Integral Group Rings of Frobenius Groups and the Conjectures of H.J. Zassenhaus

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Abstract

The conjecture of H.J. Zassenhaus for finite subgroups of units of integral group rings, restricted to p-subgroups, is proved for finite Frobenius groups when p is an odd prime. The result for 2-subgroups is established for those Frobenius groups that cannot be mapped homomorphically onto S_5. The conjecture in its full strength is proved for A_5, S_5 and $SL(2,5)$.

1 Introduction

Let $\mathcal{U}(\mathbb{Z}G)$ denote the group of units of the integral group ring of a finite group G, and set $\mathcal{U}_1(\mathbb{Z}G) = \{u \in \mathcal{U}(\mathbb{Z}G) \mid \varepsilon(u) = 1\}$, where $\varepsilon : \mathbb{Z}G \to \mathbb{Z}$ denotes the augmentation map. A well-known theorem of G. Higman (see [26, Theorem 20.9]) states that, in the case when G is abelian, $\mathcal{U}_1(\mathbb{Z}G) = G \times F$, where F is a free abelian group of finite rank. In some sence, this is a natural version of Dirichlet Unit Theorem for these orders. In particular,

this means that an element of finite order of $\mathcal{U}_1(\mathbb{Z}G)$ is trivial; i.e., it belongs to G. In the non-commutative setting H.J. Zassenhaus [30] has formulated a conjecture that extends the result above: if $x \in \mathcal{U}_1(\mathbb{Z}G)$ is an element of finite order then there exists a unit $u \in \mathbf{Q}G$ such that $u^{-1}xu \in G$. When such a unit exists, we say that x is *rationally conjugate* to an element of G. This conjecture is denoted (**ZC1**) in [26] and we shall keep this notation. A stronger version of this conjecture, denoted (**ZC3**) states that the same is true for finite subgroups of $\mathcal{U}_1(\mathbb{Z}G)$; i.e., if $\mathcal{H}$ is a finite subgroup of $\mathcal{U}_1(\mathbb{Z}G)$ then $\mathcal{H}$ is rationally conjugate to a subgroup of G.

On of the firsts positive results on (**ZC1**) was given by A.K. Bhandari and I.S. Luthar in [1] for some special families of metacyclic groups. It is known that (**ZC3**) holds for finite nilpotent groups (see A. Weiss [28], [29] and also [23], [26]) and for split metacyclic groups (see C. Polcino Milies, J. Ritter and S.K. Sehgal [22] and A. Valenti [27]). An analogue of this conjecture in the case of integral alternative loop rings has been established in [7], [8].

K.W. Roggenkamp and L. Scott were the firsts to exibit a counterexample to (**ZC3**) (see L. Klinger [16]). However, there are no known examples of either a cyclic subgroup or a p-subgroup in $U_1(\mathbb{Z}G)$ which are not rationally conjugate to a subgroup in G. Also, somewhat weaker statements of the conjecture hold for large families of finite or infinite groups (see [2], [4], [6], [11], [12], [24], [17] and [20]). In view of these facts, we consider the

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following p-subgroup version of the Conjecture of Zassenhaus.

> **(p-ZC3)** Let $\mathcal{H}$ be a finite p-subgroup of $U_1(\mathbb{Z}G)$. Then, there exists a unit $u \in \mathbb{Q}G$ such that $u^{-1}\mathcal{H}u \subset G$.

In particular, if **(p-ZC3)** is true for a group G, then any Sylow p-subgroup of $U_1(\mathbb{Z}G)$ is rationally conjugate to a subgroup of G. Conjugation of those Sylow p-subgroups which can be imbedded in group bases was investigated in [14] and [15] (see also [24]). The validity of **(p-ZC3)** has been established in [5] for nilpotent-by-nilpotent groups, for solvable groups in which any Sylow p-subgroup is either abelian or a generalized quaternion group, for solvable Frobenius groups and for solvable groups whose orders are not divisible by the fourth power of any prime rational number.

The goal of this paper is to study **(p-ZC3)** for Frobenius groups. In so doing, we shall also prove **(ZC3)** for some particular groups which appear naturally as intermediate steps in the proof of our main result. In section §2 we state some known results that shall be needed in the sequel. Then, we prove **(ZC3)** for A_5, $SL(2,5)$ and S_5 in sections §3, §4 and §5 respectively. We use these results in section §6 to establish **(p-ZC3)** with $p > 2$ for Frobenius groups and also **(2-ZC3)** for Frobenius groups which do not have S_5 as a homomorphic image.

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2 Preliminary results

Let N be a normal subgroup of a group G, set $\overline{G} = G/N$ and let $\psi : \mathbb{Z}G \rightarrow \mathbb{Z}(G/N)$ be the natural map. We recall the following results.

