Research Article

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Star-group identities on units of group algebras: The non-torsion case

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Abstract: Let G be a group, F a field and FG the corresponding group algebra. We consider an involution on FG which is the linear extension of an involution of G, e.g., $g^* = g^{-1}$ for $g \in G$. This paper is focused on the characterization of a non-torsion group G provided the group of units $U(FG)$ satisfies a $*$-group identity. The torsion case was studied in [7], and when $*$ is the classical involution, this problem was solved in the case of symmetric units in [21].

Keywords: Group algebra, involution, unit, group identity

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1 Introduction

Let FG be the group algebra of a group G over a field F and let $U(FG)$ denote its group of units. $U(FG)$ has been extensively studied in the past (see for instance [10, 16, 20]), and it turns out that, even in the case of finite groups, it is quite large and difficult to study.

In an attempt to tie the group of units to the structure of the algebra, in the 80s Hartley conjectured that for torsion groups, a group identity on $U(FG)$ would force a polynomial identity on FG. This conjecture was proved in the 90s in [4, 8, 13, 15], and it turns out that one can actually obtain a classification of torsion groups G such that the group of units $U(FG)$ satisfies a group identity.

These results were the starting point for the development of a theory of group identities on $U(FG)$: on the one hand, the investigation was carried over to non-torsion groups [8], and on the other hand, group identities on significant units of $U(FG)$ were studied, such as symmetric units or unitary units [3, 5, 12, 21].

In this last setting, since a group algebra FG is always endowed with an involution, one can consider involutions $*$ obtained as a linear extension of an involution of the group G (see [2, 9]). One such involution is for instance the so-called classical involution defined on G by $g^* = g^{-1}$ (notice that any other involution is the composition of an automorphism of G of order 1 or 2 with the inverse map). In this case, a natural extension of Hartley's conjecture and its further development is to consider group identities on symmetric units, i.e., words of the free group evaluating onto the identity when computed on symmetric units of FG.

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This program was carried out by several authors and a link between group identities on the unit group and group identities on symmetric units was established (see [10] for a comprehensive reading on the subject).

In the last few years, we have adopted a more general and comprehensive approach to the subject by considering the so-called $*$-group identities. Recall that a word in the free group with involution $*$ is a $*$-group identity for a group U with involution if it becomes the identity element when evaluated on U. In this setting, a group identity on symmetric units is a special case of a $*$-group identity. Notice that any group endowed with the canonical involution satisfies the $*$-identity $xx^* \equiv 1$.

In this paper, we consider group algebras FG endowed with an involution $*$ which is a linear extension of an involution of G. Under mild hypotheses we are able to classify the groups G such that the unit group $U(FG)$ satisfies a $*$-group identity (Theorem 3.10 and Theorem 6.8). When G is a torsion group this result was achieved in [7]. Here we deal with the non-torsion case.

2 Prerequisites and general setting

Let G be a group. Recall that G satisfies a group identity (or GI) if there is a non-trivial word $w(x_1, \dots, x_n)$ in the free group $\langle x_1, x_2, \dots\rangle$ such that $w(g_1, \dots, g_n) = 1$ for all $g_i \in G$.

Recall that an involution $*$ is an antiautomorphism of order two. A typical example is the classical involution on G given by $g^* = g^{-1}$ for all $g \in G$. The free group $\langle x_1, x_2, \dots\rangle$ has an involution induced by $x_1^* = x_2, x_3^* = x_4, \dots$. Reordering the variables, we obtain the free group with involution $\langle x_1, x_1^*, x_2, x_2^*, \dots\rangle$. We say that a group G with involution satisfies a $*$-group identity (or $*$-GI) if there exists a non-trivial word $w(x_1, x_1^*, \dots, x_n, x_n^*) \in \langle x_1, x_1^*, x_2, x_2^*, \dots\rangle$ such that $w(g_1, g_1^*, \dots, g_n, g_n^*) = 1$ for all $g_1, \dots, g_n \in G$.

Clearly, a GI is a $*$-GI (where no $*$'s appear). Moreover, since xx^* is a symmetric element, a group identity on symmetric elements of a group G yields a $*$-group identity of G.

We start by recalling some basic results that we shall use throughout the paper. In the next three lemmas, R is a unitary algebra with an involution $*$ over an infinite commutative domain A, char $A \neq 2$, and the elements of A are not zero divisors in R. Recall that $R^+ = \{a \in R \mid a^* = a\}$ is the set of symmetric elements of R and $U(R)$ is the group of units of R.

Lemma 2.1 ([7, Lemma 2]). *If $U(R)$ satisfies a $*$-GI, there exists a positive integer $N \geq 1$ such that*
(i) *if $a \in R$ and $a^2 = 0$, then $(aa^*)^N = 0$;*
(ii) *if $s, t \in R^+$, $s^2 = t^2 = 0$, then $(stsd)^N = 0$ for all $d \in R^+$.*

Lemma 2.2 ([7, Lemma 3]). *Let R be a semiprime ring with involution such that $U(R)$ satisfies a $*$-GI. Let $s \in R^+$ be such that $s^2 = 0$.*
(i) *If $t \in R^+$ is nilpotent, then $sts = 0$.*
(ii) *If $x, y \in R$ are such that $xy = 0$, then $xsy = 0$.*

Lemma 2.3 ([7, Lemma 3]). *Let R be a semiprime ring with involution such that $U(R)$ satisfies a $*$-GI. Then every symmetric idempotent of R and $A^{-1}R$ is central.*

We record an easy observation which will be used without mention.

Lemma 2.4 ([5, Lemma 2.9]). *If A is an abelian group with an involution $*$, then we have $A^2 \subset A_1A_2$, where $A_1 = \{a \in A \mid a^* = a\}$ and $A_2 = \{a \in A \mid a^* = a^{-1}\}$. If A is torsion with no elements of order 2, then $A = A_1 \times A_2$.*

The starting point of our investigation is the following result.

Theorem 2.5 ([7, Main Theorem]). *Let G be a torsion group with no elements of order two and let F be an infinite field of characteristic $p \geq 0$, $p \neq 2$. Assume that FG is endowed with an involution $*$ induced from an involution of G. Then we have the following:*
(i) *If FG is semiprime, then $U(FG)$ satisfies a $*$-GI if and only if G is abelian.*
(ii) *If FG is not semiprime, then $U(FG)$ satisfies a $*$-GI if and only if G' is of bounded p-power exponent (and hence P, the set of p-elements, is a subgroup and G/P is abelian) and FG is PI.*

Throughout the paper, we assume that F is an infinite field, char $F = p \geq 0$, $p \neq 2$, G is a group with no 2-elements such that

$$T = T(G) = \{g \in G \mid \circ(g) < \infty\}$$

is a subgroup, and FG is an algebra with an involution induced from an involution of G.

We recall the following notations:

- $G_{p'} = \{g \in G \mid g \text{ is a } p'\text{-element}\}$;
- $P = \{g \in G \mid g \text{ is a } p\text{-element}\}$;
- $P_1 = \{g \in G \mid g^p = 1\}$ (we agree that if $p = 0$, then $P = 1$, $T = T_{p'}$);
- $\Phi_p = \Phi_p(G) = \{g \in P \mid g \text{ has a finite number of conjugates in } G\}$;
- $\Delta(G, N)$ is the kernel of $FG \to F(G/N)$ and $\Delta(G)$ is $\Delta(G, G)$.

