

On the first τ -Hochschild cohomology of an algebra

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Abstract. In this paper, we introduce, according to one of the main ideas of τ -tilting theory, the τ -Hochschild cohomology in degree one of a finite-dimensional k -algebra Λ , where k is a field. We define the excess of Λ as the difference between the dimensions of the τ -Hochschild cohomology in degree one and the dimension of the usual Hochschild cohomology in degree one. One of the main results is that for a zero excess bound quiver algebra $\Lambda = kQ/I$, the Hochschild cohomology in degree 2 $\mathrm{HH}^2(\Lambda)$ is isomorphic to the space of morphisms $\mathrm{Hom}_{kQ-kQ}(I/I^2, \Lambda)$. This is useful to determine when $\mathrm{HH}^2(\Lambda) = 0$ for these algebras. We compute the excess for hereditary, radical square zero and monomial triangular algebras. For a bound quiver algebra Λ , a formula for the excess of Λ is obtained. We also give a criterion for Λ to be τ -rigid.

1. Introduction

Let A be a finite-dimensional algebra over a field k that we will call an algebra for short. Let M and N be finitely generated left A -modules, henceforth called left A -modules. Let τ denote the Auslander–Reiten translation (see, for instance, [3] or [16]) and denote $D(-) = \mathrm{Hom}_k(-, k)$. We reproduce an extract from B. Marsh’s lecture notes in Cologne [17]: “the Auslander–Reiten duality suggests that in contexts where $\mathrm{Ext}_A^1(M, N)$ appears, we might investigate replacing it with $D \mathrm{Hom}_A(N, \tau M)$ and this can be regarded as one of the main ideas of τ -tilting theory.” While D is absent in the original text, D is present in Auslander–Reiten’s duality formula for it to be functorial. Of course, adding D does not change the dimensions. Recall that M is called τ -rigid if $\mathrm{Hom}_A(M, \tau M) = 0$ (see, for instance, [16, Subsection 4.1]).

On the other hand, let $\Lambda^e = \Lambda \otimes_k \Lambda^{\mathrm{op}}$ be the enveloping algebra of an algebra Λ . Let X be a Λ -bimodule. The Hochschild cohomology of Λ with coefficients in X is $\mathrm{H}^n(\Lambda, X) = \mathrm{Ext}_{\Lambda^e}^n(\Lambda, X)$ (see [7, 15, 19]), and it is denoted by $\mathrm{HH}^n(\Lambda)$ when $X = \Lambda$. Moreover, Hochschild homology is $\mathrm{H}_n(\Lambda, X) = \mathrm{Tor}_n^{\Lambda^e}(\Lambda, X)$. Since left Λ^e -modules are the same as Λ -bimodules, in the sequel, we often replace Λ^e with $\Lambda - \Lambda$.

According to the main idea of τ -tilting theory mentioned above, we will investigate in this paper the replacement of $\mathrm{Ext}_{\Lambda - \Lambda}^1(\Lambda, X)$ by the τ -Hochschild cohomology

Mathematics Subject Classification 2020: 16E40 (primary); 16G70, 16D20, 16E30 (secondary).

Keywords: Hochschild, cohomology, quiver.

in degree one $\mathsf{H}_\tau^1(\Lambda, X) = \mathsf{D}\mathsf{Hom}_{\Lambda-\Lambda}(X, \tau\Lambda)$. Note that here τ is the Auslander–Reiten translation of left Λ^e -modules, that is, of Λ -bimodules. When $X = \Lambda$, we denote $\mathsf{HH}_\tau^1(\Lambda) = \mathsf{H}_\tau^1(\Lambda, \Lambda)$. The excess $e(\Lambda)$ is defined as $\dim_k \mathsf{HH}_\tau^1(\Lambda) - \dim_k \mathsf{HH}^1(\Lambda)$.

One of the main results of this paper is that for a zero excess bound quiver algebra $\Lambda = kQ/I$, we have $\mathsf{HH}^2(\Lambda) = \mathsf{Hom}_{kQ-kQ}(I/I^2, \Lambda)$ (see Corollary 4.8). This result will be useful in a future work to determine when an algebra with zero excess has zero Hochschild cohomology in degree 2. The algebras Λ with $\mathsf{HH}^2(\Lambda) = 0$ are important since they are rigid in the following sense. Suppose that k is algebraically closed, and let V be a k -vector space of dimension n . Let \mathcal{Alg}_n be the affine open subscheme of algebra structures with 1 of the affine algebraic scheme defined by $\mathcal{S}_n(R) = \{\text{associative } R\text{-algebra structures on } R \otimes_k V\}$, where R is a commutative k -algebra. As stated in [13, Corollary 2.5], $\mathsf{HH}^2(\Lambda) = 0$ if and only if the orbit of $\Lambda \in \mathcal{Alg}_n$ under the general linear group $\text{GL}(V)$ is an open subscheme of \mathcal{Alg}_n – that is, by definition, Λ is rigid. Moreover, P. Gabriel in [13, p. 140] mentions that it should be one of the main tasks of associative algebra to determine for every n the number of irreducible components of \mathcal{Alg}_n . The determination of open orbits makes it possible to obtain lower bounds for the number of irreducible components of \mathcal{Alg}_n , as G. Mazzola did in [18, p. 100].

The paper is organised as follows. In Section 2, we give a more detailed definition, as well as properties of the τ -Hochschild cohomology and of the excess. Let Tr be the transpose of a bimodule (see, for instance, [3]) and recall that X_Λ denotes the coinvariants of a Λ -bimodule X (see Remark 2.4). We prove that $\mathsf{HH}_\tau^1(\Lambda) = (\text{Tr}\Lambda)_\Lambda$, and we give a formula for the dimension of the vector space $\mathsf{HH}_\tau^1(\Lambda)$. Note that in this article we use the symbol $=$ for the existence of a canonical isomorphism.

