17) imply that, for t sufficiently large and provided that $\mathbf{r}_i(t) = \mathbf{r}(t)$,

$$\varphi < 2\,|\psi|,$$

and therefore, using also (3.5)

$$\ddot{r}_i(t) \geqslant \ddot{\mathbf{z}}_i \cdot \mathbf{u}_i = \varphi + \psi > \frac{\varphi}{2} > 0, \tag{3.18}$$

for t larger than some number τ.

The definition (3.6) of $r(t)$ and the continuity of the r_i's imply that for some $|s|$ smaller than some $s_o > 0$,

$$r_i(t) < r(t) \Rightarrow r_i(t+s) < r(t+s), \tag{3.19}$$

and therefore that

$$r(t+s) = \max_{r_i(t) = r(t)} r_i(t+s), \qquad |s| < s_0 \tag{3.20}$$

or equivalently

$$r(t+s) = r(t) + \max_{r_i(t) = r(t)} \dot{r}_i(t + \hat{s}(s))\,s \tag{3.21}$$

where, here and in the following, $\hat{s}(s)$ is a suitable point in the interval $(0, s)$.

From (3.21) it follows

$$\begin{cases} \lim_{s\to 0^+} \dfrac{r(t+s)-r(t)}{s} = \max_{r_i(t)=r(t)} \dot r_i(t) \\ \lim_{s\to 0^-} \dfrac{r(t+s)-r(t)}{s} = \min_{r_i(t)=r(t)} \dot r_i(t), \end{cases} \tag{3.22}$$

that is, $r(t)$ has at each $t\in\mathbb{R}$ left and right derivatives that we denote by

$$\begin{cases} \dot r^+(t) = \max_{r_i(t)=r(t)} \dot r_i(t), \\ \dot r^-(t) = \max_{r_i(t)=r(t)} \dot r_i(t). \end{cases} \tag{3.23}$$

From (3.23) it follows

$$\dot r^+(t+s) = \max_{r_i(t+s)=r(t+s)} \dot r_i(t+s). \tag{3.24}$$

On the other hand, assuming $s_0>0$ has been chosen sufficiently small, we have for $0\leqslant s<s_0$

$$r(t+s) = r(t) + \dot r^+(t)\, s + \tfrac{1}{2} \max_{i\in\mathfrak{J}^+(t)} \ddot r_i(t+\hat s(s))s^2 \tag{3.25}$$

where $\mathfrak{J}^\pm(t) = \{i\in I_o \mid r_i(t)=r(t),\ \dot r_i(t)=\dot r^\pm(t)\}$.

It follows that a necessary condition in order that $r_i(t+s)=r(t+s)$ is that

$$i\in\tilde{\mathfrak{J}}^+(t) =: \{j\in\mathfrak{J}^+(t) \mid \ddot r_j(t) = \max_{h\in\mathfrak{J}^+(t)} \ddot r_h(t)\}. \tag{3.26}$$

Therefore from (3.24) we have that for $0\leqslant s<s_0$

$$\min_{i\in\tilde{\mathfrak{J}}^+(t)} \dot r_i(t+s) \leqslant \dot r^+(t+s) \leqslant \max_{i\in\tilde{\mathfrak{J}}^+(t)} \dot r_i(t+s) \tag{3.27}$$

which implies

$$\min_{i\in\tilde{\mathfrak{J}}^+(t)} \ddot r_i(t+\tilde s(s)) \leqslant \frac{\dot r^+(t+s)-\dot r^+(t)}{s} \leqslant \max_{i\in\tilde{\mathfrak{J}}^+(t)} \ddot r_i(t+\hat s(s)) \tag{3.28}$$

and therefore that

$$\ddot r^+(t) =: \lim_{s\to 0^+} \frac{\dot r^+(t+s)-\dot r^+(t)}{s} = \max_{i\in\tilde{\mathfrak{J}}^+(t)} \ddot r_i(t). \tag{3.29}$$

A similar argument shows that for $-s_0 < s \leqslant 0$

$$\min_{i \in \tilde{\mathfrak{J}}^-(t)} \dot{r}_i^+(t+s) \leqslant \dot{r}^+(t+s) \leqslant \max_{i \in \tilde{\mathfrak{J}}^-(t)} \dot{r}_i(t+s) \tag{3.30}$$

where $\tilde{\mathfrak{J}}^-(t) = \{ i \in \mathfrak{J}^-(t) \mid \ddot{r}_i(t) = \max_{j \in \tilde{\mathfrak{J}}^-(t)} \ddot{r}_j(t) \}$. From equation (3.30) it follows

$$\lim_{s \to 0^-} \dot{r}^+(t+s) = \dot{r}^-(t), \tag{3.31}$$

and moreover that

$$\ddot{r}^-(t) =: \lim_{s \to 0^-} \frac{\dot{r}^+(t+s) - \dot{r}^-(t)}{s} = \max_{i \in \tilde{\mathfrak{J}}^-(t)} \ddot{r}_i(t). \tag{3.32}$$

LEMMA 3.2. *There is $\tau \in \mathbb{R}$ such that for any given $t > \tau$, there is a $\delta > 0$ such that $t - \delta < \alpha \leqslant t \leqslant \beta < t + \delta$ implies*

$$\dot{r}^+(\beta) - \dot{r}^+(\alpha) \geqslant -C((\beta - \alpha))(\beta - \alpha),$$

where $C(s)$, $C(0) = 0$, is a continuous function which is independent of t for t in compact sets.

Proof. From the above discussion and in particular from (3.29), (3.32) we have

$$\begin{cases} \dot{r}^+(t+s) = \dot{r}^+(t) + \ddot{r}^+(t)\, s + o(s), & 0 \leqslant s < s_0 \\ \dot{r}^+(t+s) = \dot{r}^-(t) + \ddot{r}^-(t)\, s + o(s), & -s_0 < s < 0 \end{cases} \tag{3.33}$$

where $s_0 > 0$ is a sufficiently small number and $o(s)$ satisfies a bound of the form

$$|o(s)| \leqslant C(s) s \tag{3.34}$$

where $C(s)$, $C(0) = 0$, is a continuous function independent of t for t in compact sets. This follows from the fact that the r_i's are C^2 functions and therefore $\ddot{r}_i(t+s) - \ddot{r}_i(t)$ has a bound independent of i and independent of s, t for s small and t in compact sets.

From (3.23) it follows

$$\dot{r}^+(t) - \dot{r}^-(t) \geqslant 0. \tag{3.35}$$

If this inequality holds with the sign of strict inequality, then equation (3.33) implies that if $\delta > 0$ is sufficiently small we have

$$\dot{r}^+(\beta) - \dot{r}^+(\alpha) \geqslant 0 \tag{3.36}$$

for α, β as in the statement of the lemma.

