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Abstract

Let $\mathrm{SG}(M, \mathbb{R}^p)$ be the set of special generic maps of a closed manifold M into $\mathbb{R}^p$. We give a necessary and sufficient condition for two elements of $\mathrm{SG}(M, \mathbb{R}^p)$ to belong to the same connected component. Furthermore, we relate the set of connected components to the set of regular equivalence classes of $\mathrm{SG}(M, \mathbb{R}^p)$ and use this relation to give some examples.

1 Introduction

A smooth map from a smooth n-manifold M to $\mathbb{R}^p$, $n \geq p \geq 1$, is called **special generic** when its only singularities are definite fold points. An important tool in the study of a special generic map $f : M \to \mathbb{R}^p$ is its Stein factorization. This is given by a commutative diagram

$$\begin{array}{ccc} M & & \\ q_f \downarrow & \overset{f}{\searrow} & \\ W_f & \xrightarrow{f'} & \mathbb{R}^p \end{array}$$

where W_f is the quocient space obtained by the identification to a point of each connected component of each fiber of f. When M is closed and $n > p$, W_f has a smooth structure of a compact oriented n-manifold with boundary such that f' is an orientation preserving immersion. Furthermore, q_f is smooth and maps the singular set $S(f)$ of f diffeomorphically onto ∂W_f. For the basic tools and notations we refer to [S1].

Assume that M is closed and connected and $n > p$. Let $SG(M, \mathbb{R}^p)$ be the set of special generic maps of M into $\mathbb{R}^p$. In section 2. we prove the following theorem.

Theorem 1.1 *f and g belong to the same connected component of $SG(M, \mathbb{R}^p)$ if, and only if, there are diffeomorphisms $H : M \to M$ and $h : W_f \to W_g$ such that H is smoothly isotopic to* id $: M \to M$, *$g' \circ h$ is regularly homotopic to f' and the diagram*

$$\begin{array}{ccc} M & \xrightarrow{H} & M \\ q_f \downarrow & & \downarrow q_g \\ W_f & \xrightarrow{h} & W_g \end{array}$$

commutes.

Burlet-de Rham [BR], Porto-Furuya [PF], Saeki [S1] defined some equivalence relations on the set $SG(M, \mathbb{R}^p)$.

In particular, following Porto-Furuya, we say that the special generic maps f and g from M to $\mathbb{R}^p$ are *regularly equivalent* if there are diffeomorphisms $H : M \to M$ and $h : W_f \to W_g$ such that the diagram in Theorem 1.1 commutes and $g' \circ h$ is regularly homotopic to f'.

The following theorem, due to Porto-Furuya [PF], is completed by Saeki in [S1].

Theorem 1.2 *The special generic maps $f, g : M \to \mathbb{R}^p$ are regularly equivalent if and only if, there is a smooth family $f_t, t \in [0,1]$, of special generic maps and a diffeomorphism $K : M \to M$ such that $f_0 = f$ and $f_1 = g \circ K$.*

Notice that it follows from the openness of $SG(M, \mathbb{R}^p)$ in $C^\infty(M, \mathbb{R}^p)$, with the C^∞-topology, that f and g are in the same component if, and only if, there is a smooth family f_t, $t \in [0,1]$, of special generic maps from f to g. Hence, two special generic maps f and g are regularly equivalent if, and only if, there is a diffeomorphism K such that $f \circ K$ and g belong to the same connected component of $\mathrm{SG}(M, \mathbb{R}^p)$.

Let $f \in \mathrm{SG}(M, \mathbb{R}^p)$. Consider the set

$$A_f = \{\sigma \in \pi_0(\mathrm{SG}(M, \mathbb{R}^p)) : g \text{ is regularly equivalent to } f,$$
$$\text{where g represents } \sigma\}$$

Consider the following equivalence relation on Diff M : $H \sim K$ if there are $\Phi \in$ Diff M, isotopic to $H \circ K^{-1}$, and $\varphi \in \mathrm{Diff}(W_f)$ such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{\Phi} & M \\ q_f \downarrow & & \downarrow q_f \\ W_f & \xrightarrow{\varphi} & W_f \end{array}$$

commutes and $f' \circ \varphi$ is regularly homotopic to f'. Let $G_f = \frac{\mathrm{Diff}\, M}{\sim}$ be the quotient set of this relation. Note that H and K isotopic implies that $H \sim K$. The relation between the connected components and the regular equivalence classes of $\mathrm{SG}(M, \mathbb{R}^p)$ is given by the following proposition.

Proposition 1.3 *The map* $G_f \longrightarrow A_f$ *obtained by factorizing the map* $H \mapsto f \circ H$ *is well defined and a bijection.*

In a few cases, the regular equivalence classes of $\mathrm{SG}(M, \mathbb{R}^p)$ are known (see section 3.). However, the connected components of $\mathrm{SG}(M, \mathbb{R}^p)$ have not been studied before, up to author's knowledge.