In Section 3, for a hereditary algebra Λ , we prove that the dimensions of $\mathsf{HH}_\tau^1(\Lambda)$ and $\mathsf{HH}^1(\Lambda)$ are equal. We say that an algebra Λ has the H^2 cancellation properties if $\mathsf{HH}^2(\Lambda) = 0 = \mathsf{H}^2(\Lambda, r^i)$ for all $i > 0$, where r is the Jacobson radical of Λ . For instance, hereditary algebras have the H^2 cancellation properties. We obtain that $e(\Lambda) = 0$ whenever Λ has the H^2 cancellation properties, based on a formula for the dimension of $\mathsf{HH}^1(\Lambda)$ in [8].

In Section 3, we also consider radical square zero algebras and monomial algebras whose quiver has no oriented cycles. For those algebras Λ , we prove that $\mathsf{HH}^1(\Lambda) = 0$ if and only if $\mathsf{HH}_\tau^1(\Lambda) = 0$, and this occurs precisely when Q is a tree. This extends a result of [5]. We provide examples where the excess is not zero.

In Theorem 4.5, we give a formula for the excess of a bound quiver algebra. Finally, we provide a criterion for the algebra Λ to be τ -rigid in terms of the dimension of its Hochschild cohomology in degree 2.

In a future article, we will consider higher τ -Hochschild cohomology in relation to Happel’s question (see [14]).

2. τ -Hochschild cohomology in degree one

We begin this section by briefly recalling the definition of the Auslander–Reiten translation and the duality formula which are useful for our aims; for more details, see, for instance, [3] or [1]. Let A be an algebra and M a left A -module.

First, the transpose $\text{Tr } M$ is defined as follows. Consider a minimal projective presentation of M

$$P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \longrightarrow 0.$$

Applying to d_1 the functor $\text{Hom}_A(-, A)$ which sends left A -modules to right A -modules, we get

$$\text{Hom}_A(P_0, A) \xrightarrow{d_1^*} \text{Hom}_A(P_1, A).$$

By definition, $\text{Tr } M = \text{Coker } d_1^*$.

This gives a bijection between the isomorphism classes of indecomposable non-projective left A -modules and the isomorphism classes of indecomposable non-projective right A -modules.

Next the exact functor $D = \text{Hom}_k(-, k)$ sends right A -modules to left A -modules. We obtain an exact sequence of left A -modules

$$0 \longrightarrow D \text{Tr } M \longrightarrow D \text{Hom}_A(P_1, A) \xrightarrow{Dd_1^*} D \text{Hom}_A(P_0, A).$$

Finally, by definition, $\tau M = D \text{Tr } M$.

This gives a bijection between the isomorphism classes of indecomposable non-projective left A -modules and the isomorphism classes of indecomposable non-injective left A -modules.

Suppose that M and N are left A -modules. Let $\mathcal{I} \text{Hom}_A(M, N)$ be the k -subspace of $\text{Hom}_A(M, N)$ of morphisms which factor through an injective left A -module. The quotient is denoted $\overline{\text{Hom}}_A(M, N)$. The Auslander–Reiten duality formula in [2] is

$$\text{Ext}_A^1(M, N) = D \overline{\text{Hom}}_A(N, \tau M).$$

As mentioned in the introduction, one of the main ideas of τ -tilting theory is to replace $\text{Ext}_A^1(M, N)$ with $D \text{Hom}_A(N, \tau M)$, which in a sense amounts to recover the missing morphisms which factor through injectives.

Let Λ be an algebra. To define the τ -Hochschild cohomology in degree one, recall that $H^1(\Lambda, X) = \text{Ext}_{\Lambda-\Lambda}^1(\Lambda, X)$. Note that this concerns bimodules; hence in the following, τ is the Auslander–Reiten translation for bimodules or equivalently for left Λ^e -modules.

Definition 2.1. Let Λ be an algebra, and let X be a Λ -bimodule. The first τ -Hochschild cohomology of Λ with coefficients in X is

$$H_\tau^1(\Lambda, X) = D \text{Hom}_{\Lambda-\Lambda}(X, \tau \Lambda).$$

In this paper, we will focus on the case $X = \Lambda$:

$$\mathsf{HH}_\tau^1(\Lambda) = \mathsf{D} \mathsf{Hom}_{\Lambda-\Lambda}(\Lambda, \tau\Lambda).$$

Definition 2.2. The *excess* of an algebra Λ is

$$e(\Lambda) = \dim_k \mathsf{HH}_\tau^1(\Lambda) - \dim_k \mathsf{HH}^1(\Lambda).$$

Lemma 2.3. *The excess is a non-negative integer equal to $\dim_k \mathcal{I} \mathsf{Hom}_{\Lambda-\Lambda}(\Lambda, \tau\Lambda)$.*

Proof. By definition,

$$\mathsf{HH}_\tau^1(\Lambda) = \mathsf{D} \mathsf{Hom}_{\Lambda-\Lambda}(\Lambda, \tau\Lambda),$$

while

$$\mathsf{HH}^1(\Lambda) = \mathsf{Ext}_{\Lambda-\Lambda}^1(\Lambda, \Lambda).$$

The Auslander–Reiten duality formula is

$$\mathsf{Ext}_{\Lambda-\Lambda}^1(\Lambda, \Lambda) = \mathsf{D} \left(\frac{\mathsf{Hom}_{\Lambda-\Lambda}(\Lambda, \tau\Lambda)}{\mathcal{I} \mathsf{Hom}_{\Lambda-\Lambda}(\Lambda, \tau\Lambda)} \right). \quad \blacksquare$$

Next we recall some well-known facts about invariants and coinvariants.

