RT-MAP-8409

DIRECTED CUT TRANSVERSAL PACKING FOR SOURCE-SINK CONNECTED GRAPHS

P. Feofiloff
D.H. Younger

NOVEMBRO 1984

Submitted to "Combinatorica"

DIRECTED CUT TRANSVERSAL PACKING FOR SOURCE-SINK CONNECTED GRAPHS

P. Feofiloff and D.H. Younger

Transversal Packing Conjecture: For any finite directed graph, a maximum packing of transverals of directed cuts is equal in cardinality to a minimum directed cut.

Woodall [6] describes this as the Menger dual of the directed cut packing minimax equality (Lucchesi and Younger [5], Lovász [4]). This paper gives a proof of the Conjecture for source-sink connected graphs, a proof that builds the required packing of transversals one edge at a time, by maintaining a Hall-like feasibility condition throughout the construction. A proof of this case has been given by Schrijver [7] from a different point of view.

Let G be a finite graph with vertex set VG and edge set eG. The coboundary operator δ is a function that takes any subset X of VG to the set δX of edges in G having one end in X and one end in $VG-X$. A coboundary in G is any set of edges that lies in the range of δ.

A directed graph is a graph in which each edge α is assigned a positive end (or tail) $p\alpha$ and a negative end (or head) $n\alpha$. A coboundary δX is directed if each edge in δX has its positive end in X or if each edge in δX has its negative end in X. A directed cut is a minimal non-null directed coboundary. Let C denote the collection of directed coboundaries of directed graph G.

For a subcollection B of C, a transversal of B is a subset of eG that has a nonnull intersection with each nonnull set in B. In the statement of the Conjecture, a transversal of C is called a transversal of directed cuts.

For any packing (= disjoint collection) of transversals of C and any nonnull element d in C, $|T| \le |d|$. This is elementary. The crux of the Conjecture is that every graph contains a pair T,d of equal size. That is, if eG is a k-transversal of C, then there is in eG a k-packing of transversals of C. A k-transversal is a subset r of eG such that $|r \cap d| \ge k$ for each nonnull d in C. A k-packing is a disjoint collection consisting of k elements. Following Seymour [12], we say that transversal r of C packs if for the largest integer k such that r is a k-transversal of C, there is a k-packing T of transversals of C such that $\cup T \subseteq r$. In this terminology, the Conjecture translates to: For each directed graph G, eG is a transversal of C that packs. A natural generalization has been formulated by Edmonds and Giles [2]:

Generalized Conjecture: Every transversal of C packs.

Schrijver [6] has constructed a counterexample to the Generalized Conjecture, but not to the basic Conjecture. The Generalized Conjecture is true for source-sink connected graphs. A directed graph is source-sink connected if it is acyclic and each source is joined to each sink by a directed path. A source is a vertex of invalence zero; a sink is a vertex of outvalence zero.

In this paper, we prove the source-sink connected case of the Generalized Conjecture in terms of side coboundaries of an arbitrary directed graph. We now develop this formulation.

Arguments of a directed coboundary d are defined as follows. Let X be a minimal subset of VG such that $d = \delta X$, where X contains the positive end of each edge of d. The positive (edge) argument pd of d is the set of edges of G with positive end in X. The negative argument nd is defined dually. Here we refer to directional duality, which interchanges the positive and negative ends of each edge.

Let Dn be the collection of elements d in C such that either d = ∅ or pd ∩ pc ≠ ∅ for each nonnull element c of C. Define Dp dually. The union D = Dp ∪ Dn is the collection of side coboundaries of G. Examples of side coboundaries are given in Figure 1.

Transversal Packing Theorem: Every transversal of D packs.

This Theorem implies the source-sink connected case of the Generalized Conjecture, since in a source-sink connected graph each directed coboundary is a side coboundary, i.e., C = D.

Our first step in proving this Theorem is a reduction to a Bi-transversal Theorem. Let Sp be the collection of p-minimal (= minimal positive argument) elements of Dp - {∅}. For subset t of eG, let tp denote t ∩ (∪Sp). Define Sn and tn dually. A bi-transversal of D is a set t of edges such that tp is a transversal of Dp and tn is a transversal of Dn. A bi-transversal of D is, in particular, a transversal of D. A bi-transversal r of D packs if, for the largest integer k such that r is a k-bi-transversal of D, there is in r a k-packing of bi-transversals of D.

Bi-transversal Theorem: Every bi-transversal of D packs.

In Section 3, the Transversal Packing Theorem is reduced to the Bi-Transversal Theorem.

