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A quasi transversal Hopf bifurcation

Introduction

Bifurcation of generic one parameter families of vector fields with simple recurrence are caused by, either local phenomena such as non-hyperbolicity of singularities or periodic orbits, or by semi-local phenomena, like the non transversal intersection of the invariant manifolds of two critical elements.

The persistent ones, i.e., those that remain after small perturbations, are the saddle node and the Andronov-Hopf bifurcation [1], [3], [11], [12] and [10], in case we lose hyperbolicity. When the family bifurcates by losing transversality, we get persistence by imposing quasi-transversality, that is, non transversality of the least degenerated kind [6], [7], [8].

There are different ways of defining equivalence between families X_μ and $\tilde{X}_\mu$. In the first place, for fields we have that a topological equivalence between two vector fields X, $\tilde{X}$ on M is a homeomorphism $h: M \to M$ such that h sends orbits of X onto orbits of $\tilde{X}$, preserving time orientation. If in addition h preserves time, that is $h X_t = \tilde{X}_t h$ holds, then h is called a conjugacy. A vector field X is called structurally stable if it is equivalent to any nearby vector field.

We say that a family X_μ at μ_o is mildly equivalent to

$\tilde{X}_\mu$ at $\tilde{\mu}_o$, if there exists a reparametrization (homeomorphism) $\lambda: (U \subset \mathbb{R}^n, \mu_o) \to (U, \tilde{\mu}_o)$ and a family of homeomorphisms $h_\mu: M \to M$ so that h_μ maps orbits of X_μ onto orbits of $\tilde{X}_{\lambda(\mu)}$ for μ near μ_o. If we can choose the homeomorphisms h_μ to depend continuously on μ, then we say that X_μ is continuously (or strongly) equivalent to $\tilde{X}_\mu$ (at μ_o and $\tilde{\mu}_o$).

The conditions under which two persistent families X_μ and $\tilde{X}_\mu$ are equivalent are well known. When the bifurcation is due to the loss of hyperbolicity of a singularity, the saddle-node and the Andronov Hopf bifurcation are locally strongly equivalent to any nearby family, or strongly stable.

On the other hand, some classes of families going through quasi transversal bifurcations are not strongly or even mildly stable. But Van Strien [9] characterized the conditions under which two families X_μ and $\tilde{X}_\mu$ are either mildly or strongly semi-locally equivalent, that is, in a neighborhood of the orbit of tangency. He did this in terms of moduli of stability [10],[11],[12],[13],[14].

This concept appears naturally. For example, let f be a diffeomorphism of a surface which exhibits two hyperbolic fixed points p_1, p_2, such that $W^u(p_1)$ meets $W^s(p_2)$ quasi transversally, that is, with parabolic contact. Take g near f. Then it shall be topologically equivalent to f if and only if it "looks like" f and $\lambda = \dfrac{\ln \beta^s(p_1)}{\ln \beta^u(p_2)} = \dfrac{\ln \beta^s(\tilde{p}_1)}{\ln \beta^u(\tilde{p}_2)} = \tilde{\lambda}$ where $\beta^\sigma(p_i)$ $(\beta^\sigma(\tilde{p}_i))$ is the eigenvalue associated to $W^\sigma(p_i)$ $(W^\sigma(\tilde{p}_i))$ $(\sigma = s,u;\ i=1,2)$ [9], [5]. Thus, besides f and g having the same shape, there exists a differentiable real invariant which is preserved under topological equivalence. In this context we call λ a modulus for equivalence.

Here we are interested in a class of 2-parameter families X_μ of vector fields with simple recurrences in M^3, which is persistent. It appears in a natural way: X_μ bifurcates by simultaneously losing hyperbolicity and transversality.

We will give necessary and sufficient conditions for the existence of a weak equivalence between members of this class. The notion of modulus of stability is essential for our characterization. We shall also exhibit some moduli of stability for strong equivalence.

More precisely,

Let $\mathfrak{X}_1(M^3)$ be the space of C^∞ vector fields endowed with the C^∞ Whitney topology and let $\mathfrak{X}_2(M)$ be the space of the mappings $X: I\times I \to \mathfrak{X}_1(M)$ with the usual C^∞ topology $(I = [-1,1])$. Suppose $X(\mu)$, is a 2-parameter family of vector fields such that X_μ at $\mu = (0,0)$, or shortly X_o is a field which exhibits:

a) a saddle type periodic orbit $\sigma_1 = \sigma_1(0)$ of period 1 hyperbolic, with associated eigenvalues $0 < |\beta_1| < 1 < \beta_2$, C^2 linearizable;

b) a quasi-hyperbolic singularity $p=p(0)$ of saddle type such that $\dim W^u(p(0)) = 1$, associated to the eigenvalue $\alpha_3 > 0$ and $\dim W^s(p(0)) = 2$, associated to eigenvalues $\alpha_1 \pm i\alpha_2$ $(\alpha_1=0)$. Consider $X/W^s(p(0))$ as a "vague attractor";

c) a unique orbit γ in $W^u(p(0)) \cap W^s(\sigma_1(0))$ that is, a quasi-transversal intersection.

Call $E \subset \mathfrak{X}_1(M^3)$ the set of fields like X_o and E' the set of families as the one described above.

Let us now give the semilocal versions of the notions of equivalence.

A semilocal equivalence between X and $\tilde{X}$ and $\tilde{X} \in E$ shall be an equivalence defined from a neighborhood of $\bar{\gamma}$ onto a neighborhood of $\bar{\tilde{\gamma}}$.

We can now state the following

<u>Theorem A</u>. Let $X, \tilde{X}$ be two vector fields in $E, \varepsilon - C^r$ near.

Then they are semilocally topologically equivalent iff $\beta_2 = \tilde{\beta}_2$.

In this case we say that X has modulus of stability one, and that β_2 is a modulus for X. In some sense, we are parametrizing the equivalence classes in a neighborhood of X by the parameter β_2.

The next theorem is concerned with families of vector fields. Before stating it, we give a rough description of the vector fields that belong to the families with which we shall deal for heuristic purposes.

Due to the hyperbolicity of $\sigma_1(0)$ and non singularity of $dX(p(0))$, both σ_1 and p persist for nearby vector fields. We denote then by $\sigma_1(\mu_1,\mu_2)$ and $p(\mu_1,\mu_2)$ respectively.

When $\mu_1 > 0$ there shall appear new closed orbits $\sigma_2(\mu_1,\mu_2)$, unique for each value of (μ_1,μ_2), $\mu_1 > 0$, whose unstable manifold $W^u(\sigma_2)$ meets $W^s(\sigma_1)$ for some values of μ_2. These orbits σ_2 lie very near p, according to Hopf Theorem.

We prove that the intersection of those manifolds contains alternatively 0, 1 or 2 orbits. In case there is only 1 orbit, it shall be a quadratic tangency between $W^u(\sigma_2)$ and $W^s(\sigma_1)$ and this gives rise to a modulus of stability, as stated before.