Therefore we can assume that $\dot{r}^+(\beta) - \dot{r}^+(\alpha) = 0$. This implies that $\mathfrak{J}^-(t) = \mathfrak{J}^+(t)$ and therefore we see from (3.29), (3.32) that $\ddot{r}^-(t) = \ddot{r}^+(t)$ that is $\dot{r}^+$ is differentiable at t and equation (3.33) imply

$$\dot{r}^+(\beta) - \dot{r}^+(\alpha) \geqslant \ddot{r}^+(t)(\beta - \alpha) + o(\beta - t) + o(t - \alpha). \tag{3.37}$$

For each $t \in \mathbb{R}$, $\ddot{r}^+(t)$ coincides with one of the $\ddot{r}_i(t)$ with i such that $r_i(t) = r(t)$. Therefore if t is sufficiently large we see from (3.18) that $\ddot{r}^+(t) \geqslant 0$. This, (3.35) and (3.34) imply

$$\dot{r}^+(\beta) - \dot{r}^+(\alpha) \geqslant -C((\beta - \alpha))(\beta - \alpha) \tag{3.38}$$

that concludes the proof.

LEMMA 3.3. *$\dot{r}^+$ is nondecreasing.*

Proof. We shall prove that given any two numbers $\tau < a < b$ then

$$\dot{r}^+(b) \geqslant \dot{r}^+(a).$$

Given $N > 0$, we can associate to each $t \in [a, b]$ an interval $(t - \delta, t + \delta)$ with $\delta < 1/N$ and the properties described in Lemma 3.2. So we have an open covering of $[a, b]$ and therefore we can extract a finite covering.

Let $I_i = (t_i - \delta_i, t_i + \delta_i)$, $i = 1, ..., M$, be this finite covering. Let i_1 be such that

$$t_{i_1} + \delta_{i_1} = \max\{t_i + \delta_i \mid a \in I_i\}. \tag{3.39}$$

Let $t_{i_{k+1}} + \delta_{i_{k+1}} = \max\{t_i + \delta_i \mid t_{i_k} + \delta_{i_k} \in I_{i_{k+1}}\}$.
Then we have

$$t_{i_{k+1}} > t_{i_k} \tag{3.40}$$

because otherwise we have a contradiction with the definition of t_{ik}. It follows that we can construct a finite sequence

$$a \leqslant t_{i_1} < t_{i_2} < \cdots < t_{i_M} \leqslant b$$

such that $I_{i_k} \cap I_{i_{k+1}} \neq \emptyset$. This implies that we can choose points $\alpha_k \leqslant t_{i_k} \leqslant \beta_k$, with $\alpha_k, \beta_k \in I_{i_k}$ and $\alpha_{k+1} = \beta_k$, $\alpha_1 = a$, $\alpha_M = b$. Therefore we have, using also Lemma 3.2:

$$\begin{aligned} \dot{r}^+(b) - \dot{r}^+(a) &= \sum_{k=1}^{M} \dot{r}^+(\beta_k) - \dot{r}^+(\alpha_k) \\ &\geqslant -\sum_{k=1}^{M} C((\beta_k - \alpha_k))(\beta_k - \alpha_k) \\ &> -C\left(\frac{1}{N}\right) \sum_{k=1}^{M} (\beta_\alpha - \alpha_k) \\ &= -C\left(\frac{1}{N}\right)(b-a) \end{aligned} \tag{3.41}$$

which implies

$$\dot{r}^+(b) - \dot{r}^+(a) \geqslant 0$$

because (3.41) holds for any $N > 0$. ∎

From Lemma 3.3 and a classical theorem on real functions (see for instance [HS] theorem (18.14)), we have

$$\dot{r}^+(t) \geqslant \dot{r}^+(\tau) + \int_\tau^t \ddot{r}^+(s)\, ds > \dot{r}^+(\tau) + \tfrac{1}{2} \int_\tau^t \varphi(s)\, ds. \tag{3.42}$$

If $\dot{r}^+(\tau) \geqslant 0$, this estimate and (3.17) imply

$$\lim_{t \to \infty} \dot{r}^+(t) > 0 \tag{3.43}$$

in contradiction with the definition of I_0. If instead $\dot{r}^+(\tau) < 0$, then from (3.17) and assumption (v) it follows that $\varphi(s) > C > 0$, $s > \tau$ provided that $r(s) \leqslant r(\tau)$. Therefore (3.42) implies

$$\dot{r}^+(t) > \dot{r}^+(\tau) + C(t - \tau), \qquad t > \tau, \quad r(t) \leqslant r(\tau), \tag{3.44}$$

and so the existence of $\bar{t} > \tau$ such that $\dot{r}^+(\bar{t}) \geqslant 0$. Thus we are back to the previous case and we have again (3.43) contradicting the definition of I_0.

Remark 3.4. For later reference we note that (3.1) implies

$$\lim_{t \to \pm\infty} \dot{\rho}_{ij}(t) = \lim_{t \to \pm\infty} \mathbf{u}_{ij}(t) \cdot (\dot{\mathbf{x}}_i(t) - \dot{\mathbf{x}}_j(t)) = |\mathbf{v}_i^\pm - \mathbf{v}_j^\pm| > 0. \tag{3.45}$$

4. PROOF OF THEOREM 1.1

If $\mathbf{x}_i(\cdot)$, $i=1, ..., N$ is the solution to Eq. (1.5) satisfying $\mathbf{x}_i(t_0)=\mathbf{a}_i$, $\dot{\mathbf{x}}_i(t_0)=\mathbf{b}_i$ and $\mathbf{f}_{ij}(t)$, $i \neq j$ are defined by

$$\mathbf{f}_{ij}(t) = -\frac{\partial}{\partial_{r_{h(i,j)}}} V(|\mathbf{x}_1(t)-\mathbf{x}_2(t)|, ..., |\mathbf{x}_{N-1}(t)-\mathbf{x}_N(t)|) \times \frac{\mathbf{x}_i(t)-\mathbf{x}_j(t)}{|\mathbf{x}_i(t)-\mathbf{x}_j(t)|} \tag{4.1}$$

then, from h_2 and $V>0$ it follows that Theorem 2.1 can be applied yielding the existence of the asymptotic velocities. Therefore there is a map $(\mathbf{a}, \mathbf{b}) \rightarrow \mathbf{v}^{\pm}((\mathbf{a}, \mathbf{b}))$ (here we use the notation $\mathbf{a}=(\mathbf{a}_1, ..., \mathbf{a}_N) \in \mathbb{R}^{nN}$ and similarly for $\mathbf{b}$ and $\mathbf{v}^+$) from $X \times \mathbb{R}^{nN}$ into $\mathbb{R}^{nN}$.