Remark 2.4. Let Λ be an algebra, and let X be a Λ -bimodule.

- The subspace of invariants of X is

$$\mathsf{H}^0(\Lambda, X) = X^\Lambda = \{x \in X \mid \forall \lambda \in \Lambda, \lambda x = x\lambda\} = \mathsf{Hom}_{\Lambda-\Lambda}(\Lambda, X),$$

where the last canonical isomorphism sends $\varphi \in \mathsf{Hom}_{\Lambda-\Lambda}(\Lambda, X)$ to $\varphi(1)$.

- The vector space of coinvariants of X is

$$\mathsf{H}_0(\Lambda, X) = X_\Lambda = X / \langle \lambda x - x\lambda \mid \lambda \in \Lambda \text{ and } x \in X \rangle = \Lambda \otimes_{\Lambda-\Lambda} X,$$

where the last canonical isomorphism sends $\lambda \otimes x \in \Lambda \otimes_{\Lambda-\Lambda} X$ to the class of λx .

- It is easy to show that $D(X^\Lambda) = (DX)_\Lambda$. Observe that more generally we have in all degrees

$$DH^n(\Lambda, X) = \mathsf{H}_n(\Lambda, DX).$$

Proposition 2.5. *Let Λ be an algebra. We have*

$$\mathsf{HH}_\tau^1(\Lambda) = (\mathsf{Tr} \Lambda)_\Lambda.$$

Proof. Let $Y = \mathsf{D} \mathsf{Tr} \Lambda$. According to Remark 2.4 and using that D^2 is the identity, we have the following chain of equalities and canonical isomorphisms of vector spaces:

$$\mathsf{HH}_\tau^1(\Lambda) = \mathsf{D} \mathsf{Hom}_{\Lambda-\Lambda}(\Lambda, Y) = \mathsf{D}(Y^\Lambda) = (\mathsf{D} Y)_\Lambda = (\mathsf{D} \mathsf{D} \mathsf{Tr} \Lambda)_\Lambda = (\mathsf{Tr} \Lambda)_\Lambda. \quad \blacksquare$$

In this paper, a *quiver* Q is a finite oriented graph, given by a set of vertices Q_0 , a set of arrows Q_1 and two maps called source and target $s, t : Q_1 \rightarrow Q_0$. The *quiver algebra* kQ is a vector space with basis the set B of all oriented paths in Q , including those of length 0, that is, Q_0 . The product of two paths is their concatenation if it is possible and 0 otherwise. The algebra structure of kQ is obtained by extending linearly the product on paths. Note that Q_0 is a set of orthogonal idempotents, and their sum gives the unit of kQ . The set of paths of strictly positive length $B^{>0}$ is a basis of the ideal $F = \langle Q_1 \rangle$. An ideal I is *admissible* if there exists $n \geq 2$ such that $F^n \subset I \subset F^2$. The quotient algebra kQ/I is called a *bound quiver algebra*.

An algebra Λ is called *sober* if the endomorphism algebra of each simple left Λ -module is reduced to k , which is always the case if k is algebraically closed. A well-known result of P. Gabriel is that any sober algebra is Morita equivalent to a bound quiver algebra kQ/I for a unique quiver Q . Note that the admissible ideal I is in general not unique.

Theorem 2.6. *Let $\Lambda = kQ/I$ be a bound quiver algebra, and let $Z\Lambda$ be its centre. We have*

$$\dim_k \mathrm{HH}_\tau^1(\Lambda) = \dim_k Z\Lambda - \sum_{x \in Q_0} \dim_k x\Lambda x + \sum_{a \in Q_1} \dim_k t(a)\Lambda s(a).$$

Proof. By Proposition 2.5, we have to compute $\dim_k (\mathrm{Tr} \Lambda)_\Lambda$. To begin with, we will consider $\mathrm{Tr} \Lambda$. Let $E = kQ_0$, which is a maximal commutative semisimple subalgebra of kQ . The projective minimal presentation of Λ as Λ -bimodule is known to have the following form (see [6, p. 324] and [4, p. 72]):

$$\Lambda \otimes_E kQ_1 \otimes_E \Lambda \xrightarrow{f} \Lambda \otimes_E \Lambda \longrightarrow \Lambda \longrightarrow 0, \quad (2.7)$$

where $\Lambda \otimes_E \Lambda \rightarrow \Lambda$ is given by the product of Λ . For $a \in Q_1$, we have

$$f(t(a) \otimes a \otimes s(a)) = a \otimes s(a) - t(a) \otimes a.$$

Consequently, for $\lambda, \mu \in \Lambda$, we obtain

$$f(\mu \otimes a \otimes \lambda) = \mu a \otimes s(a)\lambda - \mu t(a) \otimes a\lambda.$$

We write \otimes instead of \otimes_k . Also note that the enveloping algebra Λ^e viewed as a Λ -bimodule is isomorphic to $\Lambda \otimes \Lambda$ with action $\lambda(a \otimes b)\mu = \lambda a \otimes b\mu$.