For $\mu_1 < 0$ we also get tangencies when $W^u(p)$ meets $W^s(\sigma_1)$ again arising a modulus [14]. Then, every such family exhibits curves of vector fields with one modulus of stability each, and we have

<u>Theorem B</u>. Let $X_\mu \in E'$ be a generic 2-parameter family of vector fields. Then

a) the modulus of stability for mild topological equivalence is one, namely β_2,

b) the modulus of stability for strong topological equivalence is infinite, moreover, it is modelled over a space of germs of functions.

For the sake of completeness, let us now consider the "dual"

Hopf bifurcation, that is, a family X_μ where

a) p is a hyperbolic repellor for $\mu_1 < 0$

b) p is a "vague" repellor for $\mu_1 = 0$

c) p is a hyperbolic saddle point for $\mu_1 > 0$

then a hyperbolic periodic orbit σ_2 appears for $\mu_1 > 0$, and a curve of modulus one vector fields, namely $\{\mu_2 = 0, \mu_1 > 0\}$.

In this case we have, similarly:

Theorem A'. If X_μ and $\tilde{X}_\mu$ are C^r near enough, then:

a) $\tilde{X}_o$ is topologically equivalent to X_o

b) the families X_μ and $\tilde{X}_\mu$ are mildly equivalent

c) the modulus of stability for strong topological equivalence is infinite, and modelled over a space of germs of functions.

This paper is organized as follows:

Section 1 deals with the central bifurcation and there we prove the necessity of the modulus for those vector fields.

In Section 2 we give the bifurcation diagram and begin the proof of Theorem B. Section 3 completes the proof of Theorems A and B, by proving the sufficiency of modulus for the central bifurcation and for mild equivalence.

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§1. The vector fields in E.

In this section we study the central bifurcation $X_o = X$, not considering it as a member of a family of vector fields.

By Takens [15] pp.144-145 we can take coordinates in $\mathbb{R}^3$ such that X is locally in standard form:

$$X = X_1(x_1,x_2)\frac{\partial}{\partial x_1} + X_2(x_1,x_2)\frac{\partial}{\partial x_2} + A(x_1,x_2)x_3\frac{\partial}{\partial x_3}$$

where

1) all eigenvalues of $(\partial X_i/\partial x_j)$ in $x_1 = x_2 = 0$ have real part zero.

2) $A(0,0) > 0$.

As X has only one hyperbolic eigenvalue, the coordinates can be taken C^r, for r arbitrarily high. In this way we get a splitting of our vector field, in a neighborhood L of the singularity.

Recalling a result of Takens [16] on normal forms we take the following as the expression of $X/W^s_\mu(0)$

$$\dot\rho = b\rho^3 + o(\rho^5), \quad b < 0 \quad \text{(vague attractor)}$$
$$\dot\theta = \alpha_2 + o(\rho^2).$$

Let us search a parametric expression for the orbits of this vector field.

<u>Proposition 1</u>. The orbits $(\rho(t),\theta(t))$ of X verify:

$$|\rho^2(t)(1-2bt)-1| \le O(\sqrt{t})$$
$$|\theta(t)-\theta_o-\alpha_2 t| = O(\ln(t)).$$

<u>Proof</u>: It is clear that $\rho(t) \underset{t\to+\infty}{\to} 0$, because $b < 0$ and we can take $\rho(0)$ do as to get $b + \frac{o(\rho^5)}{\rho^3} < 0$. We suppose that this is possible with $\rho(0) = 1$, without loss of generality.

Then $\frac{\rho}{\rho^3} = b + o(\rho^2)$ and

$$\frac{1}{2} - \frac{1}{2\rho^2(t)} = bt + \int_0^t o(\rho^2(s))ds \qquad (1.1)$$

or $[1-2bt-2\int_0^t o(\rho^2(s))ds]^{-1} = \rho^2(t)$.

Clearly, $\rho^2(t) = o(t^{-1})$: in case $\int_0^t |o(\rho^2(1))|ds < \infty$ this is direct and if $\int_0^t |o(\rho^2(s))|ds \to +\infty$, an application of L'Hôpital rule to $\dfrac{\int_0^t |o(\rho^2(s))|ds}{t}$ proves this fact.

We claim that $\varepsilon(\rho(t)) = 2\rho^2(t)\int_0^t o(\rho^2(s))ds = o(t^{-1/2})$.

Indeed, if $\int_0^t |o(\rho^2(s))|ds < \infty$, this follows from the fact that $\rho^2(t) = o(t^{-1})$. On the other hand, if $\int_0^t |o(\rho^2(s))|ds \to +\infty$, we change variables so that

$$ds = \frac{ds}{d\rho}\, d\rho,$$

and $\int_0^t |o(\rho^2(s))|ds = \int_1^{\rho(t)} |o(\rho^2)| \; \frac{1}{\dot{\rho}}\, d\rho = -\int_{\rho(t)}^1 \frac{|o(\rho^2)|}{\rho^3} \; \frac{1}{b+o(\rho^2)}\, d\rho =$

$= o(\ell n\, \rho(t))$.

Returning to the expression of ε,

$$\varepsilon(\rho(t)) \leq 2\rho^2(t)o(\ell n(\rho(t))) \leq o(t^{-1/2})$$

because $\rho(t) \xrightarrow[t\to\infty]{} 0$, and $x\, \ell n\, x \xrightarrow[x\to 0]{} 0$.

So, replacing in (1.1),

$$\rho^2(t)(1-2bt)-1 = 2\rho^2(t)\int_0^t o(\rho^2(s))ds = \varepsilon(\rho(t)) = O(\sqrt{t})$$

as we wished.

In the same way we prove the result for $\theta(t)$. □

Notation: With the definitions of Proposition 1, we take $\rho(t) = \sqrt{\frac{1+\varepsilon}{1-2bt}}$ as the expression of the radial component of an orbit in $W^s(p(0))$.

Remark: The convergence is uniform, that means, it is independent of the spiral.

We shall now define some objects for each vector field, that will be used to prove our results. Take $\Lambda = \{(x_1,x_2,1)\} \subset L$, a section transversal to the flow at $(0,0,1)$.

Consider $\pi_1: L \to W^s_\mu(p)$ the trivial fibration obtained by projecting on the first coordinates. Each fiber is $\pi_1^{-1}(\mu_1,\pi_2)(x_1^o,x_2^o) = \{(x_1^o,x_2^o,x_3),\ x_3 \in \mathbb{R}\} = R$, a line. Let $F = \bigcup_{t\geq 0} X_t(R) \cap \Lambda$, this we call a linear spiral. A sector S_n of F shall be a connected component of $F - W^s(\sigma_1)$.

We reparametrize all fields near X_o, so that the periodic orbit $\sigma_1(\mu_1,\mu_2)$ has period 1. Also, we take a section Σ, transversal to all flows near X_o and near $\sigma_1(\mu_1,\mu_2)$ where the Poincaré transform is simultaneously C^2 linearized.