2. Properties of Side Coboundaries

The domain of the Theorem can be reduced easily to connected graphs. The following properties of the collection D = Dp ∪ Dn of side coboundaries of a connected graph are those used in this paper:

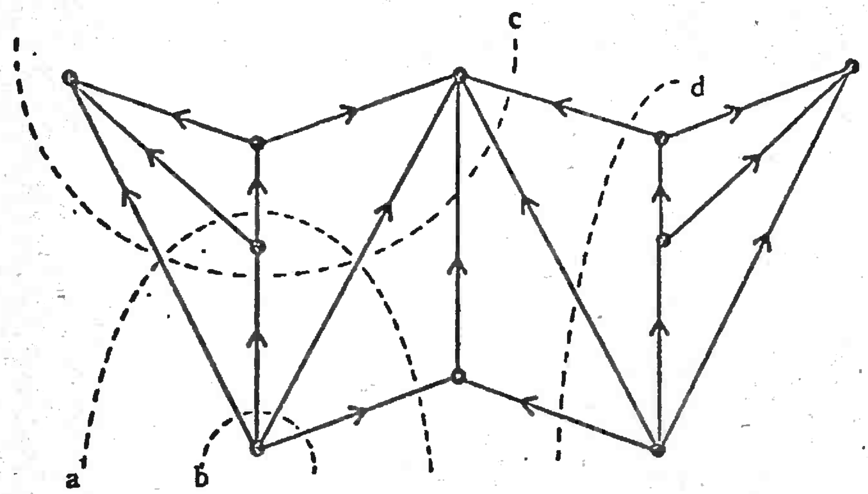

Figure 1: Side coboundaries a in Dp, b in Sp, c in Dn; directed coboundary d in C - D.

1. $\emptyset \in Dn$, where $n\emptyset = p\emptyset = \emptyset$. If Dn contains a nonnull element, so does Dp.
2. For d in Dn, $nd \cap pd = d$; if $d \neq \emptyset$, then $nd \cup pd = eG$.
3. a) For c in Dn, d in D, there exists an element $c \wedge_n d$ in Dn, the <u>n-meet</u> of c and d, such that $n(c \wedge_n d) = nc \cap nd$; if $c \wedge_n d \neq \emptyset$, then $p(c \wedge_n d) = pc \cup pd$.

   b) For c,d in Dn, there exists an element $c \vee_n d$ in Dn, the <u>n-join</u> of c and d, such that $n(c \vee_n d) = nc \cup nd$; if each of c and d is nonnull, then $p(c \vee_n d) = pc \cap pd$.
4. For c in Dn, d in D, if $nc \subseteq nd$, then $nc - c \subseteq nd - d$.

The directional duals of the above also hold.

From properties 1 and 2, each side-coboundary is the intersection of its positive and negative arguments. By property 4, there is but one element of Dn with a given negative argument. Dually for Dp. By properties 2 and 3, $c \wedge_n d = c \cap nd \cup d \cap nc$ and, if c and d are nonnull, $c \vee_n d = c \cap pd \cup d \cap pc$; if one of c and d is null, then $c \vee_n d$ is equal to the other. The <u>n-join</u> $\bigvee X$ of a subcollection X of Dn is $\emptyset$ if X is null and is $x \vee_n \bigvee(X-\{x\})$, $x \in X$, if X is nonnull. The <u>p-join</u> is defined dually.

3. <u>Reduction to the Bi-transversal Theorem</u>

We now prove the Transversal Packing Theorem assuming the Bi-transversal Theorem. Let r be a transversal of the collection D of side coboundaries of directed graph G. We interpret the case $D = \{\emptyset\}$ as satisfying the Theorem and assume hereafter that D contains a nonnull element. Let k be the largest integer for which r is a k-transversal of D. Since r is a transversal of D, $k \geq 1$.

The basis of induction is the case in which r is a k-bi-transversal of D. By the Bi-transversal Theorem, there is in r a k-packing of bi-transversals of D. Each bi-transversal is a transversal and so the assertion holds.

Assume as induction hypothesis that the assertion holds for every graph G' and every subset r' of eG' such that $|r'| < |r|$ or $|r'| = |r|$ and $|eG'| < |eG|$.

Case 1: For some edge α in r, $r - \{\alpha\}$ is a k-transversal of D.

By induction hypothesis, there is in $r - \{\alpha\}$ a k-packing of transversals of D. This k-packing satisfies the assertion for r and G.

Case 2: $|r \cap d| = k$ for some d in $D - Sp \cup Sn$.

