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Some structural properties of the topological ring of Colombeau's generalized numbers

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Abstract

Let $\overline{\mathbf{K}}$ denote the commutative ring with identity of Colombeau's generalized numbers. ($\mathbf{K}$ will denote either $\mathbf{R}$ or $\mathbf{C}$). This ring can be endowed with an ultra-metric in such a way that $\overline{\mathbf{K}}$ is a topological ring. There are many interesting questions about $\overline{\mathbf{K}}$ in the frameworks of Commutative Algebra and General Topology as well as of the superposition of these two subjects. This paper is meant to represent an initial step in this direction. In a few simple cases the study is extended to the ring of Colombeau's generalized functions on an open subset of $\mathbf{R}^n$.

Introduction

In what follows $\mathbf{K}$ will denote either $\mathbf{R}$ or $\mathbf{C}$ and $\overline{\mathbf{K}}$ will denote the commutative ring with identity of generalized numbers. For every non-void open subset Ω of $\mathbf{R}^n$ we shall denote by $\mathcal{G}(\Omega)$ the $\overline{\mathbf{K}}$ - algebra of Colombeau generalized functions on Ω (in the simplified sense).

Recently, D. Scarpalezos (see [S1] and [S2]) introduced metrizable topologies on $\overline{\mathbf{K}}$ and on $\mathcal{G}(\Omega)$ (called "sharp topologies") for which all the operations involved in the structure of the $\overline{\mathbf{K}}$ - álgebra $\mathcal{G}(\Omega)$ are continuous. The purpose of this paper is to give a contribution to the study of some properties of the algebraic and the topological structures of $\overline{\mathbf{K}}$. Sometimes, when it is natural and simple, we extend this study to $\mathcal{G}(\Omega)$. We now describe briefly the contents of this paper. In section 1 we collect the basic definitions, results and notation that we shall use in the forthcoming sections, although we omit the well known proofs. In section 2 we present the first general results. Among

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them we call the attention to the following ones: (a) The set $\text{Inv}(\overline{\mathbf{K}})$ of the units of $\overline{\mathbf{K}}$ is open, (b) $\overline{\mathbf{K}}$ is neither a local nor an integral domain. The same holds for $\mathcal{G}(\Omega)$. (c) The evaluation functions $f \in \mathcal{G}(\Omega) \longmapsto f(x) \in \overline{\mathbf{K}}(x \in \Omega)$ are surjective, continuous ring homomorphisms. This fact is used to derive several results connecting the ideals of $\overline{\mathbf{K}}$ with a large class of ideals of $\mathcal{G}(\Omega)$, (d) Every maximal ideal of $\overline{\mathbf{K}}$ is closed, (e) The nilradical of $\overline{\mathbf{K}}$ vanishes, (f) Any $x \in \overline{\mathbf{K}}$ is either an unit or a zero divisor. This result (Theorem 2.18) is very useful in section 4. In section 3 we quest to get some information about the geometry of $\overline{\mathbf{K}}$ which derives from the strange properties of ultra-metrics (the metric in $\overline{\mathbf{K}}$ is in fact an ultrametric). We exhibit an infinite set of continuous functions from $\overline{\mathbf{K}}$ into itself with interesting geometric properties and also an infinite set of homeomorphisms of $\overline{\mathbf{K}}$. We conclude this section by proving that $\overline{\mathbf{K}}$ is a fractal. Finally, in section 4, we present a more accurate look of the structure of the maximal ideals of $\overline{\mathbf{K}}$. This study is based on two basic tools. First, a careful analysis of the set of zeros of the representatives of elements of $\overline{\mathbf{K}}$. Second, the introduction of a set $\mathcal{S}$ of subsets of the domain $]0,1]$ of the representatives of elements of $\overline{\mathbf{K}}$. The family (χ_S) $(S \in \mathcal{S})$ of elements of $\overline{\mathbf{K}}$, where χ_S is representated by the characteristic function of S, is very useful to the study of a number of interesting questions. Among the several results of this section we point out to the following ones: (a) $\overline{\mathbf{K}}$ is neither noetherian nor artinian. The same holds for $\mathcal{G}(\Omega)$, (b) The Jacobson radical or $\overline{\mathbf{K}}$ vanishes, (c) The (open) set of the units of $\overline{\mathbf{K}}$ is a dense set, (d) The set of the maximal ideals of $\overline{\mathbf{K}}$ is uncountable. We also derive several characterizations of the units of $\overline{\mathbf{K}}$ as well as a complete description of the maximal ideals of $\overline{\mathbf{K}}$ (Theorem 4.20). The description of the prime ideals of $\overline{\mathbf{K}}$ appears to be more complicated than that of the maximal ones. Nevertheless, a first step in this direction is obtained in Theorem 4.19.

1 The sharp topologies of $\overline{\mathbf{K}}$ and $\mathcal{G}(\Omega)$

In this section we recall the basic definitions and results about $\overline{\mathbf{K}}$ and $\mathcal{G}(\Omega)$ (see [S1],[S2]) with the purpose of fixing the terminology. As a rule, the proofs will be ommited. The notation below will be used in the remainder of this paper.

Notation 1.1 K stands for $\mathbf{R}$ or $\mathbf{C}$.

$\mathbf{I} :=]0,1]$, $\bar{\mathbf{I}} := [0,1]$ and $\mathbf{I}_\eta :=]0,\eta[\ \forall\ \eta \in \mathbf{I}$.
$A \setminus B := \{a \in A | a \notin B\}$.
$\mathbf{K}^* := \mathbf{K} \setminus \{0\}$.
$\mathbf{N}$ denotes the set of the natural numbers and $\mathbf{N}^* := \mathbf{N} \setminus \{0\}$.
$\mathbf{R}_+ := \{x \in \mathbf{R} | x \geq 0\}$ and $\mathbf{R}_+^* := \{x \in \mathbf{R} | x > 0\}$.
Ω is a non-void open subset of $\mathbf{R}^n$ and $K \subset\subset \Omega$ means that K is a compact subset of Ω. For a given function $f : X \to \mathbf{K}$ and $\emptyset \neq Y \subset X$ we set $||f||_Y := \sup_{x \in Y} |f(x)|$.

Let $\mathcal{E}[\Omega]$ be the ring (pointwise operations) of the functions $u : \mathbf{I} \times \Omega \to \mathbf{K}$ such that $u(\varepsilon, \cdot) \in C^\infty(\Omega)$ for each $\varepsilon \in \mathbf{I}$. If $\alpha \in \mathbf{N}^n$ and $x \in \Omega$ we set $\partial^\alpha u(\varepsilon, x) := \partial^\alpha u(\varepsilon, \cdot)(x)$. Let $\mathcal{E}_M[\Omega]$ denote the subring of $\mathcal{E}[\Omega]$ consisting of those functions satisfaying the following "moderation condition":

$(M) \forall\ K \subset\subset \Omega$ and $\forall\ \alpha \in \mathbf{N}^n\ \exists\ \sigma \in \mathbf{R}$ such that $||\partial^\alpha u(\varepsilon, \cdot)||_K = o(\varepsilon^\sigma)$ as $\varepsilon \to 0$.

By $\mathcal{N}[\Omega]$ we denote the ideal of $\mathcal{E}_M[\Omega]$ of those functions satisfying the following "nullity condition"

$(N) \forall\ K \subset\subset \Omega$, $\forall\ \alpha \in \mathbf{N}^n$ and $\forall\ \sigma \in \mathbf{R}$ we have $||\partial^\alpha u(\varepsilon, \cdot)||_K = o(\varepsilon^\sigma)$ as $\varepsilon \to 0$.

The *Colombeau algebra of the generalized functions on* Ω is defined by

$$\mathcal{G}(\Omega) := \mathcal{E}_M[\Omega]/\mathcal{N}[\Omega].$$

Let $\mathcal{E}(\mathbf{K})$ be the ring (pointwise operations) of the functions $v : \mathbf{I} \to \mathbf{K}$. Let $\mathcal{E}_M(\mathbf{K})$ denote the subring of $\mathcal{E}(\mathbf{K})$ of those functions v satisfying:

$$(M')\ \exists\ \sigma \in \mathbf{R} \text{ such that } |v(\varepsilon)| = o(\varepsilon^\sigma) \text{ as } \varepsilon \to 0$$

and let $\mathcal{N}(\mathbf{K})$ be the ideal of $\mathcal{E}_M(\mathbf{K})$ of those functions v satisfying:

$$(N')\ \forall\ \sigma \in \mathbf{R} \text{ we have } |v(\varepsilon)| = o(\varepsilon^\sigma) \text{ as } \varepsilon \to 0.$$

The ring of the Colombeau generalized numbers is defined by

$$\overline{\mathbf{K}} := \mathcal{E}_M(\mathbf{K})/\mathcal{N}(\mathbf{K}).$$

It is easy to see that the above definitions are equivalent to those given in [A-B, section 8]. If $f \in \mathcal{G}(\Omega)$ (resp. $x \in \overline{\mathbf{K}}$) we shall use notation as $\hat{f}, f_*$, etc. (resp. $\hat{x}, x_*$, etc.) to denote a representative of f (resp. x). If $u \in \mathcal{E}_M[\Omega]$ (resp. $v \in \mathcal{E}_M(\mathbf{K})$) we shall denote by $\mathrm{cl}(u)$ (resp. $\mathrm{cl}(v)$) the class of u (resp. v) in $\mathcal{G}(\Omega)$ (resp. $\overline{\mathbf{K}}$). We have a natural embedding of $\mathbf{K}$ into $\overline{\mathbf{K}}$ (induced by the map $\lambda \mapsto (\varepsilon \mapsto \lambda)$) and a natural embedding of $\overline{\mathbf{K}}$ into $\mathcal{G}(\Omega)$ (induced by the map. $v \mapsto ((\varepsilon, x) \mapsto v(\varepsilon))$), which allow to write $\mathbf{K} \subset \overline{\mathbf{K}} \subset \mathcal{G}(\Omega)$. Hence $\mathcal{G}(\Omega)$ (resp. $\overline{\mathbf{K}}$) is a $\overline{\mathbf{K}}$ - algebra (resp. $\mathbf{K}$ - algebra) which is commutative and with and identity element. If $f \in \mathcal{G}(\Omega), \xi \in \Omega$ and $\hat{f}$ is any representative of f, the function $\varepsilon \in \mathbf{I} \mapsto \hat{f}(\varepsilon, \xi) \in \mathbf{K}$ belongs to $\mathcal{E}_M(\mathbf{K})$ and its class in $\overline{\mathbf{K}}$, which does not depend on the representative $\hat{f}$, is denoted by $f(\xi)$ and is called the *value of f in ξ*. An element $x \in \overline{\mathbf{K}}$ is *associated to zero,* a property which will be denoted by $x \approx 0$, if for some (or equivalently, for each) representative $\hat{x}$ of x we have $\lim\limits_{\varepsilon \to 0} \hat{x}(\varepsilon) = 0$. Two elements $x_1, x_2 \in \overline{\mathbf{K}}$ are *associated* if $x_1 - x_2 \approx 0$, a property which will be denoted by $x_1 \approx x_2$. We shall denote by $\mathrm{Inv}(\overline{\mathbf{K}})$ the set of units of $\overline{\mathbf{K}}$ and clearly $\mathbf{K}^*$ is a subgroup of the multiplicative group $\mathrm{Inv}(\overline{\mathbf{K}})$. Another interesting subgroup of $\mathrm{Inv}(\overline{\mathbf{K}})$ is $Q := \{\alpha_r | r \in \mathbf{R}\}$, where $\hat{\alpha}_r(\varepsilon) := \varepsilon^r (\varepsilon \in \mathbf{I})$. It will be convenient to set $\beta_r := \alpha_{\log r}$ for each $r \in \mathbf{R}_+^*$.

Next we recall the sharp topologies on $\overline{\mathbf{K}}$ and $\mathcal{G}(\Omega)$ (see $[S1], [S2]$).

Definition 1.2 *For a given $u \in \mathcal{E}_M(\mathbf{K})$ the valuation of u is $V(u) := \sup\{a \in \mathbf{R} |\ |u(\varepsilon)| = o(\varepsilon^a)$ as $\varepsilon \to 0\}$.*

Proposition 1.3 *For given $u, v \in \mathcal{E}_M(\mathbf{K})$ and $\lambda \in \mathbf{K}$ we have:*
(a) $V(\lambda u) = V(u)$ *if $\lambda \neq 0$;* (b) $V(uv) \geq V(u) + V(v)$*;* (c) $V(\hat{\alpha}_r u) = r + V(u)$ *for each $r \in \mathbf{R}$;* (d) $V(u+v) \geq \inf(V(u), V(v))$*;* (e) $V(u) = +\infty$ *iff $u \in \mathcal{N}(\mathbf{K})$,* (f) *$V$ is constant on each equivalence class modulo $\mathcal{N}(\mathbf{K})$.*

Proposition 1.3 shows that the function $D : \overline{\mathbf{K}} \times \overline{\mathbf{K}} \to \mathbf{R}_+$ defined by $D(x, y) := \exp(-V(\hat{x} - \hat{y}))$, where $\hat{x}$ and $\hat{y}$ are any representatives of x and y respectively, is well defined. With these notations we have:

Corollary 1.4 *The function D is an ultra-metric on $\overline{\mathbf{K}}$ which is invariant under translations.*

See [D,3.8, Probl. 4] for the more elementary properties of an ultrametric. D determines an uniform structure on $\overline{\mathbf{K}}$ called the *sharp uniform structure* on $\overline{\mathbf{K}}$ and the topology resulting from D is called the *sharp topology* on $\overline{\mathbf{K}}$ and we shall denote it by τ_s.

Notation 1.5 *Let $a \in \overline{\mathbf{K}}$ and $r \in \mathbf{R}_+^*$. In what follows $B_r(a)$ (resp. $B'_r(a)$ and $S_r(a)$) denotes the open D-ball (resp. closed D-ball and D-sphere) with center at a and radius r. In the case when $a = 0$ we shall omit it in the notation, writing B_r, B'_r and S_r. For the sake of simplicity we define*

$$||x|| := D(x, 0) \;\; \forall \; x \in \overline{\mathbf{K}}.$$

The result below is a direct consequence of Proposition 1.3.

Corollary 1.6 *For given $x, y \in \overline{\mathbf{K}}$, $r \in \mathbf{R}$, $s \in \mathbf{R}_+^*$ and $a, b \in \mathbf{K}$ we have:*
(a) $||x+y|| \leq \max(||x||, ||y||)$ *and* $||xy|| \leq ||x||||y||$; (b) $||x|| = 0$ *iff* $x = 0$, (c) $||ax|| = ||x||$ *if* $a \neq 0$; (d) $||\alpha_r x|| = e^{-r}||x||$ *and* $||\beta_s x|| = s||x||$; (e) $||a|| = 1$ *if* $a \neq 0$; (f) $||a - b|| = \delta_{ab}$ *(Kronecker δ).*

Proposition 1.7 $(\overline{\mathbf{K}}, \tau_s)$ *is a complete topological ring.*

It is known that $\overline{\mathbf{K}}$ is not a $\mathbf{K}$ - topological vector space, it is not separable and also not locally compact.

Later we shall need the following result.

Proposition 1.8 *Let $(x_m)_{m \in \mathbf{N}}$ be a sequence in $\overline{\mathbf{K}}, x \in \overline{\mathbf{K}}$ and let $\hat{x}$ and $\hat{x}_m$ be representatives of x and x_m respectively. Then:*

(a) *(x_m) is a Cauchy sequence in $(\overline{\mathbf{K}}, \tau_s)$ iff $\forall\, p \in \mathbf{N}\; \exists\, N \in \mathbf{N}$ such that, if $m \geq n \geq N$ there is $\varepsilon_{mn} \in \mathbf{I}$ verifying $|\hat{x}_m(\varepsilon) - \hat{x}_n(\varepsilon)| \leq \varepsilon^p$ whenever $0 < \varepsilon < \varepsilon_{mn}$.*
(b) *(x_m) converges to x in $(\overline{\mathbf{K}}, \tau_s)$ iff $\forall\, p \in \mathbf{N}\; \exists\, N \in \mathbf{N}$ such that, if $m \geq N$ there is $\varepsilon_m \in \mathbf{I}$ verifying $|\hat{x}_m(\varepsilon) - \hat{x}(\varepsilon)| \leq \varepsilon^p$ whenever $0 < \varepsilon < \varepsilon_m$.*

We now introduce the sharp topology in $\mathcal{G}(\Omega)$.

