A Polynomial Collocation Method for Singular Integro-differential Equations in Weighted Spaces

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Abstract

A polynomial collocation method is proposed for the numerical solution of a class of singular integro-differential equations of Cauchy type; the collocation points are chosen to be the Chebyshev nodes. Function spaces are defined and theorems concerning the boundedness of certain operators are developed. Convergence of the numerical method is demonstrated in weighted uniform normed spaces of continuous functions; convergence rates are then determined in accordance with the smoothness of the functions characterising the problem. Numerical examples are provided which go some way to confirming these estimates.

Keywords: *Cauchy singular integro-differential equations, Polynomial collocation methods and Convergence, Weighted spaces of continuous functions*

1 Introduction

This paper is devoted to the investigation of the convergence rates of a polynomial collocation method for solving singular integro-differential equations (SIDE) of the form:

$$a_1\,\varphi'(x) + \frac{b_1}{\pi}⨍_{-1}^{1} \frac{\varphi'(t)}{t-x}\,dt + \int_{-1}^{1} l_1(x,t)\varphi'(t)dt + a_2(x)\varphi(x) + \int_{-1}^{1} l_2(x,t)\varphi(t)dt = f(x),$$
$$|x| < 1, \tag{1}$$

where $a_2(x)$, $f(x)$ and $l_{1,2}(x,t)$ are Hölder continuous functions on $(-1,1)$ (on $(-1,1)^2$ in the case of $l_{1,2}$) and a_1, b_1 are given constants, such that $a_1^2 + b_1^2 = 1$. The symbol $⨍_{-1}^{1}$ denotes the Cauchy principal value and the non-homogeneous boundary conditions are given by:

$$\varphi(-1) = \xi_1 \quad \text{and} \quad \varphi(1) = \xi_2.$$

The integro-differential equation (1) is first transformed into an integral equation via a change of the dependent variable, as shall be demonstrated in Section 2. The resulting integral equation is then approximated by collocation at the zeros of first-kind Chebyshev polynomials. We shall present an analysis of the mapping properties and boundedness of the various operators involved. The convergence of the numerical method is then analyzed in weighted spaces of continuous functions, more specifically, in subspaces of Besov-type spaces. The use of Jackson's theorem for the weighted norms will prove to be essential.

In the literature, the convergence of numerical methods for the solution of SIDE has been studied in a variety of function spaces. Cuminato [7] and Nagamine & Cuminato [16], proposed a numerical method for solving equation (1) for $a_1 = 0$ and homogeneous boundary conditions $\varphi(-1) = \varphi(1) = 0$; and they also presented a convergence analysis in the uniform norm. Due the form of the term $a_2(x)\varphi(x)$ (after the change of variables), the authors were only able to demonstrate a convergence rate of $\frac{1}{2}$, irrespective of the smoothness of the functions involved. Here, we show that this restriction can be weakened and a convergence rate of $\frac{5}{2}$ can be obtained, depending on the smoothness of the functions.

One of the first studies on the convergence of polynomial approximation methods for the numerical solution of singular integral equations (SIE) in weighted uniform normed spaces was presented in Capobianco et al. [3] and extended in [10]. In [4], [5] and [13], the authors considered a SIDE with homogeneous boundary conditions and employed the identity

$$a_1\,\varphi'(x) + \frac{b_1}{\pi}\int_{-1}^{1}\frac{\varphi'(t)}{t-x}\,dt \;=\; \frac{d}{dx}\left[a_1\,\varphi(x) + \frac{b_1}{\pi}\int_{-1}^{1}\frac{\varphi(t)}{t-x}\,dt\right]$$

which can be demonstrated by integration by parts and which leads to a simplification of the analysis. However, the Besov spaces required in performing the analysis in [3] are more restrictive than those required with the analysis of this paper.

In [4], equation (1) with $a_1 = 0$ was studied in the weighted space $\mathscr{L}^2$, in [5] this analysis was performed in a weighted uniform norm and in [13] the mapping properties of the singular integral operators, as well as their upper bounds, were discussed in the weighted space $\mathscr{L}^1$ where the restrictions are less stringent.

This paper is organized as follows. First the dependent variable of the integro-differential equation is transformed so that the problem takes a homogeneous form. In Section 2 the problem is reduced to a Cauchy integral equation and in Section 3 the new dependent variable is approximated by Jacobi polynomials with the Chebyshev nodes as collocation points. Section 4 introduces the abstract spaces that will be employed throughout the paper, while Section 5 discusses the properties of the operators. This then allows the analysis of convergence to be undertaken in section 6. Finally in Section 7 the paper concludes with numerical examples which illustrate these convergence estimates.

2 Preliminaries

Before deriving a numerical method for solving (1), we shall perform the following change of variables

$$\psi(x) = \varphi(x) + \frac{(\xi_1 - \xi_2)}{2}\,x \;-\; \frac{(\xi_1 + \xi_2)}{2}$$

so that the boundary conditions take the homogeneous form:

$$\psi(-1) = 0 \quad \text{and} \quad \psi(1) = 0.$$

In the new dependent variable $\psi(x)$, equation (1) becomes

$$a_1\,\psi'(x) + \frac{b_1}{\pi}\int_{-1}^{1}\frac{\psi'(t)}{t-x}\,dt + \int_{-1}^{1} l_1(x,t)\psi'(t)\,dt + a_2(x)\psi(x) + \int_{-1}^{1} l_2(x,t)\psi(t)\,dt = \overline{f}(x), \quad (2)$$

where

$$\overline{f}(x) = f(x) + \frac{(\xi_1 - \xi_2)}{2}\left[a_1 + \frac{b_1}{\pi}\log\left|\frac{1-x}{1+x}\right| + \int_{-1}^{1} l_1(x,t)\,dt + a_2(x)\,x + \int_{-1}^{1} l_2(x,t)\,t\,dt\right] - \frac{(\xi_1 + \xi_2)}{2}\left[a_2(x) + \int_{-1}^{1} l_2(x,t)\,dt\right].$$

Note that the function $\overline{f}$ has singularities at -1 and 1, whenever $\xi_1 \neq \xi_2$.

Changing now the dependent variable according to

$$u(x) = \psi'(x) \;\text{ or, equivalently }\; \psi(x) = \int_{-1}^{x} u(t)dt$$

transforms the SIDE into a SIE. So substituting the new variable into (2), yields

$$a_1 u(x) + \frac{b_1}{\pi} \fint_{-1}^{1} \frac{u(t)}{t-x}\,dt + \int_{-1}^{1} l_1(x,t)u(t)dt + a_2(x)\int_{-1}^{x} u(t)dt + \int_{-1}^{1} l_2(x,t)\left[\int_{-1}^{t} u(s)ds\right]dt = \overline{f}(x). \tag{3}$$

For the new dependent variable $u(x)$ the boundary conditions now simply become

$$\int_{-1}^{1} u(t)\,dt = 0. \tag{4}$$

Integrating the term $\int_{-1}^{1} l_1(x,t)u(t)\,dt$, by parts we see that equation (3) can be rewritten as:

$$a_1 u(x) + \frac{b_1}{\pi} \fint_{-1}^{1} \frac{u(t)}{t-x}\,dt + a_2(x)\int_{-1}^{x} u(t)dt + \int_{-1}^{1}\left(l_2(x,t) - \frac{\partial\, l_1(x,t)}{\partial t}\right)\left(\int_{-1}^{t} u(s)ds\right)dt = \overline{f}(x) \tag{5}$$

with the boundary condition $\int_{-1}^{1} u(t)\,dt = 0$.

According to the theory of Cauchy singular integral equations (CSIE) (see [15]), the solution $u(x)$ of equation (5) has the form $u(x) = g(x)\omega^{\alpha,\beta}(x)$, where g is unknown and $\omega^{\alpha,\beta}(x) = (1-x)^{\alpha}(1+x)^{\beta}$ is a Jacobi weight with $-1 < \alpha, \beta < 1$, defined by

$$\alpha = \frac{1}{2\pi \mathrm{i}}\log\left(\frac{a_1 - \mathrm{i}b_1}{a_1 + \mathrm{i}b_1}\right) + M, \quad \beta = -\frac{1}{2\pi \mathrm{i}}\log\left(\frac{a_1 - \mathrm{i}b_1}{a_1 + \mathrm{i}b_1}\right) + N,$$

and M and N integers determined such that the index $\varkappa = -(\alpha + \beta) = -(M+N)$ is restricted to $\{-1, 0, 1\}$. Due to the form of equation (5) and due to the fact that $\psi(-1) = \psi(1) = 0$, we need only consider $\varkappa = 1$ and $-1 < \alpha, \beta < 0$ (see [15] for details).

In what follows, we shall refer to equation (5) in operator form as:

$$[\, H + (D + L)R \,]\, g = \overline{f} \tag{6}$$

where

$$\begin{cases} Hg(x) &=& a_1\, g(x)\omega^{\alpha,\,\beta}(x) + \dfrac{b_1}{\pi} \displaystyle\fint_{-1}^{1} \dfrac{g(t)\omega^{\alpha,\,\beta}(t)}{t-x}\, dt, \\ Rg(x) &=& \displaystyle\int_{-1}^{x} g(t)\omega^{\alpha,\,\beta}(t)\, dt, \\ Dh(x) &\equiv& a_2(x)h(x), \\ l(x,t) &=& -\dfrac{\partial\, l_1(x,t)}{\partial t} + l_2(x,t), \\ Lh(x) &\equiv& \displaystyle\int_{-1}^{1} l(x,t)\, h(t)\, dt. \end{cases} \tag{7}$$

In [7] it has been shown that, for $\varkappa = 1$, the operator

$$H^I g(x) = a_1 \omega^{-\alpha,\,-\beta}(x) g(x) - \frac{b_1}{\pi} \fint_{-1}^{1} \frac{\omega^{-\alpha,\,-\beta}(t) g(t)}{t-x}\, dt,$$

satisfies

$$H^I\, H\, g = g + g_0,$$

where $g_0 \in \ker H$ is an arbitrary constant. A unique solution of equation (6) exists if the operator $\{\, I + H^I\, [\, (\, D + L\,)\, R\,]\,\}$ is invertible and the condition

$$\int_{-1}^{1} \omega^{\alpha,\,\beta}(x) \{\, I + H^I\, [\, (\, D + L\,)\, R\,]\,\}^{-1} 1\, dx \neq 0,$$

is satisfied, where 1 represents the constant function identically 1. (See for instance [15] for details). This will be assumed throughout this paper.

3 The numerical method

We define the approximation of the function $g(x)$ as:

$$g_n(x) = c_0 P_0^{\alpha,\,\beta}(x) + c_1 P_1^{\alpha,\,\beta}(x) + \ldots + c_n P_n^{\alpha,\,\beta}(x), \tag{8}$$

where $\{P_j^{\alpha,\beta}\}_{j=0}^n$ is a sequence of Jacobi polynomials, orthogonal with respect to the weight $\omega^{\alpha,\beta}(x)$, and $c_0, c_1, \dots, c_n$ are unknown constants to be determined. We remark that due to the boundary condition (4) it can be shown that $c_0 = 0$.