In $W^s(\sigma_1(\mu_1,\mu_2)) \cap \Sigma$ we fix fundamental domains $D(\mu_1,\mu_2)$ which vary continuously. We also fix an open set N, $D(\mu_1,\mu_2) \subset N \subset \Sigma$ which we cal a fundamental neighborhood. It has the following property: $X_{2,\mu}(N) \cap N = \phi$, where $X_{2,\mu}$ is the time two flow of X_μ, μ small.

Call $\psi\colon \Lambda \to N \subset \Sigma$ the Poincaré transform associated to the flow, $\psi = (\psi_1,\psi_2)$, that is, $\psi_2 = \pi_2(\psi)$ where $\pi_2(\mu_1,\mu_2)\colon\Sigma\to W^u(\sigma_1(\mu_1,\mu_2))$ is the trivial fibration obtained by projecting on the second linearizing coordinate, with fiber $\pi_2^{-1}(\mu_1,\mu_2)(c)$, $c \in W^u(\sigma_1)$. Identifying $W^u(\sigma_1(\mu_1,\mu_2))$ with R, we call S_n an "upper" sector if $\pi_2(\psi(S_n)) \subset \mathbb{R}^+$.

<u>Proposition 2</u>. Let (ρ_n,θ_n) denote the maximum of $\pi_2(\psi(S_n))$, in polar coordinates. Then

$$\theta_n \to \pi/2 \pmod{2\pi}.$$

<u>Proof</u>: Let $\psi_2(x_1,x_2) = \psi_2(x_1(t),x_2(t))$ where $(x_1(t),x_2(t))$ is the parametrization of a sector.

The critical points shall be:

$$\frac{d\psi_2}{dt} = \langle\nabla\psi_2,(x_1',x_2')\rangle = \left\{\frac{(1+\varepsilon)2b}{(1-2bt)^2} + \frac{\varepsilon}{(1-bt)}\right\}\frac{1}{2}\sqrt{\frac{1-2bt}{1+\varepsilon}}\,\langle\nabla\psi_2,(\cos,\mathrm{sen})\rangle$$

$$+\ \theta'\langle\nabla\psi_2,(-\mathrm{sen},\cos)\rangle\sqrt{\frac{1+\varepsilon}{1-2bt}} =$$

$$= \sqrt{\frac{1+\varepsilon}{1-2bt}}\left\{\left[\frac{2b}{(1-2bt)} + \frac{\varepsilon'}{1+\varepsilon}\right]\frac{\langle\nabla\psi_2(\cos,\mathrm{sen})\rangle}{2} + \right.$$

$$\left. +\ [\alpha_2+O(\rho^2(t))]\ \langle\nabla\psi_2,(-\mathrm{sen},\cos)\rangle\right\} = 0.$$

Remark that $\varepsilon' \to 0$:

$$\varepsilon = \rho^2(t)\int_0^t o(\rho^2(s))ds \quad \text{and}$$

$$\frac{d\varepsilon}{dt} = 2\rho\dot{\rho}\int_0^t o(\rho^2(s))ds + \rho^2(t)o(\rho^2(t)).$$

As $\dot{\rho} = b\rho^3 + o(\rho^5)$, it follows

$$\frac{de}{dt} = 2\rho^4(b+o(\rho^2)) \int_0^t o(\rho^2(s))ds + \rho^2(t)o(\rho^2(t)) \underset{t \to +\infty}{\to} 0 .$$

Returning to the expression of $\frac{d\psi_2}{dt}$, we observe that the first term tends to zero. The term given by

$$[\alpha_2+o(\rho^2(t))]\langle\nabla\psi_2,(-\text{sen},\cos)\rangle = [\alpha_2+o(\rho^2(t))]\langle(\frac{\partial\psi_2}{\partial x_1}, \frac{\partial\psi_2}{\partial x_2}), (-\text{sen},\cos)\rangle .$$

As $\frac{\partial\psi_2}{\partial x_1}(0,0) = 0$, and $\frac{\partial\psi_2}{\partial x_2}(0,0) \neq 0$, the expression shall be zero if $\cos\theta \sim 0$, that is, $\theta \sim \frac{\pi}{2} \pmod{2\pi}$. □

Our next claim has a neat geometrical sense. We prove that if some iterates $f^{m_k}(e_{n_k})$ of a sequence of maxima of sectors S_{n_k} accumulate on $z_o \in W^u(\sigma_1)-\sigma_1$, that is, a point of the unstable manifold not in σ_1, then the maxima of $h(S_{n_k})$ shall accumulate on $h(z_o)$.

<u>Proposition 3</u>. Let (ρ_n,θ_n) be the maxima of the sectors S_n. Take $\pi_2(\psi(\rho_n,\theta_n)) = A_n\rho_n$. Suppose X is another vector field, semilocally equivalent to X by an homeomorphism h. Let us call ρ_n "the" maximum of the sector $\tilde{S}_n = h(S_n)$. If $A_{n_k}\rho_{n_k}\beta_2^{m_k} \to z_o$, then $\tilde{A}_{n_k}\rho_{n_k}\tilde{\beta}_2^{m_k} \to h(z_o)$.

<u>Proof</u>: $h(S_n)$ accumulates on an interval whose maximum is $h(z_o)$. □

Let $F = \bigcup_{t\geq 0} X_t(R) \cap \Lambda$ be parametrized by $(\rho(t),\theta(t),1)$ in polar coordinates. Consider the analogous objects for another vector field $\tilde{X}$ in E.

<u>Proposition 4</u>. Suppose the semilocal equivalence h is such that $h(\Lambda) \subset \tilde{\Lambda}$. Take $(\rho(\tilde{t}_n),\tilde{\theta}(\tilde{t}_n))$ "the" sequence of maxima of $h(F)$. Then $\lim_{n\to+\infty} \theta(t_n) = \pi/2$.

<u>Proof</u>: Near the focus, we can approximate $h(F)$ by two linear spirals, thus getting a spiral neighborhood. Then $h(F)$ is shrunk in an arbitrarily thin spiral neighborhood, which behaves as a thick linear spiral, and the result follows.

□

Nest, we prove that β_2 is a modulus of stability by topological equivalence, and the linearity of $h/W^u(\sigma_1(0))$ thus establishing the first part of Theorem A.

<u>Proposition 5</u>. Let $X, \tilde{X} \in E$ be semilocally equivalent by an homeomorphism h.

Then $\beta_2 = \tilde{\beta}_2$ and $h/W^u(\sigma_1)$ is linear.

<u>Proof</u>: Let F be a linear spiral for X, $c_1 \in W^u(\sigma_1) \cap \Sigma$, and $\{S_n\}_{n\geq 1}$ an ordered sequence of the "upper" sectors of F.

Define $N_m(F, \pi_2^{-1}(c_1), S_p) = \#\{S_j / S_j \cap X_{-m}(\pi_2^{-1}(c_1)) \neq \phi,\ j \geq p\}$.