The functor $\mathrm{Hom}_{\Lambda-\Lambda}(-, \Lambda \otimes \Lambda)$ applied to (2.7) provides the exact sequence defining $\mathrm{Tr} \Lambda$

$$\mathrm{Hom}_{\Lambda-\Lambda}(\Lambda \otimes_E \Lambda, \Lambda \otimes \Lambda) \xrightarrow{f^*} \mathrm{Hom}_{\Lambda-\Lambda}(\Lambda \otimes_E kQ_1 \otimes_E \Lambda, \Lambda \otimes \Lambda) \longrightarrow \mathrm{Tr} \Lambda \longrightarrow 0.$$

Next we use that for an E -bimodule U and a Λ -bimodule X , there is a canonical isomorphism

$$\mathrm{Hom}_{\Lambda-\Lambda}(\Lambda \otimes_E U \otimes_E \Lambda, X) = \mathrm{Hom}_{E-E}(U, X)$$

and observe that $\Lambda \otimes_E \Lambda = \Lambda \otimes_E E \otimes_E \Lambda$. We thus obtain the following exact sequence, where we kept the same notation for the Λ -bimodule morphism f^* :

$$\text{Hom}_{E-E}(E, \Lambda \otimes \Lambda) \xrightarrow{f^*} \text{Hom}_{E-E}(kQ_1, \Lambda \otimes \Lambda) \longrightarrow \text{Tr } \Lambda \longrightarrow 0. \quad (2.8)$$

In the following, we work out the exact sequence (2.8). Let $y, x \in Q_0$, and let ${}_y k_x$ be the simple E -bimodule of dimension 1 given by the idempotent $y \otimes x \in E^e$, namely ${}_y k_x = yE \otimes Ex$. Let U be an E -bimodule. Clearly, we have a canonical isomorphism

$$\text{Hom}_{E-E}({}_y k_x, U) = yUx.$$

Observe that as E -bimodules we have

$$E = \bigoplus_{x \in Q_0} {}_x k_x \quad \text{and} \quad kQ_1 = \bigoplus_{a \in Q_1} {}_{t(a)} k_{s(a)}.$$

The exact sequence (2.8) becomes, by still keeping the same notation for f^* ,

$$\bigoplus_{x \in Q_0} (x\Lambda \otimes \Lambda x) \xrightarrow{f^*} \bigoplus_{a \in Q_1} (t(a)\Lambda \otimes \Lambda s(a)) \longrightarrow \text{Tr } \Lambda \longrightarrow 0. \quad (2.9)$$

Let M be a right Λ -module and N be a left Λ -module, then $M \otimes N$ is a Λ -bimodule for the *internal* action $\lambda(m \otimes n)\mu = m\mu \otimes \lambda n$. On the other hand, $N \otimes M$ is a Λ -bimodule for the *external* action $\lambda(n \otimes m)\mu = \lambda n \otimes m\mu$. Of course, these Λ -bimodules are isomorphic through the flip map $\sigma(n \otimes m) = m \otimes n$.

We rewrite (2.9) using the flip maps

$$\sigma_x : x\Lambda \otimes \Lambda x \rightarrow \Lambda x \otimes x\Lambda \quad \text{and} \quad \sigma_a : t(a)\Lambda \otimes \Lambda s(a) \rightarrow \Lambda s(a) \otimes t(a)\Lambda,$$

thus getting an exact sequence for bimodules with external action. By abuse of notation, we still write f^* instead of $(\bigoplus_{a \in Q_1} \sigma_a) f^* (\bigoplus_{x \in Q_0} \sigma_x^{-1})$:

$$\bigoplus_{x \in Q_0} (\Lambda x \otimes x\Lambda) \xrightarrow{f^*} \bigoplus_{a \in Q_1} (\Lambda s(a) \otimes t(a)\Lambda) \longrightarrow \text{Tr } \Lambda \longrightarrow 0. \quad (2.10)$$

It is an easy but rather meticulous computation to track the morphism of Λ -bimodules f^* along the previous steps. In the end, we obtain the following formula in the context of (2.10):

$$f^*(x \otimes x) = \sum_{\substack{a \in Q_1 \\ s(a)=x}} x \otimes a - \sum_{\substack{b \in Q_1 \\ t(b)=x}} b \otimes x. \quad (2.11)$$

Recall that our aim is to compute the dimension of the coinvariants of $\text{Tr } \Lambda$, that is, of $\Lambda \otimes_{\Lambda-\Lambda} \text{Tr } \Lambda$ by Remark 2.4. The functor $\Lambda \otimes_{\Lambda-\Lambda} -$ is right exact and preserves direct sums, so we obtain the exact sequence

$$\bigoplus_{x \in Q_0} (\Lambda x \otimes x\Lambda)_{\Lambda} \xrightarrow{f_{\Lambda}^*} \bigoplus_{a \in Q_1} (\Lambda s(a) \otimes t(a)\Lambda)_{\Lambda} \longrightarrow (\text{Tr } \Lambda)_{\Lambda} \longrightarrow 0. \quad (2.12)$$

Moreover, as before, let N (resp. M) be a left (resp. right) Λ -module. Consider the Λ -bimodule with external action $N \otimes M$. We have that $(N \otimes M)_{\Lambda}$ is isomorphic to $M \otimes_{\Lambda} N$ via the flip map. Note that this is the degree 0 instance of the graded isomorphism (see, e.g., [7, Corollary 4.4, p. 170]):

$$H_*(\Lambda, N \otimes M) = \text{Tor}_*^{\Lambda}(M, N).$$

Thus,

$$(\Lambda x \otimes y\Lambda)_{\Lambda} = y\Lambda \otimes_{\Lambda} \Lambda x = y\Lambda x$$

which leads to the exact sequence

$$\bigoplus_{x \in Q_0} x\Lambda x \xrightarrow{f_{\Lambda}^*} \bigoplus_{a \in Q_1} t(a)\Lambda s(a) \longrightarrow (\text{Tr } \Lambda)_{\Lambda} \longrightarrow 0. \quad (2.13)$$

We underline that for $y, x \in Q_0$, the multiplicity of the vector space $y\Lambda x$ in the second direct sum is the number of parallel arrows from x to y .