It is easy to check that h_1, h_2, h_3 imply that $\mathbf{x}_i$, $\mathbf{f}_{ij}$ also satisfy the assumptions of Theorem 3.1 which shows that the image of $\mathbf{v}^{\pm}(\cdot)$ is in fact contained in the subset $X \subset \mathbb{R}^{nN}$ defined in the Introduction (see also Remark 3.4.).

THEOREM 4.1. *Under the assumptions of Theorem* 1.1 *the maps* $(\mathbf{a}, \mathbf{b}) \rightarrow \mathbf{v}^{\pm}(\mathbf{a}, \mathbf{b})$ *are* C^{δ}, $(\delta = l \geqslant 1, \delta=\infty, \delta=\omega)$. *Moreover these maps are onto the subset* $X \subset \mathbb{R}^{nN}$ *defined in the introduction.*

Proof. We rewrite Eqs. (1.5) in the form

$$\begin{cases} \dot{t}=1 \\ \dot{\mathbf{x}}_i = \mathbf{v}_i \\ \dot{\mathbf{v}}_i = -\dfrac{1}{m_i} \sum_{j \neq i} \dfrac{\partial V}{\partial r_{h(i,j)}} (|\mathbf{x}_1-\mathbf{x}_0|, ..., |\mathbf{x}_{N-1}-\mathbf{x}_N|) \dfrac{\mathbf{x}_i-\mathbf{x}_j}{|\mathbf{x}_i-\mathbf{x}_j|} \end{cases} \tag{4.2}$$

that generates a dynamical system on the enlarged phase space

$$\mathbb{R} \times X \times \mathbb{R}^{nN}.$$

The map defined by

$$\begin{cases} \theta = t^{-\beta}, \\ \mathbf{z}_i = t^{-1}\mathbf{x}_i \\ \mathbf{v}_i = \mathbf{v}_i, \end{cases} \tag{4.3}$$

is a diffeomorphism of $\mathbb{R}^+ \times X \times \mathbb{R}^{nN}$ that transforms (4.3) into the equivalent equations

$$\begin{cases} \dot{\theta} = -\beta\theta^{(1/\beta)+1} \\ \dot{\mathbf{z}}_i = -\theta^{1/\beta}(\mathbf{z}_i - \mathbf{v}_i) \\ \dot{\mathbf{v}}_i = -\dfrac{1}{m_i}\sum_{j\neq i}\dfrac{\partial V}{\partial r_{h(i,j)}}(\theta^{-1/\beta}\,|\mathbf{z}_1 - \mathbf{z}_2|, ..., \theta^{-1/\beta}\,|\mathbf{z}_{N-1} - \mathbf{z}_N|)\,\dfrac{\mathbf{z}_i - \mathbf{z}_j}{|\mathbf{z}_i - \mathbf{z}_j|}. \end{cases} \tag{4.4}$$

If $t = t(\tau)$, $t(\tau) \to \infty$ as $\tau \to \infty$, is a rescaling of time then clearly

$$\mathbf{v}^+((\mathbf{a}, \mathbf{b})) = \lim_{t\to\infty} \mathbf{v}(t, \mathbf{a}, \mathbf{b},) = \lim_{\tau\to\infty} \mathbf{v}(t(\tau), \mathbf{a}, \mathbf{b}).$$

Therefore we can replace system (4.4) by

$$\begin{cases} \theta' = -\beta\theta, \\ \mathbf{z}_i' = -\mathbf{z}_i + \mathbf{v}_i, \\ \mathbf{v}_i' = -\dfrac{1}{m_i}\sum_{j\neq i}\theta^{-1/\beta}\,\dfrac{\partial V}{\partial r_{h(i,j)}}(\theta^{-1/\beta}\,|\mathbf{z}_1 - \mathbf{z}_2|, ..., \theta^{-1/\beta}\,|\mathbf{z}_{N-1} - \mathbf{z}_N|)\,\dfrac{\mathbf{z}_i - \mathbf{z}_j}{|\mathbf{z}_i - \mathbf{z}_j|}. \end{cases} \tag{4.5}$$

where primes denote derivates with respect to the new time $\tau = \ln t$. The advantage of this system with respect to (4.2) is that orbits of (4.4) are bounded in the future. In fact the existence of the limit velocities implies

$$\begin{cases} \lim_{t\to\infty} t^{-\beta} = 0, \\ \lim_{\tau\to\infty} \mathbf{z}_i(\tau) = \lim_{t\to\infty} \dfrac{\mathbf{x}_i(t)}{t_i} = \mathbf{v}_i^+, \\ \lim_{t\to\infty} \mathbf{v}_i(t) = \mathbf{v}_i^+. \end{cases} \tag{4.6}$$

The motivation for introducing the transformation (4.3) is as follows: Eq. $(4.3)_2$ is suggested by $(4.6)_2$ and requires working in the enlarged phase space. To explain the role of transformation $t = \theta^{-1/\beta}$ we observe that the existence of the asymptotic velocities implies that for t large

$$\frac{\partial V}{\partial r_h}(|\mathbf{x}_1(t) - \mathbf{x}_2(t)|, ...) \cong \frac{\partial V}{\partial r_h}(t\,|\mathbf{v}_1^+ - \mathbf{v}_2^+|, ...).$$

Therefore the behaviour of the potential at $t = \infty$ determines the behaviour at 0 of the function

$$W\left(\frac{1}{t}\right) = t\,\frac{\partial V}{\partial r_h}(t\,|\mathbf{v}_1^+ - \mathbf{v}_2^+|, ...)$$

which in the case of long range potentials (decaying at ∞ like $r^{-\alpha}$, $\alpha<1$) has a singularity at 0. This singularity can be eliminated by setting $t=\psi(\theta)$ if ψ diverges to ∞ sufficiently fast as $\theta\to 0$. But, then, the right hand side of $(4.5)_1$ must be replaced by ψ/ψ_θ which has, in fact, a vanishing derivative at $\theta=0$. When this is the case, same pathological situations can arise and the conclusions of Theorem 4.1 may not hold. This is the motivation for assumption h_4.

The set K of critical points of (4.5) is defined by

$$K=\{(\theta, \mathbf{z}, \mathbf{v}) \mid \theta=0, \mathbf{z}_i=\mathbf{v}_i\}, \tag{4.7}$$

This follows from (4.5) and from h_2 and $V=0$ which imply that the right hand side of $(4.5)_3$ vanishes at $\theta=0$.