Calling $e_m = \pi_2(\psi(\rho_m, \theta_m))$, by definition,

$$e_{1+N_m} \leq c_1 \beta_2^{-m} \leq e_{N_m}.$$

Analogously $\tilde{e}_{1+N_m} \leq \tilde{c}_1 \beta_2^{-m} \leq \tilde{e}_{N_m}$ where $\tilde{c}_1 = h(c_1)$. Then

$$\frac{e_{1+N_m}}{\tilde{e}_{N_m}} \leq \frac{c_1}{\tilde{c}_1} \left(\frac{\beta_2}{\tilde{\beta}_2}\right)^{-m} \leq \frac{e_{N_m}}{\tilde{e}_{1+N_m}}.$$

By Proposition 4, the angles are approximately $\frac{\pi}{2}$, for m big enough, implying $e_{N_m} \sim \rho_{N_m} A_{N_m}$ and $\tilde{e}_{N_m} \sim \tilde{\rho}_{N_m} \tilde{A}_{N_m}$, where A and $\tilde{A}$ stand for the normal derivatives $\frac{\partial \psi_2}{\partial x_2}$ and $\frac{\partial \tilde{\psi}_2}{\partial \tilde{x}_2}$.

So the last inequality becomes

$$\frac{A_{1+N_m}}{\tilde{A}_{N_m}} \sqrt{\frac{1+\epsilon_{1+N_m}}{1+\tilde{\epsilon}_{N_m}} \left[\frac{1-2\tilde{b}\tilde{t}_{N_m}}{1-2bt_{1+N_m}}\right]} \leq \frac{c_1}{\tilde{c}_1} \left[\frac{\beta_2}{\tilde{\beta}_2}\right]^{-m} \leq$$

$$\leq \frac{A_{N_m}}{\tilde{A}_{1+N_m}} \sqrt{\frac{1+\epsilon_{N_m}}{1+\tilde{\epsilon}_{1+N_m}} \, \frac{1-2\tilde{b}\tilde{t}_{1+N_m}}{1-2bt_{N_m}}}.$$

The expression under the radical tends to $\frac{\tilde{b}\alpha_2}{b\tilde{\alpha}_2}$ when m tends to infinity.

So there exists $\delta_1, \delta_2 > 0$ such that

$$0 < \delta_1 < \left[\frac{\beta_2}{\tilde{\beta}_2}\right]^{-m} < \delta_2 \quad \text{for all } m \in \mathbb{N}.$$

Hence $\beta_2 = \tilde{\beta}_2$ establishing the first statement.

In the limit, the inequality becomes

$$\frac{A}{\tilde{A}}\sqrt{\frac{\alpha_2}{\tilde{\alpha}_2}\frac{\tilde{b}}{b}} = \frac{c_1}{\tilde{c}_1} = \frac{A}{\tilde{A}}\sqrt{\frac{a_2\tilde{b}}{\tilde{a}_2 b}}$$

thus proving the linearity of $h/W^u(\sigma_1)$. □

Remarks: 1) Comparing with [13], [2] we see that we did not take into account any consideration about the irrationality of the invariant.

2) The same method of demonstration applied to the hyperbolic case does not lead to the same conclusion.

3) $h/W^u(\sigma_1)$ is extremely rigid, i.e., it is unique, given the normal forms of X and $\tilde{X}$.

Call now $D \subset \mathfrak{X}_1(M)$ the set of vector fiels analogous to those of E, except that the singularity p is hyperbolic.

Proposition 6. Take $X \in E$, $\tilde{X} \in D$. Then X is not semilocally equivalent to $\tilde{X}$.

Proof: With the same notation of the last proposition, we will see that $\frac{N_m}{\beta_2^{2m}} \to L \neq 0$ and $\frac{N_m}{m} \to \frac{2\pi\tilde{\alpha}_1}{\tilde{\alpha}_2 \ln \tilde{\beta}_2}$ and clearly both statements cannot be true at the same time.

1) By the definition of N_m, it follows

$$A_{1+N_m}\sqrt{\frac{1+\varepsilon_{1+N_m}}{1-2bt_{1+N_m}}} \leq C_1\, \beta_2^{-m} \leq A_{N_m}\sqrt{\frac{1+\varepsilon_{N_m}}{1-2bt_{N_m}}} .$$

Quadrating the inequality and multiplying by N_m, we get

$$A_{1+N_m}^2 \frac{1+\varepsilon_{1+N_m}}{\frac{1}{N_m}-2b\frac{t_{1+N_m}}{N_m}} \leq C_1^2\frac{N_m}{\beta_2^{2m}} \leq \frac{1+\varepsilon_{N_m}}{\frac{1}{N_m}-2b\frac{t_{N_m}}{N_m}} A_{N_m}^2 .$$

As $\frac{N_m}{t_{N_m}} \to \frac{\alpha_2}{2\pi}$, in the limit the inequality becomes

$$-\frac{\alpha_2}{2b2\pi} \le \frac{c_1^2}{A^2} \lim_{m\to+\infty} \frac{N_m}{\beta_2^{2m}} \le \frac{-\alpha_2}{4\pi b}$$

or $\frac{N_m}{\beta_2^{2m}} \to \frac{-A^2\alpha_2}{4\pi c_1^2 b} = L.$

2) A calculation of the same type, applied to $\tilde{X}$, leads to

$\frac{N_m}{m} \to \frac{2\pi\tilde{\alpha}_1}{\tilde{\alpha}_2 \ln \tilde{\beta}_2}$.

Dividing both expressions, it follows that

$$\frac{m}{\beta_2^{2m}} \to \frac{L\tilde{\alpha}_2 \ln \tilde{\beta}_2}{2\pi\ \tilde{\alpha}_1} \neq 0 \quad \text{(absurd)}. \qquad \square$$

§2. <u>The families in</u> E'.

In order to give our bifurcation diagram, let us state Hopf Theorem, following [4].

<u>Hopf Theorem in $\mathbb{R}^2$</u>. Let X_λ be a C^k $(k\geq 4)$ vector field on $\mathbb{R}^2$ such that $X_\lambda(0) = 0$ for all λ and $X = (X_\lambda, 0)$ is also C^k. Let $dX_\lambda(0,0)$ have two distinct, complex conjugate eigenvalues $\alpha_1(\lambda) \pm i\,\alpha_2(\lambda)$, $\alpha_1 > 0$ for $\lambda > 0$. Also let $\frac{d}{d\lambda}\alpha_1(\lambda)\Big|_{\lambda=0} > 0$. Then

A: there is a C^{k-2} function $\lambda: (-\varepsilon,\varepsilon) \to \mathbb{R}$ such that $(x_1, 0, \lambda(x_1))$ is on a closed orbit of period $\approx 2\pi/|\alpha_2(0)|$ and radius growing like $\sqrt{\alpha_1}$, of X for $x_1 \neq 0$ and such that $\lambda(0) = 0$.

B: there is a neighborhood U of $(0,0,0)$ in $\mathbb{R}^3$ such that any closed orbit in U is one of those above. Furthermore, if 0 is a "vague attractor" for X_0, then

C: $\lambda(x_1) > 0$ for all $x_1 \neq 0$ and the orbits are attracting.

In our case $X_\mu \in \mathfrak{X}_1(M^3)$, and it is hyperbolic (expansive) along the x_3-direction.