Another easy and rather meticulous computation gives a formula for f_{Λ}^* in the context of (2.13). For $\lambda \in x\Lambda x$, we have

$$f_{\Lambda}^*(\lambda) = \sum_{\substack{a \in Q_1 \\ t(a)=x}} \lambda a - \sum_{\substack{b \in Q_1 \\ s(b)=x}} b\lambda,$$

where $\lambda a \in t(a)\Lambda s(a)$, that is, the direct summand corresponding to a . Similarly, $b\lambda \in t(b)\Lambda s(b)$, that is, the direct summand corresponding to b .

Let $C = \sum_{a \in Q_1} a \in \Lambda$. Note that for $\lambda \in \bigoplus_{x \in Q_0} x\Lambda x$, we have

$$f_{\Lambda}^*(\lambda) = \lambda C - C\lambda.$$

To show that $\text{Ker } f_{\Lambda}^* = Z\Lambda$, it is convenient as usual to consider the k -category \mathcal{C}_{Λ} associated to Λ : its set of objects is Q_0 , while the set of morphisms ${}_v\mathcal{C}_u$ from u to v is $v\Lambda u$; composition is given by the product of Λ . The centre of Λ viewed in this category is

$$\{(x\lambda_x)_{x \in Q_0} \mid {}_v\lambda_v \ {}_v\alpha_u = {}_v\alpha_u \ {}_u\lambda_u \text{ for all } {}_v\alpha_u \in {}_v\mathcal{C}_u\}.$$

On the other hand, as already observed, in the case of parallel arrows, there is one direct summand for each arrow in $\bigoplus_{a \in Q_1} t(a)\Lambda s(a)$. Note also that $Q_0 \cup Q_1$ is a set of generators of \mathcal{C}_{Λ} as an algebra. Using these three observations, the proof of $\text{Ker } f_{\Lambda}^* = Z\Lambda$ is immediate. ■

3. Hereditary, radical square zero and triangular monomial algebras

In this section, we compute the excess (see Definition 2.2) of some families of algebras.

3.1. Hereditary algebras and algebras with the H^2 cancellation properties

We first prove that the excess is zero for hereditary algebras. The proof is based on the fact that the set of morphisms which do not factor through injectives is zero, and we believe it provides a useful method in other contexts.

Later in Theorem 3.4, we generalise the result for algebras with the H^2 cancellation properties (see the introduction for the definition). Its proof relies on the fact that for an algebra Λ with the H^2 cancellation properties, a formula for the dimension of $HH^1(\Lambda)$ is known (see [8]).

Theorem 3.1. *Let Q be a finite connected quiver without oriented cycles. Let $\Lambda = kQ$ be the corresponding hereditary algebra. We have $e(\Lambda) = 0$.*

Proof. We will show that if I is an injective Λ -bimodule, then $\text{Hom}_{\Lambda-\Lambda}(I, \tau\Lambda) = 0$. *A fortiori*, $\mathcal{I} \text{Hom}_{\Lambda-\Lambda}(\Lambda, \tau\Lambda) = 0$. By Lemma 2.3, it follows that $e(\Lambda) = 0$.

We have that $\text{pd}_{\Lambda-\Lambda}\Lambda \leq 1$. Indeed, kQ is the tensor algebra $T_{kQ_0}kQ_1$. It is well known (see, for instance, [9, Theorem 2.3]) that there is a minimal projective resolution of kQ as a kQ -bimodule, as follows:

$$0 \longrightarrow kQ \otimes_{kQ_0} kQ_1 \otimes_{kQ_0} kQ \longrightarrow kQ \otimes_{kQ_0} kQ \longrightarrow kQ \longrightarrow 0. \quad (3.2)$$

We recall [3, Proposition 1.7 (a), p. 319]: let A be an algebra, and let M be an indecomposable left A -module. The projective dimension of M is at most 1 if and only if $\text{Hom}_A(DA, \tau M) = 0$. We will use this result for Λ -bimodules, that is, replacing A by the enveloping algebra of Λ . We have supposed Q connected; therefore, Λ is indecomposable as a Λ -bimodule, and the aforementioned proposition of [3] applies.

It follows that $\text{Hom}_{\Lambda-\Lambda}(D(\Lambda \otimes \Lambda), \tau\Lambda) = 0$. Of course, for an algebra A , every injective left A -module is isomorphic to a direct summand of a direct sum of copies of DA , where A is viewed as a right A -module and DA as a left A -module. ■

Corollary 3.3 ([8, 11, 14]). *Let B be the set of paths of Q , and let $|yBx|$ be the number of paths from x to y . For $\Lambda = kQ$, we have*

$$\dim_k HH^1(\Lambda) = 1 - |Q_0| + \sum_{a \in Q_1} |t(a)Bs(a)| = \dim_k HH_\tau^1(\Lambda).$$

We provide in the following a generalisation of Theorem 3.1 for algebras having the H^2 cancellation properties.

For a bound quiver algebra Λ with the H^2 cancellation properties, the dimension of $HH^1(\Lambda)$ is known by [8, p. 647]. This allows us to prove the following.