In section 3. we show that $\mathrm{SG}(S^n, \mathbb{R})$ is connected if $n > 1$ and $n \neq 4, 5$, as well as $\mathrm{SG}(S^3, \mathbb{R}^2)$, $\mathrm{SG}(S^5, \mathbb{R}^2)$ and $\mathrm{SG}(S^{11}, \mathbb{R}^p)$ for $p = 8, 9$ or 10. In general, $\pi_0(\mathrm{SG}(S^n, \mathbb{R}))$ is given by $\pi_0(\mathrm{Diff}^+ D^n)$, up to bijection. We prove that $\pi_0(\mathrm{SG}(S^1 \times S^2, \mathbb{R}^2))$ is given bijectively by $\mathbb{Z}$. We also prove that $\pi_0(\mathrm{SG}(S^4, \mathbb{R}^2))$ is given by $\pi_0(\mathrm{Diff}^+ S^4)$. Another result is that the connected components of $\mathrm{SG}(S^5, \mathbb{R}^3)$ are in 1-1 correspondence to the set of oriented diffeomorphism classes of homotopy 3-spheres.

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2 Proofs of Theorem 1.1 and Proposition 1.3

Before proving theorem 1.1, we need the following lemma.

Lemma 2.1 *Let $f_t : M \longrightarrow \mathbb{R}^p$, $t \in (-\varepsilon, 1+\varepsilon)$ for some $\varepsilon > 0$, be a smooth family of special generic maps, where M is closed and connected of dimension $n > p$. Then $F : M \times (-\varepsilon, 1+\varepsilon) \longrightarrow \mathbb{R}^p \times \mathbb{R}$ given by $F(u,t) = (f_t(u), t)$ is special generic. Furthermore, for each singular point (u_0, t_0) of F, there are local coordinates $(x_1, ..., x_n, t)$ around (u_0, t_0) and $(y_1, ..., y_p, s)$ around $F(u_0, t_0)$, where $t : M \times I \rightarrow I$ and $s : \mathbb{R}^p \times \mathbb{R} \rightarrow \mathbb{R}$ are the usual projections onto the last coordinates and such that, in these coordinates, F is given by*

$$\begin{cases} y_i \circ F &= x_i, i = 1, \ldots, p-1 \\ y_p \circ F &= x_p^2 + \cdots + x_n^2 \\ s \circ F &= t. \end{cases}$$

Proof

For notation on Singularities Theory we refer to [GG]. We first note that $(u_0, t_0) \in U$ is a singular point of F if, and only if, u_0 is a singular point of f_{t_0}. The map $G : M \times (-\varepsilon, 1+\varepsilon) \longrightarrow \mathbb{R}^p \times \mathbb{R}$ given by $G(u,t) = (f(u, t_0), t)$ is special generic. Consider the family $H_s : M \times (-\varepsilon, 1+\varepsilon) \longrightarrow \mathbb{R}^p \times \mathbb{R}$, given by $H_s(u,t) = (f(u, t_0 + s(t - t_0)), t), s \in [0,1]$. Since G is locally stable, then for small positive s, G is locally equivalent to H_s, at (u_0, t_0). On the other hand, it is easily seen that H_s is also equivalent to F at (u_0, t_0). It follows that (u_0, t_0) is a definite fold point of F and hence F is special generic.

It also follows that there are diffeomorphisms $H : \mathcal{U}' \longrightarrow \mathcal{U}''$ and $h : \mathcal{V}' \longrightarrow \mathcal{V}''$ such that $h \circ F \circ H^{-1} = G$, where $\mathcal{U}'$ and $\mathcal{U}''$ are neighborhoods of (u_0, t_0) and $\mathcal{V}'$ and $\mathcal{V}''$ are neighborhoods of $F(u_0, t_0)$.

Let $\phi = (x_1, ... x_n)$ and $\psi = (y_1, ... y_p)$ be local coordinates around u_0 and $f_{t_0}(u_0)$. Let Φ' and Ψ' be given in neighborhoods of (u_0, t_0) and $F(u_0, t_0)$, by $\Phi'(u,t) = (\phi(u), t)$ and $\Psi'(v,s) = (\psi(v), s)$.

Then, computing matrices for the derivatives of $L = \Phi' \circ H \circ \Phi'^{-1}$ and $l = \Psi' \circ h \circ \Psi'^{-1}$, we see that

$\frac{\partial(L_1,...,L_{n-1})}{\partial(x'_1,...,x'_{n-1})}(x'_0,t_0) \neq 0$ and $\frac{\partial(l_1,...,l_{p-1})}{\partial(x'_1,...,x'_{p-1})}(y'_0,t_0) \neq 0$, possibly after a linear change of coordinates in $\mathbb{R}^{n+1}$ and $\mathbb{R}^{p+1}$, where $x'_0 = \phi(u_0)$ and $y'_0 = \psi(f_{t_0}(u_0))$.