The linearization of (4.5) at $(0, \mathbf{c}, \mathbf{c})\in K$ reads

$$\begin{cases} \hat{\theta}' = -\beta\hat{\theta}, \\ \hat{\mathbf{z}}_i' = -\hat{\mathbf{z}}_i+\hat{\mathbf{v}}_i, \\ \hat{\mathbf{v}}_i' = \hat{\theta}\boldsymbol{\Phi}_i(\mathbf{c}), \end{cases} \tag{4.8}$$

where variations are denoted by ˆand

$$\boldsymbol{\Phi}_i(\mathbf{c}) = -\lim_{\theta\to 0}\frac{1}{m_i\theta}\sum_{j\neq i}\theta^{-1/\beta}\frac{\partial V}{\partial r_{h(i,j)}}(\theta^{-1/\beta}|\mathbf{c}_1-\mathbf{c}_2|, ...)\frac{\mathbf{c}_i-\mathbf{c}_j}{|\mathbf{c}_i-\mathbf{c}_j|}. \tag{4.9}$$

Remark 4.2. $\Phi_i(\mathbf{c})$ is well defined because $(0, \mathbf{c}, \mathbf{c})\in K$ implies $\mathbf{c}_i\neq\mathbf{c}_j$ $i\neq j$. On the other hand we have previously observed that the assumptions h_1-h_3 yield, via Theorem 3.1, that $\mathbf{v}_i^+\neq\mathbf{v}_j^+$, $i\neq j$, that together with (4.6) imply that all asymptotic behaviours of system (4.2) correspond to points $(0, \mathbf{v}^+, \mathbf{v}^+)\in K$.

Eigenvalues λ and corresponding generalized eigenspace W^λ for the linear vector field (4.8) are as follows

$$\lambda=0, \qquad w^0=\{(\hat{\theta}, \hat{\mathbf{z}}, \hat{\mathbf{v}}) \mid \hat{\theta}=0, \hat{\mathbf{v}}=\hat{\mathbf{z}}\}, \qquad \dim W^0=nN$$

$$\lambda=-1, \qquad \begin{cases} W^{-1}=\widetilde{W}^{-1}=\{(\hat{\theta}, \hat{\mathbf{z}}, \hat{\mathbf{v}}) \mid \hat{\theta}=0, \hat{\mathbf{v}}=0\} \\ \quad \text{if} \quad \beta\neq 1, \dim W^{-1}=nN \\ W^{-1}=\widetilde{W}^{-1}\oplus\{(\hat{\theta}, \hat{\mathbf{z}}, \hat{\mathbf{v}}) \mid \hat{\mathbf{z}}=0, \hat{\mathbf{v}}=-\hat{\theta}\boldsymbol{\Phi}_i(\mathbf{c})\} \\ \quad \text{if} \quad \beta=1, \dim W^{-1}=nN+1. \end{cases}$$

If $\beta\neq 1$

$$\lambda=-\beta, \qquad W^{-\beta}=\{(\hat{\theta}, \hat{\mathbf{z}}, \hat{\mathbf{v}}) \mid \hat{\mathbf{z}}=\hat{\theta}(\beta+1)^{-1}\boldsymbol{\Phi}(\mathbf{c}),$$

$$\hat{\mathbf{v}}=-\hat{\theta}\boldsymbol{\Phi}(\mathbf{c})\}, \qquad \dim W^{-\beta}=1.$$

Therefore in any case we have that $\mathbb{R}^{2nN+1}$ can be decomposed as $W^0 \oplus W^s$ where W^s corresponds to negative eigenvalues, $W^s = \tilde{W}^{-1} \oplus W^{-\beta}$ if $\beta \neq 1$ and $W^s = W^{-1}$ if $\beta = 1$ and $\dim W^s = nN + 1$. Moreover from the characterization of W^0 and the definition (4.7) of K it follows that K coincides with the center manifold of any $(0, c, c) \in K$. The general theory of invariant manifolds then yields for any fixed $\bar{\mathbf{c}} \in X$, the existence of an open ball $B_{\bar{\mathbf{c}}} \subset K$ centered at $(0, \bar{\mathbf{c}}, \bar{\mathbf{c}})$ and of a continuous fibration $\pi: U_{\bar{\mathbf{c}}} \to B_{\bar{\mathbf{c}}}$, $\pi = (\pi^\theta, \pi^{\mathbf{z}}, \pi^{\mathbf{v}})$, $U_{\bar{\mathbf{c}}} \subset \mathbb{R} \times X \times \mathbb{R}^{nN}$ open, such that for any $(0, \mathbf{c}, \mathbf{c}) \in B_{\bar{\mathbf{c}}}$, $\pi_{\mathbf{c}}^{-1} = \pi^{-1}((0, \mathbf{c}, \mathbf{c}))$ is the local strongly stable manifold corresponding to $(0, \mathbf{c}, \mathbf{c})$. $\pi_{\mathbf{c}}^{-1}$ is a C^δ manifold tangent to W^s. The fibration π is actually a C^δ fibration. This, as shown in Lemma 4.1 below, is a simple consequence of the fact that the set K is smooth, an open subset of a subspace.

Remark 4.3. $\pi_{\mathbf{c}}^{-1}$ contains the linear variety $(0, \mathbf{c}, \mathbf{c}) + \tilde{W}^{-1}$ as follows by observing that this variety is invariant under the flow of (4.5).

LEMMA 4.4. *Consider the equation*

$$\dot{x} = f(x) \tag{4.10}$$

where $f: \Omega \to \mathbb{R}^n$, Ω an open neighborhood of $0 \in \mathbb{R}^n$, is a C^δ function and $f(0) = 0$. Assume

(i) $R_e \lambda(L) \leqslant 0$, $L =: (\partial f / \partial x)(0)$.

(ii) *there is a C^δ imbedding $\hat{x}: \mathscr{A} \to \mathbb{R}^n$, $\mathscr{A}$ an open connected neighborhood of $0 \in \mathbb{R}^k$, $1 < k < n$, $\hat{x}(0) = 0$, such that $E = \{x \mid x \in \Omega, \exists a \in \mathscr{A} : x = \hat{x}(a)\}$ is the set of critical points of* (4.10).

(iii) *The eigenvalue $\lambda = 0$ of L has multiplicity k.*

Then there is an open ball $B \subset E$ centered at $0 \in E$ and a C^δ fibration $\Pi: U \to B$, $U \subset \Omega$ an open neighborhood of $x = 0$ such that, for each $x \in B$, Π_x^{-1} is the local stable manifold of x.

Proof. For any fixed $a \in \hat{x}^{-1}(\Omega)$ the change of variable $x = y + \hat{x}(a)$ transforms (4.10) into the equivalent equation

$$\dot{y} = g(y, a) =: f(y + \hat{x}(a)),$$

and g is a C^δ function of (y, a) in a neighborhood of $(0, 0)$. Moreover $g(0, 0) = 0$, $(\partial g / \partial a)(0, 0) = L$ and $(\partial g / \partial a)(0, 0) = L(\partial \hat{x} / \partial a)(0) = 0$.