Adjust notation so that $d \in Dp$. Let D' be the collection of side coboundaries of the graph G', where G' is obtained from G by contracting the edges of $nd - d$. Then $D' = D'p \cup D'n$, where $D'p = \{c \in Dp : pc \subseteq pd\}$ and $D'n$ contains $\emptyset$ and d and perhaps some other coboundaries not relevant here. Let D'' be the collection of side coboundaries of the graph G'' obtained from G by contracting the edges of $pd - d$. Then $D'' = D''p \cup D''n$, where $D''p = \{c \in Dp : pc \cap pd = \emptyset \text{ or } pd\}$ and $D''n = \{b \in Dn : nb \subseteq nd\}$. Now $r' = r \cap pd$ and $r'' = r \cap nd$ are k-transversals of D' and D'', respectively. Since d is, by hypothesis, not in Sp or Sn, each of $|eG'|$ and $|eG''|$ is strictly smaller than $|eG|$. By induction hypthesis, there is in r' a k-packing T' of transversals of D'. Likewise, there is in r'' a k-packing T'' of transversals of D''. Let T be $\{t' \cup t'' : t' \in T', t'' \in T'', t' \cap d = t'' \cap d\}$. Since $|r \cap d| = k$, each edge of $r \cap d$ lies in exactly one transversal of T' and in one transversal of T''. So T is a k-packing of subsets of r. We assert that each t in T is a transversal of D. This is proved, as in [10], as follows.

Each t in T is of the form $t = t' \cup t''$. Let a be any nonnull element of D, say in Dp. If $a \wedge_p d = \emptyset$, then $a \in D''p$ and $t \cap a = t'' \cap a \neq \emptyset$. If $a \wedge_p d \neq \emptyset$, then $a \wedge_p d \in D'p - \{\emptyset\}$ and $a \vee_p d \in D''p - \{\emptyset\}$, whence $t \cap (a \wedge_p d) = t' \cap (a \wedge_p d) \neq \emptyset$ and $t \cap (a \vee_p d) = t'' \cap (a \vee_p d) \neq \emptyset$. From the modularity relation $|t \cap (a \wedge_p d)| + |t \cap (a \vee_p d)| = |t \cap a| + |t \cap d|$, since $|t \cap d| = 1$, thus $t \cap a \neq \emptyset$. So t is a transversal of Dp. Likewise, t is a transversal of Dn, and thus of all of D. This case is complete.

Case 3: $|r \cap d| > k$ for each d in $D - Sp \cup Sn$, $r = rp \cup rn$, and r is not a k-bi-transversal of D.

Adjust notation so that rn is not a k-transversal of Dn. There exists a nonnull element a in Dn such that $|rn \cap a| < k$; adjust the choice of a so that it is n-minimal. Since $|r \cap a| \geq k$, there is an edge β in $(rp-rn) \cap a$. Now a, since it contains an edge of $r - rn$, does not lie in Sn. So there is an element a' in Dn such that $na' \subseteq na - \{\beta\}$. Adjust the choice of a' so that is is n-maximal. By the choice of a, $|rn \cap a'| \geq k$. Let α be any edge in $rn \cap a' - a$. We then have the following properties: for each c in Dn such that $\alpha \in nc$, $\{\alpha,\beta\}$ intersects c; for each d in Dp such that $\beta \in pd$, $\{\alpha,\beta\}$ intersects d.

Let G' be the graph obtained from G by adding a new edge γ to G with negative $n\alpha$ and positive end $p\beta$. Let $D' = D'n \cup D'p$ be the collection of side coboundaries of G'. Then $D'n$ is the same as Dn except that γ is added to each c in Dn such that $\alpha \in nc$. And $D'p$ is the same as Dp except that γ is added to each d in Dp such that $\beta \in pd$. Let $r' = (r - \{\alpha,\beta\}) \cup \{\gamma\}$. Since $|r \cap d| \geq k$ for each d in D, with equality only for d in $Sp \cup Sn$, thus

$|r' \cap d| \geq k$ for each d in D', i.e., r' is a k-transversal of D'. Since $|r'| < |r|$, by induction hypothesis there is in r' a k-packing T' of transversals of D'. Each element of T' that does not contain γ is a transversal of D. For the element t' of T' that contains γ, $t = (t'-\{\gamma\}) \cup \{\alpha,\beta\}$ is a transversal of D. So $T = (T'-\{t'\}) \cup \{t\}$ is a k-packing in r of transversals of D. Again the assertion holds.

This completes the proof of the Transversal Packing Theorem under the assumption of the Bi-transversal Theorem.

4. Constructing bi-transversals one edge at a time

In his proof of Edmonds Disjoint Branchings Theorem [1], Lovász [4] finds one branching that saves enough room in each coboundary for the remaining $k-1$ branchings. This branching is constructed one edge at a time, by successively adding a new edge emanating from the current partial branching. The property required of each new edge is that its choice leaves at least $k-1$ edges unchosen in each coboundary. Our proof also uses this one edge at a time approach.