Substituting g_n into (6), we obtain the residual equation:

$$\begin{aligned} r_n(x) &= \sum_{j=1}^{n} c_j \left\{ a_1\, \omega^{\alpha,\,\beta}(x) P_j^{\alpha,\,\beta}(x) + \frac{b_1}{\pi} \int_{-1}^{1} \frac{\omega^{\alpha,\,\beta}(t) P_j^{\alpha,\,\beta}(t)}{(t-x)}\, dt \right. \\ &+ \left. a_2(x) \int_{-1}^{x} \omega^{\alpha,\,\beta}(t) P_j^{\alpha,\,\beta}(t)\, dt + \int_{-1}^{1} l(x,t) \left(\int_{-1}^{t} \omega^{\alpha,\,\beta}(s) P_j^{\alpha,\,\beta}(s)\, ds \right) dt \right\} \\ &- \overline{f}(x). \end{aligned} \tag{9}$$

Note that the residual can be written in terms of the operators, defined in (7), as:

$$r_n(x) = \{\, [\, H + (D+L)\, R] g_n - \overline{f}\, \}(x). \tag{10}$$

From the Rodrigue's formula [17, (4.3.1)], we have

$$\omega^{\alpha,\,\beta}(x) P_j^{\alpha,\,\beta}(x) = -\frac{1}{2j} \frac{d}{dx} [\, \omega^{\alpha+1,\beta+1}(x)\, P_{j-1}^{\alpha+1,\,\beta+1}(x)\,].$$

This then implies that

$$\int_{-1}^{x} \omega^{\alpha,\,\beta}(t) P_j^{\alpha,\,\beta}(t)\, dt = -\frac{1}{2j}\, \omega^{\alpha+1,\,\beta+1}(x) P_{j-1}^{\alpha+1,\,\beta+1}(x).$$

Taking this result into (9), yields

$$\begin{aligned} r_n(x) &= \sum_{j=1}^{n} c_j \left\{ a_1\, \omega^{\alpha,\,\beta}(x) P_j^{\alpha,\,\beta}(x) + \frac{b_1}{\pi} \int_{-1}^{1} \frac{\omega^{\alpha,\,\beta}(t) P_j^{\alpha,\,\beta}(t)}{(t-x)}\, dt \right. \\ &- \left. \frac{1}{2j} \left[a_2(x) \omega^{\alpha+1,\,\beta+1}(x) P_{j-1}^{\alpha+1,\,\beta+1}(x) + \int_{-1}^{1} l(x,t)\, \omega^{\alpha+1,\,\beta+1}(t) P_{j-1}^{\alpha+1,\,\beta+1}(t)\, dt \right] \right\} \\ &- \overline{f}(x). \end{aligned} \tag{11}$$

To deal with the first two terms of (11), we applied the following result.

From [12, p. 226, (2.1)] we know that the orthogonal polynomials $\{P_n^{\alpha,\beta}\}$ and $\{P_n^{-\alpha,-\beta}\}$ with respect to weights $\omega^{\alpha,\,\beta}(x)$ and $\omega^{-\alpha,\,-\beta}(x)$ respectively, satisfy

$$a\omega^{\alpha,\,\beta}(x) P_n^{\alpha,\,\beta}(x) + \frac{b}{\pi} \int_{-1}^{1} \frac{\omega^{\alpha,\,\beta}(t) P_n^{\alpha,\,\beta}(t)}{(t-x)}\, dt = -\frac{2^{-\varkappa}}{\sin(\pi\,\alpha)}\, b P_{n-\varkappa}^{-\alpha,\,-\beta}(x), \tag{12}$$

where a and b are such that $a^2 + b^2 = 1$, $P_n^{\alpha,\,\beta} = P_n^{-\alpha,\,-\beta} \equiv 0$ for $n < 0$, and $\varkappa$ is the index of the SIE.

Applying formula (12) to equation (11) and recalling that $\varkappa = 1$, we obtain

$$r_n(x) = \sum_{j=1}^{n} c_j \Bigg\{ \frac{-b_1 P_{j-1}^{-\alpha,\,-\beta}(x)}{2\sin(\pi\,\alpha)} - \frac{1}{2j}\Bigg[a_2(x)\,\omega^{\alpha+1,\beta+1}(x) P_{j-1}^{\alpha+1,\beta+1}(x) + \int_{-1}^{1} l(x,t)\,\omega^{\alpha+1,\beta+1}(t) P_{j-1}^{\alpha+1,\beta+1}(t)\,dt \Bigg]\Bigg\} - \overline{f}(x).$$

The method of *Polynomial Collocation* consists of suitably choosing n distinct points say $x_i,\ i = 1,2,3,\ldots,n$, on $(-1,1)$ and imposing $r_n(x_i) = 0$ for $1 \le i \le n$; the resulting n equations will then allow us to calculate g_n. In this paper, we choose the collocation points as the zeros of the first kind Chebyshev polynomial of degree n, i.e. $x_i = \cos[\,(2i{+}1)\pi/2n\,]$, $1 \le i \le n$; the reason for this choice will be made clear later.

This gives rise to the system of linear equations, i.e.

$$\begin{aligned} r_n(x_i) \;&=\; \sum_{j=1}^{n} c_j \Bigg\{ \frac{-\,b_1 P_{j-1}^{-\alpha,\,-\beta}(x_i)}{2\sin(\pi\,\alpha)} - \frac{1}{2j}\Bigg[a_2(x_i)\omega^{\alpha+1,\beta+1}(x_i) P_{j-1}^{\alpha+1,\beta+1}(x_i) \\ &+\; \int_{-1}^{1} l(x_i,t)\omega^{\alpha+1,\,\beta+1}(t) P_{j-1}^{\alpha+1,\,\beta+1}(t)\,dt \Bigg]\Bigg\} - \overline{f}(x_i) = 0, \end{aligned}$$

that may be solved for the coefficients $c_j, j = 1,2,3,\ldots,n$. However, in practice, the integrals

$$\int_{-1}^{1} l(x,t)\omega^{\alpha+1,\beta+1}(t) P_{j-1}^{\alpha+1,\beta+1}(t)\,dt,$$

cannot be evaluated analytically. For this reason, we shall employ the Gauss-Jacobi quadrature formula with n nodes, i.e.,

$$\begin{aligned} &\int_{-1}^{1} l(x,t)\,\omega^{\alpha+1,\beta+1}(t) P_{j-1}^{\alpha+1,\beta+1}(t)\,dt \\ &\simeq \textstyle\sum_{k=1}^{n} l(x,t_k)\,P_{j-1}^{\alpha+1,\beta+1}(t_k)\,\lambda_k^{\alpha+1,\beta+1} := \widetilde{L}_n\,[\,P_{j-1}^{\alpha+1,\beta+1}(x)\omega^{\alpha+1,\beta+1}(x)\,], \end{aligned} \tag{13}$$

where

$$\lambda_k^{\alpha+1,\beta+1} = 2^{\alpha+\beta+3}\,\frac{\Gamma(n+\alpha+2)\Gamma(n+\beta+2)}{\Gamma(n+1)\Gamma(n+\alpha+\beta+3)}\,(1-t_k^2)^{-1}\,[\,(P_n^{\alpha+1,\,\beta+1})'(t_k)\,]^{-2} \tag{14}$$

(see [17, (15.3.1)]), where $t_k,\ k = 1,2,\ldots,n$, denote the n zeros of $P_n^{\alpha+1,\beta+1}(x)$ that we obtain with the assistance of the Mathematica software. From Theorem 1 of [9, Section 7.3], this quadrature formula has maximum degree of precision $2n-1$.

From [17, (4.21.7)], we have the identity

$$\frac{d}{dx}\,P_n^{\alpha,\,\beta}(x) = \frac{1}{2}(n+\alpha+\beta+1)\,P_{n-1}^{\alpha+1,\,\beta+1}(x),$$

that, when applied to (13), gives rise to the expression:

$$\widetilde{L}_n\,[\,P_{j-1}^{\alpha+1,\,\beta+1}(x)\omega^{\,\alpha+1,\beta+1}(x)\,] =$$

$$\frac{2^{\alpha+\beta+5}}{(n+\alpha+\beta+3)^2}\,\frac{\Gamma(n+\alpha+2)\Gamma(n+\beta+2)}{\Gamma(n+1)\Gamma(n+\alpha+\beta+3)}\sum_{k=1}^{n} l(x,t_k)\,\frac{P_{j-1}^{\alpha+1,\beta+1}(t_k)}{(1-t_k^2)}\,[P_{n-1}^{\alpha+2,\beta+2}(t_k)\,]^{-2}.$$

In order to evaluate the polynomials $P_j^{-\alpha,\,-\beta}(x), P_j^{\alpha+1,\beta+1}(x),\ \ 1 \le j \le n-1$ and $P_{n-1}^{\alpha+2,\beta+2}(x)$, we apply the recurrence relation
$P_n^{\alpha,\,\beta}(x) = \dfrac{1}{2n(n+\alpha+\beta)(2n+\alpha+\beta-2)}\{(2n+\alpha+\beta-1)[\,(2n+\alpha+\beta)(2n+\alpha+\beta-2)x+\alpha^2-\beta^2\,]\ P_{n-1}^{\alpha,\,\beta}(x) - 2(n+\alpha-1)(n+\beta-1)(2n+\alpha+\beta)P_{n-2}^{\alpha,\,\beta}(x)\}$, $n \ge 2$,

$$P_0^{\alpha,\,\beta}(x) = 1, \qquad P_1^{\alpha,\,\beta}(x) = [\,(\alpha+\beta+2)x + a - \beta\,]/2,$$

(see [17, 4.5.1)]). In particular when $\alpha = \beta = -1/2$ or $\alpha = \beta = 1/2$, Jacobi polynomials simplify to

$$P_n^{-1/2,\,-1/2}(x) = \frac{1.3\cdots(2n-1)}{2.4\cdots(2n)}\,T_n(x) \quad \text{or} \quad P_n^{1/2,\,1/2}(x) = 2\,\frac{1.3\cdots(2n+1)}{2.4\cdots(2n+2)}\,U_n(x) \quad (15)$$

respectively, where $T_n(x)$ are the Chebyshev polynomials of first kind and $U_n(x)$ are the Chebyshev polynomials of second kind, (see [17, (4.1.7)]).

The system of equations that we solve in practice then has the form $Ac = \overline{f}$, where

$$A_{i,j} = \frac{-b_1 P_{j-1}^{-\alpha,\,-\beta}(x_i)}{2\sin(\pi\,\alpha)} - \frac{1}{2j}\Bigg[a_2(x_i)\,\omega^{\alpha+1,\beta+1}(x_i)P_{j-1}^{\alpha+1,\beta+1}(x_i) + \sum_{k=1}^{n} l(x_i,t_k)P_{j-1}^{\alpha+1,\beta+1}(t_k)\lambda_k^{\alpha+1,\beta+1}\Bigg],\ 1 \le i,j \le n,$$

$c = (c_1, c_2, \cdots, c_n)^T$ and $\overline{f} = (\overline{f}(x_1), \overline{f}(x_2), \cdots, \overline{f}(x_n))^T$.