Define

$$K(u,t) = (H_1(u,t),...,H_n(u,t),t),$$
$$k(v,s) = (h_1(v,s),...,h_p(v,s),s).$$

Then K is a local diffeomorphism around $(u_0,t_0) \in M \times I$, and k is a local diffeomorphism around $F(u_0,t_0) \in \mathbb{R}^p \times \mathbb{R}$ such that

$$k(f(u,t),t) = (h_1(f(u,t),t),...,h_p(f(u,t),t),t) =$$
$$(H_1(u,t),...,H_{p-1}(u,t),H_p^2(u,t)+...H_n^2(u,t),t).$$

Taking smaller neighbourhoods, if necessary, which may be of a convenient form, we may define $\Phi = \Phi' \circ K, \Psi = \Psi' \circ k$, then $\Psi \circ f \circ \Phi^{-1}(x,t) = (x_1,...,x_{p-1},x_p^2+...+x_n^2,t)$.□

Proof of Theorem 1.1

This proof follows closely to the proof of Theorem 1.2 ([PF] and [S1]). For the only if part, we get the Stein factorization of F

$$\begin{array}{ccc} M \times I' & & \\ q_F \downarrow & \overset{F}{\searrow} & \\ W_F & \xrightarrow{F'} & \mathbb{R}^p \times \mathbb{R} \end{array}$$

and a diffeomorphism $L : W_f \times I' \longrightarrow W_F$ such that the diagrams

$$\begin{array}{ccccc} W_f \times I' & \xrightarrow{L} & W_F & \xrightarrow{F'} & \mathbb{R}^p \times I' \\ & \pi_2 \searrow & \downarrow p_2 \circ F' & \swarrow p_2 & \\ & & I' & & \end{array}$$

commute, where $\pi_2 : W_f \times I' \longrightarrow I'$ is the projection on the second factor and $L(y,0) = y$, for any $y \in W_f$.

Next we construct diffeomorphisms $\Gamma : M \times I \longrightarrow M \times I$ and $\theta : W_f \times I \longrightarrow W_f \times I$ such that the diagram

$$\begin{array}{ccc} M \times I & \xrightarrow{\Gamma} & M \times I \\ q_f \times id \downarrow & & \downarrow L^{-1} \circ q_F \\ W_f \times I & \xrightarrow{\Theta} & W_f \times I \end{array}$$

commutes.

Recall that $F' \circ L$ is a local diffeomorphism. Then, for each $(u_0, t_0) \in M \times I'$ there are special local coordinate systems $(\mathcal{U}, \Phi)$ around (u_0, t_0) and $(\mathcal{V}, \Psi)$ around $L^{-1} \circ q_F(u_0, t_0)$ such that F is locally given by

$$\Psi \circ F \circ \Phi^{-1}(x_1, ..., x_n, t) = (x_1, ..., x_p, 0, ..., 0, t)$$

or, by Lemma 2.1, by

$$\Psi \circ F \circ \Phi^{-1}(x_1, ..., x_n, t)$$
$$= (x_1, ..., x_p, x_p^2 + ... + x_n^2, t).$$

In both cases we consider the field $\frac{\partial}{\partial t}$ on $U \times J$, and thus the local field $X_{\mathcal{U}} = (\Phi_*)^{-1}(\frac{\partial}{\partial t})$ on $\mathcal{U} \subset M \times I$, as well as $\frac{\partial}{\partial s}$ on $V \times J$, and $Y_{\mathcal{V}}(\Psi_*)^{-1}(\frac{\partial}{\partial s})$ on $\in W_f \times I$. It follows that $(L^{-1} \circ q_F)_* X_{\mathcal{U}} = Y_{\mathcal{V}}$. (Here the stars indicate derivatives.)

We piece together the local fields $X_{\mathcal{U}}$ and the local fields $Y_{\mathcal{V}}$ by convenient partitions of unity subordinate to covers of $M \times I$ and $\mathbb{R}^p \times \mathbb{R}$, given by the above neighborhoods to get smooth vector fields X on $M \times I$ and Y on $W_f \times I$ such that $(L^{-1} \circ q_F)_* X = Y$. Following the integral curves of those vector fields we get the desired diffeomorphisms $\Gamma : M \times I \longrightarrow M \times I$ and $\Theta : W_f \times I \longrightarrow W_f \times I$, which are of the forms $\Gamma(u,t) = (\gamma(u,t), t)$ and $\Theta(z,s) = (\theta(z,s), s)$ and $\gamma(u,0) = (u,0)$.