A subset t of eG is central in Dn if each coboundary d in Dn that is disjoint from tn has nd disjoint from tn. Centrality in Dp is defined dually. A subset t of eG is central if it is central in in Dn and in Dp. Recall that $tn = t \cap (\cup Sn)$.

For subset t of eG, the frontier f of t in Dn is the n-maximal coboundary in Dn that is disjoint from tn.

4.1. Let t be a subset of eG central in Dn and let α be an edge of the positive argument pf of the frontier f of t in Dn. Then $t \cup \{\alpha\}$ is central in Dn.

Proof: Assume that $a \in \cup Sn$, else the centrality of $t \cup \{a\}$ follows from t central. Let d be a coboundary in Dn disjoint from $tn \cup \{a\}$. Since t is central, nd is disjoint from tn. Since $a \in pf$, thus $a \notin nf - f$. By definition of frontier, $nd \subseteq nf$, whence $nd - d \subseteq nf - f$ and so $a \notin nd - d$. So nd is disjoint from $tn \cup \{a\}$. The assertion follows.

A subcollection Z of Dn is n-disjoint if $nc \cap nd = \emptyset$ for each distinct c,d in Z. In this case, the join $\bigvee Z$ is an element of Dn equal to the union $\cup Z$, and $n(\cup Z) = \cup nd \ (d \in Z)$. Let nZ abbreviate $n(\cup Z)$.

We next define feasibility for a subset t of bi-transversal r. Let Z be an n-disjoint subcollection of $Dn - \{\emptyset\}$ and X a p-disjoint subcollection of $Dp - \{\emptyset\}$. Define (Z,X) to be a Dn-pair if $rZ \supseteq rX$, where $rZ = rn \cap (\cup Z)$ and $rX = rp \cap (\cup X)$. We verbalize $rZ \supseteq rX$ as Z shades X. Define function u on n-disjoint subcollections of $Dn - \{\emptyset\}$ by $uZ = |rZ-t| - (k-1)|Z|$. A subset t of r is feasible in Dn if every Dn-pair (Z,X) for which tp is disjoint from $\cup X$ satisfies $uZ \geq |X|$. Incidentally, all Dn-pairs to be considered satisfy $tp \cap \cup X = \emptyset$: we take this condition as understood. A Dp-pair and feasibility in Dp are defined dually. A subset t or r is feasible if it is feasible in Dn and in Dp.

4.2. Let r be a k-bi-transversal of D. The null set is central and feasible.

Proof: Centrality of the null set is immediate. For feasibility, consider any D-pair (Z,X), say a Dn-pair. Since rn is a k-transversal of Dn and t is null, $uZ = |rZ| - (k-1)|Z| \geq k|Z| - (k-1)|Z| = |Z|$. From $rZ \supseteq rX$ and rp a k-transversal of Dp, $|rZ| \geq |rX| \geq k|X|$, whereupon

$uZ = |rZ| - (k-1)|Z| \geq k|X| - (k-1)uZ$, and so $uZ \geq |X|$.

For subset t of r that is central and feasible, an augment of t is any edge α of r-t such that $t \cup \{\alpha\}$ is also central and feasible. The crux of the theory is that every central feasible subset of r that is not a bi-transversal of D has an augment; this is shown in Section 6. Given this, it follows that there is in r a feasible bi-transversal t of D. This in turn implies that r-t is a $(k-1)$-bi-transversal of D. To prove the latter, consider any nonnull element d in D, say $d \in Dn$. Then $(\{d\},\emptyset)$ is a Dn-pair, for which, by t feasible, $0 \leq u\{d\} = |r\{d\}-t| - (k-1)$. Thus $|r\{d\}-t| = |(rn-t)\cap d| \geq k - 1$, whence $r - t$ is a $(k-1)$-bi-transversal of D.

The Bi-transversal Theorem is proved next. Let r be a k-bi-transversal of D. We proceed by induction on k. For $k = 1$, the Theorem is trivially true. Assume then that $k \geq 2$. Let t be a feasible bi-transversal of D in r. Since $r - t$ is a $(k-1)$-bi-transversal, by induction hypothesis there is in r-t a $(k-1)$-packing T' of bi-transversals of D. Then $T' \cup \{t\}$ is a k-packing in r of bi-transversals of D. The Theorem follows by induction.

There remains only the proof that a central feasible subset t of r that is not a bi-transversal has an augment. Conditions on an edge α under which $t \cup \{\alpha\}$ is central are given in 4.1; conditions for feasibility are considered here. For α in r-t, a Dn-pair (Z,X) (such that $tp \cap \cup X = \emptyset$) is a blocker of α in Dn if $uZ = |X|$ and $\alpha \in rZ - rX$. A blocker of α in Dp is defined dually. From this definition, it follows that for any central feasible subset t of k-bi-transversal r of D and any edge α in $r - t$, $t \cup \{\alpha\}$ is feasible iff α has no blocker either in Dn or in Dp.