4 Function spaces

In this section we shall present the functions spaces that will be required for the subsequent analysis of convergence.

We shall consider the functions in (6)-(7), belonging to the space

$$C_{\rho,\tau} := \{\, h \in C(-1,1) : h\omega^{\rho,\tau} \in C[-1,1]\,\},$$

equipped with the norm

$$\|h\|_{\infty,\,\rho,\tau} = \max_{|x|\le 1} |(h\omega^{\rho,\tau})(x)|,$$

with $\rho,\ \tau \geq 0$. This is a Banach space.

Let Π_n denote the set of polynomials of degree not greater than n and $E_n^{\rho,\tau}(h)$ denote the error in the best weighted uniform approximation to a function h by polynomials belonging to Π_n, i.e.,

$$E_n^{\rho,\tau}(h) = \inf\{\ \|h - P_n\|_{\infty,\rho,\tau}\ :\ P_n \in \Pi_n\ \}.$$

We denote by $\mathscr{B} = \{b_n\}$ a sequence of positive real numbers such that $\lim_{n\to\infty} b_n = 0$. Now we can defined the weighted Besov space

$$C_{\rho,\tau}^{\mathscr{B}} := \left\{ h \in C_{\rho,\tau} : \|h\|_{\,\rho,\tau,\mathscr{B}} = \|h\|_{\infty,\,\rho,\tau} + \sup_{n=1,2,3,\ldots} \frac{E_n^{\rho,\tau}(h)}{b_n} < \infty \right\}.$$

According to [10, Lemma 3.3 and Remark 3.5], if $\mathscr{B} = \{b_n\}$ and $\mathscr{C} = \{c_n\}$ are such that $\lim_{n\to\infty}(b_n/c_n) = 0$, then $C_{\rho,\tau}^{\mathscr{B}}$ is compactly imbedded into $C_{\rho,\tau}^{\mathscr{C}}$ and if, $\rho_1 \leq \rho_2$ and $\tau_1 \leq \tau_2$ then the embedding $C_{\rho_1,\tau_1}^{\mathscr{B}}$ into $C_{\rho_2,\tau_2}^{\mathscr{B}}$ is continuous.

We denote by $\mathscr{H}^{r,\mu}$ the space of all real-valued functions $h \in C^r[-1,1]$, such that $h^{(r)}$, the r-th derivative of h, is Hölder continuous with index $0 < \mu \leq 1$. The space $\mathscr{H}^{r,\mu}$ endowed with the norm

$$\|h\|_{r,\mu} = \sum_{i=0}^{r} \|h^{(i)}\|_\infty + \sup_{\substack{|x|,|t|\le 1\\ x\neq t}} \{\, |x-t|^{-\mu} |\, h^{(r)}(x) - h^{(r)}(t)|\,\},$$

is a Banach space.

According to [10, Lemma 3.11], if $h \in C_{\rho,\tau}^{\mathscr{B}}$, where $\mathscr{B} = \{b_n\}$ is such that $b_n = O(n^{-\gamma})$, then $h\omega^{\rho,\tau} \in \mathscr{H}^{0,\mu}$ for $0 < \mu < 1$, where μ depends on ρ,τ and γ. In the following theorems and lemmata it will be assumed that the functions a_2, l (in both variables) and $\overline{f}$ all belong to the Besov space $C_{\rho,\tau}^{\mathscr{B}}$ for given values of ρ,τ and some $\mathscr{B}$. Therefore, we conclude that the functions of equation (6) satisfy the necessary conditions for the theory of SIE to be applied.

5 Operator properties

In this section, we shall present some mapping properties of the operators H, H^I, D, L and R, and discuss their boundedness. This will then allow us to provide characterizations of the error in the best weighted uniform polynomial approximation of the operators in (6).

Define the non-negative constants $\alpha^+, \alpha^-, \beta^+$ and β^- by: $\alpha = \alpha^+ - \alpha^-$ and $\beta = \beta^+ - \beta^-$, where $0 \leq \alpha^\pm, \beta^\pm < 1$. Notice that, since $-1 < \alpha, \beta < 0$ and $\varkappa = 1$, we have $0 \leq \alpha^+, \beta^+ < 1/2$ and $1/2 \leq \alpha^-, \beta^- < 1$.

If X and Y are Banach spaces, we denote by $\mathcal{L}(X, Y)$ the Banach space of all linear bounded operators from X into Y.

Let the sequence b_n be denoted by $b_n = \begin{cases} 1 & if \quad n = 1, \\ (\log^q n)/n^\gamma & if \quad n \geq 2, \text{ for } q \geq 0, \gamma > 0. \end{cases}$

In what follows, we write $l \in C^{\mathscr{B}}_{\rho,\tau,x} \oplus C_{\nu,\varsigma,t}$, when the function l satisfies the following conditions:

$$i)\ l(x,t)\, \omega^{\rho,\tau}(x)\, \omega^{\nu,\varsigma}(t) \in C[-1,1]^2,$$
$$ii)\ l(x,\bullet)\omega^{\nu,\varsigma}(\bullet) \in C^{\mathscr{B}}_{\rho,\tau} \text{ uniformly with respect to } t \in [-1,1].$$

The next lemma will be fundamental in the proof of the theorem following it. This theorem is an adaptation of Proposition 4.12 of [10] and is necessary for making the definition of the operator L precise.

Remark 5.1. *In the remainder of this text, we denote c to be a generic constant independent of n.*

Lemma 5.2 ([10], Lemma 4.11). *Let $l \in C^{\mathscr{B}}_{\rho,\tau,x} \oplus C_{\nu,\varsigma,t}$ with $\mathscr{B} = \{b_n\}$. Then there exists a sequence $\{P_n\}_{n=1}^\infty$ of polynomials $P_n(x,t) = \sum_{j=0}^n c_{nj}(t)x^j$ of degree not greater than n in x, such that $c_{nj}(t)\omega^{\nu,\varsigma}(t)$ is piecewise constant for all $j = 0, 1, \ldots, n$ and*

$$\sup_{x,t\in[-1,1]} |\,(l(x,t) - P_n(x,t)\,)\,\omega^{\rho,\tau}(x)\omega^{\nu,\varsigma}(t)\,| \leq c\, b_n, \qquad n = 1, 2, 3, \ldots.$$

Theorem 5.3. *Let ν, ς be nonnegative constants, such that $\nu + \alpha^-, \varsigma + \beta^- < 1$. If $l \in C^{\mathscr{B}}_{\rho,\tau,x} \oplus C_{\nu,\varsigma,t}$, then*

$$L \in \mathcal{L}\,(\,C_{\alpha^-,\beta^-}, C^{\mathscr{B}}_{\rho,\tau}\,).$$

Proof: Since $l \in C^{\mathscr{B}}_{\rho,\tau,x} \oplus C_{\nu,\varsigma,t}$, with $\mathscr{B} = \{b_n\}$ and $h \in C_{\alpha^-,\beta^-}$, we have

$$|(Lh)(x)\omega^{\rho,\tau}(x)| = \left| \int_{-1}^1 l(x,t)h(t)\omega^{\rho,\tau}(x)\,dt \right| \leq c\,\|h\|_{\infty,\alpha^-,\beta^-} \int_{-1}^1 \omega^{-\nu-\alpha^-,-\varsigma-\beta^-}(t)\,dt.$$

As $\nu+\alpha^-,\varsigma+\beta^-<1$, the integral $\int_{-1}^1 \omega^{-\nu-\alpha^-,-\varsigma-\beta^-}(t)\,dt$ is bounded.

Define

$$Q_n(x) := \int_{-1}^1 P_n(x,t)h(t)dt,$$

where $P_n(x,t)$ is given by Lemma 5.2. Then, $Q_n\in\Pi_n$ since $c_{nj}\,\omega^{\nu,\varsigma}\in\mathscr{L}^\infty(-1,1)$. So,

$$\begin{aligned}|(Lh-Q_n)(x)\omega^{\rho,\tau}(x)| &= \left|\int_{-1}^1 [\,l(x,t)-P_n(x,t)]\,h(t)\,\omega^{\rho,\tau}(x)\,dt\right|\\ &\le c\,b_n\,\|h\|_{\infty,\alpha^-,\beta^-}\int_{-1}^1 \omega^{-\nu-\alpha^-,-\varsigma-\beta^-}(t)dt \;\le\; c\,b_n\,\|h\|_{\infty,\alpha^-,\beta^-}.\end{aligned}$$

Therefore,

$$E_n^{\rho,\tau}(Lh)\le c\,b_n\,\|h\|_{\infty,\alpha^-,\beta^-} \qquad \textit{and} \qquad Lh\in C^{\mathscr{B}}_{\rho,\tau}. \quad \blacksquare$$

Theorem 5.4. *If $g\in C_{\alpha^+,\beta^+}$ then $Rg\in C_{\alpha^-,\beta^-}$.*

Proof: Since $-\alpha^-,-\beta^->-1$, we have

$$|Rg(x)| = \left|\int_{-1}^x g(t)\,\omega^{\alpha,\beta}(t)\,dt\right| \le c\,\|g\|_{\infty,\alpha^+,\beta^+}\int_{-1}^x \omega^{-\alpha^-,-\beta^-}(t)\,dt \le c\,\|g\|_{\infty,\alpha^+,\beta^+}.$$

Then, $Rg\in C_{0,0}$ implies $Rg\in C_{\alpha^-,\beta^-}$. $\blacksquare$

Lemma 5.5. *Suppose $a_2\in C^{\mathscr{B}}_{0,0}$ and $Rg\in C^{\mathscr{C}}_{\alpha^-,\beta^-}$, where $\mathscr{B}=\{b_n\}$ with $b_n=O(n^{-\gamma_1})$ and $\mathscr{C}=\{c_n\}$ with $c_n=O(n^{-\gamma_2})$. Then,*

$$D\in\mathcal{L}(\,C^{\mathscr{C}}_{\alpha^-,\beta^-},\,C^{\mathscr{D}}_{\alpha^-,\beta^-}\,),$$

where $\mathscr{D}=\{d_n\}=O(n^{-\widetilde{\gamma}})$ with $\widetilde{\gamma}=\min\{\gamma_1,\gamma_2\}$.

Proof: Since $-\alpha^-,-\beta^->-1$, we have

$$|DRg(x)| = \left|a_2(x)\int_{-1}^x g(t)\,\omega^{\alpha,\beta}(t)\,dt\right| \le c\,\|g\|_{\infty,\alpha^+,\beta^+}\int_{-1}^x \omega^{-\alpha^-,-\beta^-}(t)\,dt \le c\,\|g\|_{\infty,\alpha^+,\beta^+}.$$

Then, $D\in\mathcal{L}(\,C_{\alpha^-,\beta^-},\,C_{0,0}\,)$ implies $D\in\mathcal{L}(\,C_{\alpha^-,\beta^-},\,C_{\alpha^-,\beta^-}\,)$.