Let $H : M \longrightarrow M$ and $h : W_f \longrightarrow W_g$ be the diffeomorphisms given by $H(u) = \gamma(u,1)$ and $h(z) = L \circ \theta(z,1)$. Then H is isotopic to identity and the diagram

$$\begin{array}{ccc} M & \xrightarrow{H} & M \\ q_f \downarrow & & \downarrow q_g \\ W_f & \xrightarrow{h} & W_g \end{array}$$

commutes. Let $\alpha_s : W_f \longrightarrow \mathbb{R}^n$ be given by $\alpha_s(z) = p_1 \circ F'(L(z,s))$. Then

$$\begin{aligned} \alpha_0(z) &= p_1 \circ F'(L \circ \theta(z,0)) = f'(z), \\ \alpha_1(z) &= p_1 \circ F'(L \circ \theta(z,1)) = g' \circ h(z) \end{aligned}$$

and α_s is a regular homotopy from f' to $g' \circ h$. □

Proof of Proposition 1.3

Let $K \in \mathrm{Diff}(M)$. Suppose that $[f \circ K] = [f]$ as elements of $\pi_0(\mathrm{SG}(M, \mathbb{R}^p))$. Then, by Theorem 1.1, there is a commutative diagram

$$\begin{array}{cccc} M & \xrightarrow{H} & M & \\ q_f \downarrow & & \downarrow & q_{f\circ K} \\ W_f & \xrightarrow{h} & W_{f\circ K} & \end{array}$$

such that H is isotopic to identity and $(f \circ K)' \circ h$ is regularly homotopic to f'. Consider the commutative diagrams

$$\begin{array}{ccccccc} M & \xrightarrow{H} & M & \xrightarrow{K} & M & & \\ & & & & & \overset{f}{\searrow} & \\ q_f \downarrow & & \downarrow q_{f\circ K} & & \downarrow q_f & & \\ W_f & \xrightarrow{h} & W_{f\circ K} & \xrightarrow{k} & W_f & \xrightarrow{f'} & \mathbb{R}^p \end{array}$$

where k is the unique diffeomorphism such that the middle diagram commutes, that is, $(f \circ K)' = f' \circ k$.

Define $\Phi = K \circ H, \varphi = k \circ h$. Then Φ is smoothly isotopic to K. Furthermore, we have

$$f' \circ \varphi = f' \circ k \circ h = (f \circ K)' \circ h.$$

Thus $f' \circ \varphi$ is regularly homotopic to f'.

Conversely, if there is a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\Phi} & M \\ q_f \downarrow & & \downarrow q_f \\ W_f & \xrightarrow{\varphi} & W_f \end{array}$$

such that Φ is smoothly isotopic to K and $f' \circ \varphi$ is regularly homotopic to f' we set $H = K^{-1} \circ \Phi$ and $h = k^{-1} \circ \varphi$. Then Φ is isotopic to identity. As

$$(f \circ K)' \circ h = f' \circ k \circ h = f' \circ \varphi,$$

then $(f \circ K)' \circ h$ is regularly homotopic to f'. Hence $[f] = [f \circ K]$.

It follows that $[f \circ H] = [f \circ K]$ if and only if $H \sim K$. This means that the given maps $G_f \longrightarrow A_f$ well defined and injective. To prove that the map $A_f \longrightarrow G_f$ is also surjective, consider $[g] \in A_f$. Then by Theorem 1.2 there is a diffeomorphism $H : M \longrightarrow M$ such that $\sigma = [f \circ H^{-1}]$. Then the result follows.□

3 Examples

Our first example concerns to Morse functions with exactly two critical points. The following result may be known. However, up to author's knowledge, it was not explicitly written before.

Proposition 3.1 *$\pi_0(SG(M^n, \mathbb{R}))$ is given by $\pi_0(Diff^+ D^n)$, up to bijection. In particular, $SG(M^n, \mathbb{R})$ is connected if $n \neq 4, 5$.*

Proof

Let $f : M^n \longrightarrow \mathbb{R}$ be special generic. Then f is a Morse function with exactly two critical points. From now on we refer to [C1] and

[Mi]. It is known that in this case M^n is homeomorphic to S^n. More precisely, let us assume that the image of f is $J = [-1, 1]$, by an isotopy of $\mathbb{R}$. Then there are orientation preserving diffeomorphisms $h : S^{n-1} \to S^{n-1}$ and $H : M^n \longrightarrow D_1^n \cup_h - D_2^n$ such that $f = p_h \circ H$, where p_h is the map given by $x \mapsto 1 - \|x\|^2$ on D_1^n and $x \mapsto -1 + \|x\|^2$ on D_2^n, and $D_i^n, i = 1, 2$, are two copies of the standard unit disk. If $k : S^{n-1} \to S^{n-1}$ is also an orientation preserving diffeomorphism, then h is isotopic to k if, and only if, there is an orientation preserving diffeomorphism $T : D_1^n \cup_h -D_2^n \longrightarrow D_1^n \cup_k -D_2^n$ such that $p_h = p_k \circ T$. In fact, if h is isotopic to k, then one easily constructs such a T. For the converse, recall that the inclusion $\mathcal{O}(n) \longrightarrow \mathrm{Diff}^{\,t}(D^n)$ is a deformation retraction where $\mathrm{Diff}^{\,t}(D^n)$ is the space of the orientation preserving diffeomorphisms T of D^n such that $\|T(x)\| = \|x\|$, for any $x \in D^n$.