Also, for $g \in T$, we denote by $\widehat{g}$ the sum of all powers of g: $\widehat{g} = \sum_1^{o(g)} g^i$.

3 Semiprime algebras

Throughout this section, we assume that FG is a semiprime algebra endowed with an involution $*$ induced from an involution of G, and $U(FG)$ satisfies a $*$-GI.

Since T is a characteristic subgroup of G, it is closed under $*$. Hence $U(FT)$ satisfies a $*$-identity and Theorem 2.5 applies since T is torsion. Thus, for example, since $P \subseteq T$, we see that P is a characteristic subgroup of T and hence a normal subgroup of G. Similarly, $\Phi_p(T) = \Phi(T) \cap P$ is a characteristic subgroup of T and hence normal in G. We state this in the following.

Lemma 3.1. *The group $\Phi_p(T)$ is a normal subgroup of G.*

Lemma 3.2. (i) *If $g \in P_1$ and $h \in P$, then g normalizes $\langle h \rangle$;*
(ii) *P_1 is a normal abelian subgroup of G;*
(iii) *If g is an element of order p such that $g^* = g$ or g^{-1} and $h \in G_{p'}$, then h normalizes $\langle g \rangle$.*

Proof. The lemma is proved for torsion groups in [11, Lemma 31], but the proof works in general by using that T is a subgroup. □

A property that we shall use throughout is the following.

Remark 3.3. Let $g \in G$. If $\circ(g)$ is odd or $\circ(g) = \infty$, then $1 + g$ is not a zero divisor in FG (see [6, Lemma 2.13]). The same proof shows that $c + g$, $c \in F$, is not a zero divisor if $o(g) = \infty$.

Lemma 3.4. *Let $x \in P_1$ be such that $x^* = x^\varepsilon$, where $\varepsilon = \pm 1$, and let $y \in G \setminus T$. Then x has a finite number of conjugates under $\langle y \rangle$.*

Proof. Recall that by Lemma 3.2, P_1 is a normal abelian subgroup of G, hence we can regard P_1 as a vector space V over $GF(p)$. Also if we regard y as acting as an automorphism, then V can be considered as a module over $R = GF(p)[y, y^{-1}]$, the group algebra of $\langle y \rangle$ over $GF(p)$.

Let $I = \{r \in R \mid xr = 0\}$ be the annihilator of $x \in P_1$ in R. Then $xR \cong R/I$.

If $I \neq 0$, let $I \cap GF(p)[y] = (f(y))$, where $f(y) = b_r y^r + \cdots + b_{n-1} y^{n-1}$ with $b_r \neq 0$. Then from

$$y^{-(r+1)} f(y) = b_r y^{-1} + b_{r+1} y^0 + \cdots \in I$$

we get $y^{-1} \equiv c_0 + c_1 y + \cdots \pmod I$ for some $c_0, c_1, \ldots \in GF(p)$ with $c_0 \neq 0$. From this it follows that every element of R can be expressed (mod I) as a linear combination of $1, y, \ldots, y^{n-1}$. Thus R/I has order p^n, and so x has a finite number of conjugates under $\langle y \rangle$, as wished.

Now, assume that $I = 0$. Then $xR \cong R$ and it follows that the sum of the various subgroups of the form $\langle x^{y^i} \rangle$ is direct. For example,

$$W = \langle x, x^y, x^{y^2} \rangle = \langle x \rangle \oplus \langle x^y \rangle \oplus \langle x^{y^2} \rangle$$

and $x^{y^3} \notin W$.

Write $X = \hat{x}$. Then $X(y + y^*)X$ is a symmetric square-zero element; moreover $X^{y^2}(1 - x^{y^2}) = 0$. Hence by Lemma 2.2,

$$X^{y^2}X(y + y^*)X(1 - x^{y^2}) = 0.$$

Now, if $y^* \not\equiv y \pmod{P_1}$, then $X^{y^2}XyX(1 - x^{y^2}) = 0$, and so

$$X^{y^2}XX^y = X^{y^2}XX^y x^{y^3}.$$

It follows, from the direct sum decomposition of W that $X^{y^2}XX^y = \widehat{W}$. Thus $\widehat{W} = \widehat{W}x^{y^3}$ and this contradicts the fact that $x^{y^3} \notin W$.

On the other hand, if $y^* = y\pi$ for some $\pi \in P_1$, then

$$X^{y^2}Xy(1 + \pi)X(1 - x^{y^2}) = 0.$$

Recalling that P_1 is a normal abelian subgroup and $1 + \pi$ is not a zero divisor, we still get $X^{y^2}XyX(1 - x^{y^2}) = 0$ and a contradiction. □

Remark 3.5. If $A = A^*$ is a finite abelian subgroup normalized by $\langle y\rangle$ for some $y \in G$, then $\langle y^*\rangle$ normalizes A.

Proof. If $a \in A$, then $y^*ay^{*^{-1}} = y^*b^*y^{*^{-1}}$ for $b = a^*$. Hence $y^*ay^{*^{-1}} = (y^{-1}by)^* \in A^* = A$. □

If char $F = 0$, then FT is semiprime and by Theorem 2.5, T is abelian. The following lemma handles the case of positive characteristic.

Lemma 3.6. *The algebra FT is semiprime.*

Proof. We need to prove that $\Phi_p(T) = 1$. To this end, it suffices to prove that $\Phi_p(T) \cap P_1 = 1$.

Suppose by contradiction that $\Phi_p(T) \cap P_1 \neq 1$. Since by Lemma 3.2, P_1 is a normal abelian subgroup of G, we obtain that $\Phi_p(T) \cap P_1$ is an elementary abelian p-group.

If $\Phi_p(T) \cap P_1 = \langle g\rangle$ is cyclic, then $\langle g\rangle$ is the unique normal subgroup of G of order p in $\Phi_p(T)$, and so $\langle g\rangle$ is normal in G, contradicting the semiprimeness of FG.

Therefore, $\Phi_p(T) \cap P_1$ contains at least two cyclic factors. Being a $*$-invariant abelian subgroup of G, we may arrange by Lemma 2.4 that $\Phi_p(T) \cap P_1$ contains the factor $\langle x\rangle \times \langle x_1\rangle$, where $x^* = x$ or x^{-1}, and $x_1^* = x_1$ or x_1^{-1}.

If we show that $\langle x\rangle \times \langle x_1\rangle$ is normalized by $\langle y\rangle$ for any $y \in G \setminus T$, then we would have $\langle x\rangle \times \langle x_1\rangle \subseteq \Phi_p(G)$, which contradicts the semiprimeness of FG.

Therefore, in order to finish the proof we need only to show that $\langle x\rangle \times \langle x_1\rangle$ is normalized by $\langle y\rangle$ for any $y \in G \setminus T$.

Let $H = \langle x, y, y^*\rangle$ be the subgroup generated by x, y and y^*, and let $\langle x\rangle^H$ be the normal $*$-closure of $\langle x\rangle$ in H. Since by Lemma 3.4, $\langle x\rangle$ has only a finite number of conjugates under $\langle y\rangle$, we get that $\langle x\rangle^H$ is a finite normal abelian p-subgroup of H. Hence $\Delta(H, \langle x\rangle^H)$, the augmentation ideal of $\langle x\rangle^H$ in H, is a nilpotent ideal of FH. But then the element $(1 - x)y + y^*(1 - x^*) \in \Delta(H, \langle x\rangle^H)$ is nilpotent. Since it is symmetric, by Lemma 2.2 we get that

$$X_1((1 - x)y + y^*(1 - x^*))X_1 = 0,$$

where $X_1 = \hat{x}_1$.