<u>Corollary</u>. For small (positive) values of α_1 the unique periodic orbit σ_2 which appears near p, is hyperbolic of saddle type.

We want to distinguish a subset of E'. Namely, those families that meet E only at one point, that is, only one member of the family is simultaneously non hyperbolic and non transversal.

To do so, consider a family $X_\mu \in E'$ and call $v(\mu_1,\mu_2) = \pi_2(W^{uu}(p(\mu_1,\mu_2)) \cap N$ where $W^{uu} = W^u$ when p is a saddle type singularity.

Identify $W^u(\sigma_1) \cap \Sigma$ with a neighborhood of $o \in R$, so that $v: U \to \mathbb{R}$, $v \in C^1$.

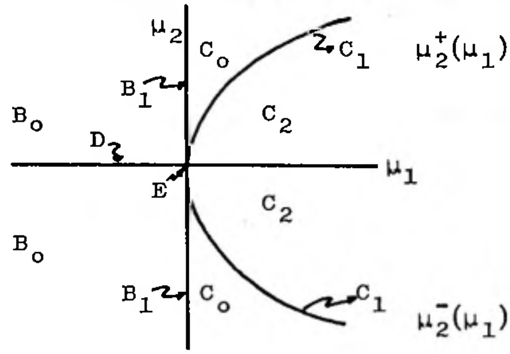
As before, $\alpha_1(\mu_1,\mu_2)$ is the real part of the complex eigenvalue of $dX(\mu_1,\mu_2)$ at $p(\mu_1,\mu_2)$.

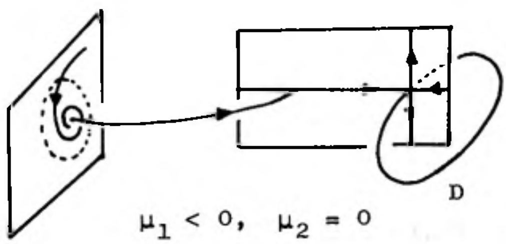
<u>Definition of E''</u>. Let $X_\mu \in E'$. We say that $X_\mu \in E''$ if $J(\alpha_1(\mu_1,\mu_2),v(\mu_1,\mu_2))(0,0)$ is non singular, where J is the Jacobian matrix of (α_1,v). For families in E'' we shall consider, from now on, $\mu_1 = \alpha_1$ and $\mu_2 = v$, on account of the Inverse Function Theorem.

E'' is open and dense in E', so we shall call $X_\mu \in E''$ a generic family.

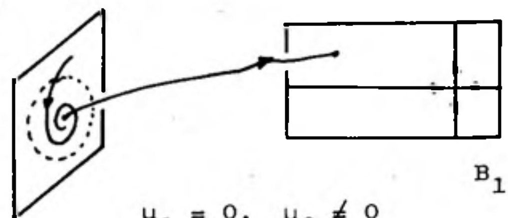
We want to establish the necessity of some moduli of stability for equivalence between members of E''.

In the first place, let us mention which kind of vector fields shall appear in a generic family (see Figure 1).

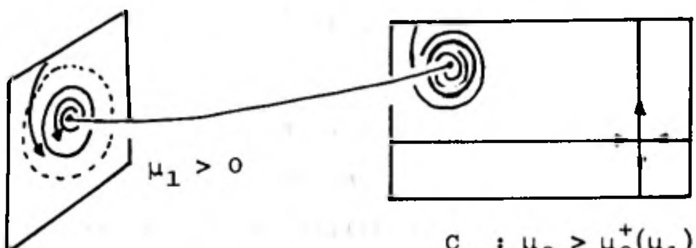




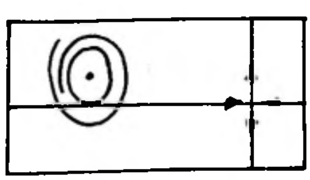
$\mu_1 < 0, \quad \mu_2 = 0$



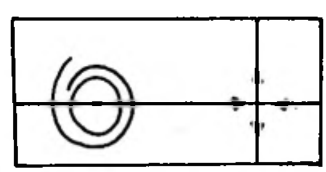
$\mu_1 = 0, \quad \mu_2 \neq 0$



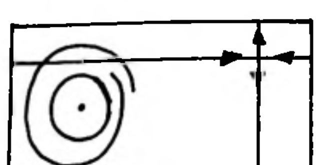
$C_o \; : \; \mu_2 > \mu_2^+(\mu_1)$



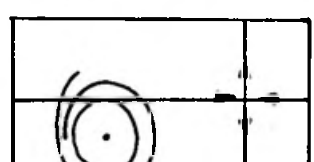
$C_1 \; : \; \mu_2 = \mu_2^+(\mu_1)$



$C_2 \; : \; \mu_2^-(\mu_1) < \mu_2 < \mu_2^+(\mu_1)$



$C_o \; : \; \mu_2 < \mu_2^-(\mu_1)$



$\mu_2 = \mu_2^-(\mu_1) \subset C_1$

1) a unique vector field in E, the central bifurcation,

2) vector fields in D, which look like those in E, but are hyperbolic,

3) vector fields in C, which exhibit a singularity of the source type, hyperbolic and a periodic orbit σ_2 very near this singularity.

This set C we subdivide into:

a) C_1, the set of fields such that $W^u(\sigma_2) \cap W^s(\sigma_1)$ consists only of one orbit, giving rise to a quasi transversal intersection.

b) C_2, the set of vector fields such that $W^u(\sigma_2) \pitchfork W^s(\sigma_1)$ along 2 orbits.

c) C_o, the set of vector fields such that $W^u(\sigma_2) \cap W^s(\sigma_1) = \emptyset$.

4) Vector fields in B, with a singularity p of saddle type, such that $W^u(p) \cap W^s(\sigma_1) = \emptyset$, which we subdivide into: a) B_o the set of fields where p is hyperbolic; b) B_1 the set of fields where p is quasi-hyperbolic.

Observe that $W^u(\sigma_2) \cap W^s(\sigma_1)$ consists alternatively of none, one or two orbits, and that there might be tangencies only for those fields of C_1.

This description follows from the next

Lemma 7 (unicity of the curve of tangencies). Let N be the fundamental neighborhood which we chose before, and consider it fibered by π_1 and π_2.

Then these fibers intersect each other along an unique orbit of tangencies $\Gamma(\mu_1,\mu_2)$ which is differentiable, and depends continuously on the parameters.

Proof: We are needing an expression of the vector field X_μ in a neighborhood of $(0,0,0)$.

Due to the hyperbolicity of $(0,0,0)$ along the x_3 direction (uniform in μ), the λ-Lemma, and Strong Stable Foliation we get a product structure for $X_\mu = X_{1,\mu}(x_1,x_2)\frac{\partial}{\partial x_1} + X_{2,\mu}(x_1,x_2)\frac{\partial}{\partial x_2} + A_\mu(x_1,x_2)x_3\frac{\partial}{\partial x_3}$ where $A(0,0) > 0$ and the eigenvalues of $(\partial X_{i,\mu}/\partial x_j)$ at $(0,0)$ are complex conjugate, whose real part is α_1. The differentiability of the manifold $\{x_3=0\}$ depends on $x_1/\alpha_3)$. If $\alpha_1 > 0$ is small enough, we get a high class of differentiability For $\alpha_1 = 0$, we can take a C^∞ central manifold because p_o is a "vague attractor" in this direction.