Assume first that $n \neq 4, 5$. Then $\pi_0(\mathrm{Diff}^+ D^n) = 0$. If M^n is diffeomorphic to S^n then h extends to an orientation preserving diffeomorphism of D^n. Hence h is isotopic to id $(n \neq 4, 5)$. It follows that $f = p_{id} \circ K$ where K is some orientation preserving diffeomorphism of S^n (if we identify S^n to $D_1^n \cup_{id} -D_2^n$, to simplicity) . Since K is diffeomorphic to a diffeomorphism Φ such that $\Phi = id$ on the lower hemisphere of S^n and such that the restriction of Φ to the upper hemisphere is isotopic to id, then Φ is isotopic to a diffeomorphism T of S^n that $p_{id} \circ T = p_{id}$. If M^n is not diffeomorphic to S^n, we get the same result as follows. Notice that it follows from Palais-Cerf Theorem (see for example [Mi]) that $D_1^n \cup_h -D_2^n$ is diffeomorphic to $D_1^n \cup_k -D_2^n$ if, and only if, $h \circ k^{-1}$ extends to D^n. Since $n \neq 4, 5$, in this case h is isotopic to k. Hence for any $f : M^n \to \mathbb{R}$, $f = p_h \circ H$, for a fixed h and some H. Similarly to the S^n case, any diffeomorphism of $D_1^n \cup_h -D_2^n$ is isotopic to a diffeomorphism Φ that is the identity on D_2^n and hence it is isotopic to a diffeomorphism T such that $p_h \circ T = p_h$. This implies that $\mathrm{SG}(M^n, \mathbb{R})$ is connected. Hence the case $n \neq 4, 5$ is done.

If $n = 4$ or 5, then any orientation preserving diffeomorphism of S^{n-1} extends to D^n $(\Gamma_n = 0$, [C1]). Hence M^n is diffeomorphic to S^n. For any $n > 1$, there is a surjective map $E : \pi_0(\mathrm{Diff}^+ D^n) \longrightarrow \pi_0(\mathrm{SG}(S^n, \mathbb{R}))$, which associates the component of $p_h \circ H$ to the isotopy class of a given $H_0 \in \mathrm{Diff}^+ D^n$, where $h = H_0|_{\partial D^n}$ and $H : S^n \longrightarrow D_1^n \cup_h -D_2^n$ is given by H_0 on D_1^n and by id on D_2^n. We have to show that E is bijective. It follows from Theorem 1.1, $p_h \circ H$ and $p_k \circ K$ are in the same component if, and only if, there are orientation preserving diffeomorphisms $L : D_1^n \cup_h -D_2^n \longrightarrow D_1^n \cup_k -D_2^n$ isotopic to $H \circ K^{-1}$ and $l : J \to J$ such that $p_k \circ L = l \circ p_h$. Then we only have to show that L is isotopic to T such that $T = id$ on D_1^n and $\|T(x)\| = \|x\|$ on D_2^n. First, by isotoping L and l, we may assume that $l(0) = 0$ (recall that for any h and any $[a, b] \subset (-1, 1)$ there is a suitable diffeomorphism of $p_h^{-1}([a, b])$ onto $S^n \times [a, b]$). Now we note that if $R : D^n \to D^n$ and $r : [0, 1] \to [0, 1]$ are orientation preserving diffeomorphisms such that $\|R(x)\|^2 = r(\|x\|^2)$, then R is isotopic to the restriction of a linear map (and hence isotopic to id). More precisely, this isotopy may be given by

$$R(x,t) = \frac{R(\sqrt{r^{-1}(t^2)}x)}{t}, 0 < t \leq 1,$$

$$R(x,0) = \sqrt{(r^{-1\prime}(0)}dL_0(x).$$

Then the assertion follows from suitable compositions, using the last isotopy. Hence $E(H_0) = E(K_0)$ if, and only if, H_0 is isotopic to K_0. This gives the result for $n = 4$ or 5 (or for S^n, $n > 1$). □

Remark 3.2 From the proof of Proposition 3.1, it follows that any two Morse functions with exactly two critical points are regularly equivalent, if $n = 4$, since $\pi_0(\mathrm{Diff}^+ S^3) = 0$. For $n = 5$, they are regularly equivalent if, and only if, they are in the same component.

For the next examples, we recall basic results in [S1]. Let $f : M \to \mathbb{R}^p$ be special generic (M closed n-dimensional and $n > p \geq 1$). Then Saeki ([S1], Prop.2.1) shows that M is diffeomorphic to $E \cup_h B$, where $\pi : E \longrightarrow W_f$ is a smooth S^{n-p} - bundle over W_f, $\pi' : B \longrightarrow \partial W_f$ is a linear D^{n-p+1} - bundle over ∂W_f and $h : \partial B \longrightarrow \partial E$ is an isomorphism of the restriction bundles $\pi : \partial E \longrightarrow \partial W_f$, $\pi' : \partial B' \longrightarrow \partial W_f$, covering the identity on ∂W_f. Furthermore, he showed that $\pi_1(W_f)$ is isomorphic to $\pi_1(M)$.