Definition 1.9 *Let* $(\Omega_m)_{m\in\mathbf{N}}$ *be an open exhaustion of* Ω *(e.g.* $\Omega = \bigcup_m \Omega_m$ *and* $\overline{\Omega}_m \subset\subset \Omega_{m+1}$ $\forall\ m \in \mathbf{N}$*). Given* $u \in \mathcal{E}_M[\Omega]$ *and* $m, p \in \mathbf{N}$ *we set* $V_{mp}(u) := \sup\{a \in \mathbf{R} | \forall\ \alpha \in \mathbf{N}^n$ *with* $|\alpha| \leq p$*, we have* $||\partial^\alpha u(\varepsilon, \cdot)||_{\Omega_m} = o(\varepsilon^a)$ *as* $\varepsilon \to 0\}$.

Results similar to Proposition 1.3 holds for V_{mp} which shows that the function $D_{mp} : \mathcal{G}(\Omega) \times \mathcal{G}(\Omega) \to \mathbf{R}_+$ given by

$$D_{mp}(f, g) := \exp[-V_{mp}(\hat{f} - \hat{g})],$$

where $\hat{f}$ and $\hat{g}$ are arbitrary representatives of f and g respectively, is well defined and is a pseudo-ultra-metric on $\mathcal{G}(\Omega)$. The uniform structure determined by the family $(D_{mp})_{(m,p)\in\mathbf{N}^2}$ is called the *sharp uniform structure* on $\mathcal{G}(\Omega)$ and the metrizable topology resulting from (D_{mp}) is called the *sharp topology* on $\mathcal{G}(\Omega)$ which we will denote by τ_Ω. Note that τ_Ω is Hausdorff in view of the result below.

Proposition 1.10 *For every* $(m, p) \in \mathbf{N}^2$ *the pseudo-ultra-metric* D_{mp} *is invariant under translations. Morever, for* $f \in \mathcal{G}(\Omega)$ *and* $\alpha \in \mathbf{N}^n$ *the following statements hold:*

(a) $f = 0$ *iff for some (or equivalently, for each) representative* $\hat{f}$ *of* f *we have* $V_{mp}(\hat{f}) = +\infty$ *(i.e* $D_{mp}(f, 0) = 0$*) for all* $(m, p) \in \mathbf{N}^2$.
(b) $D_{mp}(\partial^\alpha f, 0) \leq D_{m,p+|\alpha|}(f, 0)$ $\forall (m, p) \in \mathbf{N}^2$.

In what follows we shall assume that $\overline{\mathbf{K}}$ and $\mathcal{G}(\Omega)$ are endowed with their sharp topologies.

Proposition 1.11 $\mathcal{G}(\Omega)$ *is a complete topological ring and a* $\overline{\mathbf{K}}$ *- topological module.*

Later we shall need the following result.

Proposition 1.12 *Let* $(f_l)_{l\in\mathbf{N}}$ *be a sequence in* $\mathcal{G}(\Omega)$*,* $f \in \mathcal{G}(\Omega)$ *and let* $\hat{f}_l$ *and* $\hat{f}$ *representatives of* f_l *and* f *respectively. Then:*

(a) (f_l) *is a Cauchy sequence in* $\mathcal{G}(\Omega)$ *iff* $\forall\ \nu \in \mathrm{N}^*$, $\forall$ *finite sequence* $s = (m_k, p_k)_{1\leq k\leq \nu}$ *in* N^2 and $\forall\ a \in]0,1[\ \exists\ l_0 \in \mathrm{N}$ *such that, if* $r \geq t \geq l_0$ *there is* $\varepsilon_{rt} \in \mathrm{I}$ *verifying* $||\partial^\alpha(\hat{f}_r - \hat{f}_t)(\varepsilon,\cdot)||_{\Omega_{m_k}} \leq \varepsilon^{-\ln a}$ *whenever* $0 < \varepsilon < \varepsilon_{rt}$, $1 \leq k \leq \nu$ *and* $|\alpha| \leq p_k$.
(b) (f_l) *converges to* f *in* $\mathcal{G}(\Omega)$ *iff* $\forall\ \nu \in \mathrm{N}^*$, $\forall$ *finite sequence* $s = (m_k, p_k)_{1\leq k\leq \nu}$ *in* N^2 *and* $\forall\ a \in]0,1[\ \exists\ l_0 \in \mathrm{N}$ *such that if* $l \geq l_0$ *there is* $\varepsilon_l \in \mathrm{I}$ *verifying* $||\partial^\alpha(\hat{f}_l - \hat{f})(\varepsilon,\cdot)||_{\Omega_{m_k}} \leq \varepsilon^{-\ln a}$ where $0 < \varepsilon < \varepsilon_l$, $1 \leq k \leq \nu$ and $|\alpha| \leq p_k$.

2 Some properties of $\overline{\mathrm{K}}$

In this section we shall present some algebraic properties of $\overline{\mathrm{K}}$. We refer the reader to the previous section whose notation there established will now be used freely. As usual $\overline{X}$ stand for the topological closure of X (except in the notation $\overline{\mathrm{K}}$). We start with the following:

Lemma 2.1 (a) $0 \in \overline{\mathrm{Inv}(\overline{\mathrm{K}})}$; (b) *For each* $x \in \overline{\mathrm{K}}$, *if* $r := -V(\hat{x})$ *where* $\hat{x}$ *is an arbitrary representative of* x, *we have* $y := \alpha_r x \in S_1$; (c) *If* $x \in \mathrm{Inv}(\overline{\mathrm{K}})$ *then* $V(\hat{x}) + V(\hat{x}^{-1}) \leq 0$ *for all representatives* $\hat{x}$ *and* $\hat{x}^{-1}$ *of* x *and of* x^{-1} *respectively.*

Proof. (a) Follows at once from Corollary 1.6(d). (b) Fix any $\sigma > 0$ and a representative $\hat{x}$ of x and set $\hat{y} := \hat{\alpha}_r\hat{x}$. The definitions of $V(\hat{x})$ and r show that

$$\lim_{\varepsilon\to 0} \varepsilon^\sigma.\hat{y}(\varepsilon) = \lim_{\varepsilon\to 0} \varepsilon^{\sigma - V(\hat{x})}.\hat{x}(\varepsilon) = 0$$

and hence $V(\hat{y}) \geq 0$ since $\sigma > 0$. Now, by assuming that $V(\hat{y}) > 0$ there would exist $\sigma > 0$ such that $\lim_{\varepsilon\to 0} \varepsilon^{-\sigma}\hat{y}(\varepsilon) = 0$ which means that

$$\lim_{\varepsilon\to 0} \varepsilon^{-V(\hat{x})-\sigma}.\hat{x}(\varepsilon) = 0$$

and this is false in view of the definition of $V(\hat{x})$. We then conclude that $V(\hat{y}) = 0$ i.e. $||y|| = 1$. (c) Given two arbitrary representatives $\hat{x}$ and $\hat{y}$ of x and x^{-1} respectively, since $\hat{1} - \hat{x}\hat{y} \in \mathcal{N}(\mathrm{K})(\hat{1}(\varepsilon) := 1\ \forall\ \varepsilon \in \mathrm{I})$ if follows from Proposition 1.3 that $0 = V(\hat{1}) = V(\hat{x}\hat{y}) \geq V(\hat{x}) + V(\hat{y})$. ■

Remark The inequality $V(\hat{x}) + V(\hat{x}^{-1}) \leq 0$ can be strict or not. Indeed, if $r \in \mathbf{R}$ then $V(\hat{\alpha}_r) + V(\hat{\alpha}_{-r}) = 0$ and we have the equality. On the other hand, consider the functions $\hat{x}, \hat{y} : \mathbf{I} \to \mathbf{K}$ defined by $\hat{x}(\varepsilon) := 1$ (resp. ε) if $\varepsilon \in \mathbf{I} \cap \mathbf{Q}$ (resp. $\varepsilon \in \mathbf{I}$, ε irrational) and $\hat{y}(\varepsilon) := 1$ (resp. ε^{-1}) if $\varepsilon \in \mathbf{I} \cap \mathbf{Q}$ (resp. $\varepsilon \in \mathbf{I}$, ε irrational). Clearly, $\hat{x}, \hat{y} \in \mathcal{E}_M(\mathbf{K}), V(\hat{x}) = 0,\ V(\hat{y}) = -1$ and if $x := cl(\hat{x})$ then $cl(\hat{y}) = x^{-1}$.

Lemma 2.2 (1o.) *$x \in B_1$ iff $V(\hat{x}) > 0$ for every representative $\hat{x}$ of x.* (2o.) *If $x \in B_1$ then the following statements hold:* (a) *$x \approx 0$;* (b) *$D(1,x) = \|1-x\| \geq 1$. Hence $1 \notin \overline{B}_1, B_1 \cap B_1(1) = \emptyset,\ B_1^{'} \supset \overline{B_1}$ and $B_1^{'} \neq \overline{B}_1$.*

Proof. (1o.) Obvious in view of the definition of $V(\hat{x})$. (2o.) (a) For an arbitrary representative $\hat{x}$ of $x \in B_1$, from the definition of $V(\hat{x})$ and from (1o.) it follows that there is $a > 0$ such that $\varepsilon^{-a}|\hat{x}(\varepsilon)| \to 0$ as $\varepsilon \to 0$ and hence $|\hat{x}(\varepsilon)| \leq \varepsilon^a$ for ε small enough, i.e. $x(\hat{\varepsilon}) \to 0$ as $\varepsilon \to 0$. (b) Fix an arbitrary representantive $\hat{x}$ of x, it is enough to show that $V(\hat{1} - \hat{x}) \leq 0$ ($\hat{1}(\varepsilon) := 1\ \forall \varepsilon \in \mathbf{I}$). Indeed, by assuming that $V(\hat{1} - \hat{x}) > 0$ there would exist $b > 0$ such that $\varepsilon^{-b}|(\hat{1} - \hat{x})(\varepsilon)| \to 0$ as $\varepsilon \to 0$, which implies $\lim_{\varepsilon \to 0} \hat{x}(\varepsilon) = 1$ and this is a contradiction by virtue of (a). Thus, $D(1,x) = \|1-x\| = e^{-V(\hat{1}-\hat{x})} \geq 1$. If $1 \in \overline{B}_1$ there would exist a sequence (x_m) in B_1 which is τ_s-convergent to 1 and therefore $1 \leq D(1, x_m) \to 0$ as $m \to \infty$, which is a contradiction ■

Let A be a topological commutative ring with 1 and X be an A-topological module. We recall that a subset B of X is said to be *bounded* if for every 0-neighborhood W in X there is a 0-neighborhood V in A such that $VB \subset W$. The result below holds for A-topological modules X such that $0 \in \overline{\text{Inv(A)}}$ (([A, Corol. 1.2.6 and Prop. 2.4.6] but here we restrict our attention to the case when $A = \overline{\mathbf{K}}$ (see Lemma 2.1(a)).

Proposition 2.3 (a) *$\overline{\mathbf{K}}$ does not have proper open ideals.* (b) *No topological $\overline{\mathbf{K}}$ - modules has proper open submodules* (c) *If X is a Hausdorff topological $\overline{\mathbf{K}}$ - module then for all $x \in X$, $x \neq 0$ the set $\text{Inv}(\overline{\mathbf{K}}).x := \{\lambda x | \lambda \in \text{Inv}(\overline{\mathbf{K}})\}$ is not a bounded set. Whence, $\text{Inv}(\overline{\mathbf{K}})$ is not a bounded subset of $\overline{\mathbf{K}}$.* (d) *A given $B \subset \overline{\mathbf{K}}$ is bounded iff B is D - bounded.* (e) *The only $\overline{\mathbf{K}}$ - topological module which is bounded is (0). Whence, the only $\overline{\mathbf{K}}$ - topological module which is compact is (0).*

Proof. (a) If $\mathbf{a}$ is an open ideal of $\overline{\mathbf{K}}$ then, from Lemma 2.1(a) it follows that $\mathbf{a} \cap \mathrm{Inv}(\overline{\mathbf{K}}) \neq \emptyset$ hence $\mathbf{a} = \overline{\mathbf{K}}$. (b) If S is a proper open submodule of X, fix $x \in X \setminus S$ then the continuity of $\lambda \in \overline{\mathbf{K}} \mapsto \lambda x \in X$ implies that $\mathbf{a} := \{\lambda \in \overline{\mathbf{K}} | \lambda x \in S\}$ is a proper open ideal of $\overline{\mathbf{K}}$ which is a contradiction in view of (a). (c) Since X is Hausdorff there is a 0-neighborhood W of X such that $x \notin W$. From Lemma 2.1(a), for every 0-neighborhood V of $\overline{\mathbf{K}}$ there is $\lambda \in V \cap \mathrm{Inv}(\overline{\mathbf{K}})$ and so $x = \lambda.\lambda^{-1}x \in V.\mathrm{Inv}(\overline{\mathbf{K}}).x$. (d) Fix an arbitrary $r > 0$. Since $B_t.B_s \subset B_{ts}$ for all $t > 0$, $s > 0$, it is clear that for every $s > 0$ there exists $t > 0$ such that $B_t B_r \subset B_s$ which shows that every D-bounded set is bounded. Conversely, if $L \subset \overline{\mathbf{K}}$ is bounded, there exists $r > 0$ such that $B_r L \subset B_1$. From Lemma 2.1 (a) there is $\lambda \in B_r \cap \mathrm{Inv}(\overline{\mathbf{K}})$ and therefore $L \subset \lambda^{-1} B_1$. Since there is $t \geq 1$ such that $\lambda^{-1} B_1 \subset B_t$, it follows that L is D-bounded. (e) The first statement follows at once from (c) and the second claim follows from the first one taking into account that in a topological module every compact set is bounded ■

Remark By virtue of Proposition 1.11 the above result is applicable to $\mathcal{G}(\Omega)$.

Proposition 2.4 *The ring* $\overline{\mathbf{K}}$ *is neither a local nor an integral domain.*

Proof. In order to proof the first claim consider the following moderate functions $\hat{x}(\varepsilon) := -\mathrm{sen}(\varepsilon^{-1})$ and $\hat{y}(\varepsilon) := 1 - \hat{x}(\varepsilon)$ $(\varepsilon \in \mathbf{I})$, clearly we have $x, y \in \overline{\mathbf{K}} \setminus \mathrm{Inv}(\overline{\mathbf{K}})$, where $x := \mathrm{cl}(\hat{x})$ and $y := \mathrm{cl}(\hat{y})$. If $\overline{\mathbf{K}}$ is local and we denote by $\mathbf{m}$ the unique maximal ideal of $\overline{\mathbf{K}}$ we have $\mathbf{m} = \overline{\mathbf{K}} \setminus \mathrm{Inv}(\overline{\mathbf{K}})$ and thus $1 = x + y \in \mathbf{m}$, which is a contradiction. Let us now shows the second statement. We set $S := \{m^{-1} | m \in \mathbf{N}^*\}$, $S^c := \mathbf{I} \setminus S$ and denote by $\hat{\chi}_S$ (resp. $\hat{\chi}_{S^c}$) the characteristic function of S (resp. S^c). Clearly $\hat{\chi}_S$ and $\hat{\chi}_{S^c}$ are moderate, $\hat{\chi}_S, \hat{\chi}_{S^c} \notin \mathcal{N}(\overline{\mathbf{K}})$ and $(\hat{\chi}_S \hat{\chi}_{S^c})(\varepsilon) = 0$ for all $\varepsilon \in \mathbf{I}$ ■

In the sequel we will introduce a class of ideals of $\mathcal{G}(\Omega)$ which will be very useful in our study. If $\emptyset \neq X \subset \Omega$ and $\mathbf{a}$ is an ideal of $\overline{\mathbf{K}}$ we define the following ideal of $\mathcal{G}(\Omega)$:

$$\mathcal{G}_{X,\mathbf{a}}(\Omega) := \{f \in \mathcal{G}(\Omega) | f(x) \in \mathbf{a} \ \forall \ x \in X\}$$

If $\mathbf{a} = (0)$ (resp. $X = \{\xi\}$) we denote the above ideal by $\mathcal{G}_{X,0}(\Omega)$ (resp. $\mathcal{G}_{\xi,\mathbf{a}}(\Omega)$). Given a non void subset X of Ω and an ideal $\mathbf{a}$ of $\overline{\mathbf{K}}$ we have $(0) \neq \mathcal{G}_{\Omega,0}(\Omega) \subset \mathcal{G}_{X,\mathbf{a}}(\Omega)$.