Now, we change coordinates x_1, x_2 to polar ones. By [16] Theorem 2.1 applied to the angular part of X_μ and Proposition 2.3 of the same paper, we have $X_\mu = (\alpha_1\rho+b\rho^3)\frac{\partial}{\partial\rho} + (\alpha_2+o(\rho^2))\frac{\partial}{\partial\theta} + A(\rho,\theta)x_3\frac{\partial}{\partial x_3}$, where the coefficients and coordinates depend on (μ_1,μ_2).

Consider $\psi(\mu_1,\mu_2): \Lambda \to \Sigma$ the Poincaré transform.

Take an arc $(x_1(t),x_2(t))$ corresponding to a sector, and $a(t) = \phi_2(\rho(t)\cos\theta(t),\rho(t)\operatorname{sen}\theta(t))$.

Examine the critical points:

$$\frac{d}{dt}a(t) = \langle\nabla\psi_2(x_1(t),x_2(t)),(\dot\rho\cos\theta-\dot\theta\rho\operatorname{sen}\theta,\dot\rho\operatorname{sen}\theta+\dot\theta\rho\cos\theta)\rangle =$$

$$= \langle\nabla\psi_2(x_1(t),y_1(t)),(\cos\theta,\operatorname{sen}\theta)\rangle\dot\rho + \langle\nabla\psi_2,\dot\theta(-\operatorname{sen}\theta,\cos\theta)\rangle\rho =$$

$$= \{\langle\nabla\psi_2(\cos\theta,\operatorname{sen}\theta)\rangle[\mu_1+b\rho^2+o(\rho^4)]+\dot\theta\langle\nabla\psi_2(-\operatorname{sen}\theta,\cos\theta)\rangle\rho = 0.$$

Recall that $\frac{\partial\psi_2}{\partial x_1} = 0$ when $x_1 = x_2 = 0$ and $\mu_2 = 0$, and $\frac{\partial\psi_2}{\partial x_1}(0,0)$ is bounded away from zero. So

$$[(\mu_1+b\rho^2+o(\rho^3))\cos\theta-(\alpha_2+o(\rho^2))\operatorname{sen}\theta]\frac{\partial\psi_2}{\partial x_1} =$$

$$= -[\alpha_2\cos\theta+(\mu_1+b\rho^2+o(\rho^3))\operatorname{sen}\theta]\frac{\partial\psi_2}{\partial x_2}$$

for μ_1 and ρ small enough, we see that $\theta \sim \pi/2$, in order to satisfy the equation.

Let us examine the type of these critical points. Call H = Hessian ψ_2, and Φ the expression in brackets { }.

$$\frac{d^2}{dt^2} a(t) = \dot{\rho}\Phi + \rho \frac{d}{dt} \Phi .$$

As $\Phi = 0$ on critical points, we develop

$$\frac{d}{dt} \Phi = \dot{\theta}\langle \nabla\psi_2,(-\text{sen}\theta,\cos\theta)\rangle(\mu_1+b\rho^2+O(\rho^3))+\dot{\theta}^2\langle\nabla\psi_2,(-\cos\theta,-\text{sen}\theta)\rangle +$$
$$+ \langle\nabla\psi_2,(\cos\theta,\text{sen}\theta)\rangle(2b\rho\dot{\rho}+3\rho\dot{O}(\rho^2)+\dot{\theta}o(\rho^3))+\rho\{[(\mu_1+b\rho^2+o(\rho^3)]^2 \cdot$$
$$\cdot (\cos\theta,\text{sen}\theta)H(\cos\theta,\text{sen}\theta)^T + \dot{\theta}(\mu_1+b\rho^2+o(\rho^3))(-\text{sen}\theta,\cos\theta)H(\cos\theta,\text{sen}\theta)^T$$
$$+ (\mu_1+b\rho^2+o(\rho^3))(\cos\theta,\text{sen}\theta)H(-\text{sen}\theta,\cos\theta)^T+\dot{\theta}(-\text{sen}\theta,\cos\theta)H(-\text{sen}\theta,\cos\theta)^T\}$$
$$+ \ddot{\theta}\langle\nabla\psi_2,(-\text{sen}\theta,\cos\theta)\rangle .$$

As $\ddot{\theta} = O(\dot{\rho}\rho)$, it follows that

$$sg \frac{d}{dt} \Phi = sg[\dot{\theta}^2\langle\nabla\psi_2,(-\cos\theta,-\text{sen}\theta)\rangle+\rho\dot{\theta}(-\text{sen}\theta,\cos\theta)H(-\text{sen}\theta,\cos\theta)^T +$$
$$+ \dot{\theta}\langle\nabla\psi_2,(-\text{sen}\theta,\cos\theta)\rangle(\mu_1+b\rho^2+o(\rho^3))],$$ for ρ small.

We are only interested in the points for which $\frac{d}{dt}$ (t) = 0, that is, $\dot{\theta}\langle\nabla\psi_2,(-\text{sen}\theta,\cos\theta)\rangle = -\langle\nabla\psi_2,(\cos\theta,\text{sen}\theta)\rangle[\mu_1+b\rho^2+o(\rho^3)]$. Replacing in the expression of $sg \frac{d}{dt}\Phi$, we obtain

$$sg \frac{d}{dt} \Phi = sg\{-[\dot{\theta}^2+(\mu_1+b\rho^2+o(\rho^3))^2]\langle\nabla\psi_2,(\cos\theta,\text{sen}\theta)\rangle +$$
$$+ \rho\dot{\theta}(-\text{sen}\theta,\cos\theta)H(-\text{sen}\theta,\cos\theta)^T\}$$

and if ρ is smaller,

$$sg \frac{d}{dt} \Phi = -sg\langle\nabla\psi_2,(\cos\theta,\text{sen}\theta)\rangle = -sg \frac{\partial\psi_2}{\partial x_2} \text{sen}\theta,$$ because $\cos\theta \sim 0$.

So all the critical points for the "upper" sectors are extrema of the same type, for ρ and μ_1 small enough. In particular, if the periodic orbit $\sigma_2(\mu)$ is such that $\sigma_2(\mu) < \rho_o$, the tangency between $W^u(\sigma_2)$ and $W^s(\sigma_1)$ shall be unique or else will not exist.

From the last expression we get the continuity of $\Gamma(\mu_1,\mu_2)$.

□

We are now ready to prove the necessity of modulus for weak equivalence.

Proposition 8. Let $X_\mu, \tilde{X}_\mu \in E''$ be weakly equivalent. Then $\beta_2(0) = \tilde{\beta}_2(0)$.

Proof: For parameters in a small neighborhood of zero, each family exhibits unique vector fields X_o and $\tilde{X}_o$ that belong to E. They should be equivalent on account of the hypothesis and Proposition 6. In this case, we use Proposition 5 to prove our claim. □

Let us prove the necessity of infinite moduli for strong equivalence.