Let P_n^*,P_n^{**} be such that $E_n(a_2)=\|a_2-P_n^*\|_\infty$ and $E_n^{\alpha^-,\beta^-}(Rg)=\|Rg-P_n^{**}\|_{\infty,\alpha^-,\beta^-}$. Then,

$$\begin{aligned}E_{2n}^{\alpha^-,\beta^-}(DRg) &\le \|a_2\,Rg-P_n^*P_n^{**}\|_{\infty,\alpha^-,\beta^-}\\ &\le \|\,(a_2-P_n^*)Rg\|_{\infty,\alpha^-,\beta^-}+\|\,(Rg-P_n^{**})P_n^*\|_{\infty,\alpha^-,\beta^-}\\ &\le c\{\,E_n(a_2)\,+\,E_n^{\alpha^-,\beta^-}(Rg)\,\}\;\le\;c\,(b_n\,+\,c_n)\;\le\;c\,d_n\;\le\;c\,d_{2n}. \quad \blacksquare\end{aligned}$$

Remark 5.6. *From [10, Proposition 4.7 and Remark 4.9], $H^I \in \mathcal{L}\,(C^{\mathscr{B}}_{\alpha^-,\,\beta^-}, C^{\mathscr{B}\,\log n}_{\alpha^+,\,\beta^+})$ and $H \in \mathcal{L}\,(C^{\mathscr{B}}_{\alpha^+,\,\beta^+}, C^{\mathscr{B}\,\log n}_{\alpha^-,\,\beta^-})$. Then, from the assumptions of Theorem 5.3, for $\rho \leq \alpha^-$ and $\tau \leq \beta^-$, we obtain*

$$LR \in \mathcal{L}(\,C_{\alpha^+,\beta^+},\, C^{\mathscr{B}}_{\alpha^-,\beta^-})\ \textit{hence}\ H^I LR \in \mathcal{L}(\,C_{\alpha^+,\beta^+},\, C^{\mathscr{B}\,\log n}_{\alpha^-,\beta^-}\,)$$

and from Lemma 5.5,

$$DR \in \mathcal{L}(\,C_{\alpha^+,\beta^+},\, C^{\mathscr{D}}_{\alpha^-,\beta^-}\,)\ \textit{hence}\ H^I DR \in \mathcal{L}(\,C_{\alpha^+,\beta^+},\, C^{\mathscr{D}\,\log n}_{\alpha^-,\beta^-}\,).$$

6 Weighted uniform convergence

This section contains the main results that we shall use for proving the weighted uniform convergence of the collocation method. Some of these results are an adaptation of lemmata and theorems from [10]; others are original, developed by the authors. Firstly, some important properties of the operators defined in (7) are presented.

Let the projection operator be defined in $C[-1,1]$ by

$$(\,\mathbb{P}_{n-1}h\,)(x) = \sum_{i=1}^{n} h(x_i)p_i(x),$$

where $p_i,\ 1 \leq i \leq n$ denote the Lagrange polynomials corresponding to the abscissas x_i, the nodes of the interpolation. The *Weighted Lebesgue Constant* is defined by

$$\|\mathbb{P}_{n-1}\|_{\infty,\rho,\tau} := \sup\{\|\mathbb{P}_{n-1}h\|_{\infty,\rho,\tau} : h \in C_{\rho,\tau},\, \|h\|_{\infty,\rho,\tau} = 1\}.$$

An estimate for the weighted Lebesgue constant is given by following theorem.

Theorem 6.1 ([14], Theorem 4.3.1)**.** *Let $h \in C_{\rho,\tau}$ for $\rho, \tau \geq 0$ and $\mathbb{P}_{n-1}h(x)$ be the Lagrange interpolation polynomial on the zeros of $P_n^{\sigma,\varsigma}(x)$, $\sigma, \varsigma > -1$. If the inequalities*

$$\frac{\sigma}{2} + \frac{1}{4} \leq \rho \leq \frac{\sigma}{2} + \frac{5}{4}, \quad \frac{\varsigma}{2} + \frac{1}{4} \leq \tau \leq \frac{\varsigma}{2} + \frac{5}{4},$$

are satisfied, then

$$\|\mathbb{P}_{n-1}h\|_{\infty,\rho,\tau} = O(\log n).$$

By considering the identity (15), we can conclude that the roots of the n-th degree Chebyshev polynomial of the first kind, when chosen to be the interpolation nodes, imply that the inequalities of Theorem 6.1 are satisfied when $0 \leq \rho,\, \tau < 1$.

From the definition of $\mathbb{P}_{n-1}$, we can show that $r_n(x) = 0$ if and only if $(\,\mathbb{P}_{n-1} r_n\,)(x) = 0$. Taking this fact into account, equation (10) can be rewritten as:

$$\mathbb{P}_{n-1} H g_n + \mathbb{P}_{n-1} D R g_n + \mathbb{P}_{n-1} \widetilde{L}_n R g_n = \mathbb{P}_{n-1} \bar{f}. \tag{16}$$

From equation (12), Hg_n is polynomial of degree $n-1$ and therefore we can rewrite equation (16) in the form

$$H g_n + \mathbb{P}_{n-1} D R g_n + \mathbb{P}_{n-1} \widetilde{L}_n R g_n = \mathbb{P}_{n-1} \bar{f}.$$

We can show that the operator $\mathbb{P}_{n-1} L$ can be written as

$$(\,\mathbb{P}_{n-1} L h\,)(x) = \textstyle\int_{-1}^{1} l^{[n-1]}(x,t) h(t) dt, \quad \text{where} \quad l^{[n-1]}(x,t) = \displaystyle\sum_{j=1}^{n} \Phi_j(x) \Psi_j(t)$$

is the polynomial that interpolates $l(x,t)$ in the variable x, Ψ_j are chosen to satisfy $\displaystyle\sum_{j=1}^{n} \Phi_j(x_i) \Psi_j(t) = l(x_i, t)$, $1 \leq i \leq n$, and $\{\Phi_j\}$ is a basis for the set of polynomials of degree $n-1$ satisfying the Haar condition (see [6, p.74]).

The following results will be fundamental to the proof of convergence of the collocation method.

Lemma 6.2 ([10], Lemma 5.2). *If $h \in C^{\mathscr{B}}_{\rho,\tau}$ for $\rho \leq \alpha^-$ and $\tau \leq \beta^-$, then*

$$\|\, H^I(h - \mathbb{P}_{n-1} h)\,\|_{\infty,\alpha^+,\beta^+} \leq c \,\log n \, \|\mathbb{P}_{n-1}\|_{\infty,\rho,\tau} \, E_{n-1}^{\alpha^-,\beta^-}(h), \quad n \geq 2.$$

Lemma 6.3. *Let ν, ς be nonnegative constants such that $\nu + \alpha^- < 1$ and $\varsigma + \beta^- < 1$. If $l \in C^{\mathscr{B}}_{\rho,\tau,x} \oplus C_{\nu,\varsigma,t}$, then*

$$\|\,(\widetilde{L_n} - L) R g_n\,\|_{\infty,\rho,\tau} \;\leq\; c\; E_n^{\nu,\varsigma}(\,l(\bullet, t)\,\omega^{\rho,\tau}(\bullet)\,).$$

<u>*Proof*</u> : It is known that

$$(Rg_n)(x) = \sum_{j=0}^{n} \int_{-1}^{x} c_j P_j^{\alpha,\beta}(t) \omega^{\alpha,\beta}(t)\, dt \;=\; -\sum_{j=1}^{n} \frac{c_j}{2j}\, P_{j-1}^{\alpha+1,\beta+1}(x) \omega^{\alpha+1,\beta+1}(x)$$

and so, $Rg_n(x)\omega^{-\alpha-1,-\beta-1}(x)$ is a polynomial of degree $n-1$.

Taking t_k, $1 \le k \le n$, as the zeros of $P_n^{\alpha+1,\beta+1}(x)$, we have

$$\widetilde{L}_n Rg_n \;=\; \sum_{j=1}^{n}\left\{-\frac{c_j}{2j}\sum_{k=1}^{n} l(x,t_k)\,P_{j-1}^{\alpha+1,\beta+1}(t_k)\,\lambda_k^{\alpha+1,\beta+1}\right\},$$

where $\lambda_k^{\alpha+1,\beta+1}$ were previously defined in (14).

Then

$$|\,[(\widetilde{L}_n - L)Rg_n\,\omega^{\rho,\tau}](x)\,| \;=\; \left|\int_{-1}^{1}\left[\sum_{k=1}^{n} l(x,t_k)\omega^{\rho,\tau}(x)Rg_n(t_k)\omega^{-\alpha-1,-\beta-1}(t_k)\lambda_k^{\alpha+1,\beta+1} - l(x,t)\omega^{\rho,\tau}(x)Rg_n(t)\right]dt\right|.$$

Let P^*_{2n-1} be such that

$$E^{\nu,\varsigma}_{2n-1}(\,l(\bullet,t)\,\omega^{\rho,\tau}(\bullet)\,Rg_n\,\omega^{-\alpha-1,-\beta-1}) = \|\,l(\bullet,t)\,\omega^{\rho,\tau}(\bullet)\,Rg_n\,\omega^{-\alpha-1,-\beta-1} - \,P^*_{2n-1}\|_{\infty,\nu,\varsigma}.$$

Then, we obtain,

$$\begin{aligned}
&|\,(\widetilde{L}_n - L)Rg_n(x)\omega^{\rho,\tau}(x)\,| = \\
&\Big|\int_{-1}^{1}\sum_{k=1}^{n}[\,l(x,t_k)\omega^{\rho,\tau}(x)Rg_n(t_k)\omega^{-\alpha-1,-\beta-1}(t_k) - P^*_{2n-1}(t_k)\,]\lambda_k^{\alpha+1,\beta+1} \\
&-[\,l(x,t)\omega^{\rho,\tau}(x)Rg_n(t)\omega^{-\alpha-1,-\beta-1}(t) - P^*_{2n-1}(t)\,]\,\omega^{\alpha+1,\beta+1}(t) \\
&-[\,P^*_{2n-1}(t)\omega^{\alpha+1,\beta+1}(t) - P^*_{2n-1}(t_k)\lambda_k^{\alpha+1,\beta+1}\,]\,dt\Big| \\
&\le E^{\nu,\varsigma}_{2n-1}(\,l(\bullet,t)\,\omega^{\rho,\tau}(\bullet)\,Rg_n\,\omega^{-\alpha-1,-\beta-1})\int_{-1}^{1}[\,|\,\lambda_k^{\alpha+1,\beta+1}\,| + \omega^{\alpha+1,\beta+1}(t)\,]\,\omega^{-\nu,-\varsigma}(t)\,dt.
\end{aligned}$$

As $\nu+\alpha^- < 1$ and $\varsigma+\beta^- < 1$, we have that $\int_{-1}^{1}\omega^{-\nu,-\varsigma}(t)\,dt$ is bounded. Therefore,

$$|\,(\widetilde{L}_n - L)Rg_n(x)\omega^{\rho,\tau}(x)\,| \le c\,E^{\nu,\varsigma}_{2n-1}(\,l(\bullet,t)\,\omega^{\rho,\tau}(\bullet)\,Rg_n\omega^{-\alpha-1,-\beta-1}\,).$$