Theorem 3.4. *The excess of a bound quiver algebra $\Lambda = kQ/I$ with the H^2 cancellation properties is zero.*

Proof. Let B be the basis of paths of a bound quiver algebra.

We know from [8] that

$$\dim_k \mathbb{H}^1(\Lambda) = \dim_k Z\Lambda - \sum_{x \in Q_0} |xBx| + \sum_{x,y \in Q_0} |yBx||yQ_1x|.$$

Clearly, $|yBx| = \dim_k y\Lambda x$. Hence by Theorem 2.6, the equality of dimensions holds. \blacksquare

Lemma 3.5. *A hereditary algebra kQ has the \mathbb{H}^2 cancellation properties.*

Proof. It follows from (3.2) that $\text{pd}_{kQ-kQ} kQ \leq 1$. Then for any kQ -bimodule X , we have $\mathbb{H}^2(kQ, X) = 0$. \blacksquare

Remark 3.6. We will show in Subsection 3.2 that not only hereditary algebras have the \mathbb{H}^2 cancellation properties.

3.2. Radical square zero algebras

A *radical square zero algebra* is a bound quiver algebra of the form kQ/F^2 .

Let P and P' be two sets of paths of a quiver Q . The set of *parallel paths* is

$$P//P' = \{(p, p') \in P \times P' \mid s(p) = s(p') \text{ and } t(p) = t(p')\}.$$

For instance, $Q_1//Q_0$ corresponds to the set of loops $Q_1^l = \{a \in Q_1 \mid s(a) = t(a)\}$.

We denote by Q_i the set of paths of length i .

A c -crown is a quiver C with c vertices cyclically labelled and c arrows, each one joining each vertex with the next one in the cyclic labelling. For instance, a 1-crown is a loop, and a 2-crown is a two-way quiver $\cdot \leftrightarrows \cdot$. The behaviour of the Hochschild cohomology of kC/F^2 is exceptional (see [10]), and it will be considered separately.

Proposition 3.7. *Let Q be a connected quiver which is not a crown. The radical square zero algebra $\Lambda = kQ/F^2$ has the \mathbb{H}^2 cancellation properties if and only if $Q_2//Q_1 = \emptyset$.*

Proof. Since r is a semisimple Λ -bimodule, the complex of cochains of [10, Section 2] has zero coboundaries and $\dim_k H^2(\Lambda, r) = |Q_2//Q_1|$.

Consequently, if $\Lambda = kQ/F^2$ has the \mathbb{H}^2 cancellation properties, then $|Q_2//Q_1| = 0$.

Reciprocally, note first that if $|Q_2//Q_1| = 0$, then $|Q_1//Q_0| = 0$. From [10, Theorem 3.1], we have

$$\dim_k \mathbb{H}^2(\Lambda) = |Q_2//Q_1| - |Q_1//Q_0| = 0.$$

Hence if $Q_2//Q_1 = \emptyset$, then $\mathbb{H}^2(\Lambda, r) = 0 = \mathbb{H}^2(\Lambda)$. \blacksquare

There are zero excess algebras which do not have the \mathbb{H}^2 cancellation properties (see Corollary 3.9). We first compute the excess of a radical square zero algebra.

Proposition 3.8. *Let Q be a connected quiver which is not a crown, and let $\Lambda = kQ/F^2$. We have $e(\Lambda) = |Q_1//Q_0|$.*

Proof. We will use the formula of Theorem 2.6. First we observe the following:

- $\dim_k Z\Lambda = 1 + |Q_1//Q_0|$.
- For $x \in Q_0$, we have $\dim_k x\Lambda x = 1 + |Q_1//\{x\}|$. Therefore,

$$\sum_{x \in Q_0} \dim_k x\Lambda x = |Q_0| + |Q_1//Q_0|.$$

- For $a \in Q_1 \setminus Q_1^l$, we have $\dim_k t(a)\Lambda s(a) = |t(a)Q_1s(a)|$. Then

$$\sum_{a \in Q_1 \setminus Q_1^l} \dim_k t(a)\Lambda s(a) = |(Q_1 \setminus Q_1^l)//(Q_1 \setminus Q_1^l)|.$$

- We have $\sum_{a \in Q_1^l} \dim_k s(a)\Lambda s(a) = |Q_1^l| + |Q_1^l//Q_1^l|$ since the sum is over the loops – not over the vertices which have loops.
- Finally,

$$\sum_{a \in Q_1} \dim_k t(a)\Lambda s(a) = |Q_1//Q_0| + |Q_1//Q_1|.$$

$$\begin{aligned} \dim_k \mathsf{HH}_\tau^1(\Lambda) &= 1 + |Q_1//Q_0| - |Q_0| - |Q_1//Q_0| + |Q_1//Q_0| + |Q_1//Q_1| \\ &= 1 - |Q_0| + |Q_1//Q_0| + |Q_1//Q_1|. \end{aligned}$$

On the other hand, we know from [10, Theorem 3.1], together with the observation in the next paragraph, that the following formula holds:

$$\dim_k \mathsf{HH}^1(\Lambda) = 1 - |Q_0| + |Q_1//Q_1|. \quad \blacksquare$$

In the proof of Theorem 3.1 in [10], it is stated that “ D is injective for a positive n .” This is right for $n > 0$. However, for $n = 0$, the kernel of D has dimension 1. Hence the formula for $\dim_k \mathsf{HH}^1(\Lambda)$ in the statement of [10, Theorem 3.1] has to be modified by adding 1.