5. Meet and join

Let each of Y and Z be an n-disjoint subcollection of $Dn - \{\emptyset\}$. Since an element in Dn is determined by its negative argument, $\cup Y = \cup Z$ iff $nY = nZ$. The meet $Y \wedge Z$ of Y and Z is the collection of all nonnull coboundaries of the form $c \wedge d$, $c \in Y$, $d \in Z$. The join $Y \vee Z$ is the collection of all coboundaries of the form $\bigvee W$, where W is a minimal nonnull subcollection of $Y \cup Z$ that contains every element of this union that meets $\bigvee W$.

More relevant to the proof are the following consequences of these definitions.

5.1 i) Each of $Y \wedge Z$ and $Y \vee Z$ is an n-disjoint subcollection of $Dn - \{\emptyset\}$.

ii) $\cup(Y \wedge Z) = \cup Y \wedge \cup Z$ and $\cup(Y \vee Z) = \cup Y \vee \cup Z$

$n(Y \wedge Z) = nY \cap nZ$ and $n(Y \vee Z) = nY \cup nZ$.

iii) (Supermodularity) $|Y \wedge Z| + |Y \vee Z| \geq |Y| + |Z|$.

Proof: Parts i and ii follow from the definitions of meet and join. To prove part iii, form graph B with bipartition (Y,Z), whose edges represent the pairs (c,d), $c \in Y$, $d \in Z$, such that c meets d, i.e., $c \wedge d \neq \emptyset$. Then $|Y \vee Z|$ is equal to the number of components of B. The asserted supermodularity relation translates to

$$|eB| + \# \text{ components } B \geq |VB|,$$

a simple fact about graphs.

As an abstraction from blocker, we define a Dn- or Dp-pair (Z,X) as marginal if $uZ = |X|$.

5.2. Let t be a feasible subset of k-bi-transversal r. If (Z,X) and (Z',X') are marginal Dn-pairs, then the meet $(Z\wedge Z',X\wedge X')$ and join $(Z\vee Z',X\vee X')$ are marginal Dn-pairs

<u>Proof</u>: 1. The meet and join are Dn-pairs.

Since $rX \subseteq rZ$ and $rX' \subseteq rZ'$, thus rX and rX' are subsets of $rp \cap rn$:

$$r(X\wedge X') = rn \cap p(X\wedge X') = rX \cap rX' \subseteq rZ \cap rZ' \subseteq r(Z\wedge Z'),$$

i.e., $Z \wedge Z'$ shades $X \wedge X'$. Likewise,

$$r(X\vee X') = rX \cup rX' \subseteq rp \cap (rZ \cup rZ') \subseteq r(Z\vee Z').$$

2. The u function is submodular.

For z,z' in Dn,

$$z \wedge z' \cap z \vee z' = z \cap z'$$
$$z \wedge z' \cup z \vee z' = z \cup z'.$$

Set $z = \cup Z$ and $z' = \cup Z'$ and intersect each with rn:

$$r(Z\wedge Z') \cap r(Z\vee Z') = rZ \cap rZ'$$
$$r(Z\wedge Z') \cup r(Z\vee Z') = rZ \cup rZ'.$$

Restrict each equivalence to its edges not in t and add cardinalities:

$$|r(Z\wedge Z')-t| + |r(Z\vee Z')-t| = |rZ-t| + |rZ'-t|.$$

Subtract $k-1$ times the supermodularity relation 5.1iii to get the submodularity relation for u:

$$u(Z\wedge Z') + u(Z\vee Z') \leq uZ + uZ'.$$

3. The meet and join are marginal.

From the supermodularity relation 5.1iii , feasibility conditions, and submodularity of u,

$$\begin{aligned}|X| + |X'| &\le |X\wedge X'| + |X\vee X'| \\ &\le u(Z\wedge Z') + u(Z\vee Z') \\ &\le uZ + uZ' \\ &= |X| + |X'|.\end{aligned}$$

So $|X\wedge X'| = u(Z\wedge Z')$ and $|X\vee X'| = u(Z\vee Z')$.

6. <u>Augment Lemma</u>

<u>Augment Lemma</u>: Let t be a central feasible subset of k-bi-transversal r of D. If t is not a bi-transversal of D, then t has an augment.