Theorem 2.1 ([5, Theorem 2.2]) *Let $\mathcal{H}$ be a finite subgroup of $U_1(\mathbb{Z}G)$ such that $(|\mathcal{H}|, |N|) = 1$ and let G_0 be a subgroup of G with $(|G_0|, |N|) = 1$. Then, $\mathcal{H}$ is rationally conjugate to G_0 if and only if $\varphi(\mathcal{H})$ is conjugate to $\varphi(G_0)$ in $\mathbb{Q}\overline{G}$.*

We immediately get the following.

Corollary 2.2 ([5, Corollary 2.3]) *Suppose that* (**ZC3**) *holds for the factor group G/N. Then, any finite subgroup $\mathcal{H} \subset U_1(\mathbb{Z}G)$ whose order is relatively prime to the order of N is rationally conjugate to a subgroup of G.*

The next result is crucial for our study of Frobenius groups.

Corollary 2.3 ([5, Corolary 2.4]) *Let G be an extension of a nilpotent group N by a group X which satisfies* (**p-ZC3**). *If the orders of N and X are relatively prime, then G also satisfies* (**p-ZC3**).

The following result is a modification of Lemma 37.6 of [26] and can be found in [5] as Lemma 2.6 which we state in a slightly more general form.

Lemma 2.4 *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be finite subgroups of $U_1(\mathbb{Z}G)$ and let $\varphi : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ be an epimorphism. If $\chi(h) = \chi(\varphi(h))$ for all irreducible complex characters χ of G, then $\mathcal{H}_1$ is conjugate to $\mathcal{H}_2$ in $\mathbb{Q}G$. In particular, φ is an isomorphism.*

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Corollary 2.5 *Let $\mathcal{H}$ be a finite subgroup of $U_1(\mathbb{Z}G)$ and suppose that $\varphi : \mathcal{H} \to G$ is a homomorphism such that h is conjugate to $\varphi(h)$ in $\mathbb{Q}G$, for all $h \in \mathcal{H}$. Then, $\mathcal{H}$ and $\varphi(\mathcal{H})$ are rationally conjugate.*

Proof. Clearly, φ is a homomorphism which preserves characters so one can apply Lemma 2.4. □

For briefness, a homomorphism which satisfies the assumption of Corollary 2.5 shall be called *conjugation-like*.

For an element $\alpha = \sum\limits_{g \in G} \alpha(g)g \in \mathbb{Z}G$ we put $\tilde{\alpha}(g) = \sum\limits_{h \in C_g} \alpha(h)$ where C_g is the conjugacy class of $g \in G$. We say that $\alpha \in \mathbb{Z}G$ has the *unique trace property* if there exists an element $g \in G$, unique up to conjugacy, such that $\tilde{\alpha}(g) \neq 0$.

3 The Zassenhaus conjecture for A_5

It is well-known that A_5, the alternating group on 5 letters, is a simple group of order 60 and that its conjugacy classes are as determined in the following table.

orders of elements	1	2	3	5	5
class representatives	1	(1 2)(3 4)	(1 2 3)	(1 2 3 4 5)	(1 2 3 4 5)^2

Table 1

Part (i) of the following result is well-known. We include a proof for the sake of completeness.

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Lemma 3.1 *(i) The non-cyclic proper subgroups of A_5, up to isomorphisms, are: the Klein four group, the dihedral groups of orders 6 and 10 and the alternating group A_4.*

(ii) $U_1(\mathbb{Z}A_5)$ has no subgroups of orders 15, 20 and 30.

Proof. First we show that neither A_5 nor $U_1(\mathbb{Z}A_5)$ has subgroups of order 15, 20 and 30. In fact, according to [26, Proposition 1.9] the order of a torsion unit in $\mathbb{Z}G$ must be equal to the order of an element in G; therefore, $U_1(\mathbb{Z}A_5)$ has no elements of orders 4, 10 and 15. So it suffices to observe that a group of order 15 or 30 contains an element of order 15 and that there are no groups of order 20 without elements of orders 4 and 10 (see [3, Table 1]).

Also by [3, Table 1] there are three nonabelian groups of order 12: D_6, A_4 and $< 2,2,3 > = < S, T \mid S^3 = T^2 = (ST)^2 = 1 >$. It is easy to see that S is an element of order 6 in $< 2,2,3 >$ so both D_6 and $< 2,2,3 >$ contain elements of order 6 while A_5 does not. Finally, we note that, according to [3, Table 1], the only non-abelian groups of orders 6 and 10 respectively are the dihedral ones and that the subgroups $< (1\ 2\ 3),\ (1\ 2)(4\ 5) >$ and $< (1\ 2\ 3\ 4\ 5),\ (1\ 4)(2\ 3) >$ are precisely groups of these types. $\qquad\square$

Theorem 3.2 *The Zassenhaus conjecture (ZC3) holds for A_5*

Proof. We recall that I.S. Luthar and I.B.S. Passi have shown in [18] that (ZC1) holds for A_5, so cyclic subgroups of $U_1(\mathbb{Z}A_5)$ are rationally conjugate to subgroups in A_5.