The result below encompasses some basic facts that we shall need in the sequel.

Proposition 2.5 *For an arbitrary fixed $\xi \in \Omega$ the following stataments hold:* (a) *If $z \in \overline{\mathbf{K}} \subset \mathcal{G}(\Omega)$ then $z(\xi) = z$. Thus for every $z \in \overline{\mathbf{K}}$ there is $f \in \mathcal{G}(\Omega)$ such that $f(\xi) = z$* (b) *The map $\nu_\xi : f \in \mathcal{G}(\Omega) \mapsto f(\xi) \in \overline{\mathbf{K}}$ is a continuous surjective homomorphism of $\overline{\mathbf{K}}$ - algebras and therefore $\mathcal{G}_{\xi,\mathbf{a}}(\Omega) = \nu_\xi^{-1}(\mathbf{a})$ is a closed ideal of $\mathcal{G}(\Omega)$ for each closed ideal $\mathbf{a}$ of $\overline{\mathbf{K}}$.* (c) *If $\mathbf{a}$ is an ideal of $\overline{\mathbf{K}}$ then $\nu_\xi(\mathcal{G}_{\xi,\mathbf{a}}(\Omega)) = \mathbf{a}$. Moreover, $\mathbf{a}$ is a proper ideal iff $\mathcal{G}_{\xi,\mathbf{a}}(\Omega)$ is a proper ideal for every $\xi \in \Omega$.*

Proof. Only the continuity of ν_ξ (in (b)) requires justification since the other statements are easy to verify. It is enough to prove that if (f_l) is a sequence in $\mathcal{G}(\Omega)$ which converges to $f \in \mathcal{G}(\Omega)$ then $(f_l(\xi))$ converges to $f(\xi)$ and this follows at once from Proposition 1.8(b) and Proposition 1.12(b). ∎

Proposition 2.6 *Let $\mathbf{m}$ be an ideal of $\overline{\mathbf{K}}$. Then the following statements are equivalent:*
(i) $\mathbf{m}$ *is maximal (resp. prime).*
(ii) $\mathcal{G}_{\xi,\mathbf{m}}(\Omega)$ *is maximal (resp. prime) for every $\xi \in \Omega$.*

Proof. In view of Proposition 2.5(c) it is clear that each of the conditions (i) and (ii) implies that both ideals $\mathbf{m}$ and $\mathcal{G}_{\xi,\mathbf{m}}(\Omega)$ are proper. Now consider the following diagram

$$\begin{array}{ccc} \mathcal{G}(\Omega) & \xrightarrow{\nu_\xi} & \overline{\mathbf{K}} \\ \downarrow & & \downarrow q \\ \mathcal{G}(\Omega)/\mathcal{G}_{\xi,\mathbf{m}}(\Omega) & \overset{\nu_\xi^*}{- - \rightarrow} & \overline{\mathbf{K}}/\mathbf{m} \end{array}$$

where the vertical arrows denote the quotient maps. Since $\mathrm{Ker}(q \circ \nu_\xi) = \mathcal{G}_{\xi,\mathbf{m}}(\Omega)$, it follows that ν_ξ induces, by Proposition 2.5(b), a ring isomorphism ν_ξ^*. ∎

Proposition 2.7 (a) *If $\mathbf{a}$ and $\mathbf{b}$ are ideals of $\overline{\mathbf{K}}$ and $\mathbf{a} \neq \mathbf{b}$ then $\mathcal{G}_{\xi,\mathbf{a}}(\Omega) \neq \mathcal{G}_{\xi,\mathbf{b}}(\Omega)$ for every $\xi \in \Omega$.* (b) *If $\xi, \zeta \in \Omega$ and $\xi \neq \zeta$ then $\mathcal{G}_{\xi,\mathbf{a}}(\Omega) \neq \mathcal{G}_{\zeta,\mathbf{a}}(\Omega)$ for every proper ideal $\mathbf{a}$ of $\overline{\mathbf{K}}$.* (c) *Let $\mathbf{a}, \mathbf{b}$ be ideals of $\overline{\mathbf{K}}$ and assume that at least one of them is proper and $\xi, \zeta \in \Omega$. Then the following are equivalent:* (i) $\xi = \zeta$ *and* $\mathbf{a} = \mathbf{b}$; (ii) $\mathcal{G}_{\xi,\mathbf{a}}(\Omega) = \mathcal{G}_{\zeta,\mathbf{b}}(\Omega)$.

Proof. (a) Fix any $\xi \in \Omega$ and $x \in (\mathbf{a} \setminus \mathbf{b}) \cup (\mathbf{b} \setminus \mathbf{a})$. If, for instance, $x \in \mathbf{a} \setminus \mathbf{b}$ define $f := x \in \overline{\mathbf{K}} \subset \mathcal{G}(\Omega)$ thus $f(\xi) = x$, hence $f \in \mathcal{G}_{\xi,\mathbf{a}}(\Omega)$ and $f \notin \mathcal{G}_{\xi,\mathbf{b}}(\Omega)$. (b) Fix any proper ideal $\mathbf{a}$ of $\overline{\mathbf{K}}$, $x \in \mathbf{a}$ and $y \in \overline{\mathbf{K}} \setminus \mathbf{a}$. Let V, W be open sets containing ξ and ζ respectively and such that $V \cap W = \emptyset$ and consider $\varphi \in \mathcal{D}(V)$ and $\psi \in \mathcal{D}(W)$ with $\varphi(\xi) = \psi(\zeta) = 1$. Clearly $f := \varphi.x + \psi y \in \mathcal{G}(\Omega)$, $f(\xi) = x$ and $f(\zeta) = y$, thus $f \in \mathcal{G}_{\xi,\mathbf{a}}(\Omega)$ and $f \notin \mathcal{G}_{\zeta,\mathbf{a}}(\Omega)$. (c) Only (ii) $\Rightarrow$ (i) requires justification. If (i) is false then we have one of the following cases:

(I.) $\xi = \zeta$ and $\mathbf{a} \neq \mathbf{b}$; (II.) $\xi \neq \zeta$ and $\mathbf{a} = \mathbf{b}$; (III.) $\xi \neq \zeta$ and $\mathbf{a} \neq \mathbf{b}$.

Clearly, the case (I.) (resp. (II.)) is in contradiction with (ii) by (a) (resp. (b)). In the case (III.) we also get a contradiction in the following way. Consider V, W, φ and ψ as in the proof of (b) and since $(\mathbf{a} \setminus \mathbf{b}) \cup (\mathbf{b} \setminus \mathbf{a}) \neq \emptyset$, assume for instance that $x \in \mathbf{a} \setminus \mathbf{b}$. Define $g := (\varphi + \psi)x \in \mathcal{G}(\Omega)$, then $g(\xi) = g(\zeta) = x$ hence $g \in \mathcal{G}_{\xi,\mathbf{a}}(\Omega)$ and $g \notin \mathcal{G}_{\zeta,\mathbf{b}}(\Omega)$, which is absurd. ■

Corollary 2.8 *The ring $\mathcal{G}(\Omega)$ is not local.*

Proof. $\overline{\mathbf{K}}$ has a maximal ideal by Krull's theorem. Apply Proposition 2.6 and Proposition 2.7(b). ■

The next result will be very useful in our study of the maximal ideals of $\overline{\mathbf{K}}$.

Lemma 2.9 *Let $(a_n)_{n \in \mathbf{N}}$ be any sequence in $\mathbf{K}$ and let $x \in B_1$. Then the series $\sum_{n \geq 0} a_n x^n$ converges in $\overline{\mathbf{K}}$. In particular $\sum_{n \geq 0} x_n$ converges and we have $(1 - x)(\sum_{n \geq 0} x_n) = 1$.*

Proof. The result is obvious when $x = 0$ and hence we may suppose that $x \neq 0$ and denote by $\hat{x}$ a representative of x. For every $n \in \mathbf{N}$ we define $x_n := \sum_{k=0}^{n} a_k x^k$ and $\hat{x}_n := \sum_{k=0}^{n} a_k \hat{x}^k$. We will show that (x_m) is a Cauchy sequence. In fact, denote by $J = J(m, n) := \{k \in \mathbf{N} | n + 1 \leq k \leq m$ and $a_k \neq 0\}$. Since $V(\hat{x}) > 0$ it follows from Proposition 1.3 that $D(x_m, x_n) = \exp[-V(\sum_{k \in J} a_k \hat{x}^k)] \leq \exp[-(n + 1)V(\hat{x})] \to 0$ if $m > n \to \infty$ and hence the series converges in $\overline{\mathbf{K}}$. In particular this is true when all the a_n are equal to one. Now we set $y := \sum_{n \geq 0} x^n$. So we have $x_n := \sum_{k=0}^{n} x^k \to y$ and clearly

$x^k \to 0$ as $k \to 0$. Thus $x_n(1-x) = 1 - x^{n+1} \to 1$. Since multiplication is continuous we have $x_n(1-x) \to y(1-x)$, hence $y(1-x) = 1$. ■

Corollary 2.10 *Let $a \in \mathrm{Inv}(\overline{\mathbf{K}})$ and set $r := ||a^{-1}||^{-1}$. Then we have $B_r(a) \subset \mathrm{Inv}(\overline{\mathbf{K}})$. In particular $B_1(1) \subset \mathrm{Inv}(\overline{\mathbf{K}})$ and $\mathrm{Inv}(\overline{\mathbf{K}})$ is an open set.*

Proof. Let $z \in B_r(a)$. Then it is easily seen that $||a^{-1}(a-z)|| < 1$ and since $z = a[1 - a^{-1}(a-z)]$ it follows from the previous lemma that z is a unit. ■

In the following result we shall use the known fact that the closure of a submodule of a topological module is also a submodule. In particular the closure of an ideal of $\overline{\mathbf{K}}$ is an ideal.

Theorem 2.11 *Let a be a proper ideal of $\overline{\mathbf{K}}$. Then for each $x \in \mathrm{a}$ we have $D(1,x) \geq 1$ and $D(1,\mathrm{a}) = 1$. Hence every maximal ideal of $\overline{\mathbf{K}}$ is closed and thus it is also a rare set.*

Proof. The first claim follows from the previous lemma because $B_1(1)$ consist of units and moreover that $D(1,0) = 1$. On the other hand if m is a maximal ideal of $\overline{\mathbf{K}}$ from the first part, we have $1 \notin \overline{\mathrm{m}}$. ■

Corollary 2.12 *Let m be a maximal ideal of $\overline{\mathbf{K}}$. Then $\mathcal{G}_{\xi,\mathrm{m}}(\Omega)$ is a closed maximal ideal of $\mathcal{G}(\Omega)$ and hence a rare set.*

Proof. Apply Theorem 2.11 and Propositions 2.5(b) and 2.6. ■

Theorem 2.13 *If m is a maximal ideal of $\overline{\mathbf{K}}$ then $\mathbf{K}$ can be identified to a subfield K of $L := \overline{\mathbf{K}}/\mathrm{m}$ and $[L : K] > 1$, i.e. L is a proper field extention of $\mathbf{K}$.*

Proof. Let $\pi : \overline{\mathbf{K}} \to L$ be the canonical map. Clearly $K := \pi(\mathbf{K}) \simeq \mathbf{K}$. Suppose that $L = K$ then $\overline{\mathbf{K}} = \mathbf{K} + \mathrm{m}$. But $\mathbf{K}$ is a discrete subset of $\overline{\mathbf{K}}$ hence from Theorem 2.11 and Proposition 2.3(a) it follows at once that $\mathrm{m} \cup \mathbf{K}$ is a closed set with empty interior. Therefore there is $x \in B_1$, such that $x \notin \mathrm{m} \cup \mathbf{K}$. Write $x = k_x - m_x$ with $k_x \in \mathbf{K}$ and $m_x \in \mathrm{m}$, then $k_x \neq 0$ and hence $m_x = k_x - x = k_x(1 - k_x^{-1}.x)$ and since $k_x^{-1}.m_x \in \mathrm{m}$ we can conclude that $k_x^{-1}m_x = 1 - k_x^{-1}.x \in \mathrm{m}$. This is a contradiction by virtue of Lemma 2.9 since $||k_x^{-1}x|| = ||x|| < 1$. ■

We shall see later that in fact $\mathbf{K}$ is algebraically closed in L.

Corollary 2.14 (a) *There does not exist a surjective ring homomorphism* $A : \overline{\mathbf{K}} \to \mathbf{K}$. (b) *There does not exist a surjective ring homomorphism* $B : \mathcal{G}(\Omega) \to \mathbf{K}$ *such that* $B(\overline{\mathbf{K}}) = \mathbf{K}$.

Proof. The statement (a) follows at once from the previous theorem since the kernel of such a map would be a maximal ideal of $\overline{\mathbf{K}}$. The statements (b) follows easily from (a). ∎

Consider the subset of $\overline{\mathbf{K}}$:

$$\overline{\mathbf{K}}_{as} := \{z \in \overline{\mathbf{K}} |\ \exists\ a \in \mathbf{K} \text{ such that } z \approx a\}.$$

Note that this is the set of all elements $y \in \overline{\mathbf{K}}$ such that $\lim_{\varepsilon \to 0} \hat{y}(\varepsilon)$ exists for some, and hence all, representative $\hat{y}$ of y. It is easy to see that $\overline{\mathbf{K}}_{as}$ is in fact a subalgebra of $\overline{\mathbf{K}}$. Let $\alpha : \overline{\mathbf{K}}_{as} \to \mathbf{K}$ be the map defined by $\alpha(y) := \lim_{\varepsilon \to 0} \hat{y}(\varepsilon)$. This is a $\mathbf{K}$-algebra surjective homomorphism. We shall denote by $\overline{\mathbf{K}}_0$ its kernel which is the subring of $\overline{\mathbf{K}}$ of the elements associated to zero.

Proposition 2.15 (a) *The algebra homomorphism* α *defined above is continuous for the induced topology on* $\overline{\mathbf{K}}_{as}$.
(b) $\overline{\mathbf{K}}_{as}$ *and* $\overline{\mathbf{K}}_0$ *are open subalgebras of* $\overline{\mathbf{K}}$ *containing* B_1.

Proof. From Lemma 2.2(2o.) (a) we have $B_1 \subset \overline{\mathbf{K}}_0$ and hence $\alpha(B_1) = \{0\}$. ∎

The next lemma is an easy consequence of Corollary 1.6(d).

Lemma 2.16 *Let* R_1, R_2 *be positive real numbers and set* $r := \log R_1 - \log R_2$. *Then* $\alpha_r S_{R_1} = S_{R_2}$. *In particular, if* $x \in \overline{\mathbf{K}}^*$ *and* $r := -V(\hat{x})$, *where* $\hat{x}$ *is any representative of* x, *then* $\alpha_r.x \in S_1$. ∎

Theorem 2.17 $\overline{\mathbf{K}}$ *has no non-zero nilpotent elements and hence its nilradical is trivial. It follows that* $\overline{\mathbf{K}}$ *is contained in a product of integral domains.*

Proof. By Lemma 2.16 we just have to show that S_1, has no nilpotent elements. But this is clear since such an element must be equal to zero as can be easily seen. Therefore, if $\mathcal{P}$ denotes the set of all prime ideals of $\overline{\mathbf{K}}$ we have $\bigcap_{\mathbf{p}\in\mathcal{P}} \mathbf{p} = 0$, which is the kernel of the obvious map $\overline{\mathbf{K}} \to \prod_{\mathbf{p}\in\mathcal{P}} \overline{\mathbf{K}}/\mathbf{p}$. ■

If $x \in \overline{\mathbf{K}}$ and $\hat{x}$ is a representative of x we set (see Notation 1.1):

$$Z(\hat{x}) := \{\varepsilon \in \mathbf{I} | \ \hat{x}(\varepsilon) = 0\}$$

and we denote by $\overline{Z}(\hat{x})$ the closure of $Z(\hat{x})$ in $\overline{\mathbf{I}} = [0, 1]$.