We recall that in [2] it is proven that the fields in D have modulus of stability 1, given by $\lambda_1 = \frac{\alpha_1}{\alpha_2} \ln \beta_2$. De Melo [5] and Palis [9] proved that a vector field exhibiting a quasi transversal connection between the invariant manifolds of periodic orbits has modulus 1, namely $\lambda_2 = \ln \beta_3 / \ln \beta_2$. Van Strien proves in [13] that a one parameter family in D' has 2 moduli of stability for strong equivalence: $\lambda_{1,1} = \alpha_1/\alpha_2$: $\lambda_{1,2} = \ln \beta_2$, that is, the modulus splits. For a family in C_1 the 2 moduli are $\lambda_{1,2} = \ln \beta_2$, $\lambda_{1,3} = \ln \beta_3$. Obviously for weak equivalence we have modulus 1.

Proposition 9. Let $X_\mu \in E'$ be a generic family. Then there are infinitely many moduli of stability for continuous equivalence. Moreover, the moduli are modelled over a space of germs of functions.

Proof: Take X_μ as in the hypothesis. From the genericity of this family, $J(\alpha_1(\mu), v(\mu))(0,0)$ is non singular. So applying the Inverse Function Theorem, we can consider $\mu_1 = \alpha_1$ and $v(\mu) = \mu_2$.

Suppose then that $\tilde{X}_\mu$ is another 2-parameter family equivalent to X_μ by an equivalence $H(\mu)$, with reparametrization ρ.

According to [13] a 1-parameter family $X(\mu_1,0)$ has 2 moduli for strong equivalence:

$$\lambda_{1,1} = \mu_1/\alpha_2(\mu_1) \quad \text{and} \quad \lambda_{1,2} = \ln \beta_2(\mu_1).$$

For $\mu_2 = 0$, μ_1^o small enough we can consider that $\lambda_{1,2}$ is a function of $\lambda_{1,1}$, that is, $\lambda_{1,2} = \lambda_{1,2}(\lambda_{1,1})$.

Given $\lambda_{1,1}$ there is only one value $\rho(\mu_1^o,0)$ such that

$$\tilde{X}(\rho(\mu_1^o,0)) \in D \quad \text{and} \quad \tilde{\lambda}_{1,1} = \lambda_{1,1} .$$

Then, as the families $X(\mu_1^o,\mu_2)$ and $\tilde{X}(\rho(\mu_1^o,\mu_2))$ are equivalent, we must have $\lambda_{1,2} = \tilde{\lambda}_{1,2}$. This means that if $\lambda_{1,1} = \tilde{\lambda}_{1,1}$ consequently will follow $\lambda_{1,2} = \tilde{\lambda}_{1,2}$. Then the functions $\lambda_{1,2}$, $\tilde{\lambda}_{1,2}$ must be identical on a left neighborhood of zero.

As it was demonstrated in the last Proposition, for $c > 0$, the one parameter family $X(c,\mu_2)$ must have two vector fields $X(c,\lambda_2^+(c))$ and $X(c,\lambda_2^-(c))$ which belong to C_1. Again, for a strong equivalence there shall be 2 moduli: $\ln \beta_2$, $\ln \beta_3$, where β_3 is the contracting eigenvalue associated to $\sigma_2(\mu)$. The same reasoning leads to an analogous conclusion: a) the functions $\beta_2^+(\ln \beta_3)$ and $\tilde{\beta}_2^+(\ln \tilde{\beta}_3)$ must be identical in a right neighborhood of zero and b) $\beta_2^-(\ln \beta_3)$ and $\tilde{\beta}_2^-(\ln \tilde{\beta}_3)$ must coincide in a right neighborhood of zero. □

So we have proven Theorem B, part b).

§3. Sufficiency of moduli for weak equivalence.

Now we have to prove that β_2 is the only modulus needed to construct an equivalence between two vector fields in E.

Proposition 10. Let $X,\tilde{X} \in E$ and $\beta_2 = \tilde{\beta}_2$. Then they are semi-locally equivalent.

Proof: Use Lemma 7 to apply essentially the same technique as in [2] which we outline.

Take a fundamental domain $D^s(p_1)$ and the fibrations π_1 and π_2. There is a unique curve of tangencies $C \subset N$. Consider the same objects for $\tilde{X}$.

Define h (the topological equivalence) on D^s, $h/D^s: D^s \to \tilde{D}^s$ by a rotation of small angle which induces a correspondence between the fibers of π_1 and $\tilde{\pi}_1$. Consider $h(C) \subset \tilde{C}$, and preserve $\tilde{\pi}_2$ in a neighborhood of $\tilde{C}$. Then the correspondence between π_1 and $\tilde{\pi}_1$ induces one between π_2 and $\tilde{\pi}_2$, through the definition of h on C.

By conjugation on Σ, we define $h(X_n(C)) = \tilde{X}_n(h(C))$, where X_n is time n flow, and extend it to $W^u(\sigma_1)$ as in Proposition

Now complete $\tilde{\pi}_2$ by linear segments "in the middle" of C and $\tilde{X}_1(\tilde{C})$, in order to make it compatible with h. □

This proposition and Proposition 8 prove Theorem A.

Let us end up the proof of Theorem B.

Proposition 11. Let X_μ, $\tilde{X}_\mu \in E'$, generic, $\beta_2(0) = \tilde{\beta}_2(0)$. Then they are weakly equivalent.

Proof: On account of the genericity of X and $\tilde{X}$ we can suppose $\mu_1 = \alpha_1$, $\mu_2 = v$.

The axis $\mu_1 = 0$ corresponds to fields in B_1, if $\mu_2 \neq 0$.

The axis $\mu_2 = 0$ corresponds to fields in E (for $\mu_1 = 0$), fields in D (for $\mu_1 < 0$) and to fields in C_2 ($\mu_1 > 0$).

If $\mu_1 < 0$, $\mu_2 \neq 0$ we have fields of B_o, that is Morse-Smale ones. For $\mu_1 > 0$, there are two curves $\mu_1^+(\mu_2)$, $\mu_1^-(\mu_2)$ of vector fields such that $W^u(\sigma_2(\mu_1,\mu_2)) \cap W^s(\sigma_1(\mu_1,\mu_2))$ has only one orbit, and these manifolds meet quasi transversally along this orbit.

As $\tilde{X}_\mu$ is also generic, we get the same portrait. In order to prove our statement we shall define a reparametrization ρ.