If $y^* \not\equiv y \pmod{P_1}$, then $X_1(1 - x)yX_1 = 0$, and so $(1 - x)X_1X_1^y = 0$. This says that $x_1^y \in \langle x\rangle \times \langle x_1\rangle$. Also, exchanging the role of x and x_1, we get that $x^y \in \langle x\rangle \times \langle x_1\rangle$, as wished.

Now suppose that $y^* = y\pi$ for some $\pi \in P_1$. We consider the element $(1 - x)(y + y^*) + (y + y^*)(1 - x^*)$, which is symmetric and nilpotent since it lies in $\in \Delta(H, \langle x\rangle^H)$. Again, by Lemma 2.2 we get

$$X_1((1 - x)(y + y^*) + (y + y^*)(1 - x^*))X_1 = 0.$$

Hence

$$X_1((1 - x)y(1 + \pi) + y(1 + \pi)(1 - x^*))X_1 = 0,$$

and so

$$X_1^y((1 - x^y)(1 + \pi) + (1 + \pi)(1 - x^*))X_1 = 0.$$

Since $1 + \pi$ is not a zero divisor, we have

$$X_1^y(1 - x^y + 1 - x^\varepsilon)X_1 = 0,$$

where we recall that $x^* = x^\varepsilon$, $\varepsilon = 1$ or -1. Thus, $X_1^y(2 - x^y - x^\varepsilon)X_1 = 0$.

If $x_1^y \notin \langle x_1 \rangle$, then $X_1^y X_1 \neq 0$. Hence

$$2X_1X_1^y = x^\varepsilon(X_1X_1^y) + x^y(X_1X_1^y)$$

says that

$$2\hat{A} = x^\varepsilon\hat{A} + x^y\hat{A}, \tag{3.1}$$

where $A = \langle x_1, x_1^y \rangle$.

Since $\operatorname{char} F \neq 2$, we have no cancelation on either side of (3.1). Thus $x^\varepsilon \in \langle x_1, x_1^y \rangle$, and we get that $x^y \in \langle x_1, x \rangle$. In any case, $x_1^y \in \langle x \rangle \times \langle x_1 \rangle$. Exchanging the role of x and x_1, we get $x^y \in \langle x \rangle \times \langle x_1 \rangle$ also. Hence y normalizes $\langle x \rangle \times \langle x_1 \rangle$.

We have proved that $\langle x \rangle \times \langle x_1 \rangle$ is normalized by any element of $G \setminus T$. □

Since by the previous lemma FT is semiprime, by Theorem 2.5 we have the following corollary.

Corollary 3.7. *The group T is an abelian p'-group.*

Lemma 3.8. *Let H be a finite normal $*$-invariant abelian subgroup of G and $x \in G \setminus T$. If $K = \langle H, x, x^* \rangle$, then every idempotent of FH is central in FK.*

Proof. Since FT is semiprime, we can write $FH = e_1FH \oplus \cdots \oplus e_nFH$, where $e_1, \ldots, e_n$ are the orthogonal primitive idempotents of FH.

Since any idempotent of FH is uniquely a sum of primitive idempotents and an involution permutes the primitive idempotents, by Lemma 2.3 it is enough to show that $e_i = e_i^*$ for all $1 \leq i \leq n$.

Let e be a primitive idempotent of FH such that $e^* \neq e$, and let $f = e^* + e$. Then $f^2 = f$ is symmetric, and so central. Hence $f^x = f$, which says that $e^x + (e^*)^x = e + e^*$. If $e^x = e$, then e is central in FK and we are done. Therefore, we may assume that $e^x \neq e$.

Then from $e^x + (e^*)^x = e + e^*$ we get $e^x = e^*$ and $(e^*)^x = e$, i.e.,

$$ex = xe^* \quad \text{and} \quad e^*x = xe.$$

Consider the elements

$$s_1 = (x + x^*)e = xe + x^*e = e^*x + e^*x^* = e^*(x + x^*)$$

and

$$s_2 = (x + x^*)e^* = xe^* + x^*e^* = ex + ex^* = e(x + x^*).$$

Then s_1 and s_2 are symmetric and square-zero, and, by Lemma 2.1 s_1s_2 is nilpotent. Now,

$$s_1s_2 = e^*(x + x^*)e(x + x^*) = e^*(x + x^*)^2,$$

and so for some n, we have $0 = (s_1s_2)^n = e^*(x + x^*)^{2n}$.

Let $x^* = x\pi$ for some $\pi \in G$. Then $0 = (s_1s_2)^n = e^*(x(1 + \pi))^{2n}$. As $x(1 + \pi)$ is not a zero divisor, we get $e^* = 0$, a contradiction. □

Lemma 3.9. *T is an abelian p'-subgroup and every idempotent of FT is central in FG.*

Proof. By Lemma 3.6, FT is semiprime. Hence since G has no 2-elements, by Theorem 2.5, T is abelian. Thus T is a p'-group.

Write $T = T_1 \times T_2$ as in Lemma 2.4. If $t_i \in T_i$, $i = 1, 2$, then $\frac{1}{\circ(t_i)}\hat{t}_i$ is a symmetric idempotent, and so by Lemma 2.3 is central in G. This says that $\langle t_i \rangle$ is a normal subgroup of G. It follows that every $t = t_1t_2 \in T$ lies in a finite normal subgroup of G. Applying Lemma 3.8, we get the desired conclusion. □

Theorem 3.10. *Let F be an infinite field,* char $F \neq 2$, *and let G be a group with no 2-elements such that $T = T(G)$ is a subgroup. Assume that FG is a semiprime algebra endowed with an involution $*$ induced from an involution of G.*

If $U(FG)$ satisfies a $*$-GI, then

(i) *T is an abelian p'-subgroup such that every idempotent of FT is central in FG (and, consequently, every subgroup of T is normal in G);*

(ii) *G/T satisfies a $*$-GI;*

Conversely, if (i) *holds and G/T is a unique product group satisfying a GI, then $U(FG)$ satisfies a GI.*

Proof. The necessity of (i) is Lemma 3.9.

To prove (ii) observe that G is an invariant subgroup of $U(FG)$ under *, so G satisfies a $*$-GI and thus G/T satisfies a $*$-GI.

We shall now prove the sufficiency of (i) and (ii). To this end, suppose that G/T satisfies a group identity $w(x_1, \dots, x_n) = 1$. We recall from [8, proof of Theorem 5.7] that under assumption (i) and G/T being a unique product group, the following holds: Fix a transversal X of T in G so that whenever $x \in X$, then also $x^{-1} \in X$. Then given any finite set of units of FG, there exists a finite subgroup H of T, with $FH \equiv \oplus_i F_i$, a direct sum of fields, so that each unit can be written as

$$u = \sum e_i f_i g_i = \sum f_i g_i$$

with $f_i \in F_i$, $g_i \in X$. Here the e_i's form a complete set of primitive idempotents, $\sum e_i = 1$, and $F_i = e_i FH$. Moreover, $U(FT) \lhd U(FG)$.

For $u_1, \dots, u_n \in U(FG)$, we have that $w(u_1, \dots, u_n) \in U(FT)$, which is abelian. It follows that $U(FG)$ satisfies the GI $(w(x_1, \dots, x_n), w(y_1, \dots, y_n)) = 1$. □

We remark that the converse of the previous theorem cannot hold if one weakens the hypothesis that G/T satisfies a GI with G/T satisfying a $*$-GI. In fact, in case G/T is endowed with the classical involution, $xx^* = 1$ is a $*$-GI of G/T, but no conclusion can be inferred from it.