Corollary 3.9. *Let Q be a connected quiver which is not a crown. If the quiver has no loops and $Q_2//Q_1 \neq \emptyset$, then the radical square zero algebra kQ/F^2 does not have the H^2 cancellation properties and has zero excess.*

Proposition 3.10. *Let C be a c -crown, and let $\Lambda = kC/F^2$.*

- If $c > 1$, then $e(\Lambda) = 0$.
- If $c = 1$ and the characteristic of k is not 2, then $e(\Lambda) = 1$.
- If $c = 1$ and the characteristic of k is 2, then $e(\Lambda) = 0$.

Proof. Observe that if $c > 1$, by Theorem 2.6, we have

$$\dim_k \mathsf{HH}_\tau^1(kC/F^2) = 1 - c + c = 1.$$

If $c = 1$, then $kC/F^2 = k[x]/(x^2)$, the algebra of dual numbers, and

$$\dim_k \mathsf{HH}_\tau^1(k[x]/x^2) = 2 - 2 + 2 = 2.$$

On the other hand, it is easy to compute that if $c > 1$, then $\dim_k \mathsf{HH}^1(\Lambda) = 1$, regardless of the characteristic of k .

If the characteristic of k is not 2, then $\dim_k \mathsf{HH}^1(k[x]/(x^2)) = 1$, while if the characteristic of k is 2, then $\dim_k \mathsf{HH}^1(k[x]/(x^2)) = 2$. ■

3.3. Triangular monomial algebras

A *monomial algebra* is a bound quiver algebra $\Lambda = kQ/I$ where I is generated by a minimal set of paths denoted by Z . The algebra Λ is *triangular* if it is a quotient of a finite-dimensional hereditary algebra kQ , that is, if Q has no oriented cycles. We set the following:

- B is the set of paths of Q which do not contain any path of Z . Note that $Z = \emptyset$ if and only if B is the set of all the paths of Q . Moreover, B gives a basis of Λ .
- $(Q_1//B)_u$ is the set of pairs $(a, \varepsilon) \in Q_1//B$ such that for every $\gamma \in Z$, replacing each occurrence of a in γ by ε gives a path which is 0 in Λ . Note that $\{(a, a) \mid a \in Q_1\} \subset (Q_1//B)_u$.
- $(Q_1//B)_{nu} = (Q_1//B) \setminus (Q_1//B)_u$, that is, the set of pairs $(a, \varepsilon) \in Q_1//B$ such that there exists $\gamma \in Z$ where a occurs, verifying that at least one of the replacements of a in γ by ε gives a non-zero path in Λ .

Theorem 3.11. *Let $\Lambda = kQ/\langle Z \rangle$ be a triangular monomial algebra. We have*

$$e(\Lambda) = |(Q_1//B)_{nu}|.$$

Proof. When Q has no oriented cycles, the formula for the dimension of $\mathsf{HH}^1(\Lambda)$ given in [12] is as follows:

$$\dim_k \mathsf{HH}^1(\Lambda) = \dim_k Z\Lambda - |Q_0| + |(Q_1//B)_u|. \quad (3.12)$$

Note that since Q has no oriented cycles, for all $x \in Q_0$ we have $x\Lambda x = k$. Hence Theorem 2.6 gives

$$\dim_k \mathsf{HH}_\tau^1(\Lambda) = \dim_k Z\Lambda - |Q_0| + |(Q_1//B)|. \quad (3.13)$$

This finishes the proof. ■

Theorem 3.14. *Let Q be a connected quiver without oriented cycles, and let $\Lambda = kQ/\langle Z \rangle$ be a triangular monomial algebra. The following are equivalent:*

$$(1) \text{ } \mathsf{HH}^1(\Lambda) = 0.$$

(2) Q is a tree.

$$(3) \text{ } \mathsf{HH}_\tau^1(\Lambda) = 0.$$

Remark 3.15. The equivalence between (1) and (2) is proved without the triangular hypothesis in [5, Theorem 2.2].

Proof. For (1) implies (2), the formula (3.12) gives

$$1 - |Q_0| + |(Q_1//B)_u| = 0.$$

We have $\{(a, a) \mid a \in Q_1\} \subset (Q_1//B)_u$, hence $1 - |Q_0| + |Q_1| \leq 0$. The Euler characteristic of the underlying graph of Q is $\chi(Q) = |Q_0| - |Q_1|$, hence $\chi(Q) \geq 1$. Any finite graph has the homotopy type of a graph with 1 vertex and n loops, whose fundamental group is free on $n = 1 - \chi(Q)$ generators. We infer $n \leq 0$, hence $n = 0$, and Q is a tree.

Concerning (2) implies (3), since Q is a tree, we have $\chi(Q) = |Q_0| - |Q_1| = 1$. On the other hand, $(Q_1//B) = \{(a, a) \mid a \in Q_1\}$, and we have $|Q_1//B| = |Q_1|$. The formula (3.13) gives $\mathsf{HH}_\tau^1(\Lambda) = 0$.

The implication (3) \Rightarrow (1) follows from Lemma 2.3. \blacksquare

Example 3.16. Let Q and R , respectively, denote the quivers

$$\begin{array}{ccccc} \bullet & \xrightarrow{a} & \bullet & \xrightarrow{c} & \bullet \\ & \xrightarrow{b} & & & \end{array}$$

and

$$\begin{array}{ccccc} & & \bullet & & \\ & & \nearrow b & \searrow c & \\ \bullet & \xrightarrow{a} & \bullet & \xrightarrow{d} & \bullet \end{array}$$

The following table lists the results for the corresponding monomial algebras:

Quiver	Z	$(Q_1//B)_u$	$e(\Lambda)$	$\dim_k \mathsf{HH}^1(\Lambda)$	$\dim_k \mathsf{HH}_\tau^1(\Lambda)$
Q	$\{ca\}$	$\{(a, b)\}$	1	2	3
R	$\{ba\}$	\emptyset	0	2	2
R	$\{da\}$	$\{(d, bc)\}$	1	1	2

4. The excess

The proof of the next result relies on the calculation of the dimensions of two vector spaces and the observation that they are equal. An explicit isomorphism between these two vector spaces remains unknown to us.