Theorem 2.18 *For a given $x \in \overline{\mathbf{K}}$ we have:*
(a) $x \in \mathrm{Inv}(\overline{\mathbf{K}})$ *iff* $0 \notin \overline{Z}(\hat{x})$ *for every representative $\hat{x}$ of x.*
(b) *The following are equivalent:*
(i) $x \notin \mathrm{Inv}(\overline{\mathbf{K}})$; (ii) *There is a representative $\hat{x}$ of x such that* $0 \in \overline{Z}(\hat{x})$;
(iii) *x is a zero divisor.*

Proof. (a) Let $x \in \mathrm{Inv}\ (\overline{\mathbf{K}})$ and let $\hat{x}$ be a representative of such that $0 \in \overline{Z}(\hat{x})$. Then the class y in $\overline{\mathbf{K}}$ of the characteristic function of $Z(\hat{x})$ satisfies $xy = 0$ and $y \neq 0$, which is a contradiction. To prove the converse it is enough to prove (b) (i)$\Rightarrow$ (ii). Indeed, fix $x \notin \mathrm{Inv}(\overline{\mathbf{K}})$ and any representative x_* of x. We can then assume that $0 \notin \overline{Z}(x_*)$ and we will show that $0 \in \overline{Z}(\hat{x})$ for some representative $\hat{x}$ of x. To see this we claim that the following statement is true:

$\exists$ a sequence (ε_m) in $\mathbf{I}$ such that $\varepsilon_m \to 0$ and $|x_*(\varepsilon_m)| < \varepsilon_m^m \ \forall m \in \mathbf{N}^*$.
Note first that there exists $\varepsilon_1 \in]0, 1[$ such that $|x_*(\varepsilon_1)| < \varepsilon_1$ since otherwise $\hat{y}(\varepsilon) := (x_*(\varepsilon))^{-1}$ for $\varepsilon \in]0, 1[$ is a moderate function and his class y satisfies $xy = 1$ which is absurd. Now the same argument allows to define inductively $\varepsilon_k \in \mathbf{I}_{\eta_k}$ from $\varepsilon_j \in \mathbf{I}_{\eta_j} (1 \le j \le k-1)$, where $\eta_1 := 1$ and $\eta_j := min(\varepsilon_{j-1}, 1/j)$. Therefore, the function $\hat{x}(\varepsilon) := x_*(\varepsilon)(1 - \hat{y}(\varepsilon))$, where $\hat{y}$ is the characteristic function of the set $\{\varepsilon_m | m \in \mathbf{N}^*\}$, is a representative of x and $0 \in \overline{Z}(\hat{x})$. The implication (ii)$\Rightarrow$ (iii) is clear since if $\hat{x}$ is a representative of x such that $0 \in \overline{Z}(\hat{x})$ and $\hat{y}$ is the characteristic function of $Z(\hat{x})$ then $y := cl(\hat{y}) \in \overline{\mathbf{K}}^*$ and $xy = 0$. ■

Example 2.19 Let $x \in \overline{\mathbf{K}}$ the class of $\hat{x}(\varepsilon) := \mathrm{sen}(\varepsilon^{-1})$. Then $Z(\hat{x}) = \{(n\pi)^{-1} | n \in \mathbf{N}^*\}$ and thus x is a zero divisor. Let $\hat{x}_1$ be defined by $\hat{x}_1(\varepsilon) := \hat{x}(\varepsilon)$ if $\varepsilon \notin Z(\hat{x})$ and $\hat{x}_1((n\pi)^{-1}) := (n\pi)^{-n}$ if $n \in \mathbf{N}^*$. Then $\hat{x}_1$ is a representative of x and $Z(\hat{x}_1) = \emptyset$.

3 Some Geometric Results

In this section we prove some results that are strongly related to the topological structure of $\overline{\mathbf{K}}$ and particularly to the fact that $\overline{\mathbf{K}}$ is an ultrametric space. Among other results we present some homeomorphisms of $\overline{\mathbf{K}}$ into itself and prove that $\overline{\mathbf{K}}$ is a fractal. The forthcoming results on ultrametric spaces might be known but we gather them in the Proposition below for the sake of convenience. We shall use Notation 1.5 with the obvious adaptations.

Proposition 3.1 *Let (X,d) be an ultrametric space, $a \in X$ and $R > 0$. Then the following hold:*
(a) *Let $\xi = \lim x_n$ with $x_n \in B_R(a)$ for each $n \in \mathbf{N}$. Then one and only one of the following statements is true:*
(I.) *$\xi \neq a$ and there exists $r \in]0,R[$ such that $d(a,x_n) = r$ for all sufficiently large n.*
(II.) *$\xi = a$*
(b) *$S_R(a)$ and $B_R(a)$ are both closed and open.*
(c) *(X,d) is totally disconected.*
(d) *$B_R(x) \subset S_R(a)$ for every $x \in S_R(a)$.*
(e) *$B_R(a) = B_R(b)$ for every $b \in B_R(a)$.*
(f) *Let $r \in]0,R[$ and $x \in S_r(a)$. Then $d(x,y) = R$ for all $y \in S_R(a)$.*
(g) *If $d(x,y) \neq d(y,z)$ then $d(x,z) = \max(d(x,y), d(y,z))$.*

Proof. It is a trivial exercise on metric spaces. ■

From Corollary 1.4 it follows that Proposition 3.1 is applicable to $\overline{\mathbf{K}}$.

Fix an element $z \in \overline{\mathbf{K}}$. The maps T_z and h_z defined by $T_z(x) := x + z$ and $h_z(x) := zx$ $(x \in \overline{\mathbf{K}})$ are continuous maps from $\overline{\mathbf{K}}$ to $\overline{\mathbf{K}}$. Clearly T_z is a homeomorphism and h_z is a homeomorphism if and only if z is a unit in $\overline{\mathbf{K}}$ and in this case $h_z^{-1} = h_{z^{-1}}$. Moreover, if $a \in \overline{\mathbf{K}}$ and $R > 0$ it is clear that $T_z(B_R(a)) = B_R(T_z(a))$, $T_z(S_R(a)) = S_R(T_z(a))$ and $T_{a-z}(B_R(z)) = B_R(a) = B_R(z)$ if $z \in B_R(a)$.

Proposition 3.2 *Given $0 < r < R$ the following statements hold:* (a) *If $x \in S_r$ and $y \in S_R$ then $x + y \in S_R$;* (b) *If $a \in S_R$ then $T_a(S_r)$ is a proper subset of S_R;* (c) *If $a \in S_r$ then $T_a(S_R) = S_R(a) = S_R$.*

Proof. The statement (a) follows from Corollary 1.4 and Proposition 3.1(f). The statement (b) is clear since $a \notin T_a(S_r)$. To prove (c) it is enough to show the second identity since from (a) we have $T_a(S_R) = S_R(a) \subset S_R$. Since $b := -a \in S_R$, the argument before the Proposition shows that $T_b(S_R) = S_R(b) =\subset S_R$. Applying T_{-b} we obtain the result. ■

In the sequel we shall use some notation introduced before Definition 1.2. For $r \in \mathbf{R}$ we denote by h_{α_r} the multiplication by α_r. Recall that the map $r \in \mathbf{R} \mapsto \alpha_r \in Q$ (resp. $r \in \mathbf{R}_+^* \mapsto \beta_r \in Q$) is an isomorphism of the additive group $\mathbf{R}$ (resp. multiplicative group $\mathbf{R}_+^*$) into Q (resp. $\{\beta_r | r \in \mathbf{R}_+^*\}$). It is also obvious that for $R > 0, r > 0$ we have $h_{\beta_r}(S_R) = S_{Rr}$ and that $h_{\alpha_r}(S_R) = S_{Re^{-r}}$

The next result shows that given two spheres there exists a continuous map taking one into the other and leaving the others points of $\overline{\mathbf{K}}$ fixed.

Theorem 3.3 *Given $a, b \in \overline{\mathbf{K}}$ and $R, r \in \mathbf{R}_+^*$, there exists a continuous map $F : \overline{\mathbf{K}} \to \overline{\mathbf{K}}$ such that $F(S_r(a)) = S_R(b)$.*

Proof. Defines $s := Rr^{-1}$ and F by $F(x) := x$ if $x \notin S_r(a)$ and $F(x) := b + \beta_s(x - a)$ if $x \in S_r(a)$. Given $x \in S_r(a)$ we have $x - a \in S_r$ and hence $\beta_s(x - a) \in S_R$ thus $F(x) \in S_R(b)$. Conversely, let $y \in S_R(b) = T_b(S_R)$. If $z := y - b$ then $z \in S_R = h_{\beta_s}(S_r)$ hence there is $w \in S_r$ such that $z = \beta_s w$. Since $S_r = T_{-a}(S_r(a))$ there exists $x \in S_r(a)$ verifying $w = T_{-a}(x)$ which implies $y = F(x) \in F(S_r(a))$.

We now prove that F is a continuous map. It is enough to show that the statement $\xi = \lim x_n$ in $\overline{\mathbf{K}}$ implies $F(\xi) = \lim F(x_n)$. Firstly assume that $\xi \notin S_r(a)$. Since $S_r(a)$ is closed there exists $n_o \in \mathbf{N}$ such that $x_n \notin S_r(a)$ for each $n \geq n_0$ and thus $F(x_n) = x_n$ whenever $n \geq n_0$ and we are done. Secondly, if $\xi \in S_r(a)$, since $S_r(a)$ is open we have $x_n \in S_r(a)$ for sufficiently large $n \in \mathbf{N}$. Thus the continuity of the multiplication and addition in $\overline{\mathbf{K}}$ implies that $(F(x_n))$ converges to $F(\xi)$. ■

We now come to our main result of this section. It tell us that the group of homeomorphisms of $\overline{\mathbf{K}}$ into itself is uncountable. Before we prove it we need some notation. Let G be the group of permutations of $\mathbf{R}_+^*$ (see Notation 1.1). Given $\sigma \in G$ we define the maps j_σ and $\tilde{\sigma}$ by $j_\sigma(r) := r\sigma(r)$ and

$\tilde{\sigma}(r) := \sigma(r)^{-1}$ for each $r \in \mathbf{R}_+^*$. We also define the map $\varphi_\sigma : \overline{\mathbf{K}} \to \overline{\mathbf{K}}$ by $\varphi_\sigma(0) := 0$ and $\varphi_\sigma(x) := \beta_{\sigma(r)}x$ if $x \in S_r$ $(r \in \mathbf{R}_+^*)$. Since $\overline{\mathbf{K}}^* := \overline{\mathbf{K}} \setminus \{0\}$ is the disjoint union $\bigcup_{r \in \mathbf{R}_+^*} S_r$ it follows that φ_σ is well defined.

Theorem 3.4 (a) *If φ_σ is bijective then $\varphi_\sigma(S_r) = S_{j_\sigma(r)}$ for every $r \in \mathbf{R}_+^*$.*
(b) *φ_σ is bijective iff j_σ is bijective.*
(c) *The following are equivalent:*
(i) *φ_σ is a homeomorphism.*
(ii) *j_σ is bijective, $\lim_{r \to 0} j_\sigma(r) = 0$ and $\lim_{r \to 0} j_\sigma^{-1}(r) = 0$.*

Proof. (a) Fix an arbitrary $r \in \mathbf{R}_+^*$. Then it is clear that

$$\varphi_\sigma(S_r) \subset S_{j_\sigma(r)}.$$

Assume by contradiction that there exists $y \in S_{j_\sigma(r)} \setminus \varphi_\sigma(S_r)$. Then, since φ_σ is surjective, there is $z \in \overline{\mathbf{K}}$ such that $\varphi_\sigma(z) = y$ and $z \notin S_r$. Since $t := ||z|| > 0$ it follows that $y = \beta_{\sigma(t)}.z$ and $t \neq r$. So, from the above inclusion (applied for t replacing r) we get $y \in S_{j_\sigma(t)}$ and therefore $j_\sigma(t) = j_\sigma(r)$. Hence, we have $\varphi_\sigma(z) = \varphi_\sigma(w)$ where $w := \beta_{r/t}.z \in S_r$ and thus, the injectivity of φ_σ implies $z = w$, hence $r = t$, which is absurd.
(b) Assume that φ_σ is bijective. First we will proof that j_σ is surjective. Indeed, for a given $t \in \mathbf{R}_+^*$, from (a) we have

$$\varphi_\sigma(\overline{\mathbf{K}}^*) = \varphi_\sigma(\bigcup_{r>0} S_r) = \bigcup_{r>0} S_{j_\sigma(r)} = \overline{\mathbf{K}}^* = \bigcup_{u>0} S_u$$

hence there is $r > 0$ such that $j_\sigma(r) = t$. The injectivity of j_σ follows at once from (a) and the injectivity of φ_σ. Conversely, if j_σ is bijective, since $\tilde{\sigma} \in G$ it follows that $\sigma_1 := \tilde{\sigma} \circ j_\sigma \in G$, thus φ_{σ_1} is well defined and clearly $\varphi_{\sigma_1} \circ \varphi_\sigma = 1_{\overline{\mathbf{K}}}$ which shows that φ_σ is injective. Fix any $y \in \overline{\mathbf{K}}^*$ then $y \in S_r$ where $r := ||y|| > 0$, so there is $t \in \overline{\mathbf{R}}_+^*$, such that $j_\sigma(t) = r$. Setting $x := \beta_{\tilde{\sigma}(t)}.y$ we have $||x|| = t$ thus, in view of the definition of φ_σ we get $\varphi_\sigma(x) = y$, hence φ_σ is surjective. Note that the above argument show that:

If j_σ is bijective then φ_σ is bijective, $\varphi_\sigma^{-1} = \varphi_{\sigma_1}$ and $j_\sigma^{-1} = j_{\sigma_1}$ (3.4.1)

(c) (i) $\Rightarrow$ (ii). First we remark that it is easily seen that

$$||\varphi_\sigma(x)|| = j_\sigma(r) \quad \text{whenever} \quad x \in S_r \text{ and } r \in \mathbf{R}_+^*. \qquad (3.4.2)$$

In view of (i) and (b) it is obvious that j_σ is bijective, hence (3.4.2) holds and since $\sigma_1 \in G$ we get

$$||\varphi_{\sigma_1}(x)|| = j_{\sigma_1}(r) \text{ whenever } x \in S_r \text{ and } r \in \mathbf{R}_+^*. \qquad (3.4.2')$$

Moreover, from the last identity of (3.4.1) it follows that it remains to show only that

$$\lim_{r\to 0} j_\sigma(r) = 0 \quad\text{and}\quad \lim_{r\to 0} j_{\sigma_1}(r) = 0. \qquad (3.4.3)$$

Let (r_m) be a sequence in $\mathbf{R}_+^*$ such that $r_m \to 0$. For each $m \in \mathbf{N}$ fix an arbitrary $x_m \in S_{r_m}$ then $||x_m|| = r_m \to 0$. Thus, the continuity of φ_σ and φ_{σ_1}, (3.4.2) and (3.4.2') imply $j_\sigma(r_m) \to 0$ and $j_{\sigma_1}(r_m) \to 0$.
(ii) $\Rightarrow$ (i). Fix an arbitrary convergent sequence (x_m) in $\overline{\mathbf{K}}$ and set $\xi := \lim x_m$, then it is enough to show that

$$\lim\varphi_\sigma(x_m) = \varphi_\sigma(\xi) \quad\text{and}\quad \lim\varphi_{\sigma_1}(x_m) = \varphi_{\sigma_1}(\xi). \qquad (3.4.4)$$

If $\xi \neq 0$ then (3.4.4) follows at once from Proposition 3.1(a), the definitions of φ_σ and φ_{σ_1} and the continuity of the operations in $\overline{\mathbf{K}}$.
If $\xi = 0$, (3.4.4) is trivial if (x_m) has finite support, hence we can eliminate this case. So it remains to show only that if (x_{m_ν}) is a subsequence of (x_m) such that $x_{m_\nu} \neq 0$ for all $\nu \in \mathbf{N}$ then

$$\lim\varphi_\sigma(x_{m_\nu}) = 0 \quad\text{and}\quad \lim\varphi_{\sigma_1}(x_{m_\nu}) = 0$$

which follows immediatly from (3.4.2), (3.4.2') and (3.4.3) since $r_{m_\nu} = ||x_{m_\nu}|| \to 0$ as $\nu \to \infty$. ■

Our next result proves that $\overline{\mathbf{K}}$ is a fractal. Recall that according to B.Mandelbrot (see [E, p.179]) a *fractal* is a metric space X whose Hausdorff dimension $\dim(X)$ is strictly bigger than its small inductive dimension $\operatorname{ind}(X)$. It is well known that if X is an ultrametric space then $\operatorname{ind}(X) = 0$ and thus it remains to show that $\dim(\overline{\mathbf{K}}) \neq 0$. In fact we will show that the Hausdorff dimension of $\overline{\mathbf{K}}$ is infinite. In what follows we will use the notations of [E].