4 The general setting for the non-semiprime case

We recall that throughout F is an infinite field, char $F \neq 2$, G is a group with no 2-elements such that $T = T(G)$ is a subgroup. Also the involution on FG is induced from an involution of the group G, and here we assume that $U(FG)$ satisfies a $*$-GI.

Since T is $*$-invariant, $U(FT)$ satisfies a $*$-GI, and so by Theorem 2.5, P is a subgroup.

Let $N = N(FG)$ be the sum of the nilpotent ideals of FG. Then N is a nil ideal and by a result of Passman [14, p. 311], N is nilpotent if and only if $\Phi_p(G)$ is finite.

Next we split our investigation into two cases according as P is of bounded or unbounded exponent.

First we quote two results that were proved in [21] for the classical involution. The proofs given there are still valid for an arbitrary involution.

Lemma 4.1 ([11, Lemma 14]). *If FG has a $*$-invariant ideal that is nil but not nilpotent, then $\Phi(G)$ is of finite index in G and $\Phi(G)'$ is finite.*

Lemma 4.2 ([21, Lemma 7]). *If $N(FG)$ is a nil but not nilpotent ideal of FG, then FG satisfies a non-trivial polynomial identity (PI).*

Proof. By Lemma 4.1, $[G : \Phi] < \infty$ and $|\Phi'| < \infty$.

Since $[G : \Phi] < \infty$ in order to prove that FG satisfies a PI, it is enough to show that $F\Phi$ satisfies a PI. Since $T(\Phi)$ is a torsion group and $U(FG)$ is $*$-GI, by [7, Main Theorem], $T(\Phi)_p = T(\Phi) \cap P$ is a subgroup. Also, since $(T(\Phi)_p)' \subseteq \phi'$ is finite, $(T(\Phi)_p)'$ is a finite p-group. Factoring by $(T(\Phi)_p)'$, we may assume that $T(\Phi)_p$ is abelian. Now, since Φ' is a finite group, again by [7, Main Theorem] we get that Φ'_p is a subgroup. Hence factoring by Φ'_p, we may assume that Φ' is a finite p'-group. We have $(\Phi, T(\Phi)_p) \subseteq \Phi' \cap T(\Phi)_p = 1$.

Thus $T(\Phi)_p \subseteq \zeta(\phi)$, the center of Φ. Let $P_1 = T(\Phi_p)$. If P_1 is finite, then by [14, Theorem 8.1.12] we get that $N(FG)$ is nilpotent contrary to our assumption. Thus P_1 is infinite. Now, by [7, Lemma 4], $N(F\Phi)$ satisfies a polynomial identity. Let $f(x_1, \dots, x_n)$ be such a PI which we may assume to be multilinear. As in the proof of [8, Proposition 4.4], it follows that $F\Phi$ satisfies f.

Here are the details.

Suppose first that P_1 is of bounded exponent. Since $\Delta(\Phi, P_1)$ is a nil ideal, by [14, Lemma 8.1.7],

$$\Delta(\Phi, P_1) = F\Phi\Delta(P_1) \subseteq F\Phi J(F(P_1)) \subseteq N(F\Phi),$$

where $J(F(P_1))$ is the Jacobson radical of $F(P_1)$.

Since P_1 is infinite of bounded exponent, it contains $P_0 = \langle b_1\rangle \times \langle b_2\rangle \times \cdots$, an infinite direct product of cyclic groups of order p. Let $\{y_i\}_{i\in I}$ be a left transversal of P_0 in Φ; if $a_1, \dots, a_n \in F\Phi$, then we can write $f(a_1, \dots, a_n) = \sum_{i,s} \beta_{is} y_s q_i$, where $\beta_{is} \in F$, $q_i \in P_0$. Let $P_0 = C \times D$, where for every $\beta_{is} \neq 0$ we have $q_i \in C$. Take $b_r, b_{r+1}, \dots, b_{r+n} \in D$; then, since $(1 - b_{r+j})a_{r+j} \in \Delta(\Phi, P_1) \subseteq N(F\Phi)$, we get

$$\begin{aligned} 0 &= f((1 - b_r)a_1, (1 - b_{r+1})a_2, \dots, (1 - b_{r+n})a_n) \\ &= (1 - b_r)(1 - b_{r+1}) \cdots (1 - b_{r+n}) f(a_1, \dots, a_n). \end{aligned}$$

By our choice of D, we get that $f(a_1, \dots, a_n) = 0$. Hence f is a polynomial identity of $F\Phi$.

Next suppose that P_1 is of unbounded exponent. If we take $b \in P_1$, then, as above we get

$$\begin{aligned} 0 &= f((1 - b)a_1, (1 - b)a_2, \dots, (1 - b)a_n) \\ &= (1 - b)^n f(a_1, \dots, a_n). \end{aligned}$$

Hence $(1 - b^{p^t}) f(a_1, \dots, a_n) = 0$ for suitable $t \geq 1$ for all $b \in P_1$. Since P_1 is of unbounded exponent, this leads to the conclusion that $f(a_1, \dots, a_n) = 0$, and we are done. □

The proof of next lemma is essentially given in [21, Lemma 8], although some essential changes are needed.

Lemma 4.3. *If $I = I^*$ is a nil ideal of FG, then I satisfies the $*$-PI's $[x_1 + x_1^*, x_2]^n \equiv 0$ and $([x_1 + x_1^*, x_2]x_3)^n \equiv 0$ for some $n \geq 1$.*

Proof. Let $R = F\{X, X^*\}[[t]]$ be the algebra of formal power series over the free associative algebra with involution $F\{X, X^*\}$ on a finite set X. We regard R as an algebra with involution by setting $t^* = t$.

Let w be the $*$-GI satisfied by FG. By [7, Lemma 1] we may assume that $w = w(x, x^*)$, where $x \in X$. Hence, since the elements $1 + xt$ and $(1 + xt)^* = 1 + x^*t$ generate a free subgroup of $U(R)$ invariant under $*$, we have that $w(1 + xt, 1 + x^*t) \neq 1$. Writing out explicitly the last inequality, we get

$$\sum_{i\geq 0} f_i(x, x^*)t^i \neq 0, \tag{4.1}$$

where $f_i(x, x^*) \in F\{x, x^*\}$ is a homogeneous polynomial in x and x^* for $i \geq 1$. Moreover, there exists $r \geq 1$ such that $f_r(x, x^*)$ is a non-trivial polynomial.

Now, $1 + a\lambda \in U(FG)$ for any $a \in I$ and $\lambda \in F$. Also, a being nilpotent, $(1 + a\lambda)^{-1} = 1 - a\lambda + a^2\lambda^2 - \cdots$. Hence, since w is a $*$-GI of FG, we obtain $w(1 + a\lambda, 1 + a^*\lambda) = 1$, and from (4.1) we get

$$\sum_{i=1}^{k} f_i(a, a^*)\lambda^i = 0$$

for some $k \geq 1$, and $f_h(a, a^*) = 0$ for all $h > k$. Now since F is infinite and the index of nilpotency of λa is independent of λ, by a Vandermonde determinant argument, we obtain that $f_i(a, a^*) = 0$ for $1 \leq i \leq k$.