Proposition 4.1. *Let $\Lambda = kQ/I$ be a bound quiver algebra. We have*

$$\dim_k \mathsf{H}^1(kQ, kQ/I) = \dim_k \mathsf{HH}_\tau^1(\Lambda).$$

Proof. Let X be a kQ -bimodule. We assert that

$$\dim_k \mathsf{H}^1(kQ, X) = \dim_k X^{kQ} - \sum_{x \in Q_0} \dim_k x X x + \sum_{a \in Q_1} \dim_k t(a) X s(a). \quad (4.2)$$

Recall the projective resolution of kQ as a kQ -bimodule (3.2):

$$0 \longrightarrow kQ \otimes_{kQ_0} kQ_1 \otimes_{kQ_0} kQ \xrightarrow{g} kQ \otimes_{kQ_0} kQ \longrightarrow kQ \longrightarrow 0. \quad (4.3)$$

The functor $\mathsf{Hom}_{kQ-kQ}(-, X)$ gives the complex of cochains

$$0 \longrightarrow \mathsf{Hom}_{kQ-kQ}(kQ \otimes_E kQ, X) \xrightarrow{g^*} \mathsf{Hom}_{kQ-kQ}(kQ \otimes_E kQ_1 \otimes_E kQ, X) \longrightarrow 0,$$

where $\mathsf{Ker} g^* = \mathsf{H}^0(kQ, X)$ and $\mathsf{Coker} g^* = \mathsf{H}^1(kQ, X)$. The same way as we have obtained (2.8) and (2.9) leads to an exact sequence

$$0 \longrightarrow \mathsf{H}^0(kQ, X) \longrightarrow \bigoplus_{x \in Q_0} x X x \xrightarrow{g^*} \bigoplus_{a \in Q_1} t(a) X s(a) \longrightarrow \mathsf{H}^1(kQ, X) \longrightarrow 0,$$

which gives the equality (4.2).

We assert that if X is a kQ/I -bimodule, that is, $IX = XI = 0$, then $X^{kQ} = X^{kQ/I}$. Indeed, we have

$$X^{kQ} = \mathsf{Hom}_{kQ-kQ}(kQ, X) = \mathsf{Hom}_{kQ/I-kQ/I}(kQ/I, X) = X^{kQ/I}.$$

Note that for $X = \Lambda$, we have $\Lambda^\Lambda = Z\Lambda$. We obtain the following:

$$\dim_k \mathsf{H}^1(kQ, kQ/I) = \dim_k Z\Lambda - \sum_{x \in Q_0} \dim_k x \Lambda x + \sum_{a \in Q_1} \dim_k t(a) \Lambda s(a)$$

which is the same formula as the one for $\dim_k \mathsf{HH}_\tau^1(\Lambda)$ in Theorem 2.6. ■

Next we recall [9, Corollary 2.4].

Proposition 4.4 ([9]). *Let $\Lambda = kQ/I$ be a bound quiver algebra, and let X be a Λ -bimodule. There is an exact sequence*

$$0 \longrightarrow \mathsf{H}^1(\Lambda, X) \longrightarrow \mathsf{H}^1(kQ, X) \longrightarrow \mathsf{Hom}_{kQ-kQ}(I/I^2, X) \longrightarrow \mathsf{H}^2(\Lambda, X) \longrightarrow 0.$$

An immediate consequence of the above is a formula for the excess of an algebra, which involves the dimension of the Hochschild cohomology in degree 2.

Theorem 4.5. *Let $\Lambda = kQ/I$ be a bound quiver algebra. We have*

$$e(\Lambda) = \dim_k \text{Hom}_{kQ-kQ}(I/I^2, \Lambda) - \dim_k \text{HH}^2(\Lambda).$$

Remark 4.6. If $I = 0$, then $\text{HH}^2(kQ) = 0$, and so $e(kQ) = 0$. This confirms Theorem 3.1, as well as Theorem 3.4 for a hereditary algebra.

We infer three corollaries for a bound quiver algebra $\Lambda = kQ/I$.

Corollary 4.7. *If Λ verifies the H^2 cancelling properties, then*

$$\text{Hom}_{kQ-kQ}(I/I^2, \Lambda) = 0.$$

Corollary 4.8. *If $e(\Lambda) = 0$, then*

$$\text{HH}^2(\Lambda) = \text{Hom}_{kQ-kQ}(I/I^2, \Lambda).$$

Corollary 4.9. *The algebra Λ is τ -rigid as a Λ -bimodule if and only if*

- $\text{HH}^1(\Lambda) = 0$,
- $\dim_k \text{HH}^2(\Lambda) = \dim_k \text{Hom}_{kQ-kQ}(I/I^2, \Lambda)$.

Funding. This work was supported by the projects PIP-CONICET 11220200101855CO and USP-COFECUB. E.N. Marcos was supported by the thematic project of FAPESP 2018/23690-6 and research grants from CNPq 308706/2021-8 and 310651/2022-0. A. Solotar is a research member of CONICET (Argentina) and a Senior Associate at ICTP.

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Received 7 November 2024; revised 15 August 2025.

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