<u>Proof</u>: By hypothesis, tn is not a transversal of Dn, or we can arrange that to be the case by exchanging p for n. Thus the frontier f of t in Dn is nonnull. Extending our previous convention, let rd denote $rn \cap d$ if $d \in Dn$ and $rp \cap d$ if $d \in Dp$. For any edge α in rf, $t \cup \{\alpha\}$ is central in Dn by 4.1; it is also central in Dp. Thus it is sufficient to show that some edge of rf has no blocker. Consider an n-minimal element a of $Dn - \{\emptyset\}$ such that $ra \subseteq rf$. Let (Z,X) be a marginal Dp-pair such that $X \wedge \{a\} = \emptyset$; the null pair is one such. Adjust the choice of (Z,X) so that pZ is maximal. Now $ra \not\subseteq rZ$, else $(Z, X\cup\{a\})$ would be a Dp-pair that violates feasibility. Our first candidate for augment of t is any edge α in $ra - rZ$.

Edge α has no blocker in Dp. For consider any marginal Dp-pair (Z',X') such that $\alpha \in rZ'$. By 5.2, $(Z\vee Z', X\vee X')$ is a marginal Dp-pair. Since $\alpha \in (rZ'-rZ) \cap rn \subseteq pZ' - pZ = p(Z\vee Z') - pZ$, maximality of pZ implies that $(X\vee X') \wedge \{a\} \neq \emptyset$. Since $X \wedge \{a\} = \emptyset$, thus $X' \wedge \{a\} \neq \emptyset$. From

$r(X'\wedge\{a\}) \subseteq ra$, there follows $na \subseteq nX'$, by n-minimality of a. Since $a \in rZ' \subseteq rp$, thus $a \in rX'$. So (Z',X') is not a blocker of a. Indeed, a has no blocker in Dp.

We say that an n-disjoint subcollection Z of Dn meets coboundary d in Dn if $Z \wedge \{d\} \neq \emptyset$. A Dn-pair (Z,X) with a given property is argument-minimal with that property if every Dn-pair (Z',X') satisfying the given property and $nZ' \subseteq nZ$ and $pX' \subseteq pX$ satisfies $nZ' = nZ$ and $pX' = pX$. An internal edge of Dn-pair (Z,X) is any edge of $rn \cap (nZ-\cup Z) \cup rX$.

Returning to the proof, the only alternative to a an augment is that it have a blocker in Dn. This is then a marginal Dn-pair (Z_1,X_1) such that Z_1 meets f. Adjust (Z_1,X_1) so that it is an argument-minimal marginal Dn-pair such that Z_1 meets f. Our second candidate for augment is any internal edge β of (Z_1,X_1) in rf. The existence of β is established in part i of the following proposition, with f in the role of g and (Z_1,X_1) in the role of (Z,X).

6.1. Let g be an element of Dn disjoint from tn. Let (Z,X) be an argument-minimal marginal Dn-pair such that Z meets g.

i) There is an internal edge of (Z,X) in rg;

ii) Each such edge has no blocker in Dn.

The proof of 6.1 is given after the main argument is complete.

By part ii of 6.1, β has no blocker in Dn. The only alternative to β an augment is that it have a blocker in Dp, which can happen only if $\beta \in rX_1$. This blocker is a marginal Dp-pair (Z_2,X_2) such that Z_2 meets X_1. Adjust (Z_2,X_2) so that it is an argument-minimal marginal Dp-pair such that Z_2 meets X_1. Now $\cup X_1$ is an element of Dp disjoint from tp; by the dual of 6.1i, with UX_1 in the role of g and (Z_2,X_2) in the role of (Z,X), there is an internal edge γ of (Z_2,X_2) in rX_1. This is our third candidate for augment. It has no blocker in Dp, by 6.1ii (dual). Nor does γ have a blocker in Dn: this follows from our previous appeal to 6.1 by observing that γ is an internal edge of (Z_1,X_1) in rf. So γ is an augment of t.

The proof of the Augment Lemma is complete, except for 6.1.

Proof of 6.1: i) Let d be a coboundary in Z that meets g. Consider the Dn-pair (Z^-,X^-), where $Z^- = (Z-\{d\}) \cup \{d\wedge g\}$ and X^- is the collection of coboundaries in X that Z^- shades. Since $r(d\wedge g) = rd \cap ng \cup rg \cap nd$, thus either $nd - d$ intersects rg, in which case the assertion holds directly, or $r(d\wedge g) \subseteq rd$. In the latter case, $|X| - |X^-| \leq |rd-t| - |r(d\wedge g)-t| = uZ - uZ^- \leq |X| - |X^-|$, whence $uZ - uZ^- = |X| - |X^-|$. So (Z^-,X^-) is a marginal Dn-pair such that Z^- meets g. Since $nZ^- \subseteq nZ$ and $pX^- \subseteq pX$, argument-minimality of (Z,X) implies that $nZ^- = nZ$. In particular, $n(d\wedge g) = nd$, whence $nd \subseteq ng$. Since t is central, d is disjoint from t, whence $u\{d\} = |rd-t| - (k-1) \geq k - (k-1) = 1$. So $u(Z-\{d\}) < uZ = |X|$; since t is feasible, $Z - \{d\}$ does not shade X, i.e., $rd \cap rX \neq \emptyset$. Since $rd \cap rX \subseteq ng \cap rp \subseteq rg$, assertion i is proved.