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Let $\mathcal{H}$ be a non-cyclic subgroup of $U_1(\mathbb{Z}A_5)$ and assume first that 5 does not divide the order of $\mathcal{H}$. We claim that A_5 contains a subgroup isomorphic to $\mathcal{H}$. This is clear if $|\mathcal{H}| = 4$ or 6, so let $|\mathcal{H}| = 12$. It follows from (ZC1) that $\mathcal{H}$ has no elements of order 6, therefore, $\mathcal{H} \cong A_4$ [3, Table 1]. Thus, A_5 contains a subgroup G_0 which is an isomorphic copy of $\mathcal{H}$.

By (ZC1) we know that every element $h \in \mathcal{H}$ is rationally conjugate to an element in A_5 and Table 1 shows that this element, in turn, is conjugate in A_5 to the image of h under the isomorphism $\mathcal{H} \to G_0$. Hence, Corollary 2.5 shows that $\mathcal{H}$ is rationally conjugate to G_0.

Now, assume that 5 divides $|\mathcal{H}|$. By (ii) of Lemma 3.1 $|\mathcal{H}| \neq 15, 20, 30$. Let $|\mathcal{H}| = 10$. By [3, Table 1], up to isomorphisms, the only nonabelian group of order 10 is D_5, so there exists an isomorphism $\varphi : \mathcal{H} \to G_0 \subset A_5$. Table 1 shows that φ is conjugation-like on elements of order 2. Notice that conjugation by (2 3 5 4) induces an automorphism θ of A_5 such that $\theta(1\ 2\ 3\ 4\ 5) = (1\ 2\ 3\ 4\ 5)^2$. So if φ is not conjugation-like, then there exists an element $h \in \mathcal{H}$, of order 5, such that $\varphi(h) = (1\ 2\ 3\ 4\ 5)$ and $h \sim (1\ 2\ 3\ 4\ 5)^2$ in $\mathbb{Q}A_5$. In this case clearly $\theta \circ \varphi$ is conjugation-like. Consequently, Corollary 2.5 again implies that $\mathcal{H}$ is rationally conjugate to G_0.

Finally, since both the isomorphism problem and the automorphism problem have a possitive answer for A_5 (see [25, Theorem 4.2] and [26, Theorem 43.7]), we get that if $|\mathcal{H}| = 60$ then $\mathcal{H}$ is conjugate to A_5 in $\mathbb{Q}A_5$.

□

4 The Zassenhaus conjecture for $SL(2,5)$

Set $G = SL(2,5)$. It is well-known that the center $\mathcal{Z}(G)$ of G is cyclic of order 2, that $G/\mathcal{Z}(G) \cong A_5$ and that the Sylow 2-subgroups of G are quaternion groups of order 8 (see, for example, [21]). Let $N = \mathcal{Z}(G) = \{1, z\}$ and let $\psi : \mathbb{Z}G \to \mathbb{Z}(G/N)$ denote the natural map. It follows from [26, Theorems 45.11 and 47.3] that z is the unique element of order 2 in $\mathcal{U}_1(\mathbb{Z}G)$.

Lemma 4.1 *(i) Let $g_1, g_2 \in G$ and $o(g_1) = o(g_2)$. Then $g_1 \sim g_2$ in G if and only if $\psi(g_1) \sim \psi(g_2)$ in G/N.*

(ii) The orders of the noncyclic proper subgroups of G are 8, 12, 20 and 24.

(iii) The orders of the noncyclic finite subgroups of $\mathcal{U}_1(\mathbb{Z}G)$ are 8, 12, 20, 24 and 120.

Proof. (i) Let $g_1, g_2 \in G$ be such that $\psi(g_1)$ and $\psi(g_2)$ are conjugate in G/N. Assume first that $o(g_1) = o(g_2)$ is odd. We have that $x^{-1}g_1 x = g_2 z^m$ for some $x \in G$ and $z^m \in \{1, z\}$, but the condition on the orders implies that $z^m = 1$. It is easy to see that the statement is also true for elements of the form $g_1 z$ and $g_2 z$ with g_1, g_2 of the same odd order. Finally, if $o(g_i) = 4$, then $g_i \sim g_i^{-1} = g_i z, i = 1, 2$, as G has quaternion Sylow 2-subgroups. Hence, G has a single conjugacy class of elements of order 4.

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(ii) Since $\mathcal{Z}(G)$ is the unique subgroup of G of order 2, it is easy to see that a subgroup G_0 of G is cyclic if and only if $\psi(G_0)$ is cyclic. Moreover, since all subgroups in G of odd orders are cyclic, applying ψ we get a one-to-one correspondence between the noncyclic subgroups of G and the noncyclic subgroups of A_5. The statement now follows from (i) of Lemma 3.1.