Theorem 3.5 $\overline{\mathrm{K}}$ *is a fractal.*

To prove the theorem we need the following result:

Lemma 3.6 *Let (X, d) be an ultrametric space and let B be a borelian subset of X. Suppose that B contains an uncountable subset S such that $d(x, y) = 1$ whenever $x, y \in S$ and $x \neq y$. Then* $\dim(X) = \infty$.

Proof. Since B and X are both borelian sets, it follows, from [E,Th.6.1.6], that we only have to check that $\dim(B) = \infty$. We recall that we are using the notation of [E,Ch.6, p.147-149]. To prove our result it is enough to show that $C_{B,\varepsilon} = \emptyset$, where $\varepsilon \in]0,1[$ and $C_{B,\varepsilon}$ denotes the set of all ε-covers of B. So fix $\varepsilon \in]0,1[$ and assume that there exists an ε-cover $(A_n)_{n\in\mathrm{N}}$ of B. Since $S \subset B$ it follows that $(A_n)_{n\in\mathrm{N}}$ covers S. But since S is uncountable, some A_n must contains an infinity of elements of S. The proof will be complete if we prove that each A_n contains at most one element of S. In fact, let M denote the set of all $m \in \mathrm{N}$ such that $S \cap A_m \neq \emptyset$ then, for $m \in \mathrm{N}$ and $x_m \in S \cap A_m$ we have $d(x_m, y) \leq \varepsilon < 1$ for all $y \in A_m$, hence $A_m \subset B_1(x_m)$ for every $m \in \mathrm{N}$. Thus it is sufficient to prove that $S \cap B_1(x_m) = \{x_m\}$ for all $m \in \mathrm{N}$. Indeed, fix $m \in \mathrm{N}$ and assume that there are $u, v \in S \cap B_1(x_m)$ with $u \neq v$. From the hypothesis on S we get $1 = d(u, v) \leq \max(d(u, x_m), d(v, x_m)) < 1$, which is a contradiction. ■

Proof of Theorem 3.5 Proposition 3.1(b) implies that S_1 is a borelian subset of $\overline{\mathrm{K}}$. Moreover K^* is an uncountable subset of S_1 verifying the hypothesis of Lemma 3.6, in view of Corollary 1.6 (b) and (f). Whence $0 = \mathrm{ind}(\overline{\mathrm{K}}) < \dim(\overline{\mathrm{K}}) = \infty$. ■

4 Characteristic functions

In this section we shall take a closer look at the structure of the set of units of $\overline{\mathrm{K}}$ and the prime and maximals ideals of $\overline{\mathrm{K}}$. In fact we completely describe $\mathrm{Inv}(\overline{\mathrm{K}})$ and the set of all maximal ideals of $\overline{\mathrm{K}}$. To do this we first make a careful analysis of the set of zeros of the representatives of elements of $\overline{\mathrm{K}}$. We then study special types of characteristic functions and show that they are related to the prime and maximal ideals of $\overline{\mathrm{K}}$. Some of the main consequences of this analysis are that the set of units is an open and dense subset of $\overline{\mathbf{K}}$ and that $\overline{\mathbf{K}}$ is neither a noetherian nor an artinian ring. As a consequence the same holds for $\mathcal{G}(\Omega)$. The description of the set of all prime ideals of $\overline{\mathbf{K}}$ is, seemingly, more complicated than the same question for maximal ideals.

In what follows we use frequently the symbols $\mathbf{I}, \overline{\mathbf{I}}$ and $\mathbf{I}_\eta$ (see Notation 1.1). For every $T \subset \mathbf{I}$ let T^c be its complement in $\mathbf{I}$, $\overline{T}$ its closure in $\overline{\mathbf{I}} = [0,1]$ and denote by $\hat{\chi}_T$ the characteristic function of T with domain $\mathbf{I}$, i.e. $\hat{\chi}_T(\varepsilon) := 1$ (resp. 0) if $\varepsilon \in T$ (resp. $\varepsilon \in T^c$). This is clearly a moderate function and we denote by χ_T its class in $\overline{\mathbf{K}}$. This, by abuse of language, still will be called the characteristic function of T.

Definition 4.1 $\mathcal{S} := \{S \subset \mathbf{I} \mid 0 \in \overline{S} \cap \overline{S^c}\}$.

Clearly for a given $S \subset \mathbf{I}$ we have $S \in \mathcal{S}$ if and only if $\emptyset \neq S \cap \mathbf{I}_\eta \neq \mathbf{I}_\eta$ for each $\eta \in \mathbf{I}$. We now sum up the basic properties of the set $\mathcal{S}$.

Proposition 4.2 (a) *If $S \in \mathcal{S}$ then $\chi_S \in S_1$.*
(b) *$S \in \mathcal{S}$ if and only if $S^c \in \mathcal{S}$.*
(c) *If $S, T \in \mathcal{S}$ and $0 \in \overline{S \cap T}$ then $S \cap T \in \mathcal{S}$.*
(d) *Let $x \in \overline{\mathbf{K}}^*$ be a non-unit. Then there exists a representative $\hat{x}$ of x such that $Z(\hat{x}) \in \mathcal{S}$, and, for every representative x_* of x such that $0 \in \overline{Z}(x_*)$ we have $Z(x_*) \in \mathcal{S}$.*

Proof. The first three statements are straightforward. The first claim of (d) follows from Theorem 2.18(b) because x must be a zero divisor. Hence there is a representative $\hat{x}$ of x such that $0 \in \overline{S}$, where $S := Z(\hat{x})$, and from $0 \notin \overline{S^c}$ we would conclude that $x = 1$, which is a contradiction. ∎

Remarks (1) Note that if $S, T \in \mathcal{S}$ then χ_T and χ_S are non-trivial idempotents. Moreover, if $S \cap T = \emptyset$ then χ_S and χ_T are orthogonal idempotents, i.e., $\chi_S \cdot \chi_T = 0$. In particular this is the case when $T = S^c$. From this it follows that if $\mathbf{p}$ is a prime ideal of $\overline{\mathbf{K}}$ then either χ_S or χ_{S^c} belongs to $\mathbf{p}$
(2) If $S \in \mathcal{S}$ then $\chi_S + \chi_{S^c} = 1$ and thus the ideal generated by χ_{S^c} is equal to $\mathrm{Ann}(\chi_S)$ (the annilator of χ_S).

In view of the above remarks the following result is clear.

Proposition 4.3 *Let $S \in \mathcal{S}$. Then* $\mathrm{Ann}(\chi_S) = \overline{\mathbf{K}}\chi_{S^c}$ *and* $\overline{\mathbf{K}} = \mathrm{Ann}(\chi_S) \oplus \mathrm{Ann}(\chi_{S^c})$. *Moreover, for each prime ideal $\mathbf{p}$ of $\overline{\mathbf{K}}$ we have that either χ_S or $\chi_{S^c} = 1 - \chi_S$ belongs to $\mathbf{p}$.* ∎

Let $\sigma = (\sigma_n)_{n\in\mathbf{N}}$ be a sequence in $\mathbf{I}$ which converges to zero. It is clear that $\sigma_* := \{\sigma_n | n \in \mathbf{N}\} \in \mathcal{S}$ and thus $\sigma_*^c \in \mathcal{S}$. The set of all such sequences will be denoted by Σ. If $\sigma \in \Sigma$, in the sequel we shall still write $\hat{\chi}_\sigma, \chi_\sigma, \hat{\chi}_{\sigma^c}$ and χ_{σ^c} to denote $\hat{\chi}_{\sigma_*}, \chi_{\sigma_*}, \hat{\chi}_{\sigma_*^c}$ and $\chi_{\sigma_*^c}$ respectively. The result below is obvious:

Lemma 4.4 *There exists a sequence $(\sigma^k)_{k\in\mathbf{N}}$ in Σ such that $\sigma_*^k \cap \sigma_*^j = \emptyset$ whenever $k \neq j$.* ∎

Theorem 4.5 *$\overline{\mathbf{K}}$ is neither noetherian nor artinian. The same holds for $\mathcal{G}(\Omega)$.*

Proof. We need to produce infinitely strictly descending and ascending chains of ideals in $\overline{\mathbf{K}}$. Fix a sequence (σ^k) in Σ as in Lemma 4.4 and consider for each $n \in \mathbf{N}$ the following ideals of $\overline{\mathbf{K}}$:

$$\mathbf{a}_n := < \{\chi_{\sigma^k} | 0 \leq k \leq n\} > \text{ and } \mathbf{b}_n := < \{\chi_{\sigma^k} | k \geq n\} > .$$

Assume that there is $n \in \mathbf{N}^*$ such that $\mathbf{a}_{n-1} = \mathbf{a}_n$. Therefore $\chi_{\sigma^n} \in \mathbf{a}_{n-1}$ and $\chi_{\sigma^n} = \sum_{0\leq k\leq n-1} y_k \chi_{\sigma^k}$ with $y_k \in \overline{\mathbf{K}}$. Since the χ_{σ^j} are orthogonal idempotents it follows, after multiplying by χ_{σ^n}, that $\chi_{\sigma^n}^2 = \chi_{\sigma^n} = 0$, which is a contradiction. Hence $(\mathbf{a}_n)_{n\in\mathbf{N}}$ is a strictly ascending chain of ideals of $\overline{\mathbf{K}}$. The same argument shows that $\chi_{\sigma^n} \notin \mathbf{b}_{n+1}$, and thus $(\mathbf{b}_n)_{n\in\mathbf{N}}$ is a strictly descending chain of ideals of $\overline{\mathbf{K}}$. The claim for $\mathcal{G}(\Omega)$ follows at once from the above arguments and Proposition 2.7(a). ∎

Definition 4.6 *Given $u \in \mathcal{E}_M(\mathbf{K})$ and $a \in \mathbf{R}_+^*$ we set*
(a) $N_a(u) := \{\varepsilon \in \mathbf{I} | \, |u(\varepsilon)| < \varepsilon^a\}$.
(b) $\hat{\chi}_{a,u} := \hat{\chi}_{N_a(u)}$ *and* $\chi_{a,u} := \chi_{N_a(u)}$.

Definition 4.7 *Given $u \in \mathcal{E}_M(\mathbf{K})$ we define the following functions of $\mathbf{I}$ into $\mathbf{K}$:*

$$\theta_u(\varepsilon) := \exp(-i\mathrm{Arg}(u(\varepsilon)) \quad \text{and} \quad \theta_u^{-1}(\varepsilon) := \exp(i\mathrm{Arg}(u(\varepsilon))) \quad (\varepsilon \in \mathbf{I}).$$

In Definition 4.7, $\mathrm{Arg}(u(\varepsilon))$ denote the argument of $u(\varepsilon) \in \mathbf{K}$ with the convention that $\mathrm{Arg}(0) := 0$. In the case $\mathbf{K} = \mathbf{R}$ the images of θ_u and θ_u^{-1} are contained in $\{-1,1\}$. It is also clear that these are moderate functions and inverses of each other. Moreover, we have $u(\varepsilon).\theta_u(\varepsilon) = |u(\varepsilon)|\ \forall\varepsilon \in \mathbf{I}$. Therefore, if we define $|u|(\varepsilon) := |u(\varepsilon)|\ \forall\ \varepsilon \in \mathbf{I}$, we can write $u.\theta_u = |u|$ and thus $|u| \in \mathcal{E}_M(\mathbf{R})$.

Definition 4.8 *Given* $u \in \mathcal{E}_M(\mathbf{K})$ *we set* $\Theta_u := \mathrm{cl}(\theta_u)$ *and* $\Theta_u^{-1} := \mathrm{cl}(\theta_u^{-1})$

Since $\Theta_u.\Theta_u^{-1} = 1$ we have that they are units. Note however that these functions depend on the representative. The following result, whose proof is trivial, will be needed in the sequel.

Lemma 4.9 *Given* $u \in \mathcal{E}_M(\mathbf{K})$ *we have:*
(a) *If* $a \in \mathbf{R}_+^*$ *then* $\chi_{a,u} = 1$ *if and only if* $\mathbf{I}_\eta \subset N_a(u)$ *for some* $\eta \in \mathbf{I}$.
(b) $\chi_{a,u} = 1$ *for every* $a \in \mathbf{R}_+$ *if and only if* $u \in \mathcal{N}(\mathbf{K})$. ∎

Corollary 4.10 *Let* $x \in \overline{\mathbf{K}}^*$ *be a non-unit and assume that* x *has a representative* $\hat{x}$ *satisfying the following condition*

$$\exists\ \alpha \in \mathbf{I} \text{ such that } \hat{x}(\varepsilon) \in \mathbf{R}_+^*\ \forall\varepsilon \in \mathbf{I}_\alpha. \tag{*}$$

Then there exists $a \in \mathbf{R}_+^*$ *such that* $\chi_{a,\hat{x}} \neq 0, \chi_{a,\hat{x}} \neq 1$ *and* $N_a(\hat{x}) \in \mathcal{S}$.

Proof. The existence of $a > 0$ such that $\chi_{a,\hat{x}} \neq 1$ follows from $x \neq 0$ and Lemma 4.9(b). Hence it remains to prove that $\chi_{a,\hat{x}} \neq 0$. Otherwise there would exist $\eta \in \mathbf{I}$ such that $\hat{\chi}_{a,\hat{x}}(\varepsilon) = 0$ for each $\varepsilon \in \mathbf{I}_\eta$. Setting $\beta := \min(\alpha,\eta)$ we would have $|\hat{x}(\varepsilon)| = \hat{x}(\varepsilon) \geq \varepsilon^a$ for every $\varepsilon \in \mathbf{I}_\beta$ and thus x would be a unit, which is a contradiction. Setting $S := N_a(\hat{x})$, from $\chi_{a,\hat{x}} \neq 1$ (resp. $\chi_{a,\hat{x}} \neq 0$) it follows that $0 \in \overline{S^c}$ (resp $0 \in \overline{S}$). ∎

Lemma 4.11 *Let* x *and* $\hat{x}$ *be as in Corollary 4.10. Then there exists* $a \in \mathbf{R}_+^*$ *such that* $y := x(1-\chi_{a,\hat{x}}) + \chi_{a,\hat{x}} \in \mathrm{Inv}(\overline{\mathbf{K}})$.