ii) For any internal edge α of (Z,X) in rg, consider a marginal Dn-pair (Z',X') such that $\alpha \in rZ'$. By 5.2, $(Z\wedge Z',X\wedge X')$ is a marginal Dn-pair. Since $\alpha \in nZ \cap rZ' \subseteq r(Z\wedge Z')$, thus $Z \wedge Z'$ meets g. From $n(Z\wedge Z') \subseteq nZ$ and $p(X\wedge X') \subseteq pX$, argument-minimality of (Z,X) implies that $n(Z\wedge Z') = nZ$ and $p(X\wedge X') = pX$. From the former, $r(Z\wedge Z') = rZ$ and so $\alpha \notin nZ - \cup Z$; since α is internal in (Z,X) thus $\alpha \in rX$. From the latter, $rX = r(X \wedge X') \subseteq rX'$, whence $\alpha \in rX'$: pair (Z',X') is not a blocker of α. Indeed, α has no blocker in Dn.

With 6.1, the proof of the Bi-transversal Theorem is complete.

7. Remarks

The above proof does not translate directly to a polynomial algorithm for finding T*. One that does is given in [11]. It builds a maximum packing T* for D from maximum packings T*p and T*n of transversals of Dp and Dn, each of which is found by an analog of the Lovász algorithm [4] for disjoint branchings.

A reduction to Hall's Theorem approach was used to prove the source-sink connected case of the directed cut packing minimax equality [9]. The Hall's Theorem part of it was there treated by an alternating path approach that yielded a polynomial algorithm. This approach can be used also to relate Gupta's Theorem [3] to Hall's.

Much of this paper was written while the authors were visiting MIT. We thank Kleitman for the opportunity. This research was supported by a grant from the Natural Sciences and Engineering Research Council of Canada.

References

[1] J. Edmonds, Edge-disjoint branchings, in "Combinatorial Algorithms", (R. Rustin, ed.) Algorithmics Press, New York, 1973, 91-96.

[2] J. Edmonds and R. Giles, A min-max relation for submodular functions on graphs, in "Studies in integer programming" (P.L. Hammer, et al, eds.) Annals of Discrete Math. 1 (1977) 185-204.

[3] R.P. Gupta, A decomposition theorem for bipartite graphs, in "Theory of Graphs" (P. Rosenstiehl, ed.) Gordon and Breach, New York, 1967, 135-138.

[4] L. Lovász, On two minimax theorems in graph theory, J. Combinatorial Theory (B) 21 (1976) 96-103.

[5] C.L. Lucchesi and D.H. Younger, A minimax relation for directed graphs, J. London Math. Soc. (2) 17 (1978) 369-374.

[6] A. Schrijver, A counterexample to a conjecture of Edmonds and Giles, Discrete Math. 32 (1980) 213-214.

[7] A. Schrijver, Min-max relations for directed graphs, Annals of Discrete Math. 16 (1982) 261-280.

[8] D.R. Woodall, Menger and König systems, in "Theory and Applications of Graphs" (Proceedings of the Western Michigan Graph Theory Conference 1976) Springer-Verlag Lecture Notes in Math. 642 (1978) 620-635.

[9] D.H. Younger, From shortest paths to directed cut transversals, Annals of the New York Academy of Sciences, 319 (1979) 555-564.

[10] D.H. Younger, Maximum families of disjoint directed cut sets, in "Recent Progress in Combinatorics" (W.T. Tutte, ed.) Academic Press, 1969, 329-333.

[11] P. Feofiloff, Disjoint Transversals of Directed Coboundaries, Ph.D. Thesis, University of Waterloo, 1983.

[12] P.D. Seymour, The matroids with the max-flow min-cut property, J. Combinatorial Theory (B) 23 (1977) 189-222.