(iii) According to [26, Lemma 37.3] the order of a finite subgroup in $\mathcal{U}_1(\mathbb{Z}G)$ must divide $|G|$. Hence, all subgroups of $\mathcal{U}_1(\mathbb{Z}G)$ of odd orders are cyclic. Since z is the unique unit of order 2 in $\mathcal{U}_1(\mathbb{Z}G)$, apllying ψ we get, as above, a one-to-one correspondence between the noncyclic finite subgroups of $\mathcal{U}_1(\mathbb{Z}G)$ and the noncyclic finite subgroups of $\mathcal{U}_1(\mathbb{Z}A_5)$. Theorem 3.2 now implies the statement. □

Let a, c, d be elements of G of orders 4, 3 and 5 respectively. It follows from (i) of Lemma 4.1 that the conjugacy classes of G are as determined in the following table.

orders of elements	1	2	3	4	5	5	6	10	10
class representatives	1	z	c	a	d	d^2	cz	dz	$d^2 z$

<div align="center">Table 2</div>

Proposition 4.2 (ZC1) *holds for* $SL(2,5)$.

Proof. By Corollary 2.2 and Theorem 3.2 a torsion unit of odd order of $\mathcal{U}_1(\mathbb{Z}G)$ is rationally conjugate to an element of G. Since z is the only unit of order 2 in $\mathcal{U}_1(\mathbb{Z}G)$, the statement is also true for units of mixed orders.

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Finally, let α be a unit of $\mathcal{U}_1(\mathbb{Z}G)$ of order 4; set $\beta = \psi(\alpha)$ and $\bar{g} = \psi(g)$ for $g \in G$. By (i) of Lemma 4.1 we have that $\tilde{\alpha}(g) = \tilde{\beta}(\bar{g})$ if g has order 4. Moreover, by [26, Lemma 38.11] $\tilde{\alpha}(g) = 0$ if $o(g) \in \{3, 5, 6, 10\}$ and $\tilde{\beta}(\bar{g}) = 0$ if $o(\bar{g}) = 3$ or 5. Consequently, $\tilde{\alpha}(g) = \tilde{\beta}(\bar{g})$ for all $g \in G$. Therefore, since (ZC1) holds for G/N it follows that every element of $< \alpha >$ has the unique trace property. Hence, Lemma 41.5 of [26] implies that α is rationally conjugate to an element of G. □

Theorem 4.3 *The Zassenhaus conjecture* (**ZC3**) *holds for* $SL(2, 5)$.

Proof. Let $\mathcal{H}$ be a finite subgroup of $\mathcal{U}_1(\mathbb{Z}G)$. By Proposition 4.2 we may suppose that $\mathcal{H}$ is not cyclic. Hence $\psi(\mathcal{H})$ is also not cyclic (see the proof of (iii) in Lemma 4.1). Suppose that 5 does not divide $|\mathcal{H}|$. We show first that $\mathcal{H}$ is isomorphic to a subgroup of G.

This is clear in the case when $|\mathcal{H}| = 8$ as $\mathcal{U}_1(\mathbb{Z}G)$ has only one unit of order 2. If $|\mathcal{H}| = 12$ then $\psi(H)$ is dihedral of order 6. It is easy to see that $\mathcal{H}$ is isomorphic to $< 2, 2, 3 >$ [3, table 1]. Similarly, a subgroup G_0 of order 12 of G is also isomorphic to $< 2, 2, 3 >$. Let $|\mathcal{H}| = 24$. It follows from Theorem 3.2 that $\mathcal{H}/ < z > \cong A_4$. $\mathcal{H}$ does not have a subgroup of order 12 as A_4 does not have a subgroup of order 6. By [3, p.69] $\mathcal{H}$ is the Binary Tetrahedral Group. Also, a subgroup G_0 of order 24 of G has to be isomorphic to the Binary Tetrahedral Group.

Let $\varphi : \mathcal{H} \to G_0$ be any isomorphism. If $g \in G$ then G has only one conjugacy class of elements of order $o(g)$. Hence, by Proposition 4.2, φ is

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conjugation-like and Corollary 2.5 implies that $\mathcal{H}$ and G are conjugate in $\mathbb{Q}G$.

Suppose now that 5 divides $|\mathcal{H}|$. Since **(ZC3)** holds for G/N (Theorem 3.2), it follows that $\psi(\mathcal{H})$ is either dihedral of order 10 or isomorphic to A_5. Thus $|\mathcal{H}| = 20$ or 120. Let $|\mathcal{H}| = 20$. Up to isomorphims there are 3 nonabelian groups of order 20 [3, Table 1]; namely, the dihedral group, the K-metacyclic group and the group $< 2, 2, 5 >$. The first two groups contain more then one element of order 2 [3, p.11]; hence, $\mathcal{H}$ is isomorphic to $< 2, 2, 5 >$, as z is the unique element of order 2 in $\mathcal{H}$. A similar argument shows that a subgroup G_0 of order 20 in G is isomorphic to $< 2, 2, 5 >$. If $|\mathcal{H}| = 120$ then $\mathcal{H}$ is nonsolvable with quaternion Sylow 2-subgroups and by [21, p.122] $\mathcal{H}$ must be isomorphic to $SL(2, 5)$. Thus, in both cases there is an isomorphism $\varphi : \mathcal{H} \to G_0 \subseteq G$.

We want to show that, if necessary, we can modify φ in such a way that it becomes conjugation-like.