Let Z, $\dim Z = n - k$ (V, $\dim V = k$) the eigenspace of L corresponding to eigenvalues with negative real part (with zero real part). Then stable manifold theory implies the existence of a C^δ function $v: D \times \mathscr{B} \to V$, D an open ball centered at $0 \in Z$, $\mathscr{B} \subset \mathscr{A}$ an open ball centered at $0 \in \mathbb{R}^k$, such that for each $a \in \mathscr{B}$ the set $\{y = z + v(z, a)\}$ is the local stable manifold of $y = 0$. Moreover $v(0, 0) = (\partial v/\partial z)(0, 0) = (\partial v/\partial a)(0, 0) = 0$.

Going back to the x variable the set $\{y = z + v(z, a)\}$ becomes $\{x = \hat{x}(a) + z + v(z, a)\}$, the local stable manifold of $\hat{x}(a)$.

The equation

$$F(x, z, a) =: \hat{x}(a) + z + v(z, a) - x = 0, \tag{4.11}$$

defines a C^δ change of variables $x \to (z, a)$ in a neighborhood of $x = 0$. This follows from the implicit function theorem. In fact F is C^δ and $F(0, 0, 0) = 0$, $(\partial F/\partial z)(0, 0, 0) = I_Z$ the identity on Z, and $(\partial F/\partial a)(0, 0, 0) = (\partial \hat{x}/\partial a)(0)$ is a diffeomorphism.

Therefore we can solve (4.11) for a C^δ function $(z^*(x), a^*(x))$, $(z^*(0), a^*(0)) = (0, 0)$. This yields the fibration Π in the form

$$\Pi(x) = \hat{x}(a^*(x))$$

that shows Π is C^δ because a^* and $\hat{x}$ are both C^δ. ▮

Now fix $\bar{\mathbf{a}}, \bar{\mathbf{b}} \in X \times \mathbb{R}^{nN}$, set $\bar{\mathbf{c}} = \mathbf{v}^+(\bar{\mathbf{a}}, \bar{\mathbf{b}})$ and let $\bar{\mathbf{x}}(t, \mathbf{a}, \mathbf{b})$, $\bar{\mathbf{v}}(t, \mathbf{a}, \mathbf{b})$ be the solution of Eq. (4.2) for $(\mathbf{a}, \mathbf{b})$ in a neighborhood of $(\bar{\mathbf{a}}, \bar{\mathbf{b}})$. If t is sufficiently large the image of $(t, \bar{\mathbf{x}}, \bar{\mathbf{v}})$ under the diffeomorphism (4.3) is in the domain of π.

Therefore it results

$$\mathbf{v}^+((\mathbf{a}, \mathbf{b})) = \pi^v\left((\theta(t))^{-1/\beta}, \frac{\bar{\mathbf{x}}(t, \mathbf{a}, \mathbf{b})}{t}, \bar{\mathbf{v}}(t, \mathbf{a}, \mathbf{b})\right)$$

that implies $\mathbf{v}^+(\cdot)$ is C^δ. The statement about the ontoness follows from the observation that any $(0, \mathbf{c}, \mathbf{c}) \in K$ has a nontrivial stable manifold. ▮

To complete the proof of Theorem 1.1 let $M\colon \mathbb{R}^{nN} \to \mathbb{R}^{nN}$ be the operator defined by $M\mathbf{b} = \operatorname{col}(m_1 \mathbf{b}_1, ..., m_N \mathbf{b}_N)$ and set

$$F(\mathbf{x}, \mathbf{y}) = \begin{bmatrix} F_1(\mathbf{x}, \mathbf{y}) \\ \vdots \\ F_{nN}(\mathbf{x}, \mathbf{y}) \end{bmatrix} = M\mathbf{v}^+((\mathbf{x}, M^{-1}\mathbf{y})).$$

Let $(AF)(\mathbf{a}, \mathbf{p})$ be either $\{F_h, F_k\}$ for some h, k or the rank of dF at $(\mathbf{a}, \mathbf{p})$. Then, since the F_h are first integrals, we have

$$(AF)(\mathbf{a}, \mathbf{p}) = (AF)(\mathbf{a}(t), \mathbf{p}(t)), \qquad \forall t \tag{4.12}$$

where $\mathbf{a}(t)$, $M^{-1}\mathbf{p}(t)$ is the solution of (4.2) through $\mathbf{a}$, $M^{-1}\mathbf{p}$.

On the other hand, for t sufficiently large, it results

$$\begin{aligned} F^t(\mathbf{x}, \mathbf{y}) &=: F(\mathbf{a}(t)+\mathbf{x}, \mathbf{p}(t)+\mathbf{y}) \\ &= M\mathbf{v}^+((\mathbf{a}(t)+\mathbf{x}, M^{-1}(\mathbf{p}(t))+\mathbf{y})) \\ &= M\pi^{\mathbf{v}}\left((\theta(t))^{-1/\beta}, \frac{\mathbf{a}(t)+\mathbf{x}}{t}, M^{-1}(\mathbf{p}(t)+\mathbf{y})\right), \end{aligned}$$

which, taking also into account (4.6) and Remark 4.3, yields

$$\begin{aligned} \lim_{t\to\infty} F^t(\mathbf{x}, \mathbf{y}) &= M\pi^{\mathbf{v}}(0, \mathbf{v}^+(\mathbf{a}, M^{-1}\mathbf{p}), \mathbf{v}^+(\mathbf{a}, M^{-1}\mathbf{p})+M^{-1}\mathbf{y}) \\ &= M\mathbf{v}^+(\mathbf{a}, M^{-1}\mathbf{p})+\mathbf{y}, \end{aligned} \tag{4.13}$$

the convergence being in the C^1 sense. From this and Eq. (4.12) which implies

$$(AF)(\mathbf{a}, \mathbf{p}) = (AF^t)(0, 0) = (A \lim_{t\to\infty} F^t)(0, 0)$$

it follows

$$\begin{aligned} &(AF)(\mathbf{a}, \mathbf{p}) = 0 && \text{if} \quad AF = \{F_h, F_k\}, \\ &(AF)(\mathbf{a}, \mathbf{p}) = nN && \text{if} \quad AF \text{ is the rank of } aF. \end{aligned} \tag{4.14}$$

This completes the proof of Theorem 1.1.

Remark 4.5 (Asymptotic shape). Given two configurations $\mathbf{x} = (\mathbf{x}_1 \cdots \mathbf{x}_N)$, $\hat{\mathbf{x}} = (\hat{x}_1, ..., \hat{x}_N) \in X$ of the system we say that $\mathbf{x}$ and $\hat{\mathbf{x}}$ have the same shape or that are equivalent (and write $\mathbf{x} \sim \hat{\mathbf{x}}$) if there exists $\lambda > 0$ such that

$$\mathbf{x}_i - \mathbf{x}_j = \lambda(\hat{\mathbf{x}}_i - \hat{\mathbf{x}}_j), \qquad i \neq j$$

the set $X/\sim$ is called the set of the configuration shapes. From the ontoness of the map $(\mathbf{a}, \mathbf{b}) \to \mathbf{v}^+(\mathbf{a}, \mathbf{b})$ it follows that given any $\sigma \in X/\sim$ there exists $(\mathbf{a}, \mathbf{b})$ such that

$$\lim_{t\to\infty} \frac{\mathbf{x}(t, \mathbf{a}, \mathbf{b})}{t} = \mathbf{v}^+((\mathbf{a}, \mathbf{b})) \in \sigma.$$

that is: all asymptotic shapes are possible.