Let P^{**}_n be such that $E^{\nu,\varsigma}_n(\,l(\bullet,t)\,\omega^{\rho,\tau}(\bullet)) = \|\,l(\bullet,t)\,\omega^{\rho,\tau}(\bullet) - P^{**}_n\|_{\infty,\nu,\varsigma}$, then

$$\begin{aligned}
E^{\nu,\varsigma}_{2n-1}(\,l(\bullet,t)\,\omega^{\rho,\tau}(\bullet)\,Rg_n\,\omega^{-\alpha-1,-\beta-1}\,) &\le \|\,(\,l(\bullet,t)\,\omega^{\rho,\tau}(\bullet) - P^{**}_n\,)\,Rg_n\,\omega^{-\alpha-1,-\beta-1}\,\|_{\infty,\nu,\varsigma} \\
&\le c\,E^{\nu,\varsigma}_n(\,l(\bullet,t)\,\omega^{\rho,\tau}(\bullet)\,). \quad \blacksquare
\end{aligned}$$

The following theorem of Jackson type, provides the convergence rates of the error of the best polynomial approximation of functions in the weighted Sobolev space $W^s_{\gamma,\,\delta}$, defined by

$$W^s_{\gamma,\,\delta} := \{\,h \in C_{\gamma,\,\delta} :\, h^{(s-1)} \in A.C._{loc} \text{ and } \|\,h^{(s)}\,\phi^s\,\omega^{\gamma,\,\delta}\,\|_\infty < \infty\},$$

where $A.C._{loc}$ denotes the set of all locally absolutely continuous functions in $[-1,1]$ and $\phi(x) = \sqrt{1-x^2}$. A norm in $W^s_{\gamma,\,\delta}$ is given by

$$\| \, h \, \|_{W^s_{\gamma,\,\delta}} := \| \, h\omega^{\gamma,\,\delta} \, \|_\infty \, + \, \| \, h^{(s)} \, \phi^s \, \omega^{\gamma,\,\delta} \|.$$

Theorem 6.4 ([8], Chapter 8)**.** *Let $h \in W^s_{\gamma,\,\delta}$, with s and n positive integers. Then,*

$$E_n^{\gamma,\,\delta}(h) \leq \, \frac{c}{n^s} \, E_{n-s}^{\gamma+s/2,\,\delta+s/2}(\, h^{(s)} \,) \, \leq \, \frac{c}{n^s} \, \| \, h^{(s)} \, \phi^s \omega^{\gamma,\,\delta} \, \|_\infty.$$

We shall denote by $W^{s,\mu}_{\gamma,\delta}$ the subset of $W^s_{\gamma,\delta}$ given by

$$W^{s,\mu}_{\gamma,\delta} := \{ \, h \in W^s_{\gamma,\delta} : h^{(s)} \phi^s \omega^{\gamma,\delta} \in \mathcal{H}^{0,\mu} \, \},$$

and by Ω_ϵ, the function modulus of smoothness defined by

$$\Omega_\epsilon(\, h \,) = \sup_{|x-y| \leq \epsilon} | \, h(x) - h(y) \, |.$$

Lemma 6.5. *Let $h \in W^{s,\,\mu}_{\gamma,\,\delta}$. Then*

$$E_n^{\gamma,\,\delta}(h) \leq \frac{c}{n^{s+\mu}} \, \| h^{(s)} \, \phi^s \omega^{\gamma,\,\delta} \|_\infty.$$

<u>*Proof*</u> : We define

$$\Upsilon(x) = n \int_{x(1-1/n)-1/(2n)}^{x(1-1/n)+1/(2n)} h^{(s)}(t) \phi^s(t) \omega^{\gamma,\,\delta}(t) \, dt.$$

Then, we have

$$\begin{aligned} & | \, \Upsilon(x) - h^{(s)}(x) \phi^s(x) \omega^{\gamma,\,\delta}(x) \, | \\ & = \, n \left| \int_{x(1-1/n)-1/(2n)}^{x(1-1/n)+1/(2n)} [h^{(s)}(t) \phi^s(t) \omega^{\gamma,\,\delta}(t) - h^{(s)}(x) \phi^s(x) \omega^{\gamma,\,\delta}(x)] \, dt \right| \leq \, c \, \Omega_{3/(2n)}(h^{(s)} \phi^s \omega^{\gamma,\,\delta}). \end{aligned}$$

Let $S'(x) = h^{(s)}(x) \phi^s(x) \omega^{\gamma,\,\delta}(x)$ and $0 < \theta_1, \theta_2 < 1/(2n)$, then, we obtain

$$\begin{aligned} | \, \Upsilon(x) \, | \; & = \; n \left| \int_{x(1-1/n)-1/(2n)}^{x(1-1/n)+1/(2n)} S'(t) \, dt \right| = \, n | \, S(x - x/n + 1/(2n)) - S(x - x/n - 1/(2n)) \, | \\ & = \; \frac{1}{2} \, | \, S'(x - x/n + \theta_1) - S'(x - x/n - \theta_2) \, | \, \leq \, c \, \Omega_{1/n}(S'). \end{aligned}$$

In view of these results and Theorem 6.4, we obtain

$$\begin{aligned} E_n^{\gamma,\,\delta}(h) \; & \leq \; \frac{c}{n^s} \, \{ \, \| h^{(s)} \phi^s \omega^{\gamma,\,\delta} - \Upsilon \|_\infty + \| \Upsilon \|_\infty \, \} \\ & \leq \; \frac{c}{n^s} \, \{ \, \Omega_{3/(2n)}(h^{(s)} \phi^s \omega^{\gamma,\,\delta}) + \Omega_{1/n}(h^{(s)} \phi^s \omega^{\gamma,\,\delta}) \, \} \, \leq \, \frac{c}{n^{s+\mu}}. \end{aligned}$$

However, Theorem 6.4 is not conclusive about the order of the term $E_n^{\gamma,\delta}(h)$ when $s = 0$. In this case, we shall consider the following

$$
\begin{aligned}
|\,\Upsilon'(x)\,| &= n\,|\,(\,h\omega^{\gamma,\delta})(x - x/n + 1/(2n)) - (\,h\omega^{\gamma,\delta})(x - x/n - 1/(2n))\,| \\
&\le c\,n\,\Omega_{1/n}(h\omega^{\gamma,\delta})
\end{aligned}
$$

and $E_n^{\gamma,\delta}(\Upsilon) = \|(\Upsilon - P_n^*)\,\omega^{\gamma,\delta}\|_\infty$. Thus we get

$$
\begin{aligned}
E_n^{\gamma,\delta}(h) &\le c\,\{\,\|h\omega^{\gamma,\delta} - \Upsilon\|_\infty + \|\Upsilon(1-\omega^{\gamma,\delta})\|_\infty + \|(\Upsilon - P_n^*)\,\omega^{\gamma,\delta}\|_\infty\,\} \\
&\le c\left\{\,\Omega_{3/(2n)}(h\omega^{\gamma,\delta}) + \Omega_{1/n}(h\omega^{\gamma,\delta}) + \frac{1}{n}\,\|\Upsilon'(x)\phi(x)\omega^{\gamma,\delta}(x)\,\|_\infty\right\} \\
&\le c\,\{\,\Omega_{3/(2n)}(h\omega^{\gamma,\delta}) + \Omega_{1/n}(h\omega^{\gamma,\delta})\,\} \le \frac{c}{n^\mu}. \quad \blacksquare
\end{aligned}
$$

Lemma 6.6. *If $g \in C_{\alpha^+,\beta^+}$ the function $Rg \in \mathscr{H}^{0,q}$, where $q = \min\{1-\alpha^-, 1-\beta^-\}$.*

Proof: Let $\varepsilon > 0$, then we have

$$
\begin{aligned}
|\,Rg(x+\varepsilon) - Rg(x)\,| &= \left|\int_{-1}^{x+\varepsilon} g(t)\omega^{\alpha,\beta}(t)\,dt - \int_{-1}^{x} g(t)\omega^{\alpha,\beta}(t)\,dt\right| \\
&\le \|\,g\,\|_{\infty,\,\alpha^+,\beta^+}\int_x^{x+\varepsilon}\omega^{-\alpha^-,-\beta^-}(t)\,dt \\
&\le c\left\{B_{\frac{1+x+\varepsilon}{2}}(1-\beta^-,1-\alpha^-) - B_{\frac{1+x}{2}}(1-\beta^-,1-\alpha^-)\right\}.
\end{aligned}
$$

where $B_x(a,b)$ is the incomplete Beta function. From [1, (6.6.8) and (15.3.3)], we have

$$
\begin{aligned}
&B_x(a,b) = a^{-1}\,x^a\,{}_2F_1[a, 1-b; a+1; x] \text{ and} \\
&{}_2F_1[a,b;c;z] = (1-z)^{c-a-b}\,{}_2F_1[c-a, c-b; c; z],
\end{aligned}
$$

where ${}_2F_1[a,b;c;z]$ is the Gauss's hypergeometric function. It follows that

$$
\begin{aligned}
&B_{\frac{1+x}{2}}(\,1-\beta^-, 1-\alpha^-\,) = \\
&2^{-2+\beta^-+\alpha^-}\,(1-\beta^-)^{-1}\,\omega^{1-\alpha^-,1-\beta^-}(x)\,{}_2F_1[\,1, 2-\beta^--\alpha^-;\ 2-\beta^-;\ (1+x)/2\,].
\end{aligned}
$$

By definition ${}_2F_1[a,b;c;z] = \displaystyle\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}$, where $(a)_n = \Gamma(a+n)/\Gamma(a)$ is the Pochhammer's symbol. From this, for some $0 < \theta < 1$ we obtain

$$
\begin{aligned}
&|\,{}_2F_1[\,1, 2-\beta^- - \alpha^-;\, 2-\beta^-;\, (1+x+\varepsilon)/2\,] - {}_2F_1[\,1, 2-\beta^- - \alpha^-;\, 2-\beta^-;\, (1+x)/2\,]\,| \\
&= \left| \sum_{n=0}^{\infty} \frac{(1)_n(2-\beta^- - \alpha^-)_n}{(2-\beta^-)_n} \, \frac{(1+x+\varepsilon)^n - (1+x)^n}{2^n n!} \right| \\
&= \left| \sum_{n=0}^{\infty} \frac{(1)_n(2-\beta^- - \alpha^-)_n}{(2-\beta^-)_n} \, \frac{\varepsilon\,[\,(1+x+\theta\varepsilon)^n\,]'}{2^n n!} \right| \\
&= \left| \sum_{n=1}^{\infty} \frac{(1)_n(2-\beta^- - \alpha^-)_n}{(2-\beta^-)_n} \, \frac{\varepsilon}{2\Gamma(n)} \left(\frac{1+x+\theta\varepsilon}{2} \right)^{n-1} \right| \\
&\le \frac{\varepsilon}{2} \sum_{n=1}^{\infty} \frac{\Gamma(n+1)\Gamma(n+2-\beta^- - \alpha^-)\Gamma(2-\beta^-)}{\Gamma(n+2-\beta^-)\Gamma(2-\beta^- - \alpha^-)\Gamma(n)} = \frac{\varepsilon}{2} \, \frac{(1-\beta^-)(2-\alpha^- - \beta^-)}{(1-\alpha^-)(2-\alpha^-)} \le c\,\varepsilon.
\end{aligned}
$$

From [15, p.12], if $\sigma_1 \neq \sigma_2$ are positive numbers and $0 \le \mu \le 1$, then $|\,\sigma_1^\mu - \sigma_2^\mu\,| \le |\,\sigma_1 - \sigma_2\,|^\mu$. Therefore $\omega^{1-\alpha^-, 1-\beta^-} \in \mathcal{H}^{0,q}$ where $q = \min\{\,1-\alpha^-, 1-\beta^-\}$ and the function $B_{\frac{1+x}{2}}(1-\beta^-, 1-\alpha^-\,) \in \mathcal{H}^{0,q}$. Consequently, $Rg \in \mathcal{H}^{0,q}$. ■

The following lemma can be used to obtain a criteria for the function Rg to be in $W^s_{\alpha^-,\beta^-}$ for maximum s and at the same time to find a space $\mathcal{H}^{0,q'}$ such that the function $Rg^{(s)}\phi^s\omega^{\alpha^-,\beta^-} \in \mathcal{H}^{0,q'}$.