Let G_1 be the covering group of S_5 with generalized quaternion Sylow 2-subgroups. Then $SL(2, 5)$ is a subgroup of index 2 in G_1, $N =< z >= \mathcal{Z}(G_1)$ and $G_1/ < z > \cong S_5$. In the case when $|\mathcal{H}| = 20$ we suppose that $\psi(G_0) =< (1\ 2\ 3\ 4\ 5), (1\ 4)(2\ 3) >$ (Clearly, in the case $|\mathcal{H}| = 120$ we have $G_0 = G$). Let u be an element of G_1 such that its image in S_5 is $(2\ 3\ 5\ 4)$. Let C_g be the conjugacy class of g in G. Then by (i) of Lemma 4.1 we see that conjugation by u induces an automorphism θ of G such that

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$$\theta(G_0) = G_0, \quad \theta(C_d) = C_{d^3} = C_{d^2}, \quad \theta(C_{zd}) = C_{zd^2}$$

and

$$\theta(C_g) = C_g \quad \text{if} \quad o(g) \neq 5 \quad \text{or} \quad o(g) \neq 10.$$

It follows from the knowledge of the conjugacy classes of G (see Table 2) and Proposition 4.2 that φ is conjugation-like on elements whose orders are not divisible by 5. Moreover, if d_1 is an element of order 5 in $\mathcal{H}$ and φ is not conjugation-like, then $\varphi(d_1)$ and d_1 are not conjugate in $\mathcal{U}(\mathbf{Q}G)$. Hence $\varphi(d_1) \sim d_1^2$ and $\varphi(zd_1) \sim zd_1^2$ in $\mathbf{Q}G$. Now it is easy to see that (ZC1) for G implies that $\theta \circ \varphi : \mathcal{H} \to G_0$ is conjugation-like.

Thus, we can suppose that $\varphi(h) \sim h$ in $\mathcal{U}(\mathbf{Q}G)$ for all $h \in \mathcal{H}$ and it follows from Corollary 2.5 that $\mathcal{H}$ and G_0 are rationally conjugate. □

5 The Zassenhaus conjecture for S_5

We recall that (ZC1) has been proved for S_5 in [19].

Theorem 5.1 *The Zassenhaus conjecture (ZC3) holds for S_5.*

Proof. Set $G = S_5$. Let $\mathcal{H}$ be a finite subgroup of $\mathcal{U}_1(\mathbf{Z}G)$ and let

$$\mathcal{H}_0 = \mathcal{H} \cap (1 + \Delta(G, A_5)).$$

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Clearly, $[\mathcal{H} : \mathcal{H}_0] \leq 2$. As (ZC1) holds for S_5, every element of $\mathcal{H}$ is rationally conjugate to an element of S_5. Moreover, it is easy to see that each element of $\mathcal{H}_0$ is conjugate in $\mathcal{U}(\mathbf{Q}G)$ to an element of A_5. We know that S_5 has an absolutely irreducible representation $\Gamma : G \to GL(4, \mathbf{Q})$ with the following character χ:

	1	(1 2)	(1 2)(3 4)	(1 2 3)	(1 2 3)(4 5)	(1 2 3 4)	(1 2 3 4 5)
χ	4	2	0	1	-1	0	-1

Table 3

(see [10, p. 349]).

Since a group of order 15 or 30 has an element of order 15, we see that $|\mathcal{H}| \neq 15, 30$. If $\mathcal{H}$ is abelian we may suppose that $\mathcal{H}$ is not cyclic and choose a matrix $X \in GL(4, \mathbf{C})$ such that $X^{-1}\mathcal{H}X$ consists of diagonal matrices in $GL(4, \mathbf{C})$.

According to Corollary 2.5 it suffices to find a homomorphism $\varphi : \mathcal{H} \to G$ which is conjugation-like .

Case 1: $\mathcal{H} = <u, v>$ is isomorphic to the Klein four group. If $\mathcal{H} = \mathcal{H}_0$ then $h \sim (1\ 2)(3\ 4)$ in $\mathbf{Q}G$ for all $1 \neq h \in \mathcal{H}$ and, consequently, $u \to (1\ 2)(3\ 4), v \to (1\ 3)(2\ 4)$ defines a monomorphism which is conjugation-like. If $\mathcal{H} \neq \mathcal{H}_0$, we can choose the generators u and v such that $u \in \mathcal{H}_0$ and $v \notin \mathcal{H}_0$. By (ZC1), $u \sim (1\ 2)(3\ 4)$ and $v \sim (1\ 2)$ in $\mathbf{Q}G$. Hence, $u \to (1\ 2)(3\ 4), v \to (1\ 2)$ gives a monomorphism which is conjugation-like.

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Thus in both cases $\mathcal{H}$ is rationally conjugate to a subgroup of G.