Remark 4.6. As we have observed in the introduction it is easy to check that short range potentials of general type satisfy the condition of regularity at ∞ expressed by assumption h_4. On the other hand this condition is satisfied also by some kind of long range potentials as the one defined by (1.7) for $\gamma\alpha < 1$. But when dealing with very long range potentials as for instance

$$V = \sum_h \frac{C_h}{\ln(1 + v_h)}, \qquad C_h > 0 \tag{4.15}$$

h_4 is not satisfied and to regularize at infinity instead of the function $\psi = \theta^{-1/\beta}$ a function which exponential growth like $\psi = e^{1\theta}$, must be used. Then the right hand side of Eq. (4.5) becomes ψ/ψ_θ which has zero linear part at $\theta = 0$.

Therefore the linearized equations becames (4.8) with $\beta = 0$ and the eigenspace W^0 gains one dimension corresponding to the eigenvector $\hat{\theta} = 1$, $\hat{\underline{z}} = \hat{\underline{v}} = 0$. W^s instead loses one dimension and becomes equal to $\widetilde{W}^{-1}$. In this situation the local center manifold does not coincide anymore with the set of critical points K but extends in the direction of the θ axis. In this situation, given a critical point $(0, \mathbf{c}, \mathbf{c}) \in K$ to decide if $\mathbf{c}$ can be the AV corresponding to some initial condition, one has to analize the flow on the center manifold that is nontrivial as in the previous case.

We discuss these questions and the problem of the existence of scattering operators in a forthcoming paper [FO].

APPENDIX

Proof of Theorem 2.1. It is sufficient to show that the component $\dot{x}_i(t)$ of $\dot{\mathbf{x}}_i(t)$ on any fixed direction $\mathbf{u}$ has a limit for $t \to \pm\infty$. On the other hand the projections x_i, $\dot{x}_i$ and f_{ij} of $\mathbf{x}_i$, $\dot{\mathbf{x}}_i$ and $\mathbf{f}_{ij}$ on $\mathbf{u}$ satisfy the assumption of the theorem for $n = 1$. Therefore we only need to consider this particular case.

Remark A.1. From assumption (iii) and the hypothesis $\mathbf{x}_i(t) \neq \mathbf{x}_j(t)$, for $i \neq j$, it follows that

$$x_i(t) = x_j(t) \Rightarrow f_{ij}(t) = 0.$$

Given $\mu \in (0, m)$ and $x = (x_1, ..., x_N) \in \mathbb{R}^N$ define

$$\xi_\mu(x) = \sup \left\{ a \,\middle|\, \sum_{x_i \geqslant a} m_i \geqslant \mu \right\}. \tag{A.2}$$

Notice that, by definition, $\xi_\mu(x)$ coincides with the abscissa x_j of the left most m_j such that the sum on the right of m_j is less than μ:

$$\sum_{x_i \geqslant x_j} m_i \geqslant \mu > \sum_{x_i > x_j} m_i \Leftrightarrow \xi_\mu(x) = x_j. \tag{A.3}$$

Consider a motion $x_i(\cdot)\in C^2(\mathbb{R},\mathbb{R})$, $x(\cdot)=(x_1(\cdot),...,x_N(\cdot))$, that satisfies (i)–(iii) with some forces $f_{ij}\in C^0(\mathbb{R},\mathbb{R})$ and let $S_\mu(t)$ be the system composed of the masses on the right of $\xi_\mu(t)=\xi_\mu(x(t))$ and of a mass of value $(\mu-\sum_{x_i(t)>\xi_\mu(t)} m_i)$ located at $\xi_\mu(t)$. The first step of the proof is to show that the linear momentum of $S_\mu(t)$ is nondecreasing. This is to be expected because, due to the repelling character of the f_{ij}, the action of the forces exerted on $S_\mu(t)$ by particles on the left of $\xi_\mu(t)$ increases the linear momentum of $S_\mu(t)$. On the other hand, being the total mass of S_μ fixed to the value μ, mass can leave $S_\mu(t)$ only if at the same time an equal amount of mass enters $S_\mu(t)$ from the left and the velocity of the entering mass cannot be less than the velocity of the leaving mass. Therefore also the flux of mass through $\xi_\mu(t)$ results in increasing the linear momentum of $S_\mu(t)$. Making all this rigorous requires some work.

LEMMA A.2. *The function* $t\to\xi(t)=:\xi_\mu(x(t))$ *has right and left derivatives* $\dot\xi^\pm(t)$ *for each* $t\in\mathbb{R}$. *Moreover*

(i) $\dot\xi^\pm(t)\in\{\dot x_i(t)\mid x_i(t)=\xi(t)\}$;

(ii) *for each* $t\in\mathbb{R}$ *there exist numbers* $\ddot\xi^\pm(t)$;

$$\begin{aligned}\ddot\xi^+(t)&\in\{\ddot x_i(t)\mid x_i(t)=\xi(t),\ \dot x_i(t)=\dot\xi^+(t)\}\\ \ddot\xi^-(t)&\in\{\ddot x_i(t)\mid x_i(t)=\xi(t),\ \dot x_i(t)=\dot\xi^-(t)\}\end{aligned}\tag{A.4}$$

and a number $s_o>0$ *such that*

$$\begin{aligned}\dot\xi^+(t+s)&=\dot\xi^+(t)+\ddot\xi^+(t)\,s+o(s), && 0\leqslant s<s_0,\\ \dot\xi^+(t+s)&=\dot\xi^-(t)+\ddot\xi^-(t)\,s+o(s), && -s_0<s<0\end{aligned}\tag{A.5}$$

where $|o(s)|\leqslant C(s)$, $C(0)=0$, *a continuous function which is independent of* t *for* t *in compact sets.*

(iii) $\dot\xi^+(t)=\dot\xi^-(t)\Rightarrow\ddot\xi^+(t)=\ddot\xi^-(t)$.