Lemma 6.7. *If $g \in C_{\alpha^+,\beta^+}$, then $Rg \in W^{s,q'}_{\alpha^-,\beta^-}$ where $q' = \min\{\,1-\alpha^-, 1-\beta^-, \alpha^-, \beta^-, 1/2\}$ and*

$$
s = \begin{cases} 2 & \text{if } (\overline{f} - a_2\,)' \in C_{\alpha^-+1,\,\beta^-+1} \text{ and } \partial l(x,t)/\partial x \text{ satisfies the assumptions of} \\ & \text{Theorem 5.3, with } \rho \le 1+\alpha^-, \tau \le 1+\beta^-, \\ 1 & \text{otherwise.} \end{cases}
$$

Proof: Firstly, we shall verify for which values of s, $Rg \in W^s_{\alpha^-,\beta^-}$.

Clearly $Rg \in W^1_{\alpha^-,\beta^-}$ and,

$$
Rg^{(2)}(x) = g'(x)\omega^{\alpha,\beta}(x) + g(x)\omega^{\alpha-1,\beta-1}(x)[\,\beta - \alpha - (\alpha+\beta)x\,];
$$

hence $Rg \in W^2_{\alpha^-,\beta^-}$ if g' exists and $\|g'\omega^{\alpha^+,\beta^+}\phi^2\,\|_\infty < \infty$.

However,

$$
\begin{aligned}
Rg^{(3)}(x) \;&=\; g^{(2)}(x)\omega^{\alpha,\beta}(x) + 2g'(x)\omega^{\alpha-1,\beta-1}(x)[\,\beta - \alpha - (\alpha+\beta)x\,] \\
&+ g(x)\omega^{\alpha-2,\beta-2}(x)[\beta(\beta-1)(1-x)^2 + \alpha(\alpha-1)(1+x)^2 + 2\alpha\beta(1-x^2)\,].
\end{aligned}
$$

Hence $Rg \notin W^3_{\alpha^-,\beta^-}$ (unless $g^{(2)}$ exists, $\|g^{(2)}\omega^{\alpha^+,\beta^+}\phi^3\|_\infty < \infty$, $\|g'\omega^{\alpha^+,\beta^+}\phi\,\|_\infty < \infty$ and $\|g\omega^{\alpha^+-1/2,\beta^+-1/2}\|_\infty < \infty$. As $\alpha^+, \beta^+ < 1/2$, the last inequality requires special properties of the function g, making this a special case).

Now consider the case $s = 2$. We shall verify under which conditions $Rg \in W^2_{\alpha^-,\beta^-}$.

From (6), we obtain

$$a_1(\omega^{\alpha,\beta}(x)g(x))' = \bar{f}'(x) - a_2(x)'Rg(x) - a_2(x)\omega^{\alpha,\beta}(x)g(x) - \int_{-1}^{1} \frac{\partial}{\partial x} l(x,t)\, Rg(t)\, dt$$
$$- \frac{b_1}{\pi}\left[\frac{d}{dx}\left(\fint_{-1}^{1} \frac{\omega^{\alpha,\beta}(t)g(t)}{t-x}\, dt\right)\right].$$

Applying the Mean Value Theorem for integrals in the last integral, yields

$$\frac{d}{dx}\left(\fint_{-1}^{1} \frac{g(t)\,\omega^{\alpha,\beta}(t)}{t-x}\, dt\right) = \int_{-1}^{1} \frac{g(t)\,\omega^{\alpha,\beta}(t)}{(t-x)^2}\, dt = (g\,\omega^{\alpha^+,\beta^+})(\xi) \int_{-1}^{1} \frac{\omega^{-\alpha^-,-\beta^-}(t)}{(t-x)^2}\, dt$$
$$= (g\,\omega^{\alpha^+,\beta^+})(\xi)\Bigg\{\frac{-(-1)^{\beta^-}}{2^{\alpha^-}}\Gamma(1-\beta^-)\,(1+x)^{-1-\beta^-}\ {}_2F_1[\alpha^-, 1-\beta^-; 2+\alpha^-; (1-x)/2]$$
$$- \frac{(-1)^{\alpha^-}}{2^{\beta^-}}\Gamma(1-\alpha^-)\,(1-x)^{-1-\alpha^-}\ {}_2F_1[\beta^-, 1-\alpha^-; 2+\beta^-; (1+x)/2]\Bigg\}\,\Gamma(1+\alpha^-+\beta^-),$$

for $-1 < \xi < 1$. Therefore,

$$\left\|\phi^2\,\omega^{\alpha^-,\beta^-}\frac{d}{dx}\left(\fint_{-1}^{1}\frac{g(t)\,\omega^{\alpha,\beta}(t)}{t-x}\, dt\right)\right\|_\infty < \infty.$$

From Theorem 5.3, if $\partial\, l/\partial x \in C_{\rho,\tau,x} \oplus C_{\nu,\varsigma,t}$ with $\nu + \alpha^-,\ \varsigma + \beta^- < 1$, we have $(LRg)' \in C_{\rho,\tau}$. In this case, since $\rho \le 1+\alpha^-$, $\tau \le 1+\beta^-$, we have $\|(LRg)'\phi^2\omega^{\alpha^-,\beta^-}\| < \infty$.

Therefore, $Rg \in W^s_{\alpha^-,\beta^-}$ with

$$s = \begin{cases} 2 & if\ (\bar{f} - a_2)' \in C_{1+\alpha^-,1+\beta^-} \text{ and } \partial l(x,t)/\partial x \text{ satisfies the assumptions of Theorem} \\ & \text{5.3, with } \rho \le 1+\alpha^-, \tau \le 1+\beta^-, \\ 1 & \text{otherwise.} \end{cases}$$

From [10, Lemma 3.11], $Rg^{(s)}\phi^s\,\omega^{\alpha^-,\beta^-} \in \mathcal{H}^{0,\mu}$ for some $0 < \mu \le 1$. According to the properties of Hölder continuous functions, if a function $u(s) \in \mathcal{H}^{0,\mu}$ over some interval $s_1 \le s \le s_2$ and $f(u)$ is a function defined for values $u(s)$ in this interval, such that its derivative $f'(u)$ is bounded, then $f(u) \in \mathcal{H}^{0,\mu}$ (see [15, p. 16]) . Using these properties and the Mean Value Theorem for integrals, we obtain

$$\int_{-1}^{x} [Rg^{(1)}\phi\,\omega^{\alpha^-,\beta^-}](t)\, dt = (Rg\,\phi\,\omega^{\alpha^-,\beta^-})(x) - \int_{-1}^{x} Rg(t)\,[\omega^{\alpha^-,\beta^-}(t)\phi(t)]'\, dt$$
$$= (Rg\,\phi\,\omega^{\alpha^-,\beta^-})(x) - (Rg\,\omega^{\alpha^-,\beta^-})(\xi)\,[\beta^- - \alpha^- - (\alpha^- + \beta^-)\xi - \xi]\int_{-1}^{x}\phi^{-1}(t)\, dt$$
$$= (Rg\,\phi\,\omega^{\alpha^-,\beta^-})(x) + c\,B_{\frac{1+x}{2}}(1/2, 1/2).$$

Then, we have $Rg^{(1)}\phi\,\omega^{\alpha^-,\beta^-} \in \mathcal{H}^{0,q'}$ for $q' = \min\{1-\alpha^-, 1-\beta^-, \alpha^-, \beta^-, 1/2\}$. Analogously, we obtain

$$\int_{-1}^{x}[\,Rg^{(2)}\phi^2\,\omega^{\alpha^-,\beta^-}\,](t)\,dt \;=\; (Rg^{(1)}\phi^2\,\omega^{\alpha^-,\beta^-})(x) - \int_{-1}^{x} Rg^{(1)}(t)\,[\,\phi^2(t)\,\omega^{\alpha^-,\beta^-}(t)\,]'\,dt$$
$$= (Rg^{(1)}\phi^2\,\omega^{\alpha^-,\beta^-})(x) - (Rg^{(1)}\,\omega^{\alpha^-,\beta^-}\,)(\xi)[-2\xi\; + \beta^-(1-\xi) - \alpha^-(1+\xi)](x+1).$$

Furthermore, $Rg^{(2)}\phi^2\,\omega^{\alpha^-,\beta^-} \in \mathcal{H}^{0,q'}$ if $Rg \in W^2_{\alpha^-,\beta^-}$. ■

To prove the convergence of the collocation method, it will be necessary to show that the linear operator $[I + H^I\mathbb{P}_{n-1}(DR + LR)\,]^{-1}$ is bounded.