We have proved that $f_i(x, x^*)$ is a $*$-identity of I for any $i \geq 1$.

We first claim that there exists $l \geq 1$ such that $f_l(x, x^*)$ is not a $*$-PI for $M_2(F)$, the algebra of 2×2 matrices over F with transpose involution $*$. In fact, if we evaluate x into a square zero element a, then it is easily seen that there exists f_l such that $f_l(a, a^*) = \alpha a^{\epsilon_1}(a^*a)^l(a^*)^{\epsilon_2}$, where $\epsilon_1, \epsilon_2 \in \{0, 1\}$, $l \geq 1$ and $\alpha \in F$, $\alpha \neq 0$. Since $f_l(e_{12}, e_{21}) = \alpha e_{12}^{\epsilon_1} e_{22}^{l} e_{21}^{\epsilon_2} \neq 0$, we get that f_l is not a $*$-identity of $M_2(F)$ with transpose involution.

Next we claim that there exists $f_m(x, x^*)$ which is not a $*$-PI for the algebra of 4×4 matrices over F with symplectic involution. In fact, as above we get down to the evaluation of a polynomial f_m into square-zero elements, and we get

$$f_m(e_{12} + e_{34}, e_{43} + e_{21}) = \alpha(e_{12} + e_{34})^{\epsilon_1}(e_{44} + e_{22})^l(e_{43} + e_{21})^{\epsilon_2} \neq 0.$$

The claim is proved also in this case.

We finally claim that there exists a polynomial $g_j(x_1, x_1^*, x_2, x_2^*)$ with $x_1, x_2 \in X$ such that the sum of all monomials of g where no $*$ appears is not a polynomial identity for $M_2(F)$.

Since the elements $(1 + x_1 t)(1 + x_2^* t)$ and $(1 + x_2 t)(1 + x_1^* t)$ generate a free subgroup of $U(R)$ invariant under $*$, we have that $w((1 + x_1 t)(1 + x_2^* t), (1 + x_2 t)(1 + x_1^* t)) \neq 1$.

As above, for any $a, b \in I$ we get

$$\sum_i g_i(a, a^*, b, b^*)\lambda^i = 0,$$

and $f_i(x, x^*)$ is a $*$-identity of I for any $i \geq 1$.

Now take $a, b \in I$ such that $a^2 = b^2 = 0$. Then there exists a polynomial $g_j(x_1, x_2, x_1^*, x_2^*)$ containing a monomial $g = \alpha a^{\epsilon_1}(ba)^l b^{\epsilon_2}$ with $\alpha \in 2\mathbb{Z}$ and $\epsilon_1, \epsilon_2 \in \{0, 1\}$, where a and b appear with greatest occurrences.

Now set $B = F\{X, X^*\}/U$, where U is the T^*-ideal of the free algebra $F\{X, X^*\}$ generated by the polynomials f_i and g_i, $i \geq 1$. Since X is finite, B is a finitely generated algebra with involution satisfying non-trivial $*$-polynomial identities. By Amitsur's theorem [1], B satisfies an ordinary polynomial identity.

Let P be a prime ideal of B.

If P is invariant under $*$, then B/P has an induced involution and f_i vanishes in B/P for all $i \geq 1$. Since B/P is prime, it satisfies all $*$-PIs of $n \times n$ matrices with either transpose or symplectic involution for some $n \geq 1$ (see [17, Corollary 3.1.63]). Since f_l and f_m do not vanish on $M_2(F)$ with $*$ = transpose or on $M_4(F)$ with $*$ = symplectic, respectively, it follows that B/P satisfies all $*$-PIs of $M_2(F)$, $*$ being symplectic. Thus $[x_1 + x_1^*, x_2]$ is a $*$-PI for B/P.

If P is not invariant under $*$, we consider the ideal $(P + P^*)/P$ of B/P. Since for all $a \in P^*$ we have $a + P = a + a^* + P$, we can choose representatives of the elements of $(P + P^*)/P$ to be in P^*. It follows that when evaluating f_j on $(P + P^*)/P$, we are evaluating $x_1^* = x_2^* = 0$. Hence it follows that $(P + P^*)/P$ satisfies the identity $g(x_1, x_2)$ described above. But $g(e_{12}, e_{21}) = \alpha e_{12}^{\epsilon_1}(e_{22})^l e_{21}^{\epsilon_2} \neq 0$ says that g is not an identity of 2×2 matrices over F. It follows that $(P + P^*)/P$ must be commutative. Since B/P is prime, it follows that also B/P is commutative and $[x_1, x_2]$ is a PI for B/P.

We have proved that for all $a, b \in B$ there holds $[a + a^*, b] \in \bigcap_{P=\text{prime}} P = L$, where L is the lower nil radical of B. Since B is a finitely generated PI-algebra, by [18, Theorem 6.3.39], L is nilpotent. Thus $[x_1 + x_1^*, x_2]^k \equiv 0$ and $([x_1 + x_1^*, x_2]x_3)^k \equiv 0$ are $*$-PI for B for some $k \geq 1$. Thus such polynomials belong to U, and so they are also $*$-PIs for I. □

5 *P* of bounded exponent

Throughout this section, we also assume that $U(FG)$ satisfies a $*$-GI. As we remarked above, P is a (normal) subgroup of G and, if FG is not semiprime, $P \neq 1$.

Lemma 5.1. *Suppose that P is of bounded exponent. Then $\Delta(G, P)$ is nil of bounded degree.*

Proof. If P is finite, $\Delta(G, P)$ is nilpotent, so let P be infinite. Then N is nil but not nilpotent. Hence by Lemma 4.2, FG satisfies a PI. By [14, Corollary 3.10], G has a normal p-abelian subgroup A of finite index such that A' is a finite p-group.

In order to prove the lemma, we can factor by the nilpotent ideal $\Delta(G, A')$, and so we may assume that $A' = 1$. Thus G has a normal abelian subgroup A of finite index.

Write $A_p = A \cap P$. Since A_p is of bounded p-power exponent, $\Delta(A, A_p)$ is nil of bounded degree. But then by [15, Lemma 3.2], $\Delta(G, A_p)$ is nil of bounded degree. Observe that $(G : A) < \infty$ implies that $(P : A_p) < \infty$.

It follows that the ideal

$$\frac{\Delta(G,P)}{\Delta(G,A_p)} \leq \frac{FG}{\Delta(G,A_p)} \simeq F(G/A_p)$$

is nilpotent. In fact, for $g \in G$ and $c \in P$, we have $g(1-c) \equiv \bar{g}(1-\bar{c})$ and P/A_p is finite.

It follows that $\Delta(G,P)$ is nil of bounded degree. □

Theorem 5.2. *Let F be an infinite field,* char $F = p \geq 0$, $p \neq 2$. *Let G be a group with no 2-elements and suppose that $T = T(G)$ is a subgroup. If $U(FG)$ satisfies a $*$-GI and P is of bounded exponent, then*
(i) *T/P is abelian and all idempotents of $F(T/P)$ are central in $F(G/P)$;*
(ii) *G/T satisfies a $*$-GI.*
Conversely, if (i) *holds and G/T is a unique product group satisfying a GI, then $U(FG)$ satisfies a GI.*

Proof. The necessity follows since $F(G/P) \simeq FG/\Delta(G,P)$ is semiprime and $U(F(G/P))$ satisfies a $*$-GI as units of $F(G/P)$ can be lifted to those of FG.