Proof. Let $a \in \mathbf{R}_+^*$ be as in Corollary 4.11 and set $\hat{y} := \hat{x}(1-\hat{\chi}_{a,\hat{x}}) + \hat{\chi}_{a,\hat{x}}$. It is enough to note that $\hat{y}(\varepsilon) \geq \varepsilon^a$ for all $\varepsilon \in \mathbf{I}_1$ and thus $1/\hat{y} \in \mathcal{E}_M(\mathbf{K})$. ∎

In the result below we denote by $\mathrm{Rad}(\overline{\mathbf{K}})$ the Jacobson's radical of $\overline{\mathbf{K}}$.

Theorem 4.12 *Let* $x \in \overline{\mathbf{K}}^*$ *be a non-unit. Then there exists a maximal ideal* $\mathbf{m}$ *of* $\overline{\mathbf{K}}$ *such that* $x \notin \mathbf{m}$. *Hence* $\mathrm{Rad}(\overline{\mathbf{K}}) = 0$.

Proof. We first suppose that x has a representantive $\hat{x}$ verifying the condition $(*)$ of Corollary 4.10. Hence there is $a \in \mathbf{R}_+^*$ such that $\chi_{a,\hat{x}} \notin \{0,1\}$ and since $\chi_{a,\hat{x}}(1-\chi_{a,\hat{x}}) = 0$ it follows that $\chi_{a,\hat{x}} \notin \mathrm{Inv}(\overline{\mathbf{K}})$. By Krull's theorem there exists a maximal ideal $\mathbf{m}$ of $\overline{\mathbf{K}}$ such that $\chi_{a,\hat{x}} \in \mathbf{m}$. We claim that $x \notin \mathbf{m}$. Otherwise we would have $y = (1-\chi_{a,\hat{x}})x + \chi_{a,\hat{x}} \in \mathbf{m}$, which is a contradiction in view of Lemma 4.11. To prove the general case let $\hat{x}$ be a representative of x. Then $\Theta_{\hat{x}} \in \mathrm{Inv}(\overline{\mathbf{K}})$ and $\hat{z} := \hat{x}.\theta_{\hat{x}} = |\hat{x}|$ is a representative of $z := x.\Theta_{\hat{x}}$ which satifies condition $(*)$ of Corollary 4.10 (with $\hat{z}$ replacing $\hat{x}$). From the first case we can conclude that there exists a maximal ideal $\mathbf{m}$ of $\overline{\mathbf{K}}$ such that $z \notin \mathbf{m}$ and thus $x \notin \mathbf{m}$. ■

If $u \in \mathcal{E}_M(\mathbf{K})$ then the class $cl(u)$ of u depends only on the behavior of u in an arbitrary small interval $\mathbf{I}_\eta (\eta \in \mathbf{I})$. So whenever we write $1/u$ it means that u does not have zeros in a such interval $\mathbf{I}_\eta$, since the values of $1/u$ in $\mathbf{I}_\eta^c$ are irrelevant from the viewpoint of moderation. More precisely, in the above conditions, $1/u$ will denote the function $v : \mathbf{I} \to \mathbf{K}$ defined by $v(\varepsilon) := u(\varepsilon)^{-1} \;\; \forall\, \varepsilon \in \mathbf{I}_\eta$ and $v(\varepsilon)$ arbitrary for $\varepsilon \in \mathbf{I}_\eta^c$. Of course $1/u$ can fail to be moderate (for instance, take $u(\varepsilon) := \exp(-1/\varepsilon) \;\forall\, \varepsilon \in \mathbf{I}_\eta$ and $u(\varepsilon) := 0 \;\forall\, \varepsilon \in \mathbf{I}_\eta^c$ for any $\eta \in \mathbf{I}$).

Theorem 4.13 *For a given $x \in \overline{\mathbf{K}}$ the following are equivalent:*
(i) $x \in \mathrm{Inv}(\overline{\mathbf{K}})$.
(ii) *For every representative $\hat{x}$ of x there exist $a \in \mathbf{R}_+^*$ and $\alpha \in \mathbf{I}$ such that*

$$|\hat{x}(\varepsilon)| \geq \varepsilon^a \quad \forall\, \varepsilon \in \mathbf{I}_\alpha \tag{4.13.1}$$

(iii) $1/\hat{x} \in \mathcal{E}_M(\mathbf{K})$ *for each representative $\hat{x}$ of x.*
(ii') *There are a representative $\hat{x}$ of x, $a \in \mathbf{R}_+^*$ and $\alpha \in \mathbf{I}$ such that* (4.13.1) *holds.*
(iii') *There is a representative $\hat{x}$ of x such that* $1/\hat{x} \in \mathcal{E}(\mathbf{K})$.

Proof. We shall prove the following implications: (i) $\Rightarrow$ (ii) $\Rightarrow$ (ii') $\Rightarrow$ (iii') $\Rightarrow$ (i) and (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iii')$\Rightarrow$ (i). Note that (ii')$\Rightarrow$ (iii') and (ii) $\Rightarrow$ (iii) are trivial since

$$(4.13.1) \Longleftrightarrow \varepsilon^{a+1}|1/\hat{x}(\varepsilon)| \leq \varepsilon \;\forall\, \varepsilon \in \mathbf{I}_\alpha \Longrightarrow 1/\hat{x} \in \mathcal{E}_M(\mathbf{K}).$$

The implication (iii') $\Rightarrow$ (i) follows from $((1/\hat{x}).\hat{x})(\varepsilon) = 1 \ \forall \ \varepsilon \in \mathbf{I}_\alpha$ for some $\alpha \in \mathbf{I}$. So it remains to show that (i) $\Rightarrow$ (ii). Fix an arbitrary representative $\hat{x}$ of x and let us first assume the following additional hypothesis:

$$\text{there exists } \eta \in \mathbf{I} \text{ such that } \hat{x}(\varepsilon) > 0 \text{ for each } \varepsilon \in \mathbf{I}_\eta. \tag{4.13.2}$$

If (ii) would be false then it is clear that we get

$$\forall m \in \mathbf{N}, m \geq p, \ \exists \ \varepsilon_m \in \mathbf{I}_{1/m} \text{ such that } \ 0 < \hat{x}(\varepsilon_m) < \varepsilon_m^m \tag{4.13.3}$$

where $p \in \mathbf{N}$ is fixed such that $p^{-1} < \eta$. Obviously $S := \{\varepsilon_m | m \in \mathbf{N}$ and $m \geq p\} \in \mathcal{S}$, $\hat{x}\hat{\chi}_S \in \mathcal{N}(\mathbf{K})$ and thus $x = \mathrm{cl}(x_*)$ where $x_* := \hat{x} - \hat{x}\hat{\chi}_S$. But this is a contradiction in view of Theorem 2.18(a) since $0 \in \overline{S} \subset \overline{Z}(x_*)$, which proves (ii) in this case. In the general case we apply Theorem 2.18(a) to conclude that $0 \notin \overline{Z}(\hat{x})$ and thus there is $\eta \in \mathbf{I}$ such that $\hat{x}(\varepsilon) \neq 0$ for all $\varepsilon \in \mathbf{I}_\eta$. Since clearly $z := x.\Theta_{\hat{x}} \in \mathrm{Inv}(\overline{\mathbf{K}})$ and $\hat{z} := \hat{x}.\theta_{\hat{x}} = |\hat{x}|$, it follows that $z \in \mathrm{Inv}(\overline{\mathbf{K}})$ has a representative $\hat{z}$ which satisfies (4.13.2) (with $\hat{z}$ replacing $\hat{x}$). So, by the preceding case we can conclude that there exist $a > 0$ and $\alpha \in \mathbf{I}$ such that $\hat{z}(\varepsilon) \geq \varepsilon^a$ for every $\varepsilon \in \mathbf{I}_\alpha$, which shows that $\hat{x}$ satifies (4.13.1) ∎

Notation 4.14 *In what follows the symbol $P(\mathcal{S})$ denotes the set of all subsets of $\mathcal{S}$. If $\mathcal{S}_0 \in P(\mathcal{S})$ and $\mathcal{S}_0 \neq \emptyset$ we will denote by $g(\mathcal{S}_0)$ the ideal of $\overline{\mathbf{K}}$ generated by the set of all χ_S such that $S \in \mathcal{S}_0$, i.e.*

$$g(\mathcal{S}_0) := < \{\chi_S | S \in \mathcal{S}_0\} > .$$

Definition 4.15 *We denote by $P_*(\mathcal{S})$ the set of all $\mathcal{F} \in P(\mathcal{S})$ verifying the two following conditions:*
(I) *For every $S \in \mathcal{S}$, either S or S^c belongs to $\mathcal{F}$ but not both.*
(II) *If $S \in \mathcal{F}$ and $T \in \mathcal{F}$ then $S \cup T \in \mathcal{F}$.*

Lemma 4.16 *If $\mathcal{F} \in P_*(\mathcal{S})$ and $S, T \in \mathcal{F}$ then $\chi_{S \cap T}$ and $\chi_{S \cup T}$ belong to $g(\mathcal{F})$.*

Proof. From the definiton of $g(\mathcal{F})$ it follows that χ_S and χ_T belong to $g(\mathcal{F})$ and from Definition 4.15(II) it is clear that $\chi_{S \cup T} \in g(\mathcal{F})$. Obviously $\chi_{S \cap T} = \chi_S . \chi_T \in g(\mathcal{F})$. ∎

Lemma 4.17 *If $\mathcal{F} \in P_*(\mathcal{S})$ then $g(\mathcal{F})$ is a proper ideal of $\overline{\mathbf{K}}$.*

Proof. Suppose that $1 \in g(\mathcal{F})$, then we can write

$$1 = \sum_{i=1}^{n} a_i \chi_{T_i} \tag{4.17.1}$$

where $a_i \in \overline{\mathbf{K}}$ and $T_i \in \mathcal{F}$ $(1 \leq i \leq n)$. Now $\bigcup_{i=1}^{n} T_i \in \mathcal{F}$ and thus $S := \bigcap_{i=1}^{n} T_i^c \in \mathcal{S}$. Since $\chi_S \cdot \chi_{T_i} = 0$ for each $i = 1, 2, .., n$, from (4.17.1) it follows that

$$\chi_S = \sum_{i=1}^{n} a_i \chi_S \cdot \chi_{T_i} = 0$$

which is absurd since $S \in \mathcal{S}$. (see Proposition 4.2(a)). ■

Remark 4.18 (1) Note that in the proof of Lemma 4.17 we did not use condition (I) in Definition 4.15.
(2) If $\mathcal{F} \in P_*(\mathcal{S})$ then from Lemma 4.17 it follows that:
(a) $S \in \mathcal{F}$ if and only if $S \in \mathcal{S}$ and $\chi_S \in g(\mathcal{F})$
(b) If $F \in \mathcal{F}, S \in \mathcal{S}$ and if there exists $\eta \in \mathbf{I}$ such that $S \cap \mathbf{I}_\eta = F \cap \mathbf{I}_\eta$ then $S \in \mathcal{F}$.
(c) If $S \in \mathcal{S}, F \in \mathcal{F}$ and $S \subset F$ then $S \in \mathcal{F}$.

Theorem 4.19 *For every prime ideal $\mathbf{p}$ of $\overline{\mathbf{K}}$ there exists a unique $\mathcal{F}_\mathbf{p} \in P_*(\mathcal{S})$ such that $g(\mathcal{F}_\mathbf{p}) \subset \mathbf{p}$. In particular, $P_*(\mathcal{S}) \neq \emptyset$.*

Proof. Let $\mathbf{p}$ be an arbitrary prime ideal of $\overline{\mathbf{K}}$. Thus $\mathbf{p}$ is a proper ideal and since $\chi_S + \chi_{S^c} = 1$ for each $S \in \mathcal{S}$, it follows from Proposition 4.3 that

$$\mathcal{F}_\mathbf{p} := \{S \in \mathcal{S} \mid \chi_S \in \mathbf{p}\}$$

satisfies condition (I) of Definition 4.15. Now, for given $S, T \in \mathcal{F}_\mathbf{p}$ we have $\chi_S, \chi_T \in \mathbf{p}$ and thus $\chi_{S \cap T} = \chi_S \cdot \chi_T \in \mathbf{p}$. Therefore $\chi_{S \cup T} = \chi_S + \chi_T - \chi_{S \cap T} \in \mathbf{p}$. So it is enough to show that $S \cup T \in \mathcal{S}$ and clearly $0 \in \overline{S \cup T}$. Hence if we assume that $S \cup T \notin \mathcal{S}$ then there is $\eta \in \mathbf{I}$ such that $\mathbf{I}_\eta \subset S \cup T$ and so $\chi_{S \cup T} = 1$. Thus $1 \in \mathbf{p}$, which is absurd. We then conclude that $\mathcal{F}_\mathbf{p} \in P_*(\mathcal{S})$ and clearly $g(\mathcal{F}_\mathbf{p}) \subset \mathbf{p}$. That $\mathcal{F}_\mathbf{p}$ is unique follows from the facts that χ_S and χ_{S^c} are orthogonal idempotents with sum equal to 1 and that $\mathbf{p}$ is a proper ideal. ■

Theorem 4.19 associates to every prime ideal $\mathbf{p}$ of $\overline{\mathbf{K}}$ a well defined $\mathcal{F}_{\mathbf{p}} \in P_*(\mathcal{S})$ characterized by the inclusion $g(\mathcal{F}_{\mathbf{p}}) \subset \mathbf{p}$. In what follows the symbol $\overline{g(\mathcal{F})}$, where $\mathcal{F} \in P_*(\mathcal{S})$, denotes the τ_s-closure of $g(\mathcal{F})$ in $\overline{\mathbf{K}}$.

Now we can prove the main result of this section. It completely describes the maximal ideals of $\overline{\mathbf{K}}$.

Theorem 4.20 (a) *For every maximal ideal* $\mathbf{m}$ *of* $\overline{\mathbf{K}}$ *we have* $\mathbf{m} = \overline{g(\mathcal{F}_{\mathbf{m}})}$.
(b) *For every* $\mathcal{F} \in P_*(\mathcal{S})$ *the ideal* $\mathbf{m} := \overline{g(\mathcal{F})}$ *is maximal and* $\mathcal{F} = \mathcal{F}_{\mathbf{m}}$.

The proof of Theorem 4.20 will rest on the three following lemmas. For a given $x \in \overline{\mathbf{K}}^*, x \notin \mathrm{Inv}(\overline{\mathbf{K}})$, we know by Theorem 2.18 that there is a representative $\hat{x}$ of x such that $0 \in \overline{Z}(\hat{x})$. In the next result we state some useful properties of these sets $Z(\hat{x})$.

Lemma 4.21 *Let* $x \in \overline{\mathbf{K}}^*$ *be a non-unit. Then*
(a) $Z(\hat{x}) \in \mathcal{S}$ *for each representative* $\hat{x}$ *such that* $0 \in \overline{Z}(\hat{x})$.
(b) *For every representative* $\hat{x}$ *of* x *such that* $0 \in \overline{Z}(\hat{x})$ *there is* $a \in \mathbf{N}^*$ *such that* $N_a(\hat{x}) \in \mathcal{S}$. (see Definition 4.6)

Proof. Fix an arbitrary representative $\hat{x}$ of x such that $0 \in \overline{S}$, where $S := Z(\hat{x})$. The statement (a) is clear since the assumption $0 \notin \overline{S^c}$ implies $x = 0$ which is absurd. In order to prove (b) note that $S \subset T_\nu := N_\nu(\hat{x})$ for all $\nu \in \mathbf{N}^*$, thus from (a) it follows that $0 \in \overline{T_\nu}$ for each $\nu \in \mathbf{N}^*$. From the assumption $0 \notin \overline{T_\nu^c}$ for every $\nu \in \mathbf{N}^*$ it is easily seen that for each $\nu \in \mathbf{N}^*$ there is $\eta(\nu) \in \mathbf{I}$ such that $|\hat{x}(\varepsilon)| < \varepsilon^\nu$ for all $\varepsilon \in \mathbf{I}_{\eta(\nu)}$, which implies $x = 0$, which is again a contradiction. ∎

The next result gives a condition to decide whether $x \in g(\mathcal{F})$ for given $x \in \overline{\mathbf{K}}^*$ and $\mathcal{F} \in P_*(\mathcal{S})$.