Case 2: The order of $\mathcal{H}$ is 6. Since $\mathcal{H}$ is not cyclic, $\mathcal{H}$ is isomorphic to S_3. Let $\mathcal{H} = < u, v >$ with $u^3 = v^2 = 1$. By **(ZC1)** $u \sim (1\ 2\ 3)$ in $\mathbf{Q}G$. Put

$$u \to (1\ 2\ 3\ ), v \to \begin{cases} (1\ 2) & \text{if } v \sim (1\ 2) \text{ in } \mathbf{Q}G, \\ (1\ 2)(4\ 5) & \text{if } v \sim (1\ 2)(4\ 5) \text{ in } \mathbf{Q}G. \end{cases}$$

It is easy to see that we get a monomorphism which is conjugation-like.

Case 3: The order of $\mathcal{H}$ is 8. Suppose that $\mathcal{H} = < u > \times < v >$ where $o(u) = 4$ and $o(v) = 2$. By **(ZC1)** $uv \sim u \sim (1\ 2\ 3\ 4)$ in $\mathbf{Q}G$ and $\chi(uv) = \chi(u) = \chi[(1\ 2\ 3\ 4)] = 0$. As A_5 does not contain elements of order 4, $u \notin \mathcal{H}_0$. Consequently, $[\mathcal{H} : \mathcal{H}_0] = 2$ and we may suppose that $v \in \mathcal{H}_0$. Since $\chi(u^2) = 0$, we can choose a $X \in GL(4, \mathbf{C})$ such that $X^{-1}\Gamma(u)X = diag(i, -i, 1, -1)$. We have $\chi(v) = 0$ as $v \sim (1\ 2)(3\ 4)$ in $\mathbf{Q}G$. Hence, since $\chi(uv) = 0$ and $X^{-1}\Gamma(v)X \neq X^{-1}\Gamma(u^2)X = diag(-1, -1, 1, 1)$, we obtain that $X^{-1}\Gamma(v)X = diag(1, 1, -1, -1)$. Then $X^{-1}\Gamma(u^2 v)X = diag(-1, -1, -1, -1)$ and $\chi(u^2 v) = -4$ which is impossible as $u^2 v$ is rationally conjugate to an element of G.

Let $\mathcal{H} = < u > \times < v > \times < w >$ where $u^2 = v^2 = w^2 = 1$. If $\mathcal{H} = \mathcal{H}_0$, then $\chi(h) = 0$ for all $1 \neq h \in \mathcal{H}_0$ and it is easy to check that $X^{-1}\Gamma(\mathcal{H})X$ contains $diag(-1, -1, -1, -1)$, a contradiction. Thus, $[\mathcal{H} : \mathcal{H}_0] = 2$ and we can assume that $u, v \in \mathcal{H}_0, w \notin \mathcal{H}_0$. By **(ZC1)** we can see that for every $1 \neq h \in \mathcal{H}$ we have $\chi(h) = 0$, or 2. We can choose X such that $X^{-1}\Gamma(u)X = diag(1, 1, -1, -1)$. If $X^{-1}\Gamma(v)X = diag(-1, -1, 1, 1)$, then

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$X^{-1}\Gamma(uv)X = diag(-1,-1,-1,-1)$, contradicting the fact that $\chi(uv) = 0$. Thus, we may suppose that $X^{-1}\Gamma(v)X = diag(1,-1,1,-1)$. Then $X^{-1}\Gamma(w)X = diag(1,1,1,-1)$ because otherwise $\chi(uw) = -2$ or $\chi(vw) = -2$ which is not possible. But then $X^{-1}\Gamma(uvw)X = diag(1,-1,-1,-1)$ and $\chi(uvw) = -2$, a contradiction.

If $\mathcal{H}$ is quaternion of order 8 then $\mathcal{H}$ has six elements of order 4 which are rationally conjugate to (1 2 3 4) by (ZC1). This is impossible as $[\mathcal{H} : \mathcal{H}_0] \leq 2$ and $\mathcal{H}_0$ does not contain elements of order 4. Thus $\mathcal{H}$ must be dihedral. Since $\mathcal{H}$ contains elements of order 4, $[H : \mathcal{H}_0] = 2$ and we can choose the generators u, v of G such that $u^4 = v^2 = 1$ and $v \sim (1\ 2)(3\ 4)$ in $\mathbf{Q}G$. It is easy to check that $u \to (1\ 2\ 3\ 4), v \to (1\ 2)(3\ 4)$ defines an isomorphism $\mathcal{H} \to G_0 = <(1\ 2\ 3\ 4),(1\ 2)(3\ 4)>$ which is conjugation-like and, consequently, $\mathcal{H}$ and G_0 are conjugate in $\mathbf{Q}G$.

Case 4: The order of $\mathcal{H}$ is 10. By [3, Table 1] $\mathcal{H}$ is dihedral. Let $\mathcal{H} = <u, v>$ with $u^5 = v^2 = 1$. By (ZC1) $u \sim (1\ 2\ 3\ 4\ 5)$ in $\mathbf{Q}G$. We have that $u \not\in \mathcal{H}_0$ and we can choose v such that $v \in \mathcal{H}_0$. Hence v is rationally conjugate to (1 4)(2 3) and it is easy to check that $u \to (1\ 2\ 3\ 4\ 5), v \to (1\ 4)(2\ 3)$ defines an isomorphism $\mathcal{H} \to G_0 = <(1\ 2\ 3\ 4\ 5),(1\ 4)(2\ 3)>$ which is conjugation-like. Consequently, $\mathcal{H}$ and G_0 are conjugate in $\mathbf{Q}G$.