Conversely it follows from Theorem 3.10 that $U(F(G/P))$ satisfies a $*$-GI. So FG satisfies a $*$-GI by Lemma 5.1. □

6 *P* of unbounded exponent

We assume throughout that $U(FG)$ satisfies a $*$-GI. By Lemma 4.2, FG satisfies a PI. So G contains a p-abelian subgroup A of finite index. This A can be assumed to be normal and $*$-invariant, which we shall assume throughout.

We start with a lemma whose proof is essentially given in [8, Lemma 5.1].

Lemma 6.1. *Suppose that N is nil but not nilpotent. Then $\Delta(G,P)$ is a locally nilpotent ideal satisfying the identities $[x_1 + x_1^*, x_2]^n \equiv 0$ and $([x_1 + x_1^*, x_2]x_3)^n \equiv 0$ for some $n \geq 1$. Moreover, $\Delta(G,P)$ is nil of bounded degree if and only if P is of bounded exponent.*

Proof. Since A' is a finite p-group, we may factor G by A', and so we may assume that A is abelian.

Let $A_p = A \cap P$. Then $\Delta(A_p)$ is locally nilpotent and, since A is abelian, $\Delta(A_p)FA = \Delta(A,A_p)$ is also locally nilpotent. Let $\Phi : FG \to M_n(FA)$ be the embedding given in [14, Lemma 5.1.10], where $n = [G:A]$. Notice that $\Delta(A_p)FG = \Delta(G,A_p)$ is mapped under Φ into $M_n(\Delta(A_p)FA) = M_n(\Delta(A,A_p))$. Since $\Delta(A,A_p)$ is locally nilpotent, also $M_n(\Delta(A,A_p))$ is locally nilpotent; hence $\Delta(G,A_p)$ is locally nilpotent. Notice that in case P is of bounded exponent, then $\Delta(A_p)$ is nil of bounded degree. Since $\Delta(A,A_p)$ is G-stable, by [15, Lemma 3.2] we conclude that $\Delta(G,A_p)$ is nil of bounded degree.

Consider now the ideal $\Delta(G,P)/\Delta(G,A_p)$ of $FG/\Delta(G,A_p) \cong F(G/A_p)$. Since $[P:A_p] < \infty$, it follows that $\Delta(G,P)/\Delta(G,A_p)$ is nilpotent; hence $\Delta(G,P)$ is locally nilpotent. In case $\Delta(G,P)$ is nil of bounded degree, P is of bounded exponent. By Lemma 4.3, $\Delta(G,P)$ satisfies the identities of the desired form. □

We use the following well-known fact in the proof of the next lemma without mention.

Remark 6.2. If a is a torsion element of order n of G, then

$$\begin{aligned}\operatorname{ann}_{FG}(a-1) &= \{\alpha \in FG \mid (a-1)\alpha = 0\}\\ &= \hat{a}FG\\ &= \Big\{\sum c_i\hat{a}g_i \mid c_i \in F,\ \{g_i\} \text{ is a set of coset representatives of } \langle a\rangle \text{ in } G\Big\}.\end{aligned}$$

Proof. Let $\alpha = \sum_i b_{h_i}h_i$, where $b_{h_i} \in F$, $h_i \in G$ with $(a-1)\alpha = 0$. Then $b_{h_i} = b_{ah_i}$. So also $b_{h_i} = b_{a^2h_i} = \cdots$. Therefore, $\alpha = \hat{a}\beta$ for some $\beta \in FG$. It is also clear that such an element annihilates $a-1$. □

Lemma 6.3. *Let P be of unbounded exponent. Suppose that A is of prime index q in G. Then $(G, A\cap P)^{p^N} = 1$ for some $N \geq 0$.*

Proof. We assume, as we may, that $A' = 1$, and so A is abelian. Let $x \in G \setminus A$. Either $xx^* \in A$, and so $x^* = x^{-1}c$ for some $c \in A$, or $xx^* \notin A$, and so the order of $xx^* \pmod A$ is q. Thus after change of notation, we have that either $x^* = x$ or $x^* = x^{-1}c$ for some $c \in A$.

Also $A \cap P = (A \cap P)_1 \times (A \cap P)_2$. Hence it is enough to prove that $(x, a)^{p^N} = 1$, where $a \in (A \cap P)_1$ or $a \in (A \cap P)_2$.

Let $a \in A \cap P$ be such that $a^* = a$. Then

$$[a-1, (a-1)x] = (a-1)[a-1, x] = (a-1)[a, x] = (a-1)((x, a)-1)ax,$$

and multiplying on the right by $(ax)^{-1}(a-1)$, by Lemma 6.1 we get that $(a-1)^2((x,a)-1)$ is of bounded nilpotent degree. Thus $(a^{p^N}-1)((x,a)^{p^N}-1) = 0$ for some $N \geq 1$ (notice that $2p^{N-1} < p^N$). Out of this equality we get that either $a^{p^N} = 1$ or $(x,a)^{p^N} = 1$. In any case $(x,a)^{p^N} = 1$, and we are done in this case.

Now suppose that $a^* = a^{-1}$. Then

$$\begin{aligned}[a-1+a^{-1}-1, (a-1)x] &= -[(a-1)(a^{-1}-1), (a-1)x]\\ &= -(a-1)[(a-1)(a^{-1}-1), x]\\ &= -(a-1)[a+a^{-1}, x]\\ &= (a-1)(xa+xa^{-1}-ax-a^{-1}x)\\ &= (a-1)(a^x+a^{-x}-a-a^{-1})x.\end{aligned}$$

If we multiply on the right by $x^{-1}(a-1)$, by Lemma 6.1 we get that

$$(a^{p^N}-1)((a^x)^{p^N}+(a^{-x})^{p^N}-a^{p^N}-(a^{-1})^{p^N}) = 0,$$

for some $N \geq 0$. Out of this we get that either $a^{p^N} = 1$ or $a^{p^N} = (a^x)^{p^N}$ or $a^{p^N} = (a^{-x})^{p^N}$. In the first two cases we get $(x,a)^{p^N} = 1$, as wished.

In order to simplify the notation, we write $a^{p^N} = b$ and we are left with the possibility

$$b^x = b^{-1} \text{ and } b^* = b^{-1}. \tag{6.1}$$

We split the proof according as $x^* = x$ or $x^* = x^{-1}c$.

Suppose first that $x^* = x$. Then

$$\begin{aligned}[(b-1)x + x(b^{-1}-1), x(b-1)] &= (b-1)x^2(b-1) + x^2(b-1)^2 - x^2(b^{-1}-1)^2 - x^2(b^{-1}-1)^2\\ &= 2x^2((b-1)^2-(b^{-1}-1)^2)\\ &= 2x^2((b-1)^2-(1-b)^2b^{-2})\\ &= 2x^2(b-1)^2(1-b^{-2})\\ &= 2x^2b^{-2}(b-1)^3(b+1).\end{aligned}$$

Since the commuting elements x^2, b, $(b+1)$ are not zero divisors, we deduce that $(b-1)^{p^N} = 0$ for some N. Thus $b^{p^n} = 1$, and we are done in this case.