Remark 3.3 For a fixed f, consider the set B_f of the special generic maps g such that W_g is diffeomorphic to W_f. Assume that for each orientation preserving diffeomorphism $h : W_f \longrightarrow W_g$ there is a diffeomorphism $H : M \longrightarrow M$ such that $q_g \circ H = h \circ q_f$. In this case the regular equivalence classes are classified by the image-equivalence classes of orientation preserving immersions $\alpha : W_f \to \mathbb{R}^n$. Here $\alpha, \beta : W_f \to R^n$ are *image equivalent* if there is a diffeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ such that $\beta \circ h$ is regularly homotopic to α. This is the n-dimensional analogue of the definition of image equivalence by Kauffman[K]. In fact, under these conditions, $g = g' \circ q_g = g' \circ h \circ q_f \circ H^{-1}$ and hence g is regularly equivalent to $g' \circ h \circ q_f$ where $g' \circ h : W_f \longrightarrow \mathbb{R}^2$ is an orientation preserving immersion. Consider the maps $\alpha \circ q_f$ and $\beta \circ q_f$, where α and β are orientation preserving immersions of W_f into $\mathbb{R}^n$. If there is a diffeomorphism $h : W_f \to W_f$ such that $\beta \circ h$ is regularly homotopic to α then $\beta \circ h \circ q_f$ is regularly equivalent to $\alpha \circ q_f$. Since $\beta \circ h \circ q_f = \beta \circ q_f \circ H$, where $h \circ q_f = q_f \circ H$, then $\beta \circ h \circ q_f$ is regularly equivalent to $\beta \circ q_f$. Hence $\alpha \circ q_f$ and $\beta \circ q_f$ are regularly equivalent. Conversely, if $\alpha \circ q_f$ and $\beta \circ q_f$ are regularly equivalent then there exists h such that α and $\beta \circ h$ are regularly homotopic, by the difinition of regular equivalence.

Remark 3.4 If $p = 2$, then, for each f, W_f is an oriented 2-manifold with boundary and ∂W_f is diffeomorphic to a disjoint union of a finite number of copies of S^1. Hence, when $m = 3$, E is isomorphic to $W_f \times S^1$, ∂B is isomorphic to $\partial W_f \times S^1$ and B to $\partial W_f \times D^2$. Then the conditions in Remark 3.2 hold [BR]. Hence the regular equivalence classes are the union of the image equivalence classes of immersions $\alpha : W \to \mathbb{R}^2$, over the differentiable classes of surfaces W such that $W = W_f$ for some f. The image equivalence classes of immersions of an oriented 2-dimensional manifold with boundary into $\mathbb{R}^2$ were studied by Kauffman [K]. For example, if W is diffeomorphic to $S^1 \times J, J = [-1, 1]$, Kauffman showed that the image equivalence classes are given by Z_+. We note also that any two orientation preserving immersions of D^2 into $\mathbb{R}^2$ are regularly equivalent.

Proposition 3.5 *$SG(S^3, \mathbb{R}^2)$ is connected.*

Proof

The result follows from Remark 3.4 and Proposition 1.3 since W_f is diffeomorphic to D^2 and $\pi_0(\mathrm{Diff}^+ S^3) = 0$. □

Proposition 3.6 *$\pi_0(SG(S^1 \times S^2), \mathbb{R}^2$ is in one-to-one correspondence to $\mathbb{Z}$.*

Proof

It follows from Remark 3.4, that the set of the regular equivalence classes in $\mathrm{SG}(S^1 \times S^2, \mathbb{R}^2)$ are in $1-1$ correspondence to the image-equivalence classes of orientation preserving immersions of $S^1 \times J$ in $\mathbb{R}^2$, $J = [-1, 1]$. In [K], Kauffman proves that the image equivalence classes of orientation preserving immersions $\alpha : S^1 \times J \to \mathbb{R}^2$ are in $1-1$ correspondence to $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, where α corresponds to n if the degree of $\alpha |_{S^1 \times \{0\}}$ is $\pm n$.

Notice also that $\pi_0(\mathrm{Diff}^+(S^1 \times J)$ is isomorphic to $\mathbb{Z}_2$, generated by the diffeomorphism $h_1 : S^1 \times J \to S^1 \times J$ given by the standard reflection $e^{i\theta} \to e^{-i\theta}$ on S^1 and the reflection $(t \to -t)$ on J. Furthermore, if $\alpha : S^1 \times J \to R^2$ is an orientation preserving immersion, then the degree of $\alpha \circ h_1 |_{S^1 \times \{0\}}$ is equal to minus the degree of $\alpha |_{S^1 \times \{0\}}$. Thus $\alpha \circ h_1$ is not regularly homotopic to α, unless α corresponds to $n = 0$. Hence, if h is isotopic to h_1, then $\alpha \circ h$ is not regularly homotopic to α.