Case 5: The order of $\mathcal{H}$ is 12. If $\mathcal{H}$ is abelian, then $\mathcal{H} = <u> \times <v>$ where $u^2 = 1, v^6 = 1$. Since A_5 has no elements of order 6, it follows from (ZC1) that $\mathcal{H}_0$ has no elements of order 6. Hence $v, uv, uv^2 \not\in \mathcal{H}_0$, a

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contradiction with $[\mathcal{H} : \mathcal{H}_0] \le 2$. Thus $\mathcal{H}$ is not abelian and by [3, Table 1], up to isomorphisms, $\mathcal{H}$ is either D_6, $< 2, 2, 3 >$, or A_4.

Let $\mathcal{H} \cong\, < 2, 2, 3 >$. Then $\mathcal{H} =< u, v >$ where $u^6 = 1, v^2 = u^3, v^{-1}uv = u^{-1}$. Since A_5 has no elements of orders 6 nor 4, (ZC1) implies that $u, v, uv \notin \mathcal{H}_0$, a contradiction with $[\mathcal{H} : \mathcal{H}_0] \le 2$.

Let $\mathcal{H}$ be dihedral. Then $\mathcal{H} =< u, v >$ where $u^6 = v^2 = 1$ and $v^{-1}uv = u^{-1}$. By (ZC1) $u \sim (1\,2\,3)(4\,5)$ and $v \sim (1\,2)$ or $v \sim (1\,2)(4\,5)$ in $\mathbf{Q}G$. Put

$$u \to (1\,2\,3\,)(4\,5), \quad v \to \begin{cases} (1\,2) & \text{if } v \sim (1\,2) \text{ in } \mathbf{Q}G, \\ (1\,2)(4\,5) & \text{if } v \sim (1\,2)(4\,5) \text{ in } \mathbf{Q}G. \end{cases}$$

It is easy to see that we get a monomorphism $\mathcal{H} \to G$ which is conjugation-like and therefore, $\mathcal{H}$ is rationally conjugate to a subgroup of G.

Finally, let $\mathcal{H} \cong A_4$. Since A_4 does not have a subgroup of index 2, $\mathcal{H} = \mathcal{H}_0$. Hence, (ZC1) implies that every non-identity element of $\mathcal{H}$ is rationally conjugate to $(1\,2)(3\,4)$ or $(1\,2\,3)$. Therefore, every isomorphism $\mathcal{H} \to A_4 =< (1\,2\,3), (1\,2)(3\,4) >$ is conjugation-like and, consequently, $\mathcal{H}$ and A_4 are conjugate in $\mathbf{Q}G$.

Case 6: The order of $\mathcal{H}$ is 20. The only group of order 20 without elements of order 10 is the K-metacyclic group (see [3, Table 1 and §1.8]). Hence $\mathcal{H} =< u, v >$ where $u^5 = v^4 = 1$ and $v^{-1}uv = u^2$. By (ZC1), $u \sim (1\,2\,3\,4\,5)$ and $v \sim (2\,3\,5\,4)$ in $\mathbf{Q}G$. It is easy to check that $u \to (1\,2\,3\,4\,5), v \to (2\,3\,5\,4)$ defines a monomorphism which is conjugation-like; therefore, $\mathcal{H}$ and $G_0 =< (1\,2\,3\,4\,5), (2\,3\,5\,4) >$ are rationally conjugate.

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Case 7: The order of $\mathcal{H}$ is 24. By Case 3 the Sylow 2-subgroups of $\mathcal{H}$ are dihedral. Hence, $\mathcal{H}$ has an element of order 4. As A_5 has no elements of order 4, $\mathcal{H} \neq \mathcal{H}_0$ and $|\mathcal{H}_0| = 12$. Moreover, since $\mathcal{H}_0$ has no elements of order 6, $\mathcal{H}_0$ is isomorphic to A_4. Consequently, the largest normal $2'$-subgroup of $\mathcal{H}$ is trivial and [9, p.462] implies that $\mathcal{H} \cong PGL(2,3) \cong S_4$ (see also [3, p.96]).

Let $\varphi : \mathcal{H} \to S_4 = < (1\ 2\ 3\ 4), (1\ 2) > \subset S_5$ be an arbitrary isomorphism. We want to show that φ is conjugation-like. By Case 5, φ is conjugation-like on $\mathcal{H}_0$. Take $h \in \mathcal{H}, h \notin \mathcal{H}_0$. Since S_5 has only one conjugacy class of elements of order 4, we may assume that h has order 2. As $h \notin \mathcal{H}_0$, **(ZC1)** implies that h is rationally conjugate to an odd permutation of S_5. Since $\varphi(\mathcal{H}_0) = A_4$, $\varphi(h)$ is also odd. It follows that φ is conjugation-like and, consequently, $\mathcal{H}$ and S_4 are conjugate in $\mathbf{Q}G$.