Now suppose that $x^* = x^{-1}c$, $b^* = b^{-1}$ and $b^x = b^{-1}$. Then

$$\begin{aligned}[x(b-1)+(b^{-1}-1)x^{-1}c, b-1] &= [x(b-1)+(b^{-1}-1)x^{-1}c, b]\\ &= [x(b-1)-b^{-1}(b-1)x^{-1}c, b]\\ &= x(b-1)b - b^{-1}(b-1)x^{-1}cb - bx(b-1) + (b-1)x^{-1}c\\ &= x(b-1)(b-b^{-1}) - (b-1)(b^{-1}c^xb^{-1}x^{-1} - c^xx^{-1})\\ &= x(b-1)(b-b^{-1}) - (b-1)(b^{-2}-1)c^xx^{-1}\\ &= (b^{-1}-1)(b^{-1}-b)x + b^{-2}(1+b)(b-1)^2c^xx^{-1}\\ &= b^{-2}(1-b)(1-b^2)x + b^{-2}(1+b)(b-1)^2c^xx^{-1}\\ &= b^{-2}(1+b)(1-b)^2(x^2+c^x)x^{-1}.\end{aligned}$$

If we multiply the last expression on the right by $x(b-1)$, by Lemma 6.1 we get that

$$b^{-2}(1+b)(1-b)^3(x^2+c^x) \tag{6.2}$$

is nilpotent of bounded degree.

Since $(1+b)$ and (x^2+c^x) are not zero divisors and b commutes with (x^2+c^x), from (6.2) we deduce that $(1-b)^{p^N}=0$ for some N. □

Lemma 6.4. *Let P be of unbounded exponent. Suppose that A is of prime index q in G. Then $(G,A)^{p^N}=1$ for some $N\geq 0$.*

Proof. We can assume that A is abelian. Since by Lemma 6.3 we have $(G,A\cap P)^{p^N}=1$, factoring with $(G,A\cap P)$, we may assume that $A\cap P$ is central in G. Also $[P:A\cap P]<\infty$ as $[G:A]<\infty$. Hence, since P is of unbounded exponent, then also $A\cap P$ is of unbounded exponent. Also P central by-finite implies by a theorem of Schur [19, p. 39] that P' is a finite group. Hence, taking the quotient with P', we may assume that P is abelian.

Write $A\cap P=S=S_1\times S_2$, where S_i has the meaning of Lemma 2.4, i.e., if $t\in S_1$, then $t^*=t$, and if $t\in S_2$, then $t^*=t^{-1}$. Since S is of unbounded p-power exponent, either S_1 or S_2 must be of unbounded exponent, and we examine the two cases separately.

Suppose first that S_2 is of unbounded exponent and let $t\in S_2$ be of large order. Recall that $t^*=t^{-1}$ and t is central in G.

Take $a\in A$ such that $a^*=a$. Then for any $x\in G$ we have

$$\begin{aligned}[(t+t^{-1})a,x(t-1)]&=(t-1)(t+t^{-1})[a,x]\\&=t^{-1}(t^2+1)(t-1)[x,a]\\&=t^{-1}(t^2+1)(t-1)((x,a)-1)ax.\end{aligned}$$

If we multiply on the right by $(t-1)(ax)^{-1}$, by Lemma 6.1 we get that $(t-1)^2((x,a)-1)$ is bounded nilpotent. Taking t of large enough order, we obtain that $(x,a)^{p^N}=1$, as wished.

Take now a such that $a^*=a^{-1}$. Then

$$\begin{aligned}[(t-t^{-1})(a-a^{-1}),x(t-1)]&=(t-1)(t-t^{-1})[a-a^{-1},x]\\&=t^{-1}(t+1)(t-1)^2(((x,a)-1)ax-((x,a^{-1})-1)a^{-1}x)\\&=t^{-1}(t+1)(t-1)^2(((x,a)-1)-((x,a^{-1})-1)a^{-1}xx^{-1}a^{-1})ax.\end{aligned}$$

If we now multiply on the right by $(t-1)(ax)^{-1}$, by Lemma 6.1 we get that

$$(t-1)^3((x,a)-1-(x,a^{-1})a^{-2}+a^{-2})$$

is nilpotent of bounded p-power exponent. Taking t of large enough order, we get

$$(x,a)^{p^N}-1-(x,a^{-1})^{p^N}a^{-2p^n}+a^{-2p^N}=0.$$

If $(x,a)^{p^N}=1$, we are done. On the other hand, if $(x,a)^{p^N}=(x,a^{-1})^{p^N}$ and $a^{-2p^N}=1$, then $a^{p^N}=1$ and we are also done.

Suppose now that S_1 is of unbounded exponent and let $t\in S_1$. Recall that in this case $t^*=t$.

If $a\in A$ is such that $a^*=a$ and $x\in G$, then

$$[(t-1)a,x(t-1)]=-(t-1)^2[x,a]=(t-1)^2((x,a)-1)ax,$$

and we are done as above.

Suppose now that a is such that $a^*=a^{-1}$ and consider the element

$$[(t-1)(a+a^{-1}),x(a-1)(t-1)]=(t-1)^2[a+a^{-1},x(a-1)].$$

As in the proof of Lemma 6.3, we are left with

$$a^{p^N}=b,\quad b^x=b^{-1},\quad b^*=b^{-1},\quad t^*=t$$

(see (6.1)).

In the two cases $x^* = x$ or $x^{-1}c$, with $c \in A$, we proceed as in the proof of Lemma 6.3 by considering

$$[(t-1)((b-1)x + x(b^{-1}-1)), (t-1)x(b-1)]$$

and

$$[(t-1)x(b-1) + (b^{-1}-1)x^{-1}c, (t-1)(b-1)],$$

respectively, and conclude that $b^{p^N} = 1$, completing the proof. □

Lemma 6.5. *Let P be of unbounded exponent. Suppose that G has a normal $*$-invariant abelian subgroup A of finite index in G. Let H be the normal $*$-closure of $\langle A, x\rangle$, where x is of prime order $q \pmod{A}$, and $x^* = x$ or $x^* = x^{-1}c$, with $c \in A$. Then $(H, A)^{p^N} = 1$ for some $N \geq 0$.*

Proof. First notice that as $x^* = x^\epsilon c$, every element of H is a product of conjugates of $x \pmod{A}$. Let a and d be arbitrary elements of A and fix $g \in G$. Then $(gd, a) = (g, a)$ and, in particular, $(x^*, a) = (x^\varepsilon, a)$. As $(x^g, a) = (x^g, b^g)$ for some $b \in G$, we have that $(x^g, a) = (x, b)^g$. Since A is of index q in $\langle A, x\rangle$, the previous lemma implies that (x, b), and hence also (x^g, a), to the power p^N is equal to 1. A similar argument holds for (x^{-g}, a).

Finally, since every element of H is a product of conjugates of powers of x and of conjugates of elements in A, it suffices to show that if $(y, a)^{p^N} = 1$ for some y, then also $(x^g y, a)^{p^N} = 1$.

In fact, $(x^g y, a) = (x^g, a)^y (y, a) = (x^g, b^g)^y (y, a) = (x, b)^{gy}(y, a)$ for some $b \in A$. As both commutators are in A, which is commutative, it readily follows that $(y, a)^{p^N} = 1$, and the proof is complete. □

Lemma 6.6. *Let G be solvable and let F and $*$ be as stated at the beginning of Section 4. Suppose A is a p-abelian subgroup of finite index and P is of unbounded exponent. Then $(G')^{p^N} = 1$ for a fixed N.*

Proof. We shall use induction on $(G : A)$.

Let $G \triangleright G_1 \triangleright \cdots \triangleright A$, where $G/A \triangleright G_1/A \triangleright \cdots$ is the solvable series of G/A.