Proof. Given $t\in\mathbb{R}$ there is $s_0>0$ such that $0<s<s_0$ and the fact that the $x_j(t)$ are C^2 imply that

$$\begin{cases}x_j(t+s)>x_i(t+s),\ x_j(t)\neq x_i(t)\Rightarrow x_j(t)>x_i(t),\\ x_j(t+s)>x_i(t+s),\ x_j(t)=x_i(t),\ \dot x_j(t)\neq\dot x_i(t),\ \Rightarrow\dot x_j(t)>\dot x_i(t)\\ x_j(t+s)>x_i(t+s),\ x_j(t)=x_i(t),\ \dot x_j(t)=\dot x_i(t),\\ \qquad\ddot x_j(t)\neq\ddot x_i(t)\Rightarrow\ddot x_j(t)>\ddot x_i(t).\end{cases}\tag{A.6}$$

From this and the definition of $\xi_\mu(x)$ it follows that for each t there is $h^+\in\{1,...,N\}$ and $s_o>0$ such that for each $0\leqslant s<s_o$.

$$\xi(t+s)\in\{x_i(t+s)\mid x_i^{(k)}(t)=x_{h^+}^{(k)}(t),\ k=0,1,2\}.\tag{A.7}$$

This implies that for each $t \in \mathbb{R}$, $\xi(\cdot)$ has a right derivative $\dot{\xi}^+(t)$ and $\dot{\xi}^+(t) = \dot{x}_{h^+}(t)$. From (A.7) it follows then that, for $0 \geqslant s < s_o$,

$$\dot{\xi}^+(t+s) \in \{\dot{x}_i(t+s) \mid x_i^{(k)}(t) = x_{h^+}^{(k)}(t), k = 0, 1, 2\} \tag{2.5}$$

Therefore $\dot{\xi}^+(\cdot)$ has a right derivative $\ddot{\xi}^+(t) = \ddot{x}_{h^+}(t)$ at $t \in \mathbb{R}$ and the first equality in (A.5) follows. For $-s_o < s < 0$ in Eqs. (A.6) one must change $\dot{x}_j(t) > \dot{x}_i(t)$ to $\dot{x}_j(t) < \dot{x}_i(t)$ in the second line and therefore Eqs. (A.7), (A.8) still hold true for $-s_o < s < 0$, provided h^+ is replaced by some $h^- \in \{1, ..., N\}$ which can also coincide with h^+. From Eq. (A.7) with h^- instead of h^+ it follows that $\xi(\cdot)$ has a left derivative $\dot{\xi}^-(t) = \dot{x}_{h^-}(t)$ at $t \in \mathbb{R}$ and also that

$$\lim_{s \to 0^-} \dot{\xi}^+(t+s) = \dot{\xi}^-(t); \tag{2.9}$$

moreover, the second equation (A.5) holds with $\ddot{\xi}^-(t) = \ddot{x}_{h^-}(t)$. To prove (iii) we note that, for $0 < |s|$ suficiently small, $x_j(t) = x_i(t)$, $\dot{x}_j(t) = \dot{x}_i(t)$, $\ddot{x}_j(t) > \ddot{x}_i(t) \Rightarrow \ddot{x}_j(t+s) > x_i(t)$ and therefore that

$$x_{h^+}(t) = x_{h^-}(t), \qquad \dot{x}_{h^+}(t) = \dot{x}_{h^-}(t),$$

implies

$$\xi(t+s) \in \{x_i(t+s) \mid x_i^{(k)}(t) = x_{h^+}^{(k)}(t), k = 0, 1, 2\}$$

both for $0 \leqslant s < s_o$ and $-s_0 < s < 0$. ∎

Remark A.3. The numerical range of the function $\mu \to \dot{\xi}^+(t) = (d/dt^+)\, \xi_\mu(\mathbf{x}(t))$ coincides with $\{\dot{x}_1(t), ..., \dot{x}_N(t)\}$. In fact we have seen that $\dot{\xi}^+(t) = \dot{x}_{h^+}(t)$ and on the other hand the definition of ξ_μ implies that as μ describes the interval $(0, m)$ $\dot{x}_{h^+}(t)$ coincides one after the other with all the $\dot{x}_i(t)$.

Let $p_i^{k,t}$ be the Taylor polinomial of degree $0 \leqslant k \leqslant 2$ of the function $x_i(\cdot)$ at t and let

$$I^{\pm} = I^{\pm}(t) = \{i \mid p_i^{2,t}(s) > p_{h^{\pm}}^{2,t}(s), \pm s > 0\} \tag{A.10}$$

Let $q: \mathbb{R} \to \mathbb{R}$ be defined by

$$q(t) = q_\mu(t) = \left(\mu - \sum_{i \in I^+(t)} m_i\right) \dot{\xi}^+(t) + \sum_{i \in I^+(t)} m_i \dot{x}_i(t) \tag{A.11}$$

Define also

$$q(t^-)=\Big(\mu-\sum_{i\in I^-(t)} m_i\Big)\,\dot{\xi}^-(t)+\sum_{i\in I^-(t)} m_i\dot{x}_i(t), \tag{A.12}$$

$$\dot{q}^\pm(t)=\Big(\mu-\sum_{i\in I^\pm(t)} m_i\Big)\,\ddot{\xi}^\pm(t)+\sum_{i\in I^\pm(t)} m_i\ddot{x}_i(t) \tag{A.13}$$

Remark A.4. The function $\mu\to q_\mu(t)$ is piecewise linear and the slope of the function in each interval is given by the value of $\dot{\xi}^+(t)$ in that interval. This and Remark A.2 imply that the knowledge of the function $\mu\to q_\mu(t)$ entails that of the set $\{\dot{x}_1(t), ..., \dot{x}_N(t)\}$.

LEMMA A.5. *The function* $q\colon \mathbb{R}\to\mathbb{R}$ *defined by* (A.11) *has the following properties*

$$\lim_{s\to o^-} q(t+s)=q(t^-) \tag{A.14}$$

and

$$q(t)-q(t^-)\leqslant 0. \tag{A.15}$$

There exists $s_o>0$ *such that*

$$\begin{cases} q(t+s)=q(t)+\dot{q}^+(t)s, & 0\leqslant s<s_o,\\ q(t+s)=q(t^-)+\dot{q}^-(t)s, & -s_o<s<0. \end{cases} \tag{A.16}$$

The function q is differentiable at t whenever $\dot{\xi}^+(t)=\dot{\xi}^-(t)$.