Case 8: The order of $\mathcal{H}$ is 40. We shall show that this case does not occur. If $O(\mathcal{H})$, the largest normal $2'$-subgroup of $\mathcal{H}$ is the identity then, by [9, p.462], $PSL(2,p^n) \subseteq \mathcal{H}$ for some odd prime p. Thus, $|H| = 40$ implies that $p = 5, n = 1$ and, consequently, $|PSL(2,5)| = 60$ is a divisor of 40, which is impossible. Therefore, $O(\mathcal{H}) \neq 1$ and $\mathcal{H} \cong C_5 \rtimes D_4$. It is easy to see that the centralizer of C_5 in D_4 is not identity so $\mathcal{H}$ has an element of order 10, a contradiction.

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Case 9: The order of $\mathcal{H}$ is 60. Then $\mathcal{H}$ is not solvable as by Hall's theorem a solvable group of order 60 contains an element of order 15. Hence, $\mathcal{H}$ is isomorphic to A_5 and $\mathcal{H} = \mathcal{H}_0$. It follows from (ZC1) that every element of $\mathcal{H}$ is rationally conjugate to an even permutation. Let $\varphi : \mathcal{H} \to A_5 \subset S_5$ be an isomorphism. According to Table 1 we only need to worry about elements of order 5. The conjugation by $(2\ 4\ 5\ 3) \in S_5$ induces an automorphism ψ of A_5 such that $\psi(1\ 2\ 3\ 4\ 5) = (1\ 2\ 3\ 4\ 5)^2$. Hence, if $\varphi(h)$ is not rationally conjugate to h for some element $h \in \mathcal{H}$ of order 5, then $\psi \circ \varphi : \mathcal{H} \to A_5 \subset S_5$ is conjugation-like and, consequently, $\mathcal{H}$ and $A_5 \subset S_5$ are conjugate in $\mathbf{Q}G$.

Case 10: $|\mathcal{H}| = |G|$. Since both the isomorphism problem and the automorphism problem are positive for S_5 (see [26, Theorem 43.6] and [25, Theorem 4.3]), $\mathcal{H}$ and G are rationally conjugate. $\qquad\square$

6   The Frobenius Groups

Theorem 6.1 *Let G be a Frobenius group. Then*

(i) G satisfies (p-ZC3) if $p > 2$.

(ii) G satisfies (2-ZC3) if G can not be mapped homomorphically onto S_5.

Proof. In view of [5, Corollary 5.2] we may suppose that G is not solvable. By [9, pp. 38, 339] G is the semidirect product of a normal nilpotent subgroup N (the Frobenius kernel) by a group X (the Frobenius complement)

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of order relatively prime to $|N|$. According to Corollary 2.3, it is sufficient to prove (p-ZC3) for X.

We know by Zassenhaus' Theorem [21, p.204] that a nonsolvable Frobenius complement X contains a subgroup X_0 with $[X : X_0] \leq 2$ and $X_0 = SL(2,5) \times M$ where M is a metacyclic group whose Sylow subgroups are cyclic and $|M|$ is prime to 2, 3 and 5. Moreover, the Sylow 2-subgroups of X are generalized quaternion groups [21, p.194]. It is easy to see that the Sylow subgroups of $X/SL(2,5)$ are cyclic and, consequently, by [5, Theorem 5.1] we have that $X/SL(2,5)$ also satisfies (p-ZC3). Hence, if $p > 5$, the result follows from Theorem 2.1.

Let $p \leq 5$. If $X = X_0$ then $X/M \cong SL(2,5)$ and since p does not divide $|M|$ we are done by Corollary 2.2 and Theorem 4.3. Let $X \neq X_0$ and set $Y = X/M$. Then Y has generalized quaternion Sylow 2-subgroups and, up to isomorphisms, $SL(2,5)$ is a subgroup of index 2 in Y. Since $SL(2,5)/N \cong A_5$, where $N = \{1, z\}$ is the center of $SL(2,5)$ then again, up to isomorphisms, A_5 is a subgroup of index 2 in $\bar{Y} = Y/N$. It is easy to see that $\bar{Y}$ has dihedral Sylow 2-subgroups and that $C_{\bar{Y}}(A_5) = 1$. Hence,

$$\bar{Y} = N_{\bar{Y}}(A_5)/C_{\bar{Y}}(A_5) \subseteq Aut(A_5) \cong S_5$$

and, consequently, the equality $|\bar{Y}| = |S_5|$ implies that $\bar{Y} \cong S_5$.

Let K be the inverse image of $N \subset Y = X/M$ in X. Then $X/K \cong S_5$ and using Corollary 2.2 and Theorem 5.1 we complete the proof of (i). For (ii), by our assumption no factor group of X is isomorphic to S_5. Hence $X = X_0$ and we are done by the preceeding paragraph. □

7 ACKNOLEDGEMENTS

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