If $[G : G_1] < [G : A]$, then $(G_1')^{p^N} = 1$ for some N. Factoring by the normal $*$-invariant subgroup G_1', we can assume that G_1 is abelian. Thus $(G')^{p^M} = 1$ for some M, and we are done.

Hence we may assume that $[G : G_1] = [G : A]$, which implies $G_1 = A$ and G/A is abelian. Now, we find an element $x \in G$ such that the order of $x \pmod{A}$ is a prime. By change of notation we may assume that $x^* = x$ or $x^{-1}c$ for some $c \in A$. Then $G \triangleright \langle A, x\rangle \triangleright A$. By induction again $G = \langle A, x\rangle$. Then by Lemma 6.4, $(G, A)^{p^N} = 1$ for some N. Factoring by (G, A), we conclude that A is central. Hence G is abelian and we are done. □

Lemma 6.7. *If P is of unbounded exponent, then $(G')^{p^N} = 1$ for some $N \geq 0$.*

Proof. We may suppose that A is abelian. If $x \in G \setminus A$, then either $xx^* \in A$ or $xx^* \notin A$. In either case, changing notation, we can find an element $x \in G \setminus A$ of prime order such that either $x^* = x \pmod{A}$ or $x^* = x^{-1}c$, with $c \in A$.

Let $H = \langle A, x\rangle$ and let K be the normal closure of H in G (notice that K is also $*$-invariant). We shall prove the lemma by induction on $[G : A]$.

If $[K : A] < [G : A]$, then $(K')^{p^N} = 1$, and by factoring G by K', since $[G : K] < [G : A]$, we get that $(G')^{p^M} = 1$ for some M. Therefore, we may assume that $G = K$, the normal closure of H.

By Lemma 6.5, $(G, A)^{p^N} = 1$ for some fixed N. Factoring by (G, A), we can assume that A is central. Thus G is an FC-group, and consequently G/T is torsion-free abelian. Since T is $*$-invariant and has no element of order two, it follows from Theorem 2.5 that T is solvable. Hence G is solvable and the result follows from Lemma 6.6. □

Theorem 6.8. *Let F be an infinite field,* char $F = p \geq 0$, $p \neq 2$. *Let G be a group with no 2-elements and suppose that $T = T(G)$ is a subgroup and that FG is not semiprime.*

(i) *If $U(FG)$ satisfies a $*$-GI and P is of bounded exponent, then*
  (I) *T/P is abelian and all idempotents of $F(T/P)$ are central in $F(G/P)$;*
  (II) *G/T satisfies a $*$-GI.*

  Conversely, if (I) *holds and G/T is a unique product group satisfying a GI, then $U(FG)$ satisfies a GI.*

(ii) *If $U(FG)$ satisfies a $*$-GI and P is of unbounded exponent, then*
(a) *G contains a p-abelian subgroup of finite index;*
(b) *G' is of bounded p-power exponent.*
Conversely, if G satisfies (a) *and* (b) *and G/T is a unique product group, then $U(FG)$ satisfies a GI.*

Proof. (i) When P is of bounded exponent, the result is the content of Theorem 5.2.

(ii) If P is of unbounded exponent, then (a) and (b) follow from Lemmas 4.2 and 6.7.

Conversely, suppose that G satisfies (a) and (b) and G/T is a unique product group. Let A be a normal subgroup of finite index with a finite p-group A'. Since $\Delta(G, A')$ is nilpotent, we may factor with A' and assume that A is abelian. Since the subgroup $(G, A) \subseteq G' \cap A$ is a normal abelian subgroup of bounded p-power exponent, $\Delta((G, A))$ is nil of bounded exponent. It follows from [15, Lemma 3.2] that $\Delta(G, (G, A))$ is nil of bounded exponent. Hence we can factor by (G, A) and assume that A is central in G with $[G : A] < \infty$. But then, by a theorem of Schur [19, p. 39], it follows that G' is a finite group. We deduce from hypothesis b) that G' is a finite p-group. Consequently, $U(FG)$ satisfies a GI. □

Corollary 6.9. *Let F be an infinite field,* char $F = p \geq 0$, $p \neq 2$. *Let G be a group with no 2-elements and suppose that $T = T(G)$ is a subgroup and that FG is not semiprime. If P is of unbounded exponent, then $U(FG)$ satisfies a $*$-GI if and only if $U^+(FG)$ satisfies a GI under the assumption that G/T is a unique product group.*

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References

[1] S. A. Amitsur, Identities in rings with involution, *Israel J. Math.* **7** (1969), 63–68.
[2] O. Broche, E. Jespers, C. Polcino Milies and M. Ruiz, Antisymmetric elements in group rings. II, *J. Algebra Appl.* **8** (2009), 115–127.
[3] F. Catino, G. T. Lee and E. Spinelli, Group algebras whose symmetric elements are Lie metabelian, *Forum Math.* **26** (2014), 1459–1471.
[4] A. Giambruno, E. Jespers and A. Valenti, Group identities on units of rings, *Arch. Math. (Basel)* **63** (1994), no. 4, 291–296.
[5] A. Giambruno, C. Polcino Milies and S. K. Sehgal, Group identities on symmetric units, *J. Algebra* **322** (2009), 2801–2815.
[6] A. Giambruno, C. Polcino Milies and S. K. Sehgal, Lie properties of symmetric elements in group rings, *J. Algebra* **321** (2009), 890–902.
[7] A. Giambruno, C. Polcino Milies and S. K. Sehgal, Star-group identities and groups of units, *Arch. Math. (Basel)* **95** (2010), 501–508.
[8] A. Giambruno, S. K. Sehgal and A. Valenti, Group identities on units of group algebras, *J. Algebra* **226** (2000), 488–504.
[9] E. Jespers and M. Ruiz Marin, On symmetric elements and symmetric units in group rings, *Comm. Algebra* **34** (2006), 727–736.
[10] G. T. Lee, *Group Identities on Units and Symmetric Units of Group Rings*, Algebr. Appl. 12, Springer, London, 2010.
[11] G. T. Lee, A survey on $*$-group identities on units of group rings, *Comm. Algebra* **40** (2012), 4540–4567.
[12] G. T. Lee, S. K. Sehgal and E. Spinelli, Group rings whose unitary units are nilpotent, *J. Algebra* **410** (2014), 343–354.
[13] C. H. Liu and D. S. Passman, Group algebras with units satisfying a group identity. II, *Proc. Amer. Math. Soc.* **127** (1999), no. 2, 337–341.
[14] D. S. Passman, *The Algebraic Structure of Group Rings*, Wiley-Interscience, New York, 1977.
[15] D. S. Passman, Group algebras whose units satisfy a group identity. II, *Proc. Amer. Math. Soc.* **125** (1997), 657–662.
[16] C. Polcino Milies and S. K. Sehgal, *An Introduction to Group Rings*, Kluwer Academic, Dordrecht, 2002.
[17] L. H. Rowen, *Polynomial Identities in Ring Theory*, Academic Press, New York, 1980.
[18] L. H. Rowen, *Ring Theory. Vol. II*, Academic Press, New York, 1988.
[19] S. K. Sehgal, *Topics in Groups Rings*, Marcel Dekker, New York, 1978.
[20] S. K. Sehgal, *Units in Integral Group Rings*, Longman Scientific & Technical, New York, 1993.
[21] S. K. Sehgal and A. Valenti, Group algebras with symmetric units satisfying a group identity, *Manuscripta Math.* **119** (2006), 243–254.