Lemma 6.8. *Let the operator $[\,I{+}H^I(D{+}L)R\,]$ be continuously invertible into C_{α^-,β^-} and let the assumptions of the Theorem 5.3 and Lemma 5.5 be satisfied. Then for sufficiently large n, $[I + H^I\mathbb{P}_{n-1}(D+L)R\,]^{-1}$ exists and*

$$\|\,[I + H^I\mathbb{P}_{n-1}(D+L)R\,]^{-1}\|_{\infty,\alpha^+,\beta^+}$$
$$\leq \frac{\|\,[\,I + H^I(D+L)R\,]^{-1}\|_{\infty,\alpha^+,\beta^+}}{1 - \|\,[\,I + H^I(D+L)R\,]^{-1}\|_{\infty,\alpha^+,\beta^+}\,\|H^I[\,\mathbb{P}_{n-1}(D+L\,)\;-\;(D+L)\,]R\,\|_{\infty,\alpha^-,\beta^-}}.$$

<u>*Proof*</u> : From [10, proposition 4.7 and remark 4.9], Theorem 5.3 and Lemma 5.5, the operator $H^I(D{+}L)Rg$ and consequently $H^I\mathbb{P}_{n-1}(D{+}L)Rg$ are bounded in C_{α^+,β^+}. From Lemma 6.2 , we obtain $H^I\mathbb{P}_{n-1}(D{+}L)Rg \;\longrightarrow\; H^I(D{+}L)Rg$ uniformly as $n \to \infty$. Then for n sufficiently large,

$$\|\,[\,I{+}H^I\mathbb{P}_{n-1}(D{+}L)R\,] - [\,I{+}H^I(D{+}L)R\,]\,\|_{\infty,\alpha^-,\beta^-}\,\|\,[\,I{+}H^I(D{+}L)R\,]^{-1}\,\|_{\infty,\alpha^+,\beta^+} \leq\; 1.$$

The result then follows from [2, p. 15]. ■

Theorem 6.9. *Let $a_2 \in W^{r,\upsilon}_{0,0}$, $\overline{f} \in W^{r',\eta}_{\alpha^-,\beta^-}$, $Rg \in W^{s,q'}_{\alpha^-,\beta^-}$ and $LRg \in W^{\widetilde{r},\nu}_{\alpha^-,\beta^-}$. Suppose also that $l(\bullet,t)\,\omega^{\rho,\tau}(\bullet) \in W^{\overline{r},\mu}_{\nu,\varsigma}$ uniformly with respect to $x \in [-1,1]$, for ν,ς nonnegative constants such that $\nu + \alpha^- < 1$ and $\varsigma + \beta^- < 1$. Then for sufficiently large n,*

$$\|\,g - g_n\,\|_{\infty,\alpha^+,\beta^+} \;\leq\; c\,\frac{\log^2 n}{n^p}\,, \qquad p = \min\{r+\upsilon, r'+\eta, s+q', \widetilde{r}+\nu, \overline{r}+\mu\}.$$

Proof: Applying the operator H^I to the left of equations (6) and (16) and recalling that $H^I H g = g + g_0$, we see, respectively that

$$[I + H^I(D+L)R]\, g \;=\; H^I \overline{f} + g_0 \text{ and } [I + H^I \mathbb{P}_{n-1}(D + \widetilde{L}_n)R]\, g_n = \; H^I \mathbb{P}_{n-1}\overline{f} + g_0.$$

From Section 2, g_0 is uniquely determined and $[I+H^I(D+L)R]^{-1}$ exists. From Lemma 6.3, for n sufficiently large, $\widetilde{L}_n Rg \longrightarrow L\, Rg$ and therefore in Lemma 6.8 we can substitute L for $\widetilde{L}_n$ demonstrating that the discrete equation has a unique solution. Furthermore,

$$\begin{aligned}[I + H^I \mathbb{P}_{n-1}(DR + LR)\,](g - g_n) \;&=\; H^I\{\,(\,\overline{f} - \mathbb{P}_{n-1}\overline{f}\,) - (\,DR - \mathbb{P}_{n-1}DR\,)g \\ &-\;\; (\,LR - \mathbb{P}_{n-1}LR\,)g - \mathbb{P}_{n-1}\,(\,LR - \widetilde{L}_n R\,)\, g_n\,\}\end{aligned}$$

and applying Lemma 6.8, we obtain

$$\begin{aligned}\|g - g_n\|_{\infty,\alpha^+,\,\beta^+} \le c\,\{\,\|H^I\,(\overline{f} - \mathbb{P}_{n-1}\overline{f})\|_{\infty,\alpha^+,\beta^+} + \|H^I\,(DR - \mathbb{P}_{n-1}DR)g\|_{\infty,\alpha^+,\beta^+} \\ + \|H^I\,(\,LR - \mathbb{P}_{n-1}LR\,)g\|_{\infty,\alpha^+,\beta^+} \;+\; \|H^I\,\mathbb{P}_{n-1}\,(\,LR - \widetilde{L}_n R\,)g_n\|_{\infty,\alpha^+,\beta^+}\,\}.\end{aligned}$$

From [10, Corollary 4.5], $\|H^I P_n\|_{\infty,\alpha^+,\beta^+} \le c\,\|P_n\|_{\infty,\alpha^-,\beta^-}\, \log n$, for $n \ge 2$. From this and from Lemma 6.2, we obtain

$$\begin{aligned}\|g - g_n\|_{\infty,\alpha^+,\,\beta^+} \;&\le c\,\log n\,\|\,\mathbb{P}_{n-1}\|_{\infty,\alpha^-,\beta^-}\{E_{n-1}^{\alpha^-,\beta^-}(\overline{f}) + E_{n-1}^{\alpha^-,\beta^-}(DRg) + E_{n-1}^{\alpha^-,\beta^-}(LRg)\} \\ &+\,\log(n-1)\,\|\,\mathbb{P}_{n-1}\,(LRg_n - \widetilde{L}_n Rg_n)\,\|_{\infty,\alpha^-,\beta^-} \\ &\le\; c\,\log n\,\|\,\mathbb{P}_{n-1}\|_{\infty,\alpha^-,\beta^-}\,\{\,E_{n-1}^{\alpha^-,\beta^-}(\overline{f}) \\ &+ E_{n-1}^{\alpha^-,\beta^-}(DRg) + E_{n-1}^{\alpha^-,\beta^-}(LRg\,) \;+\; \|\,(\,LR - \widetilde{L}_n R)g_n\,\|_{\infty,\alpha^-,\beta^-}\,\}.\end{aligned}$$

Since $LRg \in W^{\widetilde{r},\nu}_{\alpha^-,\beta^-}$ and $l(\bullet,t)\,\omega^{\rho,\tau}(\bullet) \in W^{\overline{r}+\mu}_{\nu,\varsigma}$ uniformly with respect to $x \in [-1,1]$ we can deduce that $l \in C^{\mathscr{B}}_{\alpha^-,\,\beta^-,x} \oplus C_{\nu,\varsigma,t}$ for $\mathscr{B} = \{b_n\} = O(n^{-\widetilde{r}-\nu})$. Therefore Lemma 6.3 can be applied.

From Theorem 6.1 we obtain:

$$\begin{aligned}\|g - g_n\|_{\infty,\alpha^+,\,\beta^+} \le c\log^2 n\{E_{n-1}^{\alpha^-,\beta^-}(\overline{f}) + E_{n-1}^{\alpha^-,\beta^-}(DRg) + E_{n-1}^{\alpha^-,\beta^-}(LRg) \\ + E_n^{\nu,\varsigma}(\,l(\bullet,t)\,\omega^{\rho,\tau}(\bullet))\}.\end{aligned}$$

Applying Lemma 6.5 and the Lemma 5.5, we have

$$\begin{aligned}\|g - g_n\|_{\infty,\alpha^+,\,\beta^+} \le\; c\,\log^2 n\,\Bigg\{\frac{1}{n^{r'+\eta}}\,\|\overline{f}^{\,(r')}\phi^{r'}\omega^{\alpha^-,\beta^-}\|_\infty + \frac{1}{n^{s+q'}}\,\|\,(Rg)^{(s)}\phi^s\omega^{\alpha^-,\beta^-}\|_\infty \\ + \frac{1}{n^{r+v}}\,\|a_2^{(r)}\phi^r\omega^{\alpha^-,\beta^-}\|_\infty + \frac{1}{n^{\widetilde{r}+\nu}}\,\|\,(LRg)^{(\widetilde{r})}\,\phi^{\widetilde{r}}\omega^{\alpha^-,\beta^-}\|_\infty \\ + \frac{1}{n^{\overline{r}+\mu}}\,\|\,(\,l(\bullet,t)\,\omega^{\rho,\tau}(\bullet))^{(\overline{r})}\phi^{\overline{r}}\omega^{\nu,\zeta}\|_\infty\Bigg\}.\end{aligned}$$

Then,

$$\|g - g_n\|_{\infty,\alpha^+,\beta^+} \leq \frac{c}{n^p}\log^2 n, \qquad p = \min\{r' + \eta, s + q', r + \upsilon, \widetilde{r} + \nu, \overline{r} + \mu\}.$$

7 Numerical Examples

In this Section we present four numerical examples that illustrate and, in two of the cases, match exactly, the theoretical results of this work. The numerical method, derived and analyzed in this paper, approximates $g(x)$ by a polynomial $g_n(x)$ - see equation (8). Then to obtain an approximation of the function $\varphi(x)$, we calculate $\varphi_n(x) = \int_{-1}^{x} g_n(t)\omega^{\alpha,\beta}(t)\,dt$. The results of the numerical examples will be presented as the difference between φ and φ_n, i.e., $e_n = ||\varphi - \varphi_n||_\infty$ while the convergence rate τ, to be used in the numerical examples below, is defined by:

$$\tau = \frac{\log \frac{e_n}{e_{2n}}}{\log(2)}.$$

Comparative numerical results for the first example will be provided for three methods, namely: the method presented in this work, the method of Capobianco et al. [5] and Multhopp's method. The latter method would appear only to be found in reference [11] and can only be applied to SIE of Prandtl's type with homogeneous boundary conditions, due to the form of the identities used to evaluate the terms in equation (6). Since Multhopp's method may be regarded as a special case of that of Capobianco et. al. [5], when $\alpha = \beta$ and the Multhopp's method is not applicable when $\alpha, \beta \neq \pm 1/2$, the results for this method will only be shown for the first example.

Example 7.1.

Consider the SIDE

$$-\frac{1}{\pi}\int_{-1}^{1}\frac{\varphi'(t)}{t-x}\,dt + \varphi(x) + \frac{1}{\pi}\int_{-1}^{1}\frac{(1-x)^8}{\sqrt{1.00000001+t}}\varphi(t)\,dt$$
$$= \arcsin(x) - 0.386319(1-x)^8. \qquad (17)$$

The analytical solution of this equation is $\varphi(x) = \arcsin(x)$ which satisfies the inhomogeneous boundary conditions $\varphi(-1) = -\pi/2$ and $\varphi(1) = \pi/2$. Applying a change of variables, as described in Section 2, allows equation (17) to be rewritten as

$$-\frac{1}{\pi}\int_{-1}^{1}\frac{\psi'(t)}{t-x}\,dt+\psi(x)+\frac{1}{\pi}\int_{-1}^{1}\frac{(1-x)^8}{\sqrt{1.00000001+t}}\,\psi(t)\,dt=$$
$$\arcsin(x)-\frac{\pi x}{2}-\frac{1}{2}\log\left|\frac{1+x}{1-x}\right|+0.084985(1-x)^8,$$

with the solution $\psi(x) = -\pi x/2 + \arcsin(x)$; the boundary conditions now become $\psi(-1) = \psi(1) = 0$. For the new equation in the transformed variables the derivative of the solution is of the form $\psi'(x) = \omega^{\alpha,\beta}(x)g(x)$, where $g(x) = 1 - \pi\sqrt{1-x^2}/2$ and $\omega^{\alpha,\beta}(x) = 1/\sqrt{1-x^2}$. The functions, a_2, $LRg \in W^{\infty}_{1/2,1/2}$, $l(\bullet,t)\,\omega^{1/2,1/2}(\bullet) \in W^{\infty}_{0,0}$ and $\overline{f} \in W^{1,1/2}_{1/2,1/2}$. From Lemma 6.7 we obtain $Rg \in W^{2,1/2}_{1/2,1/2}$. Then from Theorem 6.9, we obtain the estimate for the asymptotic error as $\log^2 n/n^{1.5}$.