On the other hand, it follows from Gluck's Theorem [G], that $\pi_0(\mathrm{Diff}^+(S^1 \times S^2))$ is generated by of the following diffeomorphisms :

$H_1 : S^1 \times S^2 \to S^1 \times S^2$ given by the standard reflection on S^1 and the antipodal map on S^2,
the map $H_2 : S^1 \times S^2 \to S^1 \times S^2$ that rotates $S^2 \subset \mathbb{R}^3$ around the z-axis by an angle θ for each $e^{i\theta} \in S^1 \subset \mathbb{C}$,
the composite $H_3 = H_2 \circ H_1$,
the identity.

All these diffeomorphisms maps fibers of q onto fibers of q. This means that for each $H = H_i, i = 1, 2, 3$, there is a diffeomorphism $h : S^1 \times J \to S^1 \times J$ such that the diagram

$$
\begin{array}{ccc}
S^1 \times S^2 & \xrightarrow{H} & S^1 \times S^2 \\
q \downarrow & & \downarrow q \\
S^1 \times J & \xrightarrow{h} & S^1 \times J
\end{array}
$$

commutes. If $H = H_1$ then $h = h_1$. Let H be a diffeomorphism isotopic to H_1, that maps fibers of q to fibers of q. Then h is isotopic to h_1. In fact, assume by contradiction that h is isotopic to identity. Then $H(S^1 \times 0) = S^1 \times 0$, and $H : S^1 \times 0 \longrightarrow S^1 \times 0$ is orientation preserving. Then H, and hence H_1, is isotopic to a diffeomorphism K, such that $K = id : S^1 \times 0 \longrightarrow S^1 \times 0$. In this case, $K_* = id : \pi_1(S^1 \times S^2, b) \longrightarrow \pi_1(S^1 \times S^2, b)$, and $H_{1*} = -id : \pi_1(S^1 \times S^2, b) \longrightarrow \pi_1(S^1 \times S^2, b) \simeq \mathbb{Z}$, where $b = (1, 0)$. This is a contradiction. It follow that h is not isotopic to identity and hence, it is isotopic to h_1.

If $H = H_2$, then $h = \mathrm{id} : S^1 \times J \to S^1 \times J$. Hence, by Proposition 1.3, $\pi_0(\mathrm{SG}(S^1 \times S^2, \mathbb{R}^2))$ is in $1-1$ correspondence to $\mathbb{Z}$. □

Remark 3.7 If $f : S^n \to \mathbb{R}^2$ is special generic, then W_f is diffeomorphic to D^2 and hence E is isomorphic to $D^2 \times S^{n-2}$, B is isomorphic to $S^1 \times D^{n-1}$ and ∂B to $S^1 \times S^{n-2}$. If $n - 2 = 1, 2$ or 3, then $\mathcal{O}(n-1) \longrightarrow \mathrm{Diff}^+ S^{n-2}$ is a weak homotopy equivalence (see [C1] and [H]). Then the arguments in the Remark 3.2 also apply to these cases. Since any two orientation preserving immersions of D^2 are regularly homotopic, then there is a unique regular equivalence class. If also $\pi_0(\mathrm{Diff}^+ S^n) = 0$ then $\mathrm{SG}(S^n, \mathbb{R}^2)$ is connected, by Theorem 1.1. This is the case of $n = 3, 5$. Since the group $\pi_0(\mathrm{Diff}^+ S^4)$ is unknown, it is necessary to study this group in relation to the fibers of the standard projection $q : S^4 \longrightarrow D^2$. We get the next propositions.

Proposition 3.8 *$SG(S^5, \mathbb{R}^2)$ is connected.*

Proposition 3.9 *$\pi_0(SG(S^4, \mathbb{R}^2))$ is given by $\pi_0(Diff^+ S^4)$, up to bijection .*

Proof

Let $q : S^4 \to \mathbb{R}^2$ be the standard projection. Then, for a suitable collar neighborhood of ∂D^2 in D^2, then q decomposes to q_0 :

$S^1 \times D^3 \to S^1 \times I$, $I = [0,1]$, and $q_1 : D^2 \times S^2 \to D^2$, where q_1 is the trivial bundle projection and q_0 is given by $q_0(\theta, x) = (\theta, \|x\|^2)$, and where S^4 is diffeomorphic to $S^1 \times D^3 \cup_h D^2 \times S^2$, $h : \partial(S^1 \times D^3) \simeq S^1 \times S^1 \to \partial(D^2 \times S^1) \simeq S^1 \times S^1$ is the trivial bundle isomorphism covering $id : S^1 \times 1 \simeq S^1 \to \partial D^2 \simeq S^1$. From Remark 3.7 and Proposition 1.3, the connected components of $\mathrm{SG}(S^4, \mathbb{R}^2)$ are given by the equivalence classes of diffeomorphisms of S^4, where $H \sim K$ if, and only if, there are orientation preserving diffeomorphisms Φ of S^4, isotopic to $H \circ K^{-1}$, and ϕ of D^2 such that $q \circ \Phi = \phi \circ q$. Hence, by a first isotopy of φ which lifts to Φ, we may assume that $\varphi = id$ outside the collar neighborhood. Then, the commutative diagram