Proof. From (A.7) it follows that for $0\leqslant s<s_o$, $s_o>0$ sufficiently small, $h^+(t+s)$ is in the set

$$J^+=\{i\mid p_i^{2,t}(\cdot)=p_{h^+}^{2,t}(\cdot)\}$$

and therefore that the difference between the sets $I^+(t+s)$ and $I^+(t)$ is contained in J^+. It follows

$$q(t+s)=\Big(\mu-\sum_{i\in I^+(t)} m_i\Big)\,\dot{\xi}^+(t+s)+\sum_{i\in I^+(t)} m_i\dot{x}_i(t+s)+o(s),\qquad 0\leqslant s<s_o. \tag{A.17}$$

Similarly, for $-s_o < s < 0$, the difference between $I^+(t+s)$ and $I^-(t)$ is in the set $J^- = \{i \mid P_i^{2,t}(\cdot) = P_{h^-}^{2,t}(\cdot)\}$ and therefore we have

$$q(t+s) = \left(\mu - \sum_{i \in I^-(t)} m_i\right) \dot{\xi}^+(t+s) + \sum_{i \in I^-(t)} m_i \dot{x}_i(t+s) + o(s),$$
$$-s_o < s < 0. \tag{A.18}$$

Eq. (A.13) with the + sign follows from (A.17) and Lemma A.2. Taking the limit for $s \to 0^-$ in (A.18) yields Eq. (A.14). Eq. (A.13) with the − sign follows from (A.14), (A.18) and Lemma A.2. By Lemma A.2, the condition $\dot{\xi}^+(t) = \dot{\xi}^-(t)$ implies $\ddot{\xi}(t) = \ddot{\xi}^-(t)$ and therefore $I^+(t) = I^-(t)$. It follows $q(t) = q(t^-)$ and $\dot{q}^+(t) = \dot{q}^-(t)$. That is q is differentiable at t. It remains to show that (A.15) holds. First of all we note that we can replace in (A.11), (A.12) $I^\pm(t)$ with the sets

$$\hat{I}^\pm(t) = \{i \mid p_i^{1,t}(s) > p_{h^\pm}^{1,t}(s), \ \pm s > 0\}; \tag{A.19}$$

in fact $i \in I^\pm(t) \backslash \hat{I}^\pm(t) \Rightarrow \dot{x}_i(t) = \dot{x}_{h^\pm}(t) = \dot{\xi}^\pm(t)$. In the remaining part of the proof we drop the hat and use the notation $I^\pm(t)$ for the set defined by (A.20). We consider separately the cases: $\dot{\xi}^+(t) - \dot{\xi}^-(t) \geqslant 0$, $\dot{\xi}^+(t) - \dot{\xi}^-(t) < 0$.

Case $\dot{\xi}^+(t) - \dot{\xi}^-(t) \geqslant 0$. Eqs. (A.11), (A.12) imply

$$q(t) - q(t^-) = \left(\mu - \sum_{I^+ \cap I^-} m_i\right) (\dot{\xi}^+(t) - \dot{\xi}^-(t)) - \sum_{I^+ \backslash I^-} m_i(\dot{\xi}^+(t) - \dot{x}_i(t))$$
$$+ \sum_{I^- \backslash I^+} m_i(\dot{\xi}^-(t) - \dot{x}_i(t)). \tag{A.20}$$

The first term on the right of Eq (A.20) is clearly nonnegative. The other two terms are also nonnegative; in fact, in the case under consideration,

$$i \in I^+ \backslash I^- \Rightarrow x_i(t) = \xi(t), \qquad \dot{x}_i(t) \leqslant \dot{\xi}^+(t)$$
$$i \in I^- \backslash I^+ \Rightarrow x_i(t) = \xi(t), \dot{x}_i(t) \leqslant \dot{\xi}^+(t).$$

Therefore Eq. (A.20) yields

$$q(t) - q(t^-) \geqslant \left(\mu - \sum_{I^+ \cup I^-} m_i\right) (\dot{\xi}^+(t) - \dot{\xi}^-(t))$$

and therefore $q(t) - q(t^-) \leqslant 0$. In fact we have

$$\sum_{I^+ \cup I^-} m_i \geqslant \mu,$$

because $\dot{\xi}^+(t) < \dot{\xi}^-(t)$ implies

$$x_i(t) = \xi(t) \Rightarrow i \in I^+ \cap I^-. \quad \blacksquare$$

LEMMA A.6. *The function q is nondecreasing.*

We first show that if is differentiable at t then

$$\dot{q}(t) = \dot{q}^+(t) = \dot{q}^-(t) \geqslant 0.$$

Let $I_\xi(t) = \{i \mid x_i^{(k)}(t) = x_{h^+}^{(k)}(t),\ k = 0, 1, 2\}$ then the definitions of ξ and h^+ imply

$$m_\xi =: \sum_{i \in I_\xi} m_i \geqslant \mu - \sum_{i \in I^+} m_i \tag{A.21}$$

where here and in the following we do not indicate the dependence on t. We have

$$m_\xi \ddot{\xi} = \sum_{i \in I_\xi} m_i \ddot{x}_i = \sum_{\substack{i \in I_\xi \\ j \in I^+}} f_{ij} + \sum_{\substack{i \in I_\xi \\ j \notin I^+}} f_{ij} \geqslant \sum_{\substack{i \in I_\xi \\ j \in I^+}} f_{ij} \tag{A.22}$$

$$\sum_{j \in I^+} m_j \ddot{x}_j = \sum_{\substack{j \in I^+ \\ i \in I^+}} f_{ij} + \sum_{\substack{j \in I^+ \\ i \in I_\xi}} f_{ij} + \sum_{\substack{j \in I^+ \\ i \notin I^+ \cup I_\xi}} f_{ij} \geqslant \sum_{\substack{j \in I^+ \\ i \in I_\xi}} f_{ij} \tag{A.23}$$

where we have used assumptions (i), (ii), (iii), and Remark A.1. From Eqs. (A.22), (A.23), and (A.13) it follows

$$\dot{q}(t) = \dot{q}^+(t) \geqslant \left[1 - \frac{(\mu - \sum_{i \in I^+} m_i)}{m_\xi} \right] \sum_{\substack{j \in I^+ \\ i \in I_\xi}} f_{ij} \geqslant 0. \tag{A.24}$$

In fact we have

$$\sum_{\substack{i \in I^+ \\ i \in I_\xi}} f_{ij} \geqslant 0$$

by respulsivity ((iii)) and

$$\left[1 - \frac{(\mu - \sum_{i \in I^+} m_i)}{m_\xi} \right] \geqslant 0$$

by (A.21). From this point, on the basis of Eqs. (A.15) and (A.16), the same arguments as in Lemma to 3.2 and 3.3 conclude the proof. $\blacksquare$

From Lemma A.6 and the fact that $q_\mu(t)$ is bounded because the $\dot{x}_i(t)$ are, it follows that there exists w: $(0, m) \to \mathbb{R}$ such that

$$\lim_{t \to \infty} q_\mu(t) = w_\mu.$$

Remark A.4 then implies that the set $\{\dot{x}_1(t), ..., \dot{x}_N(t)\}$ has a limit $\{c_1, ..., c_n\}$ as $t \to \infty$. This and the continuity of the $\dot{x}_i s$ imply that in fact each one of the $\dot{x}_i(t)$ converges to one of the c_i as $t \to \infty$. The existence of the limit for $t \to -\infty$ is provided in the same way. ∎

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