Department of Combinatorics & Optimization
University of Waterloo
Waterloo, Ontario, Canada N2L 3G1

"RELATÓRIO TÉCNICO"
DEPARTAMENTO DE MATEMÁTICA APLICADA
TÍTULOS PUBLICADOS

RT-MAP-7701 - Ivan de Queiroz Barros
On equivalence and reducibility of Generating Matrices of RK-Procedures - Agosto 1977

RT-MAP-7702 - V.W. Setzer
A Note on a Recursive Top-Down Analizer of N.Wirth - Dezembro 1977

RT-MAP-7703 - Ivan de Queiroz Barros
Introdução a Aproximação Ótima - Dezembro 1977

RT-MAP-7704 - V.W. Setzer, M.M. Sanches
A linguagem "LEAL" para Ensino básico de Computação - Dezembro 1977

RT-MAP-7801 - Ivan de Queiroz Barros
Proof of two Lemmas of interest in connection with discretization of Ordinary Differential Equations - Janeiro 1978

RT-MAP-7802 - Silvio Ursic, Cyro Patarra
Exact solution of Systems of Linear Equations with Iterative Methods
Fevereiro 1978

RT-MAP-7803 - Martin Grötschel, Yoshiko Wakabayashi
Hypohamiltonian Digraphs - Março 1978

RT-MAP-7804 - Martin Grötschel, Yoshiko Wakabayashi
Hypotraceable Digraphs - Maio 1978

RT-MAP-7805 - W. Hesse, V.W. Setzer
The Line-Justifier: an example of program development by transformations
Junho 1978

RT-MAP-7806 - Ivan de Queiroz Barros
Discretização
Capítulo I - Tópicos Introdutórios
Capítulo II - Discretização
Julho 1978

RT-MAP-7807 - Ivan de Queiroz Barros
(Γ',Γ) - Estabilidade e Métodos Preditores-Corretores - Setembro 1978

RT-MAP-7808 - Ivan de Queiroz Barros
Discretização
Capítulo III - Métodos de passo progressivo para Eq. Dif. Ord. com condições iniciais - Setembro 1978

RT-MAP-7809 - V.W. Setzer
Program development by transformations applied to relational Data-Base queries - Novembro 1978

RT-MAP-7810 - Nguiffo B. Boyom, Paulo Boulos
Homogeneity of Cartan-Killing spheres and singularities of vector fields - Novembro 1978

TÍTULOS PUBLICADOS

RT-MAP-7811 - D.T. Fernandes e C. Patarra
Sistemas Lineares Esparsos, um Método Exato de Solução - Novembro 1978

RT-MAP-7812 - V.W. Setzer e G. Bressan
Desenvolvimento de Programas por Transformações: uma Comparação entre dois Métodos - Novembro 1978

RT-MAP-7813 - Ivan de Queiroz Barros
Variação do Passo na Discretização de Eq. Dif. Ord. com Condições Iniciais - Novembro 1978

RT-MAP-7814 - Martin Grötschel e Yoshiko Wakabayashi
On the Complexity of the Monotone Asymetric Travelling Salesman Polytope I: HIPOHAMILTONIAN FACETS - Dezembro 1978

RT-MAP-7815 - Ana F. Humes e E.I. Jury
Stability of Multidimensional Discrete Systems: State-Space Representation Approach - Dezembro 1978

RT-MAP-7901 - Martin Grötschel, Yoshiko Wakabayashi
On the complexity of the Monotone Asymetric Travelling Salesman Polytope II: HYPOTRACEABLE FACETS - Fevereiro 1979

RT-MAP-7902 - M.M. Sanches e V.W. Setzer
A portabilidade do Compilador para a Linguagem LEAL - Junho 1979

RT-MAP-7903 - Martin Grötschel, Carsten Thomassen, Yoshiko Wakabayashi
Hypotraceable Digraphs - Julho 1979

RT-MAP-7904 - N'Guiffo B. Boyom
Translations non triviales dans les groupes (transitifs) des transformations affines - Novembro 1979

RT-MAP-8001 - Ângelo Barone Netto
Extremos detectáveis por jatos - Junho 1980

RT-MAP-8002 - Ivan de Queiroz Barros
Medida e Integração
Cap. I - Medida e Integração Abstrata - Julho 1980

RT-MAP-8003 - Routo Terada
Fast Algorithms for NP-Hard Problems which are Optimal or Near-Optimal with Probability one - Setembro 1980

RT-MAP-8004 - V.W. Setzer e R. Lapyda
Uma Metodologia de Projeto de Bancos de Dados para o Sistema ADABAS
Setembro 1980

RT-MAP-8005 - Imre Simon
On Brzozowski's Problem: $(1 \cup A)^m = A^*$ - Outubro 1980

RT-MAP-8006 - Ivan de Queiroz Barros
Medida e Integração
Cap. II - Espaços Lp - Outubro 1980

TÍTULOS PUBLICADOS

RT-MAP-8408 - Paulo Feofiloff
Disjoint Transversals of Directed Coboundaries - Novembro 1984

RT-MAP-8409 - Paulo Feofiloff e D.H. Younger
Directed cut transversal packing for source-sink connected graphs - Novembro 1984