For $\mu_1 < 0$, $\mu_2 = 0$, observe that there are fields in D or else Morse-Smale. In case $X(\mu_1,0) \in D$, its modulus is $\frac{\mu_1}{\alpha_2(\mu_1,0)\ell n\ \beta_2(\mu_1,0)}$, which is a decreasing function of μ_1, for μ_1 small enough, due to the genericity of X_μ. So, for $(\mu_1,0)$, define $\rho(\mu_1,0)$ as the only value of $\tilde{\mu}_1$ such that

$$\frac{\mu_1}{\alpha_2(\mu_1,0)\ell n\ \beta_2(\mu_1,0)} = \frac{\tilde{\mu}_1}{\tilde{\alpha}_2(\tilde{\mu}_1,0)\ell n\ \tilde{\beta}_2(\tilde{\mu}_1,0)}.$$

For $\mu_1 > 0$, we have to take $\mu_1^+(\mu_2)$ onto $\tilde{\mu}_1^+(\tilde{\mu}_2)$ and $\mu_1^-(\mu_2)$ onto $\tilde{\mu}_1^-(\tilde{\mu}_2)$, respecting the modulus $\ell n\ \beta_3/\ell n\ \beta_2$, as proven in [10,11]. By a reparametrization ρ on $\tilde{\mu}_1 > 0$, in a way that

$$(\ell n\ \beta_3/\ell n\ \beta_2)(\mu_1^+(\mu_2),\mu_2) = (\ell n\ \tilde{\beta}_3/\ell n\ \tilde{\beta}_2)(P(\mu_1^+(\mu_2),\mu_2))$$

and

$$(\ell n\ \beta_3/\ell n\ \beta_2)(\mu_1^-(\mu_2),\mu_2) = (\ell n\ \tilde{\beta}_3/\ell n\ \tilde{\beta}_2)(P(\mu_1^-(\mu_2),\mu_2))$$

we accomplish our goal. Note that here we use the fact that

$$\beta_3^+ = \beta_3(\mu_1^+(\mu_2),\mu_2) \quad \text{and} \quad \beta_3^- = \beta_3(\mu_1^-(\mu_2),\mu_2)$$

are monotonous for families in E''. □

Let us now prove that β_3^+ and β_3^- are monotonous.

<u>Proposition 12</u>. The set of values (μ_1,μ_2) in U for which $W^u(\sigma_2(\mu_1,\mu_2)) \cap W^s(\sigma_1(\mu_1,\mu_2))$ is non transversal is formed by two curves $\mu_1^- = \mu_1(\mu_2)$ for $\mu_2 < 0$ and $\mu_1^+ = \mu_1(\mu_2)$ for $\mu_2 > 0$. Both are differentiable.

<u>Proof</u>: Let $(x_1(\mu_1,\mu_2),x_2(\mu_1,\mu_2)) \in \Lambda \cap W^u(\sigma_2)$ be such that $\psi_2(x_1(\mu_1,\mu_2),x_2(\mu_1,\mu_2))$ is a maximum (minimum). The dependence on (μ_1,μ_2) is differentiable.

For these points let us consider the equation

$\lambda(\mu_1,\mu_2) = \psi_2(x_1(\mu_1,\mu_2),x_2(\mu_1,\mu_2),\mu_1,\mu_2) = 0$ where we are stressing the dependence of ψ_2 upon (μ_1,μ_2).

Differentiating λ with respect to μ_1, we obtain:

$$\frac{\partial\lambda}{\partial\mu_1} = \frac{\partial\psi_2}{\partial x_1}\frac{\partial x_1}{\partial\mu_1} + \frac{\partial\psi_2}{\partial x_2}\frac{\partial x_2}{\partial\mu_1} + \frac{\partial\psi_2}{\partial\mu_1}.$$

We know that $\frac{\partial\psi_2}{\partial x_1} \sim 0$, $\frac{\partial\psi_2}{\partial\mu_1}$ is bounded. Besides both $\frac{\partial x_1}{\partial\mu_1}$ and $\frac{\partial x_2}{\partial\mu_1}$ have the same order and tend to infinity when μ_1 approaches zero.

Hence we can express μ_1 as a function of μ_2, that is, $\mu_1^- = \mu_1^-(\mu_2)$ because $\frac{\partial\lambda}{\partial\mu_1}$ preserves its sign for μ_1 sufficiently small and for $\lambda(\mu_1,\mu_2) = 0$, the curve of maxima with value zero. By the Implicit Function Theorem, we get:

$$\frac{d\mu_1^-}{d\mu_2} = \frac{-\frac{\partial\psi_2}{\partial\mu_2} - \frac{\partial\psi_2}{\partial x_1}\frac{\partial x_1}{\partial\mu_1} - \frac{\partial\psi_2}{\partial x_2}\frac{\partial x_2}{\partial\mu_2}}{\frac{\partial\psi_2}{\partial x_1}\frac{\partial x_1}{\partial\mu_1} + \frac{\partial\psi_2}{\partial x_2}\frac{\partial x_2}{\partial\mu_1} + \frac{\partial\psi_2}{\partial\mu_1}}. \qquad \square$$

<u>Proposition 13</u>. $\beta_3(\mu_1^-(\mu_2),\mu_2))$ is monotonous in a left neighborhood of zero.

<u>Proof</u>: We recall that for a periodic orbit σ_2 of X, with period T, and for the Poincaré transformation $P: \Sigma_o \to \Sigma_o$ on a transversal section Σ_o, we have

$$P'(q) = \exp[\int_0^T \operatorname{div} X(\sigma_2(t))dt] = \beta_3, \qquad q \in \sigma_2 \cap \Sigma_o.$$

Our goal shall be achieved if we see that $\frac{d}{d\mu_2}\beta_3^-$ preserves its sign.

This is the case:

$$\frac{d}{d\mu_2}\beta_3^- = \rho'(q)\frac{d}{d\mu_2}\int_0^T \operatorname{div} X(\sigma_2(t))dt =$$

$$= \rho'(q)\frac{dT}{d\mu_2}(\mu_1^-(\mu_2),\mu_2)\operatorname{div} X(\sigma_2(T)) + \int_0^T \frac{d}{d\mu_2}\operatorname{div}(X(\sigma_2(t)))dt .$$

Then the sign of $\int_0^T \frac{d}{d\mu_2}\operatorname{div}(X(\sigma_2(t))dt$ shall determinate the sign we are looking for

$$\int_0^T \frac{d}{d\mu_2}\operatorname{div}(X(\sigma_2(t)))dt =$$

$$= \int_0^T \left[\frac{d\mu_1^-}{d\mu_2}\frac{\partial}{\partial\mu_1}\operatorname{div}(X(\sigma_2(t))) + \frac{\partial}{\partial\mu_2}\operatorname{div}(X(\sigma_2(t)))\right]dt,$$

where $\int_0^T \frac{\partial}{\partial\mu_1}\operatorname{div}(X(\sigma_2(t)))\frac{\partial\mu_1}{\partial\mu_2}dt$ is dominating. As $\frac{\partial}{\partial\mu_1}\operatorname{div}(X(\sigma_2(t))) = 1 + o(r,\mu_1,\mu_2)$ and $T \sim 2\pi/\alpha_2$, we conclude that the positive sign is preserved. ☐

__Remark__. $\frac{\partial x_1}{\partial\mu_1}$ and $\frac{\partial x_2}{\partial\mu_1}$ have the same order because the periodic orbit σ_2 is almost a circle. They tend to infinity, because for the Hopf bifurcation the radius of σ_2 is $r \sim c\sqrt{|\mu_1|}$.

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