Lemma 4.22 *Given* $x \in \overline{\mathbf{K}}^*$ *and* $\mathcal{F} \in P_*(\mathcal{S})$ *the following statements are equivalent:*
(i) $x \in g(\mathcal{F})$.
(ii) *There exists a representative* $\hat{x}$ *of* x *such that* $Z(\hat{x})^c \in \mathcal{F}$ *and thus* $\chi_{Z(\hat{x})^c} = 1 - \chi_{Z(\hat{x})} \in g(\mathcal{F})$.

Proof. (i) $\Longrightarrow$ (ii): From Lemma 4.17 it follows that $x \notin \mathrm{Inv}(\overline{\mathbf{K}})$. Whence, by Theorem 2.18, there exists a representative x_* of x such that $0 \in \overline{Z}(x_*)$. By Lemma 4.21, we have $S := Z(x_*) \in \mathcal{S}$ and hence the result is proved if $S^c \in \mathcal{F}$. So we can assume that $S^c \notin \mathcal{F}$ which implies $S \in \mathcal{F}$. By (i) we can write

$$x = \sum_{i=1}^{t} a_i \chi_{S_i}$$

where $S_i \in \mathcal{F}$ and $a_i \in \overline{\mathbf{K}}$ for all $i = 1, 2, .., t$. Since $x\chi_S = 0$ we get $x = x(1 - \chi_S) = x\chi_{S^c} = \sum_{i=1}^{t} a_i \chi_{S_i \cap S^c}$ and obviously we can assume without loss of generality that

$$\chi_{S_i \cap S^c} \neq 0 \quad \forall\, i = 1, 2, .., t. \tag{4.22.1}$$

Now, we set $U := \cup_{i=1}^{t} S_i \cap S^c$ and $T := S^c \setminus U$. Thus $S_i \cap S^c \cap T = \emptyset$ for each $i = 1, 2, .., t$, which implies $x\chi_T = \sum_{i=1}^{t} a_i \chi_{S_i \cap S^c \cap T} = 0$. Therefore $\hat{x} := x_* - x_* \chi_T$ is a representative of x and $R := Z(\hat{x}) = S \cup T$. Hence it is enough to show that $R^c \in \mathcal{F}$. From (4.22.1) it follows that $0 \in \overline{S_i \cap S^c}$ $(1 \leq i \leq t)$ and since $0 \in \overline{S_i^c} \subset \overline{(S_i \cap S^c)^c}$ we get $S_i \cap S^c \in \mathcal{S} (1 \leq i \leq t)$. This, together with $S_i \in \mathcal{F}, (1 \leq i \leq t)$, implies (see Remark 4.18(c)) $S_i \cap S^c \in \mathcal{F}, (1 \leq i \leq t)$, hence $R^c = U \in \mathcal{F}$.
(ii)$\Longrightarrow$ (i): Let $\hat{x}$ be a representative of x such that $Z(\hat{x})^c \in \mathcal{F}$ and $S := Z(\hat{x})$. From $x\chi_S = 0$ it follows that $x = x - x\chi_S = x\chi_{S^c} \in g(\mathcal{F})$ since $S^c \in \mathcal{F}$. ■

The following approximation lemma is the last result that we need in order to prove our main result.

Lemma 4.23 *Let $x \in \overline{\mathbf{K}}^*$ be a non-unit. Consider the two following statements:*
(a) There exist a representative $\hat{x}$ of x and $a > 0$ such that $S := Z(\hat{x}) \in \mathcal{S}$ and $|\hat{x}(\varepsilon)| \geq \varepsilon^a$ for each $\varepsilon \in S^c$.
(b) For every representative $\hat{x}$ of x such that $0 \in \overline{Z}(\hat{x})$ there exist a sequence $(S_m)_{m \geq 1}$ in $\mathcal{S}$ and a sequence $(a_m)_{m \geq 1}$ in $\mathbf{N}^*$ verifying the following:
(I.) $S_{m+1} \subset S_m$ and $a_{m+1} > a_m$ for each $m \in \mathbf{N}^*$.
(II.) $|\hat{x}(\varepsilon)| < \varepsilon^{a_m}$ $\forall\, \varepsilon \in S_m$ and $|\hat{x}(\varepsilon)| \geq \varepsilon^{a_m}$ $\forall\, \varepsilon \in S_m^c$, whenever $m \in \mathbf{N}^*$
(III.) $x\chi_{S_m} \to o$ for $m \to \infty$.

Then if x does not satisfy (a) *necessarily it satifies* (b).

Proof. Let $\hat{x}$ be a representative of x such that $0 \in \overline{Z}(\hat{x})$. By Lemma 4.21(b) there exists $a_1 \in \mathbf{N}$ such that $S_1 := N_{a_1}(\hat{x}) \in \mathcal{S}$ and

$$|\hat{x}(\varepsilon)| < \varepsilon^{a_1} \;\forall\, \varepsilon \in S_1 \;\text{ and }\; |\hat{x}(\varepsilon)| \geq \varepsilon^{a_1} \;\forall\, \varepsilon \in S_1^c \tag{4.23.1}$$

Set $x_2 := x\chi_{S_1}$ and $\hat{x}_2 := \hat{x}\hat{\chi}_{S_1}$. We shall prove that $x_2 \neq 0$ and that x_2 is not invertible. In fact, notice first that if $x_2 = 0$ then $x_* := \hat{x} - \hat{x}_2$ is a representative of x and $Z(x_*) = S_1$. But then, by (4.23.1), x_* satisfies condition (a), which is a contradition. That x_2 is a zero divisor follows by Theorem 2.18 since $S_1^c \subset Z(\hat{x}_2)$. Hence we may apply Lemma 4.21(b) to x_2. It follows that there exists $a_2 \in \mathbf{N}$ such that $T_2 := N_{a_2}(\hat{x}_2) \in \mathcal{S}$ and

$$|\hat{x}_2(\varepsilon)| < \varepsilon^{a_2} \;\;\forall\, \varepsilon \in T_2 \;\text{ and }\; |\hat{x}_2(\varepsilon)| \geq \varepsilon^{a_2} \;\;\forall\, \varepsilon \in T_2^c. \tag{4.23.2}$$

Let $S_2 := S_1 \cap T_2$. We claim that $S_2 \in \mathcal{S}$ and $a_2 > a_1$. Indeed, the definition of $\hat{x}_2$ and (4.23.2) imply that

$$T_2^c \subset S_1. \tag{4.23.3}$$

Hence it follows that

$$|\hat{x}(\varepsilon)| = |\hat{x}_2(\varepsilon)| \;\;\forall\, \varepsilon \in T_2^c \tag{4.23.4}$$

and thus from (4.23.1), (4.23.3) and (4.23.2) we get

$$\varepsilon^{a_2} \leq |\hat{x}_2(\varepsilon)| = |\hat{x}(\varepsilon)| < \varepsilon^{a_1} \;\;\forall\, \varepsilon \in T_2^c,$$

and therefore $a_2 > a_1$. Since $S_1, T_2 \in \mathcal{S}$ we have $0 \in \overline{S_1^c} \cup \overline{T_2^c} = \overline{S_2^c}$ and hence we must have $S_2 \in \mathcal{S}$. Otherwise we would have $0 \notin \overline{S_2}$ and hence there would exist $\eta \in \mathbf{I}$ such that $S_2 \cap \mathbf{I}_\eta = \emptyset$, which implies, as it is easily seen from (4.23.3) and the definition of S_2, that $|\hat{x}(\varepsilon)| \geq \varepsilon^{a_1} \;\forall\, \varepsilon \in S_1^c$ and $|\hat{x}(\varepsilon)| \geq \varepsilon^{a_2} \;\forall\, \varepsilon \in T_2^c \cap \mathbf{I}_\eta = S_1 \cap \mathbf{I}_\eta$. Thus $|\hat{x}(\varepsilon)| \geq \varepsilon^{a_2} \;\forall\, \varepsilon \in \mathbf{I}_\eta$, i.e., x is a unit which is a contradiction.
Now it is clear that by arguing as well as for $S_m, a_m, (m = 1, 2)$ we can construct inductively the sequences $(S_m)_{\geq 1}$ and $(a_m)_{m \geq 1}$ as in (b) verifying

conditions (I.) and (II.). So it remains to check only that condition (III.) holds. Indeed, from (II.) it follows that for each $m \in \mathbf{N}$ we have $|(\hat{x}\hat{\chi}_{S_m})(\varepsilon)| < \varepsilon^{a_m}$ for every $\varepsilon \in \mathbf{I}$ which implies that $||x\chi_{S_m}|| \leq e^{-a_m} \to 0$ as $m \to \infty$. ■

Proof of Theorem 4.20. (a) Let $\mathbf{m}$ be a maximal ideal of $\overline{\mathbf{K}}$. By Theorem 4.19 there exists $\mathcal{F} = \mathcal{F}_{\mathbf{m}}$ such that $g(\mathcal{F}) \subset \mathbf{m}$. Let $x \in \mathbf{m} \setminus g(\mathcal{F})$. We shall construct a sequence (x_m) in $g(\mathcal{F})$ such that $x_m \to x$ in $\overline{\mathbf{K}}$. We first show that x satifies condition (b) of Lemma 4.23. Indeed, assume that x satisfies condition (a) of Lemma 4.23, then there exist a representative $\hat{x}$ of x and $a > 0$ such that $S := Z(\hat{x}) \in \mathcal{S}$ and $|\hat{x}(\varepsilon)| \geq \varepsilon^a$ for each $\varepsilon \in S^c$. Assuming first that $S \in \mathcal{F}$ it follows that $y := \Theta_{\hat{x}}.x + \chi_S \in \mathbf{m}$, which is absurd because y is a unit, as it is easily seen by applying Theorem 4.13 to the representative $\hat{y} := \theta_{\hat{x}}.\hat{x} + \hat{\chi}_S$. Next assume $S^c \in \mathcal{F}$; in this case it follows that $\chi_{S^c} = 1 - \chi_S \in g(\mathcal{F})$ and since $x\chi_S = 0$ we get $x = x\chi_{S^c} \in g(\mathcal{F})$, another contradiction. We can then conclude that x satisfies the condition (b) of Lemma 4.23. Fix a representative $\hat{x}$ of x such that $0 \in \overline{Z}(\hat{x})$ and consider the two sequences (S_m) and (a_m) associates to $\hat{x}$ as in condition (b) of Lemma 4.23. We will show that $S_m^c \in \mathcal{F}$ for each $m \geq 1$. In fact if this were not the case then there would exists $\nu \geq 1$ such that, as before, $y := \Theta_{\hat{x}}.x.(1 - \chi_{S_\nu}) + \chi_{S_\nu}$ would be a unit belonging to $\mathbf{m}$, a contradiction. Hence it follows that $(x\chi_{S_m^c})_{m\geq 1}$ is a sequence in $g(\mathcal{F})$ and, from Lemma 4.23, we have $x\chi_{S_m^c} = x - x\chi_{S_m} \to x$ as $m \to \infty$.
(b) Note that, by Lemma 4.17, $g(\mathcal{F})$ is a proper ideal of $\overline{\mathbf{K}}$ and therefore it is contained in a maximal ideal $\mathbf{m}$ of $\overline{\mathbf{K}}$. Moreover, Theorem 4.19 implies $\mathcal{F} = \mathcal{F}_{\mathbf{m}}$ and then the conclusion follows at once from (a). ■

We now recollect the facts already shown about maximal and primes ideals of $\overline{\mathbf{K}}$.

Remark 4.24 (a) For each prime ideal $\mathbf{p}$ of $\overline{\mathbf{K}}$ there is one and only one $\mathcal{F}_{\mathbf{p}} \in P_*(\mathcal{S})$ such that $g(\mathcal{F}_{\mathbf{p}}) \subset \mathbf{p}$ (Theorem 4.19).
(b) To each $\mathcal{F} \in P_*(\mathcal{S})$ it is associated the maximal ideal $\mathbf{m} := \overline{g(\mathcal{F})}$ (Theorem 4.20(b)). Moreover, the non-void set determined by any $\mathcal{F} \in P_*(\mathcal{S})$:

$P_{\mathcal{F}} := \{\mathbf{p} |\ \mathbf{p}$ is a prime ideal of $\overline{\mathbf{K}}$ and $g(\mathcal{F}) \subset \mathbf{p}\}$,

has the following properties:
(I.). $\mathbf{p} \subset \mathbf{m}\ \forall\ \mathbf{p} \in P_{\mathcal{F}}$ (where $\mathbf{m} := \overline{g(\mathcal{F})}$);

(II.) $\mathcal{F} = \mathcal{F}_{\mathrm{p}}\ \forall\ \mathrm{p} \in P_{\mathcal{F}}$ (in particular, $\mathcal{F} = \mathcal{F}_{\mathrm{m}}$);
(III.) If $\mathbf{q}$ is a minimal prime ideal contained in $\mathbf{m}$, then $\mathbf{q} \in P_{\mathcal{F}}$ and thus $g(\mathcal{F}) \subset \mathbf{q} \subset \mathbf{m}$ and $\mathcal{F}_{\mathrm{q}} = \mathcal{F}$.
(c) If $\mathbf{p}$ is any prime ideal then given any $\mathcal{F} \in P_*(\mathcal{S}), \mathbf{p}$ is never properly contained in $g(\mathcal{F})$.

Indeed, only the second statement of (b) and (c) require some justification, being clear that $P_{\mathcal{F}} \neq \emptyset$ since $\mathbf{m} \in P_{\mathcal{F}}$. If $\mathbf{p} \in P_{\mathcal{F}}$ and $\mathbf{p} \not\subset \mathbf{m}$, since $\mathbf{p}$ is contained in a maximal ideal $\mathbf{n}$ we must have $\mathbf{m} \neq \mathbf{n}$. But from $\mathbf{n} \supset \mathbf{p} \supset g(\mathcal{F})$ it follows that $\mathbf{m}$ and $\mathbf{n}$ are prime ideals containing $g(\mathcal{F})$. Hence, by (a), we have $\mathcal{F} = \mathcal{F}_{\mathrm{m}} = \mathcal{F}_{\mathrm{n}}$ and then, by Theorem 4.20(a), we get $\mathbf{m} = \mathbf{n}$, which is absurd and proves (I.). That (II.) holds follows from the unicity in (a). Let $\mathbf{q}$ be as in (III.). From (a) it follows that $g(\mathcal{F}') \subset \mathbf{q}$ where $\mathcal{F}' = \mathcal{F}_{\mathrm{q}}$, hence $\mathbf{q} \in P_{\mathcal{F}'}$. Thus, by (b) we get $\mathbf{q} \subset \mathbf{n} := \overline{g(\mathcal{F}')}$ and $\mathbf{n}$ is maximal. Since $\mathbf{q} \subset \mathbf{m} \cap \mathbf{n}, \mathbf{m}$ and $\mathbf{n}$ are prime and contain $g(\mathcal{F}')$ we can conclude that $\mathcal{F}' = \mathcal{F}_{\mathrm{m}} = \mathcal{F}_{\mathrm{n}} = \mathcal{F}$ and therefore $g(\mathcal{F}) \subset \mathbf{q}$. Thus $\mathbf{q} \in P_{\mathcal{F}}$. We will show that (c) holds. Assume that $\mathbf{p}$ is a prime ideal, that $\mathcal{F} \in P_*(\mathcal{S})$ and that $g(\mathcal{F})$ contains properly $\mathbf{p}$. For a given $S \in \mathcal{F}$, since χ_S and χ_{S^c} are orthogonal idempotents with sum equals to 1, it follows that $\chi_{S^c} \notin \mathbf{p}$, since otherwise we would have $1 = \chi_S + \chi_{S^c} \in g(\mathcal{F})$, which is absurd by Lemma 4.17. So $\chi_S \in \mathbf{p}$ for each $S \in \mathcal{F}$ and thus $g(\mathcal{F}) \subset \mathbf{p}$, which is a contradiction.