$$\begin{array}{ccc} S^4 & \xrightarrow{\Phi} & S^4 \\ q \downarrow & & \downarrow q \\ D^2 & \xrightarrow{\varphi} & D^2 \end{array}$$

decomposes to the following commutative diagrams

$$\begin{array}{ccc} S^1 \times D^3 & \xrightarrow{\Phi_0} & S^1 \times D^3 \\ q_0 \downarrow & & \downarrow q_0 \\ S^1 \times I & \xrightarrow{\varphi_0} & S^1 \times I \end{array}$$

and

$$\begin{array}{ccc} D^2 \times S^2 & \xrightarrow{\Phi_1} & D^2 \times S^2 \\ q_1 \downarrow & & \downarrow q_1 \\ D^2 & \xrightarrow{\varphi_1} & D^2 \end{array}$$

such that $\varphi_0(\theta, 1) = (\theta, 1)$, for each $\theta \in S^1$, and $\varphi_1 = id$. Then, since $\Phi_1|_{0 \times S^2}$ isotopes to id we may isotope Φ on S^4 such that $\Phi_1 = id$ and $\Phi_0|S^1 \times S^1 = id$. Set $\Phi_0 = (\Psi_1, \Psi_2)$ and $\varphi_0 = (\psi_1, \psi_2)$, and define $(\Psi_2)_\theta : D^3 \to D^3$ and $(\psi_2)_\theta : I \to I$ by $(\Psi_2)_\theta(x) = \Psi_2(\theta, x)$ and $(\psi_2)_\theta(t) = \psi_2(\theta, t)$. Then we may isotope $(\Psi_2)_\theta$ and $(\psi_2)_\theta$ in a similar way as in Proposition 3.1 to get isotopies of Φ_0 and φ_0 fixing $S^1 \times 1$ and such that, at the end, $(\psi_2)_\theta(t) = t$ for each $t \in I$.

If we further lift to $S^1 \times D^3$ a convenient isotopy of $S^1 \times I$ given by isotopies on $S^1 \times t$ fixed for $t = 1$, then we get $\varphi_0 = id$. Now, since Diff $^+(D^3 \mathrm{rel} \partial D^3)$ is homotopically equivalent to a point [H], then Φ_0 is isotopic to id by an isotopy that fixes $\partial(S^1 \times D^3)$. Hence Φ is isotopic to id. Hence $H \sim K$ if, and only if, H is isotopic to K. This implies that $\pi_0(\mathrm{SG}(S^4, \mathbb{R}^2))$ is given by $\pi_0(\mathrm{Diff}^+ S^4)$. □

Proposition 3.10 *Assume that $p \geq 6$ and $n-p = 1, 2$ or 3. If $\pi_0(Diff^+ S^n) = 0$, then $\pi_0(SG(S^n, \mathbb{R}^p))$ is connected. In particular, $\pi_0(SG(S^{11}, \mathbb{R}^p))$ is connected for $p = 8, 9, 10$.*

Proof

Arguments in Remark 3.7 also apply to these cases. Since $\pi_0(\mathrm{Diff}^+ S^n) = 0$, then the result follows.□

Proposition 3.11 *Assume that $p \geq 6$ and that $n - p = 1$ or 2. Then $\pi_0(SG(S^n, \mathbb{R}^p))$ is given by $\pi_0(Diff^+ S^n)$.*

Proof

This proof is similar to that of Proposition 3.9. Recall that Diff $^+(D^{n-p+1} \mathrm{rel} \partial D^{n-p+1})$ is contractible if $n - p + 1 = 1$ and is homotopically equivalent to a point if $n - p + 1 = 2$ and hence $\pi_{p-1}(\mathrm{Diff}^+(D^{n-p+1} \text{ rel } \partial D^{n-p+1})) = 0$. □

Remark 3.12 Saeki proved that the set of regular equivalence classes of $\mathrm{SG}(S^4, \mathbb{R}^3)$, as well as $\mathrm{SG}(S^5, \mathbb{R}^3)$, in his sense, is in 1-1 correspondence to the set of diffeomorphism classes of homotopy 3-spheres [S2]. Notice that in Saeki's definitions of regular equivalence classes a diffeomorphism of $\mathbb{R}^p$ is added. Hence the set of regular equivalence classes, in the sense we defined it, is in 1-1 correspondence to the set of oriented diffeomorphism classes of homotopy 3-spheres.

Proposition 3.13 *$\pi_0(SG(S^5, \mathbb{R}^3))$ is in one-to-one correspondence with the set of oriented diffeomorphism classes of homotopy 3-spheres.*

Proof

Since $\pi_0(\text{Diff}^+ S^5) = 0$ then, by theorem 2.1, the elements of $\pi_0(\text{SG}(S^5, \mathbb{R}^3))$ are the regular equivalence classes of special generic maps from S^5 to $\mathbb{R}^2$. Thus, by [S2] and Remark 3.12, $\pi_0(\text{SG}(S^5, \mathbb{R}^3))$ is $1-1$ correspondence to the set of oriented diffeomorphism classes of homotopy 3-spheres. □

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