This example presents a problem with inhomogeneous boundary conditions that, when transformed to one with homogenous boundary conditions gives rise to a problem with data having *log*-type singularities at the ends. The theory presented in this work predicts the rate of convergence to be $\log^2 n/n^{1.5}$ as shown above. On the other hand from Capobianco et al. [5] it can be shown that the predicted rate of convergence for their method for this example is $\log n/n^{0.5-\varepsilon}$, due to the way the term $Hg(x)$ is discretized in their method.

Table 1 displays the error and an approximation to the convergence rate, calculated from $\tau = \ln\left(\frac{e_n}{e_{2n}}\right)/\log(2)$ for both methods and also for Multhopp's method. The method of Capobianco [5] selected the collocation points to be the roots of the second kind Chebyshev polynomials, as required by the theory in [5]. This is the case for Examples 7.1 and 7.2. The convergence rates obtained in practice are better than those predicted by the theory, since the theory only provides a lower bound for the convergence rate. In all cases the convergence rates appear to be converging to 2. As already stated the Multhopp method is a special case of the method in Capobianco et al. [5] when in the latter method the collocation nodes are the roots of the second kind Chebyshev polynomials, and so its behaviour is similar.

n	16	32	64	128	256
$e_n^{(1)}$	3.4401×10^{-3}	8.0605×10^{-4}	1.9641×10^{-4}	4.8849×10^{-5}	1.2164×10^{-5}
$\tau^{(1)}$	2.0935	2.0370	2.0075	2.0057	
$e_n^{(2)}$	2.9066×10^{-3}	7.6525×10^{-4}	1.9415×10^{-4}	4.8806×10^{-5}	1.2202×10^{-5}
$\tau^{(2)}$	1.9788	1.9253	1.9920	1.9999	
$e_n^{(3)}$	2.9066×10^{-3}	7.6525×10^{-4}	1.9415×10^{-4}	4.8806×10^{-5}	1.2202×10^{-5}
$\tau^{(3)}$	1.9788	1.9253	1.9920	1.9999	

Table 1: (1)-collocation method; (2)-Method presented in [5]; (3)-Multhopp method.

The next example provides an instance where the predicted theoretical convergence rates are attained in practice.

Example 7.2.

Consider the SIDE

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\varphi'(t)}{t-x}\,dt+\varphi(x)+\int_{-1}^{1}l(x,t)\,\varphi(t)\,dt=f(x), \tag{18}$$

with homogeneous boundary conditions $\varphi(-1)=\varphi(1)=0$, where

$$l(x,t)=(1-x^2)^{0.2}\log\left|\frac{1+x}{1-x}\right|\frac{(1+t)^2}{1.0000001-t}.$$

and $f(x) = -1+\sqrt{1-x^2}+7.85088(1-x^2)^{0.2}\log\left|\frac{1+x}{1-x}\right|$. The exact solution to this equation is $\phi(x)=\sqrt{1-x^2}$.

Thus we see that, $LRg \in W^{1,0.2-\varepsilon}_{1/2,1/2}$, $l(\bullet,t)\,\omega^{1/2,1/2}(\bullet)\in W^{\infty}_{0,0}$ and $f\in W^{1,0.2-\varepsilon}_{1/2,1/2}$. The theoretical estimate is then $\log^2 n/n^{1.2-\varepsilon}$. The results are displayed in Table 2 and they seem to be converging to $1.2-\varepsilon$, for both method, as predicted.

n	64		128		256		512		1024
$e_n^{(1)}$	1.9703×10^{-2}		9.6064×10^{-3}		4.6402×10^{-3}		2.1912×10^{-3}		1.0024×10^{-3}
$\tau^{(1)}$		1.0363		1.0498		1.0825		1.1283	
$e_n^{(2)}$	1.9701×10^{-2}		9.6062×10^{-3}		4.6402×10^{-3}		2.1913×10^{-3}		1.0024×10^{-3}
$\tau^{(2)}$		1.0362		1.0498		1.0825		1.1283	

Table 2: (1)-collocation method; (2)-Method presented in [5].

Example 7.3.

Consider the singular integro-differential equation

$$\frac{1}{\sqrt{2}}\,\varphi'(x) - \frac{1}{\sqrt{2}\,\pi}\int_{-1}^{1}\frac{\varphi'(t)}{t-x}\,dt + \,(1-x)^{3/4}(1+x)^{1/4}\varphi(x) +$$
$$\int_{-1}^{1}\frac{\sqrt{1-x}(1-x)^{3/4}(1+x)^{1/4}}{1.00000001+t}\,\varphi(t)\,dt \;=\; -1+\frac{x^2}{2} - 1.11072\sqrt{1-x}(1-x)^{3/4}(1+x)^{1/4},$$

with the boundary conditions $\varphi(-1)=\varphi(1)=0$.

The analytical solution for this equation is $\varphi(x) = -1/2(1-x)^{1/4}(1+x)^{3/4}$. This example illustrates the case where the coefficient $a_1 \neq 0$ in equation (1), thus giving $\alpha \neq \beta$. Here $\alpha = 1/4$ and $\beta = 3/4$.

We note that $a_2 \in W^{1,1}_{3/4,1/4}$, $LRg \in W^{1,1/2}_{3/4,1/4}$, $f \in W^{1,1/2}_{3/4,1/4}$ and $Rg \in W^{2,1/4}_{3/4,1/4}$. However, since $l(\bullet,t)\,\omega^{3/4,1/4}(\bullet) \in W^{\infty}_{0,0}$, the estimate for the asymptotic error is $\log^2 n/n^{3/2}$. The results are displayed in Table 3. For Examples 7.3 and 7.4 the method in [5] was applied by collocating at the roots of the polynomials $P_n^{-1/4,-3/4}$, as required by their theory.

n	32		64		128		256		512
$e_n^{(1)}$	1.4238×10^{-3}		5.1384×10^{-4}		1.8357×10^{-4}		6.5201×10^{-5}		2.3079×10^{-5}
$\tau^{(1)}$		1.4703		1.4850		1.4933		1.4983	
$e_n^{(2)}$	1.4238×10^{-3}		5.1385×10^{-4}		1.8357×10^{-4}		6.5201×10^{-5}		2.3079×10^{-5}
$\tau^{(2)}$		1.4704		1.4850		1.4933		1.4983	

Table 3: (1)-collocation method; (2)-Method presented in [5].

Example 7.4.

In this example we consider the SIDE

$$\frac{1}{\sqrt{2}}\,\varphi'(x) - \frac{1}{\sqrt{2}\,\pi} \int_{-1}^{1} \frac{\varphi'(t)}{t-x}\,dt + \frac{\varphi(x)}{(1-x)^{3/4}(1+x)^{1/4}}$$
$$+\,\frac{1}{\pi}\int_{-1}^{1} \frac{|x|-|t|}{(1-x)^{3/4}(1+x)^{1/4}}\,\varphi(t)\,dt \;=\; \frac{1}{(1-x)^{3/4}(1+x)^{1/4}},$$

with boundary conditions $\varphi(-1) = \varphi(1) = 0$.

The analytical solution for this equation is unknown. To calculate the error we assume the numerical solution, obtained by the collocation method of this work with $n = 600$, is the exact solution.

We note that $a_2 \in W^{0,1}_{3/4,1/4}$, $LRg \in W^{0,1}_{3/4,1/4}$, $f \in W^{0,1}_{3/4,1/4}$ and $Rg \in W^{1,1/4}_{3/4,1/4}$. However, since $l(\bullet, t)\,\omega^{3/4,1/4}(\bullet) \in W^{0,1}_{0,0}$, the method of this paper can be applied.

n	32	64	128	256	512
$e_n^{(1)}$	4.4399×10^{-2}	1.9684×10^{-2}	8.0810×10^{-3}	2.6223×10^{-3}	7.2955×10^{-4}
$e_n^{(2)}$	1.3371×10^{-1}	5.7343×10^{-2}	3.5402×10^{-2}	3.3568×10^{-2}	3.4298×10^{-2}

Table 4: (1)-collocation method; (2)-Method presented in [5]

For the method in Capobianco et. al. [5] and [4] the data do not satisfy their assumptions and hence, strictly speaking their method cannot be applied to this problem. Nonetheless in table 4 we present the numerical results for the error for both methods. We see that the method of Capobianco does not appear to converge for this example as n is increased.

8 Conclusions

In this paper we have employed polynomial collocation to solve a SIDE with a Cauchy kernel. Nonhomogeneous boundary conditions meant that there would be discontinuities in the right-hand-side function data. A convergence analysis of the collocation method in a weighted uniform norm was given. Moreover, an estimate for the convergence rate of the method, that depends on the regularity of the function data involved was derived. The main difficulty posed for convergence was due to the discontinuity in the kernel, as is apparent from the numerical examples presented in Section 7.

Nagamine & Cuminato in [16] presented a uniform convergence analysis of the collocation method applied to (1) in the special case when $a_1 = 0$; and $\alpha = \beta = -1/2$, and $\varkappa = 1$. The maximum estimate for the convergence rate obtained was 1/2. Here, we were able to prove that the estimate can attain 5/2 depending on the regularity of the data functions concerned. This is due to the fact that the analysis of this paper is less restrictive on the regularity of the functions involved.

In [4], [5] and [13], the identity

$$a_1\,\varphi'(x) + \frac{b_1}{\pi}\int_{-1}^{1}\frac{\varphi'(t)}{t-x}\,dt \;=\; \frac{d}{dx}\left[a_1\,\varphi(x) + \frac{b_1}{\pi}\int_{-1}^{1}\frac{\varphi(t)}{t-x}\,dt\right],$$

valid when $\varphi(-1) = \varphi(1) = 0$, was employed. In this case, $\varphi(x) = g(x)\omega^{\alpha,\beta}(x)$ with $0 < \alpha, \beta < 1$, leading to $\alpha^- < \alpha^+$ and $\beta^- < \beta^+$, according to the definiton in our paper. For this reason the space C_{α^-,β^-}, to which a_2, f and l (with respect to x) must belong, is more restrictive than the one employed here. For some cases, this leads to an estimate for the convergence rate which is smaller than that obtained from Theorem 6.9, as discussed in the numerical examples. As a result of the more restrictive conditions, the theoretical convergence assumptions for the method of Capobianco et. al. [5] are more stringent than those for the method derived in this work, as was argued above. Nonetheless we have not managed to find a numerical example where this is shown up clearly. Even when the theoretical prediction for the convergence rate was lower in Capobianco's method, as in the case of Example 7.1, the practical calculations did not show it. Despite this, example 7.4 provides an instance where it seems that our method converges but the method of Capobianco does not.

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