From the point of view of this work there are several interesting questions about prime and maximal ideals of $\overline{\mathbf{K}}$. We list some of them: (1) $g(\mathcal{F}) \neq \overline{g(\mathcal{F})}$ for each (or for some) $\mathcal{F} \in P_*(\mathcal{S})$? (2) $g(\mathcal{F})$ is a prime ideal for each (or for some) $\mathcal{F} \in P_*(\mathcal{S})$? (3) Are there prime ideals in $\overline{\mathbf{K}}$ others than the maximal ones? (4) $g(\mathcal{F}_1)$ and $g(\mathcal{F}_2)$ are isomorphic whenever $\mathcal{F}_1$ and $\mathcal{F}_2$ belong to $P_*(\mathcal{S})$? (5) What can we say about the Krull dimension of $\overline{\mathbf{K}}$?

Another interesting consequence of the approximation Lemma 4.23 is the following:

Theorem 4.25 Inv($\overline{\mathbf{K}}$) *is an open dense subset of* $\overline{\mathbf{K}}$.

Proof. By Corollary 2.10 it is enough to show that Inv($\overline{\mathbf{K}}$) is a dense subset of $\overline{\mathbf{K}}$. To this end we fix any $x \in \overline{\mathbf{K}} \setminus \mathrm{Inv}(\overline{\mathbf{K}})$ and construct a sequence $(x_{m)m\geq 1}$ in Inv($\overline{\mathbf{K}}$) which is τ_s-convergent to x. Assume first that x satisfies the condition (a) of Lemma 4.23 then, with this notation, we set $x_m := x(1-\chi_S) + \alpha_m\chi_S = x + \alpha_m\chi_S (m \geq 1)$. From Theorem 4.13 it follows at once that $x_m \in \mathrm{Inv}(\overline{\mathbf{K}})$ for each $m \geq 1$ and it is clear that $x_m \to x$ since $||x_m - x|| = ||\alpha_m\chi_S|| = ||\alpha_m|| = e^{-m} \to 0$ as $m \to \infty$. Next we can assume that x satisfies the condition (b) of Lemma 4.23. So, for a fixed representative $\hat{x}$ of x such that $0 \in \overline{Z}(\hat{x})$ we can consider the sequences $(S_m)_{m\geq 1}$ and $(a_m)_{m\geq 1}$ associated to $\hat{x}$ as in Lemma 4.23(b). From the proof of this result we get

$$|(\hat{x}\chi_{S_m^c}(\varepsilon)| = |\hat{x}(\varepsilon)| \geq \varepsilon^{a_m} \ \forall \ \varepsilon \in S_m^c \ \ (m \geq 1). \qquad (4.25.1)$$

We define $x_m := x(1-\chi_{S_m}) + \alpha_m\chi_{S_m}$ $(m \geq 1)$ and its representative $\hat{x}_m := \hat{x}(1-\hat{\chi}_{S_m}) + \hat{\alpha}_m\hat{\chi}_{S_m}$. Hence $x_m \to x$ as $m \to \infty$ since $||x_m - x|| = ||(\alpha_m - x)\chi_{S_m}|| \leq \max\{||\alpha_m\chi_{S_m}||, ||x\chi_{S_m}||\} = \max\{||\alpha_m||, ||x\chi_{S_m}||\} = \max\{e^{-m}, ||x\chi_{S_m}||\} \to 0$ as $m \to \infty$. Moreover, for each $m \geq 1$, the inequalities (4.25.1) together with Theorem 4.13 show that $x_m \in \mathrm{Inv}(\overline{\mathbf{K}})$ for each $m \geq 1$. ∎

Now let us recall some algebraic terminology before the next results. Let K be a field and L an extension of K. An element $\alpha \in L$ is said to be *algebraic on* K if there exists $p \in K[x], p \neq 0$ such that $p(\alpha) = 0$. We denote by $A_L(K)$ the set of all $\alpha \in L$ which are algebraic on K. K is said to be *algebraically closed in* L if $A_L(K) = K$. L is said to be *a transcendent* (resp. *an algebraic*) *extension of* K if $L \neq A_L(K)$ (resp. $L = A_L(K)$)

Let $\mathbf{m}$ be a maximal ideal of $\overline{\mathbf{K}}$. From Theorem 2.13 we know that $\mathbf{K}$ can be identified to a subfield of $\overline{\mathbf{K}}/\mathbf{m}$ and we have the following result:

Proposition 4.26 $\mathbf{K}$ *is algebraically closed in* $\overline{\mathbf{K}}/\mathbf{m}$ *for every maximal ideal* $\mathbf{m}$ *of* $\overline{\mathbf{K}}$.

Proof. Firstly assume that $\mathbf{K} = \mathbf{C}$ and fix $\alpha \in \overline{\mathbf{C}}/\mathbf{m}$ which is algebraic over $\mathbf{C}$. Then $\mathbf{C}(\alpha)$ is an algebraic extension of $\mathbf{C}$ which is algebraically closed. Thus $\mathbf{C}(\alpha) = \mathbf{C}$ and therefore $\alpha \in \mathbf{C}$.

Secondly assume that $\mathbf{K} = \mathbf{R}$ and let $\alpha \in \overline{\mathbf{R}}/\mathbf{m}$ be algebraic over $\mathbf{R}$. If $\alpha \notin \mathbf{R}$, it follows that $\mathbf{R}(\alpha) \simeq \mathbf{C}$ since $\mathbf{C}$ is, up to isomorphism, the unique algebraic extension of $\mathbf{R}$. Hence $i := \sqrt{-1} \in \overline{\mathbf{R}}/\mathbf{m}$. Thus there exists $x \in \overline{\mathbf{R}}$

such that $\mu := x^2 + 1 \in \mathbf{m}$. Since μ is a zero divisor, from Theorem 2.18, it follows that μ has a representative $\hat{\mu}$ such that $0 \in \overline{Z(\hat{\mu})}$. Hence there is a sequence $(\varepsilon_n)_{n\geq 1}$ in $\mathbf{I}$ such that $\hat{\mu}(\varepsilon_n) = 0$ for all $n \geq 1$ and $\varepsilon_n \to 0$ as $n \to \infty$. Let $\hat{x}$ be any representative of x so we can write

$$\hat{x}(\varepsilon)^2 + 1 = \hat{\mu}(\varepsilon) + \hat{\omega}(\varepsilon), \qquad (\varepsilon \in \mathbf{I}),$$

where $\hat{\omega} \in \mathcal{N}(\mathbf{R})$. But, for $\varepsilon = \varepsilon_n (n \geq 1)$, the above identity leads to a contradiction since $\hat{x}(\varepsilon) \in \mathbf{R}$ for all $\varepsilon \in \mathbf{I}$ and $\hat{\omega}(\varepsilon_n) \to 0$ as $n \to \infty$. ■

Proposition 4.27 (a) *If* $\mathcal{F}_1, \mathcal{F}_2 \in P_*(S)$ *and* $\mathcal{F}_1 \neq \mathcal{F}_2$ *then* $g(\mathcal{F}_1) + g(\mathcal{F}_2) = \overline{\mathbf{K}}$.
(b) *If* $\mathcal{F} \in P_*(S)$ *then* $\overline{\mathbf{K}}/g(\mathcal{F})$ *is a local ring whose maximal ideal is* $\overline{g(\mathcal{F})}/g(\mathcal{F})$.
(c) *The field* $\overline{\mathbf{K}}/\overline{g(\mathcal{F})}$ *has characteristic* 0 *for every* $\mathcal{F} \in P_*(S)$.

Proof. (a) There is $S \in \mathcal{F}_1$ such that $S \notin \mathcal{F}_2$. Thus $S^c \in \mathcal{F}_2$ and hence $1 = \chi_S + \chi_{S^c} \in g(\mathcal{F}_1) + g(\mathcal{F}_2)$.
(b) Set $B := \overline{\mathbf{K}}/g(\mathcal{F})$ and denote by $\varphi : \overline{\mathbf{K}} \to B$ the quotient map. Let $\mathbf{m}_0$ be a maximal ideal of B. Then $\mathbf{m} := \varphi^{-1}(\mathbf{m}_0)$ is a maximal ideal of $\overline{\mathbf{K}}$ containing $g(\mathcal{F})$. Hence, from theorem 4.20 it follows that $\mathcal{F} = \mathcal{F}_{\mathbf{m}}$ and $\mathbf{m} = \overline{g(\mathcal{F})}$. Thus

$$\mathbf{m}_0 = \varphi(\mathbf{m}) = \varphi(\overline{g(\mathcal{F})}) = \overline{g(\mathcal{F})}/g(\mathcal{F})$$

is the unique maximal ideal of B.
(c) Fix any $\mathcal{F} \in P_*(S)$. From Theorem 4.20 we know that $\mathbf{m} := \overline{g(\mathcal{F})}$ is a maximal ideal of $\overline{\mathbf{K}}$. Hence from Theorem 2.13 we can conclude that $\overline{\mathbf{K}}/\mathbf{m}$ is an extension of $\mathbf{K}$. Therefore the statement follows at once from the fact that $\mathbf{K}$ has characteristic 0. ■

Theorem 4.28 *The set of maximal ideals of* $\overline{\mathbf{K}}$ *is uncountable*

Proof. Let $B = (x_\lambda)_{\lambda\in\Lambda}$ be a transcendence basis of $\mathbf{R}$ over $\mathbf{Q}$. Then Λ is an uncountable set and B is also a transcendence basis of $\mathbf{C}$ over $\mathbf{Q}$. For each $\lambda \in \Lambda$ there is $p(\lambda) \in \mathbf{N}^*$ such that $\dfrac{|x_\lambda|}{p(\lambda)} \leq 1$. Hence

$$\sigma^\lambda := \left(\frac{|x_\lambda|}{m}\right)_{m\geq p(\lambda)}$$

is a strictly decreasing sequence in $\mathbf{I}$ which converges to zero.
It follows that

$$\sigma_*^\lambda := \{\frac{|x_\lambda|}{m} \,|m \geq p(\lambda)\} \in \mathcal{S}$$

and moreover

$$\lambda, \mu \in \Lambda \text{ and } \lambda \neq \mu \Rightarrow \sigma_*^\lambda \cap \sigma_*^\mu = \emptyset. \tag{4.28.1}$$

Indeed, if we assume that (4.28.1) is false then there would exist $m \geq p(\lambda)$ and $n \geq p(\mu)$ such that

$$\frac{|x_\lambda|}{m} = \frac{|x_\mu|}{n},$$

and hence $x_\lambda = \pm \frac{m}{n} x_\mu$, which is absurd. For each $\lambda \in \Lambda$ we set

$$S_\lambda := (\sigma_*^\lambda)^c \text{ and } \chi_\lambda := \chi_{S_\lambda}.$$

From (4.28.1) it follows that

$$\lambda, \mu \in \Lambda \text{ and } \lambda \neq \mu \Rightarrow \chi_\lambda + \chi_\mu \in \mathrm{Inv}(\overline{\mathbf{K}}) \tag{4.28.2}$$

since $\lambda \neq \mu$ implies $S_\lambda \cup S_\mu = \mathbf{I}$ and hence $(\hat{\chi}_\lambda + \hat{\chi}_\mu)(\varepsilon) \geq 1$ for each $\varepsilon \in \mathbf{I}$. Since $\chi_\lambda \notin \mathrm{Inv}(\overline{\mathbf{K}})$ for every $\lambda \in \Lambda$, there is a maximal ideal $\mathbf{m}_\lambda$ of $\overline{\mathbf{K}}$ such that $\chi_\lambda \in \mathbf{m}_\lambda$ and from Theorem 4.19 it follows that there exists $\mathcal{F}_\lambda \in P_*(\mathcal{S})$ such that $g(\mathcal{F}_\lambda) \subset \mathbf{m}_\lambda$, hence $\chi_\lambda \in \mathbf{m}_\lambda = \overline{g(\mathcal{F}_\lambda)} (\lambda \in \Lambda)$. As a consequence, from the definition of $\mathcal{F}_\lambda$ (see the proof of Theorem 4.19) we get

$$S_\lambda \in \mathcal{F}_\lambda \quad \forall \ \lambda \in \Lambda. \tag{4.28.3}$$

Now it is clear from (4.28.2) and (4.28.3) that the map

$$\lambda \in \Lambda \longmapsto \mathcal{F}_\lambda \in P_*(\mathcal{S})$$

is injective and hence, the map

$$\lambda \in \Lambda \longmapsto \overline{g(\mathcal{F}_\lambda)} \in M,$$

where M denotes the set of all maximal ideals of $\overline{\mathbf{K}}$, is injective. In fact, if $\lambda, \mu \in \Lambda, \lambda \neq \mu$ and $\mathbf{m} := \overline{g(\mathcal{F}_\lambda)} = \overline{g(\mathcal{F}_\mu)}$, we would have

$$g(\mathcal{F}_\lambda) \subset \mathbf{m}, \ g(\mathcal{F}_\mu) \subset \mathbf{m} \text{ and } \mathcal{F}_\lambda \neq \mathcal{F}_\mu,$$

which is a contradiction in view of the uniqueness in Theorem 4.19. ■

We finish giving an application of Proposition 4.26. First we need some notation. For a commutative ring A with identity let $\mathcal{B}(A)$ denote the set of all idempotents of A, i.e., $\mathcal{B}(A) = \{e \in A | e^2 = e\}$.

Proposition 4.29 $\mathcal{B}(\overline{\mathbf{C}}) = \mathcal{B}(\overline{\mathbf{R}})$.

Proof. Since $\overline{\mathbf{R}} \subset \overline{\mathbf{C}}$ it is enough to prove that $\mathcal{B}(\overline{\mathbf{C}}) \subset \mathcal{B}(\overline{\mathbf{R}})$ To this end we fix $e \in \mathcal{B}(\overline{\mathbf{C}})$ and let $\hat{e} = \hat{a} + i\hat{b}$ denote a representative of e, where $\hat{a}$ and $\hat{b}$ denote the real and imaginary part of $\hat{e}$. Since $e^2 = e$, there is $\hat{u} = u_1 + iu_2 \in \mathcal{N}(\mathbf{C})$ such that $\hat{e}^2 = \hat{e} + \hat{u}$ and therefore we can write

$$\hat{e}^2 = \hat{a}^2 - \hat{b}^2 + 2\hat{a}\hat{b}i \text{ and } \hat{e} + \hat{u} = (\hat{a} + u_1) + i(\hat{b} + u_2).$$

It follows that $\hat{a}^2 - \hat{b}^2 = \hat{a} + u_1$ and $2\hat{a}\hat{b} = b + u_2$ and hence, if $a = cl(\hat{a}) \in \overline{\mathbf{R}}$, $b = cl(\hat{b}) \in \overline{\mathbf{R}}$, from the identities above we get in $\overline{\mathbf{R}}$:

$$a^2 - b^2 = a \quad \text{and} \quad 2ab = b.$$

Therefore $b = 2ab = 2a(2ab) = 4a^2b = 4(a+b^2)b = 4ab+4b^3 = 2(2ab)+4b^3 = 2b + 4b^3$, whence

$$b(4b^2 + 1) = 0.$$

Assume $b \neq 0$. Since the nilradical of $\overline{\mathbf{R}}$ vanishes we have that $b^2 \neq 0$, and since the Jacobson radical also vanishes there exists a maximal ideal $\mathbf{m}$ such that $b^2 \notin \mathbf{m}$. Consider the canonical map $\pi : x \in \overline{\mathbf{R}} \longrightarrow \overline{x} \in \overline{\mathbf{R}}/\mathbf{m}$. Identifying $\mathbf{R}$ with its image in $\overline{\mathbf{R}}/\mathbf{m}$ (see Theorem 2.13) we conclude that $4\overline{b}^2 + 1 = 0$. Hence $\overline{b}$ is algebraic over $\mathbf{R}$. But, by Proposition 4.26, the latter is algebraicly closed in $\overline{\mathbf{R}}/\mathbf{m}$ and hence $\overline{b} \in \mathbf{R}$, a contradiction since the polinomial $p(x) = 4x^2 + 1$ is irreducible over $\mathbf{R}$. So we must have that $b = 0$ and the result is proved. ■

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